ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES

778

TREATISES ON THEORETICAL PHYSICS

Published under the direction of

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Member of the Institute
Nobel Laureate

XXVII

CONTRIBUTION TO L. DE BROGLIE'S THEORY OF LIGHT

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PARIS
HERMANN AND CO., EDITORS
6 Rue de la Sorbonne, 6
—
1939

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INTRODUCTION

The new conception of light that has been developed by L. de Broglie in some recent articles is based upon the general principles of wave mechanics. In it, the state of the photon is represented by a set of 16 wave functions Φ_{ik} ; some operators are associated with classical physical quantities. In addition, L. de Broglie has defined some electromagnetic fields and potentials that satisfy Maxwell's equations *in vacuo*. Those tensors are attached to the transition of the photon from the state Φ to the "annihilated" state Φ^0. In the same fashion, he obtained two invariants and a non-Maxwellian vector, and it will obviously be important to exhibit their physical significance.

In total, two invariants, two vectors, and one anti-symmetric tensor of rank two are expressed linearly as functions of Φ_{ik}. Inversely, it is possible to express the Φ_{ik} as linear functions of the components of those tensors. That remark permits us to write the mean-value densities that are encountered in the aforementioned wave mechanics of the photon by means of those tensors and to compare those densities to the corresponding quantities in Maxwell's theory of electromagnetism. One characteristic fact is: The mean densities depend upon not only Maxwell's electromagnetic fields, but also upon some potential functions and other derivatives.

The study of electric dipole and quadrupole waves that is made in the third chapter exhibits the importance of the choice of potential in the definition of dipolar and quadrupolar photons, at least when one would like to introduce classical Maxwellian formulas such as the Abraham-Sommerfeld relation for the dipole wave. We calculate both the flux of moment of impulse and energy at infinity for quadrupole emission; the relations that will be obtained will account for the conservation of moment of impulse (about the z-axis). Finally, in the case of monochromatic circular waves, one sees very simply that the moment of impulse (normal to the wave) for the photon is equal to $\pm h / 2\pi$.

I have carried out this work in the capacity of a recipient of the Fonds National belge de la Recherche Scientifique. The administrative council of the Solvay Institute of Physics has afforded me several subsidies that have permitted me to travel to Paris to study.

I would like to thank those two institutions.

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FIRST CHAPTER

PHOTON EQUATIONS *IN VACUO*

THE FUNDAMENTAL EQUATIONS (BRIEF SUMMARY)

According to L. De Broglie ([1]), the fundamental equations for the photon *in vacuo* are:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} = \left(A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + \chi \mu_0 c\, A_4 \right) \Phi\,, \tag{I}$$

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} = \left(B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z} + \chi \mu_0 c\, B_4 \right) \Phi\,. \tag{II}$$

That system of 32 equations is equivalent to the system that is composed of the 16 equations of evolution:

$$\frac{1}{\chi}\frac{\partial \Phi}{\partial t} = (H^{(a)} + H^{(b)})\, \Phi \tag{III}$$

and the 16 conditional equations:

$$0 = (H^{(a)} - H^{(b)})\, \Phi, \tag{IV}$$

with the notations:

$$\begin{aligned} H^{(a)} &\equiv \frac{c}{2\chi}\left(A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + \chi \mu_0 c\, A_4 \right) \\ H^{(b)} &\equiv \frac{c}{2\chi}\left(B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z} + \chi \mu_0 c\, B_4 \right). \end{aligned} \tag{1}$$

The operator $H^{(a)} + H^{(b)}$ is the Hamiltonian operator. These 32 first-order partial differential equations in the 16 unknown functions Φ_{ik} (i, k = 1, …, 4) are compatible. They are invariant under any Lorentz transformation, since the Φ_{ik} vary like the products $\psi_i \psi_k^*$, in which the ψ_i are Dirac wave functions.

([1]) Louis de Broglie, *Nouvelles recherches sur la lumiere,* Actualités scientifiques et industrielles. Exposés de physique théorique, Fasc. XX; pp. 6 and 7.

We shall adopt the notations that were used in that work almost completely. That will dispense with the necessity of redefining the matrices *A* and *B*. Recall that the coordinate trihedron is tri-rectangular and that $\chi = 2i\pi / h$.

TENSORIAL QUANTITIES THAT ARE ASSOCIATED WITH THE ANNIHILATION OF THE PHOTON

The annihilated state of the photon is represented by the invariant wave function:

$$\Phi_{ik}^0 = (\alpha_4)_{ik} . \tag{2}$$

The functions (2) satisfy the system (III), but not the system (IV), at least when μ_0 is non-zero. We shall return to that point at the beginning of the following chapter. L. de Broglie defined five tensorial quantities that are linked with the passage from the photon from the state Φ to the annihilated state Φ^0. The five tensors are ([1]), with the usual notations of tensor calculus:

An invariant:

$$I = K\mu_0\, c\, I_{(1)} = -\frac{2\pi}{h} K\mu_0 c\,\Phi^0\,\frac{A_4B_4 + B_4A_4}{2}, \tag{3}$$

a space-time vector with contravariant components:

$$P^1 = -\,K\,\Phi^0\,\frac{A_4B_1 - B_4A_1}{2}\,\Phi, \;\ldots, \qquad P^4 = K\,\Phi^0\,\frac{A_4 - B_4}{2}\,\Phi, \tag{4}$$

A second-rank antisymmetric tensor with contravariant components:

$$\left\{\begin{aligned} U^{14} &= -\frac{2\pi}{h}K\mu_0 c\,\Phi^0\,\frac{A_4 iB_1B_4 - B_4 iA_1A_4}{2}, \\ U^{23} &= -\frac{2\pi}{h}K\mu_0 c\,\Phi^0\,\frac{A_4 s_1^{(B)}B_4 - B_4 s_1^{(A)}A_4}{2}, \end{aligned}\right. \tag{5}$$

A completely-antisymmetric tensor of rank three – or, what amounts to the same thing – a vector with covariant components:

$$Q^1 = K\,\Phi^0\,\frac{A_4 s_1^{(B)} - B_4 s_1^{(A)}}{2}\,\Phi, \;\ldots, \qquad Q^4 = K\,\Phi^0\,\frac{A_4 s_4^{(B)} - B_4 s_4^{(A)}}{2}\,\Phi, \tag{6}$$

A completely-antisymmetric tensor of rank four – or, what amounts to the same thing – an invariant:

$$J = K\mu_0 c\, I_{(2)} = -\frac{2\pi}{h}K\mu_0 c\,\Phi^0\,\frac{A_4B_1B_2B_3B_4 + B_4A_1A_2A_3A_4}{2}\,\Phi . \tag{7}$$

([1]) *Loc. cit.*, pp. 43.

Here, the square of the elementary interval is $ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2$, and the variance is intended to mean with respect to any Lorentz transformation; we likewise utilize the notation $x \equiv x^1$, $y \equiv x^2$, $z \equiv x^3$, $ct \equiv x^4$.

In the next paragraph, one will see that the tensor (5) satisfies Maxwell's equations, and that in addition:

$$U_{ik} = \frac{\partial P_i}{\partial x_k} - \frac{\partial P_k}{\partial x_i}. \tag{8}$$

Those properties lead one to consider (5) and (4) to be the components of the electromagnetic field and potential, respectively. Meanwhile, one must not lose sight of the fact that those fields and potentials will generally be complex, here. Upon denoting the vector and scalar potentials by **A** and V, resp., and the electric and magnetic fields by **E** and **H**, resp., one will have:

$$\begin{aligned} &P^1 = A_x, \quad \ldots, \quad P^4 = V, \\ &U^{14} = -F_x, \quad \ldots, \quad U^{23} = H_x, \quad \ldots \end{aligned} \tag{9}$$

The two invariants (3) and (7) and the vector (6) are non-Maxwellian quantities whose sense remains to be specified.

The explicit form of the tensors (3) and (7) as functions of the Φ_{ik}, as well as their variance, is obtained easily upon remarking that:

$$I_{(1)} = \sum_{iklm} \delta_{ik} \left(\frac{A_4 - B_4}{2}\right)_{ik,lm} \Phi_{lm},$$

$$\mathbf{A} = -K \sum_{iklm} \delta_{ik} \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)_{ik,lm} \Phi_{lm}, \qquad V = K \sum_{iklm} \delta_{ik} \left(\frac{1+1}{2}\right)_{ik,lm},$$

etc., …, with:

$$\delta_{ik} = \begin{cases} 0 & \text{if} \quad i \neq k, \\ 1 & \text{if} \quad i = k. \end{cases}$$

One must then have, for example:

$$I_{(1)} = \sum_{kl} (\alpha_4)_{kl} \Phi_{lk},$$

$$A_x = -K \sum_{kl} (\alpha_1)_{kl} \Phi_{lk}, \qquad A_y = -K \sum_{kl} (\alpha_2)_{kl} \Phi_{lk}, \qquad \ldots,$$

in which α_4, α_1, α_2, … are Dirac matrices. One will then explicitly find the invariant ([1]):

$$I = \frac{2\pi}{h} K\mu_0\, c\, I_{(1)} = \frac{2\pi}{h} K\mu_0\, c\, (-\Phi_{11} - \Phi_{22} + \Phi_{33} + \Phi_{44}), \tag{10}$$

the electromagnetic potential:

$$\begin{aligned}
A_x &= -K\ (\Phi_{41} + \Phi_{32} + \Phi_{23} + \Phi_{14}),\\
A_y &= -Ki\ (\Phi_{41} - \Phi_{32} + \Phi_{23} - \Phi_{14}),\\
A_z &= -K\ (\Phi_{41} + \Phi_{32} + \Phi_{23} + \Phi_{14}),\\
V &= K\ (\Phi_{11} + \Phi_{22} + \Phi_{33} + \Phi_{44}),
\end{aligned} \tag{11}$$

the electric and magnetic fields:

$$\begin{aligned}
E_x &= -\frac{2\pi}{h} K\, \mu_0\, ci\, (-\Phi_{41} - \Phi_{32} + \Phi_{23} + \Phi_{14}),\\
E_y &= +\frac{2\pi}{h} K\, \mu_0\, c\ (-\Phi_{41} + \Phi_{32} + \Phi_{23} - \Phi_{14}),\\
E_z &= -\frac{2\pi}{h} K\, \mu_0\, ci\, (-\Phi_{31} + \Phi_{42} + \Phi_{12} - \Phi_{24}),\\
H_x &= -\frac{2\pi}{h} K\, \mu_0\, c\, (-\Phi_{21} - \Phi_{12} + \Phi_{43} + \Phi_{34}),\\
H_y &= +\frac{2\pi}{h} K\, \mu_0\, ci\, (-\Phi_{21} + \Phi_{12} + \Phi_{43} - \Phi_{34}),\\
H_z &= -\frac{2\pi}{h} K\, \mu_0\, c\ (-\Phi_{11} + \Phi_{22} + \Phi_{33} - \Phi_{44}),
\end{aligned} \tag{12}$$

the vector:

$$\begin{aligned}
Q_x &= K\,\sigma_x = Q_1 = K\,(\Phi_{21} + \Phi_{12} + \Phi_{43} + \Phi_{34}),\\
Q_y &= K\,\sigma_y = Q_2 = K\,(\Phi_{21} - \Phi_{12} + \Phi_{43} - \Phi_{34}),\\
Q_z &= K\,\sigma_z = Q_3 = K\,(\Phi_{11} - \Phi_{22} + \Phi_{33} - \Phi_{44}),\\
Q_4 &= K\,\sigma_4 = K\,(\Phi_{31} + \Phi_{42} + \Phi_{13} + \Phi_{24}),
\end{aligned} \tag{13}$$

the invariant:

$$J = \frac{2\pi}{h} K\, \mu_0\, c\, I_{(2)} = \frac{2\pi}{h} K\, \mu_0\, ci\, (-\Phi_{11} + \Phi_{22} + \Phi_{33} - \Phi_{44}). \tag{14}$$

The sixteen linear combinations of the sixteen Φ_{ik} are linearly-independent ([2]). One can then solve them inversely and express the sixteen Φ_{ik} as functions of the sixteen components of the tensors (10) and (13). Here is the result of that solution:

([1]) *Loc. cit.*, pp. 43, form. 89. Our notations are somewhat different from those of L. DE BROGLIE. We point out that there are some printing errors in formulas (89); one can easily correct them with the aid of our expressions (12).

([2]) The linear combinations that are obtained by changing the signs of the terms of type (*B*) in (3), …, (7) will give zero identically.

$$
\begin{aligned}
4\,\Phi_{11} &= \frac{1}{K}\left(V + \frac{h}{2\pi\mu_0 c} H_z\right) - I_1 + \sigma_z\,, \\
4\,\Phi_{22} &= \frac{1}{K}\left(V - \frac{h}{2\pi\mu_0 c} H_z\right) - I_1 - \sigma_z\,, \\
4\,\Phi_{33} &= \frac{1}{K}\left(V - \frac{h}{2\pi\mu_0 c} H_z\right) + I_1 + \sigma_z\,, \\
4\,\Phi_{44} &= \frac{1}{K}\left(V + \frac{h}{2\pi\mu_0 c} H_z\right) + I_1 - \sigma_z\,, \\[2ex]
4\,\Phi_{13} &= \frac{1}{K}\left(-A_z + i\frac{h}{2\pi\mu_0 c} E_z\right) - iI_2 + \sigma_4\,, \\
4\,\Phi_{31} &= \frac{1}{K}\left(-A_z - i\frac{h}{2\pi\mu_0 c} E_z\right) + iI_2 + \sigma_4\,, \\
4\,\Phi_{24} &= \frac{1}{K}\left(A_z - i\frac{h}{2\pi\mu_0 c} E_z\right) + iI_2 + \sigma_4\,, \\
4\,\Phi_{42} &= \frac{1}{K}\left(A_z + i\frac{h}{2\pi\mu_0 c} E_z\right) + iI_2 + \sigma_4\,, \\[2ex]
4\,\Phi_{12} &= \frac{h}{2\pi K\mu_0 c}(H_x + iH_y) + \sigma_x + i\sigma_y\,, \\
4\,\Phi_{14} &= \frac{h}{2\pi K\mu_0 c}\, i\,(E_x + iE_y) + \frac{1}{K}(A_x + iA_y), \\
4\,\Phi_{32} &= \frac{h}{2\pi K\mu_0 c}(-E_x - iE_y) - \frac{1}{K}(A_x + iA_y), \\
4\,\Phi_{34} &= \frac{h}{2\pi K\mu_0 c}(-H_x - iH_y) + \sigma_x + i\sigma_y\,, \\[2ex]
4\,\Phi_{21} &= \frac{h}{2\pi K\mu_0 c}(H_x - iH_y) + \sigma_x - i\sigma_y\,, \\
4\,\Phi_{23} &= \frac{h}{2\pi K\mu_0 c}(E_x - iE_y) - \frac{1}{K}(A_x - iA_y), \\
4\,\Phi_{41} &= \frac{h}{2\pi K\mu_0 c}(-E_x + iE_y) - \frac{1}{K}(A_x - iA_y), \\
4\,\Phi_{43} &= \frac{h}{2\pi K\mu_0 c}(-H_x + iH_y) + \sigma_x - i\sigma_y\,.
\end{aligned}
\tag{15}
$$

The formulas permit one to replace the wave functions with the components of the five tensorial quantities (3), …, (7) in any expression in the theory of the photon.

TENSORIAL FORM FOR THE PHOTON EQUATIONS

The system (I), (II) can be replaced with an equivalent system of 32 equations that are obtained by forming 32 linearly-independent linear combinations of equations (I), (II). Those combinations will obviously be chosen in such a fashion that they refer to only the 16 tensorial quantities (3), …, (7) as unknown functions. They split into two distinct groups: A first group of 16 relations of Maxwellian form in which only the (complex) electromagnetic field and potential occur and a second group of 16 equations in which only non-Maxwellian quantities ([1]) occur.

In order to indicate the combinations in question in a simple fashion, we introduce the notation:

$$(I)_{ik} \equiv \frac{1}{c}\frac{\partial \Phi_{ik}}{\partial t} - \left(A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + \kappa\mu_0 c\, A_4 \right) \Phi_{ik} \, .$$

The symbol $(II)_{ik}$ will have an analogous significance; we set:

$$\sum_{iklm} \delta_{ik} (A)_{ik,lm} (I)_{lm} \equiv \delta A \cdot (I)$$

in addition.

The 16 combinations that are symbolized by:

$$\begin{aligned}
&\frac{2\pi}{h} K\mu_0 c\delta \{ (i\mathbf{A}\, A_4) \cdot (I) + (i\mathbf{B}\, B_4) \cdot (II) && \} \quad (a) \\
&\frac{2\pi}{h} K\mu_0 c\delta \; \{ A_4 \cdot (I) + B_4\, (II) && \} \quad (b) \\
&\frac{2\pi}{h} K\mu_0 c\delta \; \{ \mathbf{s}^{(A)} A_4 \cdot (I) + \mathbf{s}^{(B)} B_4 \cdot (II) && \} \quad (c) \\
&\frac{2\pi}{h} K\mu_0 c\delta \; \{ A_1 A_2 A_3 A_4 \cdot (I) + B_1 B_2 B_3 B_4 \cdot (II) && \} \quad (d) \\
& && \qquad\qquad (16) \\
&K\delta \{ \mathbf{A} \cdot (I) + \mathbf{B} \cdot (II) && \} \quad (e) \\
&K\delta \{ \mathbf{s}^{(A)} \cdot (I) + \mathbf{s}^{(B)} \cdot (II) && \} \quad (f) \\
&K\delta \{ (I) + (II) && \} \quad (g) \\
&K\delta \{ \mathbf{s}^{(A)} \cdot (I) + \mathbf{s}^{(A)} \cdot (II) && \} \quad (h)
\end{aligned}$$

then provide Maxwell's equations:

([1]) *Loc. cit.*, pp. 46-49.

$$\left\{\begin{aligned} &\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \left(\frac{2\pi\mu_0 c}{h}\right)^2 \mathbf{A}, && (a) \\ &\operatorname{div}\mathbf{E} = -\left(\frac{2\pi\mu_0 c}{h}\right)^2 V, && (b) \end{aligned}\right.$$

$$\left\{\begin{aligned} &\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, && (c) \\ &\operatorname{div}\mathbf{H} = 0, && (d) \end{aligned}\right.$$

$$(17)$$

$$\left\{\begin{aligned} &\mathbf{E} = -\operatorname{grad}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, && (e) \\ &\mathbf{H} = \operatorname{rot}\mathbf{A}, && (f) \end{aligned}\right.$$

$$\begin{aligned} &\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0, && (g) \\ &0 = 0, && (h) \end{aligned}$$

respectively.

The non-Maxwellian equations are obtained by forming the 16 combinations of equations (I), (II) that are deduced from (16) by changing the signs in all terms of the type (B); for example, instead of the first combination (16.a), one will write:

$$K\mu_0 c\ \delta\{A_4 \cdot (I) - B_4 \cdot (II)\},$$

etc., … One will then get, in succession:

$$\left\{\begin{aligned} &\operatorname{grad}I = 0, && (a) \\ &\frac{1}{c}\frac{\partial I}{\partial t} = 0, && (b) \end{aligned}\right.$$

$$\left\{\begin{aligned} &\operatorname{grad}J = \left(\frac{2\pi\mu_0 c}{h}\right)^2 \mathbf{Q}, && (c) \\ &\frac{1}{c}\frac{\partial J}{\partial t} = \left(\frac{2\pi\mu_0 c}{h}\right)^2 Q_4, && (d) \end{aligned}\right.$$

$$(18)$$

$$\left\{\begin{aligned} &\operatorname{rot}I = 0, && (e) \\ &\operatorname{grad}Q_4 - \frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} = 0, && (f) \end{aligned}\right.$$

$$I = 0, \qquad (g)$$

$$\operatorname{div}\mathbf{Q} - \frac{1}{c}\frac{\partial Q_4}{\partial t} = J. \qquad (h)$$

The identity (17.*h*) proves that there are only 31 linearly-independent equations in the 32 equations (I), (II).

Upon using the notations (3) and (7), it is easy to write (17), (18) in the tensorial form and in an arbitrary coordinate system x^i for which the square of the elementary interval is:

$$ds^2 = \sum_{ik} g_{ik}\,dx^i dx^k .$$

Set $g = \| g_{ik} \|$. Equations (17) then become:

$$\begin{gathered}
\sum \frac{\partial \sqrt{-g}\,U^{ik}}{\partial x^k} = -\left(\frac{2\pi\mu_0 c}{h}\right)^2 \sqrt{-g}\,P^i , \\
\sum_k \frac{\partial U_{\overline{ik}}}{\partial x^k} = 0, \\
U_{ik} = \frac{\partial P_i}{\partial x^k} - \frac{\partial P_k}{\partial x^i} , \\
\sum_k \frac{\partial}{\partial x^k}(\sqrt{-g}\,P^k) = 0,
\end{gathered} \tag{17′}$$

in which $\overline{ik}$ denotes an even permutation of the numbers 1, 2, 3, 4. The group (18) becomes:

$$\begin{gathered}
\frac{\partial I}{\partial x^i} = 0, \\
\frac{\partial J}{\partial x^i} = \left(\frac{2\pi\mu_0 c}{h}\right)^2 Q_i , \\
I = 0, \\
\sum_k \frac{\partial}{\partial x^k}(\sqrt{-g}\,Q^k) = -\sqrt{-g}\,J .
\end{gathered}$$

We finally note the important role that is played by the hypothesis that $\mu_0 c / h$ is non-zero. If one annuls the factor of $\mu_0 c / h$ in (I) and (II) *a priori* then one will be led to some results that differ noticeably from the preceding ones. Hence, upon forming the same combinations as the ones that were indicated in (16), one will once more obtain Maxwell's equations for the electric and magnetic field, but not the relations (17.*e*, *f*, *g*) that define the fields as functions of the potentials.

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CHAPTER II

QUADRATIC QUANTITIES ATTACHED TO THE PHOTON

———

OPERATORS AND MEAN VALUES

All of the operators that were encountered up to now by L. de Broglie in his theory of the photon had the form:

$$F^{(a)} + F^{(b)} ; \tag{1}$$

i.e., they consisted of the sum of a (linear, Hermitian) operator of type (a) and a (linear, Hermitian) operator of type (b) ([1]). The proper values and proper functions of those operators are defined by the simultaneous equations ([2]):

$$F^{(a)}\varphi^{\,i} = f_i^{(a)}\varphi^i , \qquad F^{(b)}\varphi^{\,i} = f_i^{(b)}\varphi^i . \tag{2}$$

In particular, consider the Hamiltonian operator $H^{(a)} + H^{(b)}$; the proper values and proper functions will then be given by:

$$H^{(a)}\varphi^{\,i} = h_i^{(a)}\varphi^i , \qquad H^{(b)}\varphi^{\,i} = h_i^{(b)}\varphi^i . \tag{3}$$

We now refer to the system (I), (II) or (III), (IV). In the case of a monochromatic wave:

$$\Phi = \varphi\, e^{2i\pi Wt/h} ,$$

equations (I), (II) will become:

$$H^{(a)}\varphi = \tfrac{1}{2}\,W\varphi, \qquad H^{(b)}\varphi = \tfrac{1}{2}\,W\varphi. \tag{4}$$

The conditional equations (IV) then serve to define the pure states, for which:

$$h_i^{(a)} = h_i^{(b)} . \tag{5}$$

That is not the case for the invariant annihilation solution (I.2) ([3]). In that case, one will have:

$$H^{(a)}\Phi_{ik}^0 = \tfrac{1}{2}\,\mu_0 c^2 A_4 \Phi_{ik}^0 = \tfrac{1}{2}\,\mu_0 c^2 (\alpha_4)_{ik}\,\Phi_{ik}^0 , \tag{6}$$

———

([1]) An operator of type (a) acts exclusively upon the first indices of Φ_{ik}, and an operator of type (b) acts exclusively upon the second ones.

([2]) *Loc. cit.*, pp. 28.

([3]) (I.2) denotes formula (2) of the first chapter.

$$H^{(a)}\Phi^0_{ik} = \tfrac{1}{2}\mu_0 c^2 B_4 \Phi^0_{ik} = -\tfrac{1}{2}\mu_0 c^2 (\alpha_4)_{ik}\Phi^0_{ik} = -H^{(a)}\Phi^0_{ik} . \tag{7}$$

Thanks to (6) and (7), one sees that the annihilation solution (I.2) is a solution for which $H^{(a)}$ and $H^{(b)}$ have proper values $\frac{1}{2}\mu_0 c^2$ and $-\frac{1}{2}\mu_0 c^2$.

On the other hand, as L. de Broglie showed, the general definition of the mean-value densities will eliminate those difficulties. We assume that the mean-value density of an operator $F^{(a)} + F^{(b)}$ for a wave Φ is defined by ([1]):

$$\Phi^\times (F^{(a)} + F^{(b)})\, \Phi. \tag{8}$$

Hence, it results that the mean value of that operator will be:

$$\bar{F} = \int \Phi^\times (F^{(a)} + F^{(b)})\, \Phi\, d\tau.$$

The integration is extended over the entire space of variables x, y, z, and thus, any geometric space, in general.

The definition (8) seems to be imposed for the current density and spin vector (see below). Moreover, the tensors that are obtained in that fashion will always have the appropriate variance, at least for all of the physical quantities that will be studied in this article.

SIXTEEN FUNDAMENTAL OPERATORS AND THEIR MEAN-VALUE DENSITIES

Since the A matrices satisfy the same commutation relations as the α matrices of Dirac's theory, one can form a table of 16 operators with the aid of the A's and the identity matrix that is analogous to the table of sixteen fundamental Dirac operators; one proceeds similarly with the B's. One can then construct the following table of 16 fundamental operators [of type (1)] from the theory of the photon:

$$\tfrac{1}{2}(A_4 + B_4), \tag{10}$$

$$\tfrac{1}{2}(A_1 + B_1), \quad \tfrac{1}{2}(A_2 + B_2), \quad \tfrac{1}{2}(A_3 + B_3), \quad 1, \tag{11}$$

$$\left\{ \begin{array}{lll} \tfrac{1}{2}(s_1^{(A)}A_1 + s_1^{(B)}B_1) & \tfrac{1}{2}(s_2^{(A)}A_4 + s_2^{(B)}B_4) & \tfrac{1}{2}(s_3^{(A)}A_4 + s_3^{(B)}B_4) \\ \tfrac{1}{2}(iA_1A_4 + iB_1B_4) & \tfrac{1}{2}(iA_2A_4 + iB_2B_4) & \tfrac{1}{2}(iA_3A_2 + iB_3B_4) \end{array} \right. \tag{12}$$

$$\tfrac{1}{2}(s_1^{(A)} + s_1^{(B)}),\ \tfrac{1}{2}(s_2^{(A)} + s_2^{(B)}),\ \tfrac{1}{2}(s_3^{(A)} + s_3^{(B)}),\ \tfrac{1}{2}(s_4^{(A)} + s_4^{(B)}), \tag{13}$$

$$\tfrac{1}{2}(A_1 A_2 A_3 A_4 + B_1 B_2 B_3 B_4). \tag{14}$$

([1]) *Loc. cit.*, pp. 29-30.

L. de Broglie showed that the four operators (11) (with the first three multiplied by – c) can be taken to be the components of the operator that corresponds to the current-density quadri-vector. Meanwhile, the mean-value densities of the current density cannot be defined as in the Dirac theory. One will obtain acceptable definitions by setting ([1]):

$$\rho\,\mathbf{u} = -\,c\,\Phi^{\times}\,\frac{B_4\mathbf{A}+A_4\mathbf{B}}{2}\,\Phi, \qquad \rho = c\,\Phi^{\times}\,\frac{B_4+A_4}{2}\,\Phi, \tag{15}$$

which conforms to (8).

Those four expressions are, in fact, the components of a quadri-vector; in addition, they satisfy the equation of continuity:

$$\operatorname{div}\rho\,\mathbf{u} + \frac{\partial\rho}{\partial t} = 0.$$

Do the same thing with all of the terms in the table (10)-(14). We will then have five real tensors, namely:

An invariant:

$$\Omega_{(1)} = \Phi^{\times}\,B_4\,A_4\,\Phi, \tag{16}$$

A vector with contravariant components:

$$U^1 = \rho\,u_x = -\,c\,\Phi^{\times}\frac{B_4A_1+A_4B_1}{2}\,\Phi, \quad \ldots, \qquad U^4 = c\rho = -\,c\,\Phi^{\times}\frac{B_4+A_4}{2}\,\Phi, \tag{17}$$

An antisymmetric tensor of rank two:

$$\begin{aligned} M^{23} &= \Phi^{\times}\frac{s_1^{(A)}B_4A_1+s_1^{(B)}A_4B_1}{2}\,\Phi, \ \ldots \\ M^{41} &= \Phi^{\times}\frac{iA_1A_4B_4+iB_1B_4A_4}{2}\,\Phi, \ \ldots \end{aligned} \tag{18}$$

An antisymmetric tensor of rank three with components S^{ijk}, or – what amounts to the same thing – a vector with covariant components N_i :

$$\begin{aligned} -\,S^{234} = N_1 &= \frac{h}{2\pi}\,\Phi^{\times}\frac{s_1^{(A)}B_4+s_1^{(B)}A_4}{2}\,\Phi, \ \ldots \\ S^{123} = N_4 &= \frac{h}{2\pi}\,\Phi^{\times}\frac{s_4^{(A)}B_4+s_4^{(B)}A_4}{2}\,\Phi, \ \ldots \end{aligned} \tag{19}$$

([1]) *Loc. cit.*, pp. 22.

A completely-antisymmetric tensor of rank four, or – what amounts to the same thing – an invariant:

$$\Omega_{(2)} = \Phi^{\times}\frac{A_1A_2A_3A_4B_4 + B_1B_2B_3B_4A_4}{2}\Phi.$$

The expressions (16) to (20) define five fundamental tensors from the theory of the photon. Here are the explicit forms for some of their components as functions of Φ_{ik} , $\Phi_{ik}^{\times}$:

$$\begin{aligned}\Omega_{(1)} = &-|\Phi_{11}|^2 - |\Phi_{22}|^2 - |\Phi_{33}|^2 - |\Phi_{44}|^2 + |\Phi_{13}|^2 + |\Phi_{24}|^2 + |\Phi_{31}|^2 + |\Phi_{42}|^2 \\ &-|\Phi_{12}|^2 - |\Phi_{34}|^2 + |\Phi_{14}|^2 + |\Phi_{32}|^2 - |\Phi_{21}|^2 - |\Phi_{43}|^2 + |\Phi_{41}|^2 + |\Phi_{23}|^2,\end{aligned} \tag{21}$$

$$\left\{\begin{aligned}\rho &= -|\Phi_{13}|^2 - |\Phi_{24}|^2 + |\Phi_{31}|^2 + |\Phi_{42}|^2 - |\Phi_{14}|^2 + |\Phi_{32}|^2 - |\Phi_{23}|^2 + |\Phi_{41}|^2, \\ \rho u_x &= -\frac{c}{2}\{(\Phi_{11}^{\times} + \Phi_{44}^{\times})(\Phi_{41} - \Phi_{14}) + (\Phi_{22}^{\times} + \Phi_{33}^{\times})(\Phi_{32} - \Phi_{23}) \\ &\quad + (\Phi_{11}^{\times} + \Phi_{44}^{\times})(\Phi_{41} - \Phi_{14}) + (\Phi_{22}^{\times} + \Phi_{33}^{\times})(\Phi_{32} - \Phi_{23})\} + \text{conj.},\end{aligned}\right. \tag{22}$$

$$\begin{aligned}2M^{14} &= i\,(\Phi_{11}^{\times} + \Phi_{44}^{\times})\,(\Phi_{41} + \Phi_{14}) + i\,(\Phi_{22}^{\times} + \Phi_{33}^{\times})\,(\Phi_{32} + \Phi_{23}) \\ &\quad + i\,(-\Phi_{31}^{\times} - \Phi_{24}^{\times})\,(\Phi_{34} + \Phi_{24}) + i\,(-\Phi_{13}^{\times} - \Phi_{42}^{\times})\,(\Phi_{43} + \Phi_{12}) + \text{conj.}, \\ 2M^{23} &= (\Phi_{11}^{\times} - \Phi_{22}^{\times})\,(\Phi_{12} - \Phi_{21}) + (\Phi_{33}^{\times} - \Phi_{44}^{\times})\,(\Phi_{34} - \Phi_{43}) \\ &\quad + (\Phi_{31}^{\times} - \Phi_{42}^{\times})\,(\Phi_{41} - \Phi_{32}) + (\Phi_{13}^{\times} - \Phi_{24}^{\times})\,(\Phi_{14} - \Phi_{23}) + \text{conj.},\end{aligned} \tag{23}$$

$$\left\{\begin{aligned}\frac{2\pi}{h}N_z &= |\Phi_{11}|^2 - |\Phi_{22}|^2 - |\Phi_{33}|^2 + |\Phi_{44}|^2 - |\Phi_{14}|^2 + |\Phi_{32}|^2 - |\Phi_{41}|^2 + |\Phi_{23}|^2, \\ \frac{2\pi}{h}N_4 &= \tfrac{1}{2}\{(\Phi_{11}^{\times} - \Phi_{33}^{\times})(\Phi_{31} + \Phi_{13}) + (\Phi_{22}^{\times} - \Phi_{44}^{\times})(\Phi_{42} + \Phi_{24}) \\ &\quad + (\Phi_{12}^{\times} - \Phi_{34}^{\times})(\Phi_{32} + \Phi_{14}) + (\Phi_{21}^{\times} - \Phi_{43}^{\times})(\Phi_{23} + \Phi_{41})\} + \text{conj.},\end{aligned}\right. \tag{24}$$

$$\begin{aligned}2\Omega_{(2)} &= i\,(\Phi_{11}^{\times} - \Phi_{33}^{\times})(\Phi_{13} - \Phi_{31}) + i\,(\Phi_{22}^{\times} - \Phi_{44}^{\times})(\Phi_{24} - \Phi_{42}) \\ &\quad + i\,(\Phi_{21}^{\times} - \Phi_{43}^{\times})(\Phi_{23} - \Phi_{41}) + i\,(\Phi_{12}^{\times} - \Phi_{34}^{\times})(\Phi_{32} - \Phi_{14}) + \text{conj.}\end{aligned} \tag{25}$$

These are bilinear functions of Φ_{ik} , $\Phi_{ik}^{\times}$ that are very asymmetric in them. The formulas will simplify remarkably when one utilizes the five tensors (I.3-7) instead of the Φ_{ik}, which is always possible, thanks to (15). One will then obtain ([1]):

([1]) The electromagnetic fields are expressed in Heaviside-Lorentz units; as L. de Broglie showed, it is then convenient to take $K = 1 / |\chi|\,\mu_0$. In order to abbreviate the notation, we have suppressed the arrows above vectorial quantities. The bracket [ ] denotes a vector product.

$$\mu_0\, c^2\, \Omega_{(1)} = \tfrac{1}{4}\Big\{|\mathbf{E}|^2 - |\mathbf{H}|^2 + \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|\mathbf{A}|^2 - |V|^2) - \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|K\boldsymbol{\sigma}|^2 - |K\sigma_4|^2) - (|I|^2 - |J|^2)\Big\}, \tag{21′}$$

$$\left\{\begin{aligned} \rho\mathbf{u} &= \frac{2i\pi}{4h}\left\{[\mathbf{A}^\times\mathbf{H}] + V^\times\mathbf{E} + J^\times\mathbf{Q}\right\} + \text{conj.},\\ \rho &= \frac{2i\pi}{4h}\left\{(\mathbf{A}^\times\mathbf{H}) - J^\times Q_4\right\} + \text{conj.},\end{aligned}\right. \tag{22′}$$

$$\left\{\begin{aligned} \mu_0 c^2 M^{23} &= \tfrac{1}{4}\Big\{[\mathbf{E}^\times\mathbf{E}]_x - [\mathbf{H}^\times\mathbf{H}]_x \\ &\qquad + \left(\frac{2\pi\mu_0 c}{h}\right)^2 [\mathbf{A}^\times\mathbf{A}]_x - \left(\frac{2\pi\mu_0 c}{h}\right)^2 [\mathbf{Q}^\times\mathbf{Q}]\Big\} + \text{conj.},\\ \mu_0 c^2 M^{41} &= \tfrac{i}{4}\Big\{[\mathbf{E}^\times\mathbf{H}]_x + [\mathbf{H}^\times\mathbf{E}]_x \\ &\qquad + \left(\frac{2\pi\mu_0 c}{h}\right)^2 (A_x^\times V - V^\times A_x) - \left(\frac{2\pi\mu_0 c}{h}\right)^2 (Q_4^\times Q_x - Q_x^\times Q_4)\Big\},\end{aligned}\right. \tag{23′}$$

$$\left\{\begin{aligned} \mathbf{N} &= \frac{1}{4c}\left\{-[\mathbf{A}^\times\mathbf{E}] + V^\times\mathbf{E} - I^\times\mathbf{Q}\right\} + \text{conj.},\\ N_4 &= \frac{1}{4c}\left\{-(\mathbf{A}^\times\mathbf{H}) - I^\times Q_4 \qquad\right\} + \text{conj.},\end{aligned}\right. \tag{24′}$$

$$\mu_0\, c^2\, \Omega_{(2)} = \{-(\mathbf{H}^\times\,\mathbf{E}) - I^\times J\} + \text{conj.} \tag{25′}$$

We now make some remarks:

1. The variance of the forms (21′), …, (25′) is immediately obvious. It will appear even more explicitly when one utilizes tensorial language in Minkowski space-time. For example, one will then have:

$$U^k = -\frac{2i\pi}{4h}\left\{\sum U^{kl} P_l^\times + J^\times Q^k\right\} + \text{conj.} \tag{26}$$

and

$$S^{ijk} = -\frac{1}{4c}\{U^{ij}P_\times^k + U^{jk}P_\times^i + U^{ki}P_\times^j + I^\times Q_l\} + \text{conj.} \tag{27}$$

2. The presence of factor $i = \sqrt{-1}$, is quite characteristic, especially in (26); it points to a very clear difference between Maxwell's classical theory and the theory of the photon. Indeed, it shows that whereas equations (I.17) are formally Maxwell's equations,

the potentials and fields that are considered here are complex, in general. Moreover, that conforms to the fact that in order to represent a monochromatic wave, one must take $e^{2i\pi Wt/h}$, and not $\sin 2\pi Wt / h$.

3. It results from the preceding formulas that the physical quantities of the theory of the photon depend upon the electromagnetic potentials explicitly. Now, in Maxwell's theory, one is accustomed to considering the potentials to be auxiliary functions that appear in physically-meaningful quantities in the form of electric and magnetic fields, which permits one to replace **A** and V with:

$$\boldsymbol{\mathcal{A}} = \mathbf{A} + \operatorname{grad} f, \qquad \varphi = V - \frac{1}{c}\frac{\partial f}{\partial t}, \tag{28}$$

with

$$\sum \frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0. \tag{28$'$}$$

The relations (28), (28′) will no longer be rigorously valid when one supposes that $\mu_0 \neq 0$, because if **A**, V are solutions of (I.17.*a*-*g*) then that will no longer be true for $\boldsymbol{\mathcal{A}}$, φ. Nevertheless, a determination that is similar to (28), (28′) will persist in the theory of the photon.

In order to see that, it suffices to consider the simple case of a monochromatic plane wave that propagates along Oz. The most general solution is then obtained by setting ([1]):

$$\begin{aligned}
&P = e^{\frac{2i\pi}{h}(Wt - pz)}, && \Delta = W/c + \mu_0\, c,\\
&A_x = (b - a)\,\frac{2hc}{2\pi\Delta}\,P, && A_y = i\,(b + a)\,\frac{2hc}{2\pi\Delta}\,P,\\
&A_z = (c' + c'')\,\frac{2hc}{2\pi\Delta}\,P, && V = (c' + c'')\,\frac{2hc}{2\pi\Delta}\cdot\frac{pc}{W}\cdot P,
\end{aligned}$$

and

$$\begin{aligned}
&E_x = -2i\,\frac{W}{h}(b - a)\,P, \quad && E_y = 2i\,\frac{W}{h}(b + a)\,P, \quad && E_z = -2i\,(c' + c'')\frac{W}{\Delta}\frac{\mu_0^2\, c^2}{W^2}\,P,\\
&H_x = -2\,\frac{cp}{\Delta}(b - a)\,P, && H_y = -2i\,\frac{cp}{\Delta}(b - a)\,P, && H_z = 0.
\end{aligned} \tag{29$'$}$$

On the other hand, one will obtain the same fields as in (29′) with the potentials:

$$\mathcal{A}_x = A_x, \qquad \mathcal{A}_y = A_y, \qquad \mathcal{A}_z = 0, \qquad V = 0, \tag{30}$$

except that the component $\mathcal{E}_z$ of the electric field is zero here.

([1]) *Loc. cit.*, pp. 18, 19.

Now, suppose that A_z and V have the same order of magnitude as A_x or A_y. E_z will then have the same order of magnitude as $E_x \dfrac{\mu_0^2 c^4}{W^2}$ or $E_y \dfrac{\mu_0^2 c^4}{W^2}$. Now, one certainly has $\mu_0^2 c^4 / W^2 \ll 1$ for the usual electromagnetic waves. Therefore, the two potentials (29) and (30) correspond to fields that differ by only terms that are negligible.

The theory of the photon must eliminate that indeterminacy, because in that theory, the state of photon is represented by the 16 wave functions Φ_{ik}, and one can no longer say *a priori* that two systems Φ and Φ' that differ by only the values of their potentials will always define the same state of the photon. One might expect from this that the study of interaction phenomena between matter and radiation would define the electromagnetic potentials uniquely, because the physical quantities (21) and (25) must take on values that would satisfy the usual conservation theorems. We especially see that one will get satisfactory results for the theorem of the conservation of moments of impulse only for certain forms of potentials.

4. The mean-value densities (21′)-(25′) are all composed of a sum of two groups of terms: Only Maxwellian quantities appear in the first group, while only non-Maxwellian quantities appear in the second one. That situation will no longer present itself for densities that correspond to the operators that are obtained by changing the signs of all operators of type (*b*) in the definitions (10)-(14).

IMPULSE

One makes the impulse world-vector correspond to the operator:

$$I^1 = -\frac{1}{\chi}\frac{\partial}{\partial x}, \quad I^2 = -\frac{1}{\chi}\frac{\partial}{\partial y}, \quad I^3 = -\frac{1}{\chi}\frac{\partial}{\partial z}, \quad I^4 = +\frac{1}{\chi}\frac{1}{c}\frac{\partial}{\partial t}, \tag{31}$$

while the current-density vector will correspond to the operator:

$$u^1 = -c\,\frac{A_1 + B_1}{2}, \ \ldots, \quad u^4 = c \cdot 1, \tag{32}$$

as we have recalled above.

One forms the tensor-operator:

$$\tau^{ij} = I^i u^j, \tag{33}$$

with these two operators (31), (32), whose mean-value density is:

$$t^{ij} = \Phi^\times I^i \,(B_4\, u_{(a)}^j + A_4\, u_{(b)}^j)\ \Phi, \tag{34}$$

conforming to (8), in which $u_{(a)}^1 = -c\,A_1 / 2$, …, etc., and $u_{(b)}^1 = -c\,B_1 / 2$, …, etc. That asymmetric tensor is indeed twice contravariant, as the position of the indices would suggest. By virtue of (31), the functions:

$$\pi^a = \frac{1}{c}\, t^{a4} \qquad (a = 1, 2, 3) \tag{35}$$

are the components of the impulse mean-value vector-density, whereas t^{44} is the (photonic) energy mean-value density. As in classical theory, the components t^{ab} can represent the impulse flux per unit area and time. On the other hand, it is possible to define a symmetric tensor T^{ij} that is equal to the classical Maxwell tensor (with complex electromagnetic fields), plus some terms that refer to non-Maxwellian quantities and some terms that are negligible in comparison to the Maxwellian terms. That real tensor is given by:

$$T^{ij} = \mu_0\, c^2\, \Phi^\times\, \frac{C_i D_j + C_j D_i}{2}\, \Phi, \tag{36}$$

with the notations:

$$\begin{aligned} C_1 &= -A_1, \quad C_2 = -A_2, \quad C_3 = -A_3, \quad C_4 = 1, \\ D_1 &= -B_1, \quad D_2 = -B_2, \quad D_3 = -B_3, \quad D_4 = 1. \end{aligned} \tag{37}$$

For example, one has, explicitly:

$$\begin{aligned} T^{14} &= -\mu_0\, c^2\, \Phi^\times\, \frac{A_1 + B_1}{2}\, \Phi \\ &= -\mu_0\, c^2\, \tfrac{1}{2}\, \{ (\Phi^\times_{11} + \Phi^\times_{44})(\Phi_{14} + \Phi_{41}) + (\Phi^\times_{22} + \Phi^\times_{33})(\Phi_{23} + \Phi_{32}) \\ &\qquad + (\Phi^\times_{12} + \Phi^\times_{43})(\Phi_{13} + \Phi_{42}) + (\Phi^\times_{21} + \Phi^\times_{43})(\Phi_{24} + \Phi_{31}) + \text{conj.}\} \\ &= \tfrac{1}{4}\, \{[\mathbf{E}^\times\, \mathbf{H}]_x + [\mathbf{E}\mathbf{H}^\times]_x - \left(\frac{2\pi\mu_0 c}{h}\right)^2 (V A_x + A^\times_x V) - \left(\frac{2\pi\mu_0 c}{h}\right)^2 (Q^\times_x Q_4 + Q^\times_4 Q_x)\ \}, \end{aligned} \tag{38}$$

$$\begin{aligned} T^{14} &= \mu_0\, c^2\, \Phi^\times\, \Phi \\ &= \tfrac{1}{4}\, \{\, |\,\mathbf{E}\,|^2 + |\,\mathbf{H}\,|^2 + 2\left(\frac{2\pi\mu_0 c}{h}\right)^2 |\,V\,|^2 + 2\left(\frac{2\pi\mu_0 c}{h}\right)^2 |\,Q_4\,|^2 \\ &\qquad + \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|\,\mathbf{A}\,|^2 - |\,V\,|^2) + \left(\frac{2\pi\mu_0 c}{h}\right)^2 (|\,\mathbf{Q}\,|^2 - |\,Q_4\,|^2) + |\,I\,|^2 + |\,J\,|^2\}. \end{aligned} \tag{39}$$

By virtue of (38), we consider:

$$G^a = T^{a4} / c \tag{40}$$

to be the components of the Maxwellian electromagnetic impulse in the geometric space x, y, z, while T^{44} will be the Maxwellian electromagnetic energy.

The tensors (34) and (36) are coupled to each other by relations that we shall now address. To that end, we introduce the real three-index tensor:

$$A^{ijk} = -\,\Phi^{\times}\,\frac{i\,A_4\,C_i\,D_j\,C_k + i\,B_4\,D_i\,C_j\,D_k}{2}\,\Phi \qquad (i \neq k)\;\; k = 1, 2, 3. \tag{41}$$

That tensor is antisymmetric with respect to the extreme indices; hence, $A^{ijk} = -\,A^{kji}$.

Theorem:

By virtue of the photon equations, one will have:

$$t^{\,ij} = T^{\,ij} + \frac{1}{2}\frac{hc}{2\pi}\sum_k \frac{\partial A^{ijl}}{\partial x^k} + \frac{1}{2\chi}\varepsilon_{(i)}\frac{\partial U^i}{\partial x^j} \qquad (i, j, k = 1, \ldots, 4), \tag{42}$$

in which $\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon_{(3)} = -\,1$, *and* $\varepsilon_{(4)} = 1$.

In order to prove (42), it suffices to multiply (I) and (II) by $C_i\,D_j\,A_4$ and $C_j\,D_i\,B_4$, respectively, and add the corresponding sides. Upon left-multiplying the relations thus-obtained by $\Phi^{\times}$, one will easily get – for example, for $i = 4$, $j = 1$:

$$t^{\,41} = -\,\mu_0\,c^2\,\Phi^{\times}\,\frac{A_1 + B_1}{2}\,\Phi + \frac{1}{2}\frac{hc}{2\pi}\left\{\frac{\partial}{\partial x}\Phi^{\times}\frac{B_1\,i\,A_4A_1 + A_1\,i\,B_4B_1}{2}\Phi\right.$$
$$\left. + \frac{\partial}{\partial y}\Phi^{\times}\frac{B_1\,i\,A_4A_2 + A_1\,i\,B_4B_2}{2}\Phi + \frac{\partial}{\partial z}\Phi^{\times}\frac{B_1\,i\,A_4A_3 + A_1\,i\,B_4B_3}{2}\Phi\right\} + \frac{1}{2\kappa c}\frac{\partial U^1}{\partial t},$$

and for $i = 1$, $j = 4$:

$$t^{\,14} = -\,\mu_0\,c^2\,\Phi^{\times}\,\frac{A_1 + B_1}{2}\,\Phi + \frac{1}{2}\frac{hc}{2\pi}\left\{\frac{\partial}{\partial y}\Phi^{\times}\frac{iA_4A_1A_2 + iAB_4B_1B_2}{2}\Phi\right.$$
$$\left. + \frac{1}{c}\frac{\partial}{\partial t}\Phi^{\times}\frac{i\,A_4A_1 + i\,B_4B_1}{2}\Phi - \frac{\partial}{\partial z}\Phi^{\times}\frac{i\,A_4\,A_3A_1 + i\,B_1B_3B_1}{2}\Phi\right\} - \frac{1}{2\chi}\frac{\partial U^4}{\partial x}.$$

One easily proves that by virtue of the equations of the photon:

$$\sum_j \frac{\partial t^{ij}}{\partial x^j} = 0 \qquad i, j = 1, \ldots, 4. \tag{43}$$

In the theory of the photon, these equations correspond to the theorem of the conservation of impulse and energy of Maxwell's theory.

One will likewise see that by virtue of the photon equations:

$$\sum_j \frac{\partial T^{ij}}{\partial x^j} = 0. \tag{44}$$

That property expresses the theorem of the conservation of electromagnetic impulse and energy.

It results from (43), (44), and (42) that:

$$\sum_j \frac{\partial A^{ijk}}{\partial x^j} = 0, \tag{45}$$

which one can verify directly, moreover.

Remarks:

1. The Maxwellian tensor T^{ij} does not have the canonical type (8) for the mean-value densities. According to the viewpoint that was adopted here, it does not define an essential physical quantity then. However, relations (42) show that there exist close links between t^{ij} and T^{ij}, as one would obviously expect. Here are two propositions that will make those links appear more clearly: In order to simplify the presentation in what follows, we shall consider only the case of a monochromatic wave of frequency $\nu = 2\pi W / h$. In the general case, the supplementary terms that are introduced into the formulas are due to the fact that the mean-value densities depend upon time.

Multiply both sides of (42) by the volume element $d\tau$ and integrate over all space. Upon supposing that, of course, the integrals mean something, we will thus obtain:

$$\int t^{ij}\,d\tau = \int T^{ij}\,d\tau + \frac{1}{2}\frac{hc}{2\pi}\int \sum \frac{\partial A^{ija}}{\partial x^a}\,d\tau + \frac{1}{2\chi}\varepsilon_{(i)}\int \frac{\partial U^j}{\partial x^i}\,d\tau\,. \tag{46}$$

If the functions A^{ija} and U^j are continuous in all of space then the last two integrals will reduce to surface integrals by an application of Green's theorem. If one then supposes that the Φ_{ik} tend to zero very quickly so those integrals will disappear at infinity then (46) will give:

$$\int t^{ij}\,d\tau = \int T^{ij}\,d\tau \tag{47}$$

when the integrals are taken over all space. The mean value of τ^{ij} will then remain equal to the integral over all space of the T^{ij} component of Maxwell's tensor.

For the energy, one will have:

$$\int t^{44}\,d\tau = \int T^{44}\,d\tau\,;$$

the mean value of the energy is always positive then.

2. The second property envisioned concerns the flux of impulse and energy across a closed surface. We first remark that, by virtue of (42):

$$t^{ij} - t^{ji} = \frac{1}{2}\frac{hc}{2\pi}\sum_k \frac{\partial}{\partial x^k}(A^{ijk} - A^{jik}) + \frac{1}{2\chi}\left(\varepsilon_{(i)}\frac{\partial U^j}{\partial x^i} - \varepsilon_{(j)}\frac{\partial U^i}{\partial x^j}\right). \tag{48}$$

However, one proves directly by using the photon equations that:

$$t^{ij} - t^{ji} = \frac{c}{2}\sum_k \frac{\partial S^{ikj}}{\partial x^k} + \frac{1}{2\chi}\left(\varepsilon_{(i)}\frac{\partial U^j}{\partial x^i} - \varepsilon_{(j)}\frac{\partial U^i}{\partial x^j}\right); \tag{48′}$$

so

$$\frac{1}{2}\frac{hc}{2\pi}\sum_k \frac{\partial(A^{ijk} - A^{jik})}{\partial x^k} = \frac{c}{2}\sum_k \frac{\partial S^{ikj}}{\partial x^k}\ . \tag{49}$$

Thanks to (49), (42) will give:

$$t^{ij} = T^{ji} + \frac{c}{2}\sum_b \frac{\partial S^{ibj}}{\partial x^b} + \frac{c}{2}\frac{h}{2\pi}\sum_b \frac{\partial A^{jib}}{\partial x^b} + \frac{1}{2\chi}\varepsilon_{(i)}\frac{\partial U^j}{\partial x^i}; \tag{48′}$$

hence, in particular, one will have:

$$t^{4a} = T^{4a} + \frac{c}{2}\sum_b \frac{\partial S^{4ba}}{\partial x^b} + \frac{c}{2}\frac{h}{2\pi}\sum_b \frac{\partial A^{a4b}}{\partial x^b},$$

or even, upon taking into account the definitions (19) and setting:

$$X_1 = \frac{c}{2}\frac{h}{2\pi}A^{243}, \ \ldots$$

one will have:

$$t^{4a} = T^{4a} + \frac{c}{2}\,\mathrm{rot}_a\,\mathbf{N} + \mathrm{rot}_a\,\mathbf{X}. \tag{50′}$$

Let F be a closed surface. Upon letting n_a denote the components of an exterior semi-normal to F and using Stokes's theorem, one will get:

$$\int_F \left(\sum t^{4a} n_a\right) dS = \int_F \left(\sum T^{4a} n_a\right) dS\ . \tag{51}$$

In classical electromagnetic theory, the left-hand side of (51) is the energy flux that crosses F per unit time outwards. The relation (51) will then show us that we should interpret the integral:

$$\int_F \left(\sum t^{4a} n_a\right) dS$$

as the energy flux per unit time that crosses F outwards.

3. We finally prove that under certain conditions, one will likewise have:

$$\int_F \left(\sum t^{4a} n_a\right) dS = \int_F \left(\sum T^{4a} n_a\right) dS \tag{52}$$

for a monochromatic wave.

By virtue of (51):

$$t^{ab} = T^{ab} + \frac{c}{2}\sum_d \frac{\partial S^{adb}}{\partial x^d} + \frac{c}{2}\frac{h}{2\pi}\sum_d \frac{\partial A^{bad}}{\partial x^d} - \frac{1}{2\chi}\frac{\partial U^b}{\partial x^a}\,.$$

Upon utilizing Stokes's theorem, the difference between the two sides of (52) will then be:

$$\frac{c}{2}\int_F [\mathrm{grad}\, N_4 \mathbf{n}]\, dS + \frac{1}{2\chi}\int_F \left(\sum_b \frac{\partial U^b}{\partial x^a} n_b\right) dS\,.$$

In order to obtain (52), it is necessary and sufficient that one must have:

$$\Phi_F\,(\mathrm{grad}\, N_4 \cdot \mathbf{n})\, dS = 0 \tag{53}$$

and

$$\int_F \left(\sum_b \frac{\partial U^b}{\partial x^a} n_b\right) dS = 0. \tag{54}$$

The relations (52) express the idea that the flux of impulse that crosses F per unit time is equal to the electromagnetic impulse flux that crosses F outward per unit time.

The integrals:

$$\int \left(\sum_b t^{ab} n_b\right) dS \tag{55}$$

will be real only if one has (54). We can free ourselves of that condition by adopting the real tensor:

$$\tfrac{1}{2}(t^{ij} + \text{conj.})$$

as the definition of the energy-impulse tensor, instead of (34).

MOMENTS OF IMPULSE

The moment of impulse vector operator with respect to the coordinate origin O is defined by:

$$\mathbf{m}^0 = [\mathbf{r}\ \mathbf{I}], \qquad (55)\ [\text{sic}]$$

in which $\mathbf{r}$ is a (spatial) vector with its origin at O and its extremity at x, y, z, and:

$$I_x = I^1 = -\frac{1}{\chi}\frac{\partial}{\partial x}, \qquad I_y = I^2, \qquad I_z = I^3. \tag{56}$$

Conforming to (8) and (55), the mean-value vector density of the orbital moment is then written:

$$\mathbf{M}^0 = \Phi^\times \frac{A_4 + B_4}{2} [\mathbf{r}\ \mathbf{I}]\ \Phi. \tag{57}$$

In space-time, one will form the operator:

$$m_0^{ijk} = x^i\ \tau^{kj} - x^k\ \tau^{ij} \tag{58}$$

and the "mean density of m_0^{ijk}" tensor:

$$M_0^{ijk} = -M_0^{kji} = \Phi^\times \frac{A_4 + B_4}{2} (x^i\ \tau^{kj} - x^k\ \tau^{ij})\ \Phi = x^i\ t^{kj} - x^k\ t^{ij}. \tag{59}$$

The functions $\frac{1}{c} M_0^{a4b}$ will then be the mean densities of the orbital moment about the (ab) axis. In addition, since the t^{ab} represent the impulse flux, the components:

$$M_0^{adb} = x^a\ t^{bd} - x^b\ t^{ad} \tag{60}$$

can represent the orbital moment flux per unit time and area. In the preceding paragraph, we were especially led to consider the closed surface integral:

$$\int \left(\sum_b t^{ab} n_b \right) dS$$

to be the impulse flux with axis (a) per unit time. We then also interpret the integrals:

$$\int \left(\sum_b t^{ab} n_b \right) dS \tag{61}$$

as the flux of orbital moment about the (ab) axis that crosses the closed surface considered outwards per unit time. In order for these integrals to have any physical sense, it is necessary that they should be real, to begin with. That will generally be true only for a closed surface that tends to infinity. The integrals (61) will then define the flux at infinity.

One knows that $\mathbf{m}^0$ is not a first integral in the theory of the photon. That is why one must add the spin operator $\mathbf{m}^s$ whose components are:

$$m_x^s = \frac{h}{2\pi} \frac{iA_2A_3 + iB_2B_3}{2}, \qquad m_y^s = \frac{h}{2\pi} \frac{iA_3A_1 + iB_3B_1}{2}, \qquad m_z^s = \frac{h}{2\pi} \frac{iA_1A_2 + iB_1B_2}{2} \tag{62}$$

to $\mathbf{m}^0$.

The operator:

$$\mathbf{m} = \mathbf{m}^0 + \mathbf{m}^s \tag{63}$$

is a first integral; it is the operator that corresponds to the total moment of the photon. The mean density of the total moment will then be:

$$\mathbf{M} = \mathbf{M}^0 + \mathbf{M}^s, \tag{64}$$

in which N_x, N_y, N_z are the mean densities of spin (19). In tensor form, one will write:

$$M^{a4b} = M_0^{a4b} + c\, S^{a4b}, \tag{65}$$

instead of (64).

We must naturally complete the definition of the tensor M^{ijk} whose components M^{a4b} are (65). The tensor M_0^{ijk} is given by (59). On the other hand, there exists a tensor S^{ijk} [see (19)] whose components ($a4b$) are the spin densities. However, that tensor is completely antisymmetric. All of its non-zero components (adb) are equal in absolute value. They can then be interpreted, in full generality, as the components of the flux of spin per unit area and time, because it is easy to see – in the case of the plane wave, for example – that this interpretation will lead to some inadmissible consequences.

The components (adb) of the tensor:

$$M^{ijk} = M_0^{ijk} + c\, S^{ijk} \tag{66}$$

do not therefore define the flux of total moment, in full generality. Meanwhile, in the following chapter, we shall show that the S^{adb} can be utilized in the case of multipole radiation, since one utilizes the M_0^{adb} in (61). The integrals:

$$\int S^{123}\mathbf{n}\, dS \tag{67}$$

then give the flux of spin at infinity per unit time, and the functions:

$$\sum_d M^{adb} n_d = \sum_d (M_0^{adb} + cS^{adb})\, n_d \tag{68}$$

will be quantities that will yield the flux of total moment at infinity when one integrates them over an arbitrary closed surface that tends to infinity.

Remarks.

1. The classical Maxwellian theory defined the density and flux of electromagnetic moment of impulse, and those definitions led to some important problems that had well-verified conclusions. Now, the electromagnetic tensor T^{ij} (36) is expressed as a function of the electric and magnetic fields in the same manner as for the Maxwell tensor. We then also consider the densities and flux of electromagnetic moments of impulse that are defined by the ($a4b$) and (adb) components of the tensor:

$$M_e^{ijk} = x^i\, T^{kj} - x^k\, T^{ij}\,. \qquad \text{(70) [}sic\text{]}$$

Thanks to (42) and (49), one will find that:

$$\overset{1i2}{M_e}$$

$$= M_i^{1i2} + \frac{c}{2}\sum_k \frac{\partial}{\partial x^k}(y\, S^{1ki} - z\, s^{2ki}) + \frac{h}{2\pi}\cdot\frac{c}{2}\left(y\sum_k \frac{\partial A^{i1k}}{\partial x^k} - x\sum_k \frac{\partial A^{i2k}}{\partial x^k}\right) - \frac{1}{2\chi}\mathrm{rot}_{12}\,\rho\mathbf{u}\,. \quad (71)$$

Relations such as (71) permit one to compare the classical electromagnetic theory to the theory of the photon, as far as moments of impulse are concerned. In the next chapter, we shall see that formulas (70) and (59) yield the same results for the calculation of the flux (at infinity) of moments of impulse in the case of electric dipole and quadrupole radiation. By contrast, in the case of a monochromatic plane wave, the results will be different; however, it is the expression (59) of the theory of the photon that gives the good values.

2. We finally point out the following relations: By virtue of (45), one will have:

$$\sum_i \frac{\partial}{\partial x^i}\frac{1}{2}(M_0^{1i2} + \text{conj.}) = \tfrac{1}{2}(t^{21} - t^{12}) + \text{conj.} \quad (72)$$

On the other hand:

$$\sum_i \frac{\partial}{\partial x^i} cS^{1i2} = (t^{12} - t^{21}) + \text{conj.}\,; \quad (73)$$

hence:

$$\sum_i \frac{\partial}{\partial x^i}\frac{1}{2}(M^{1i2} + \text{conj.}) = \tfrac{1}{2}(t^{12} - t^{21}) + \text{conj.}$$

and

$$\sum_i \frac{\partial}{\partial x^i}\left[\frac{1}{2}(M_0^{1i2} + \text{conj.}) + \frac{c}{2}S^{1i2}\right] = 0.$$

The physical significance of these relations is not apparent to us.

L. DE BROGLIE'S THEORY OF LIGHT AND TH. DE DONDER AND J. M. WHITTAKER THEORY OF PHOTONIC FIELDS

Following L. de Broglie, in the first chapter, we recalled that the system of fundamental equations (I), (II) of the photon *in vacuo* is equivalent to two groups of equations: A first group of equations (I.17) that has a Maxwellian form and a second group (II.18).

Now, one must remember that research has been ongoing for several years ([1]) into how one might establish the fundamental equations of wave mechanics in Maxwellian

([1]) We cite C. G. DARWIN, Proc. Roy. Soc. A**118** (1928), pp. pp. 654. – TH. DE DONDER, Bull. Acad. Roy. Belg. (1928), pp. 307. – J. M. WHITTAKER, Proc. Roy. Soc. A**121** (1928), pp. 548.

form. J. M. Whittaker (cf., previous footnote) succeeded in writing the fundamental equations of wave mechanics of the electron in the following form:

$$\sum_k \frac{\partial\sqrt{-g}\,U^{ik}}{\partial x^k} = \sqrt{-g}\,\sum_k g^{ik}\,\frac{\partial I}{\partial x^k} - \left(\frac{2\pi m_0 c}{h}\right)^2 \sqrt{-g}\,P^i + \sqrt{-g}\,\sum_k \zeta_k U^{ik} - \zeta^i I \qquad (a)$$

$$\sum_k \frac{\partial U_{\overline{ik}}}{\partial x^k} = \sqrt{-g}\,\sum_k g^{ik}\,\frac{\partial J}{\partial x^k} - \left(\frac{2\pi m_0 c}{h}\right)^2 \sqrt{-g}\,Q^i + \sqrt{-g}\,\sum_k \zeta_k U_{\overline{ik}} - z^i J$$

(77)

$$\sum_i \frac{\partial\sqrt{-g}\,P^i}{\partial x^i} - \sum_i \zeta_i P^i = \sqrt{-g}\,I\,,$$

$$(b)$$

$$\sum_i \frac{\partial\sqrt{-g}\,Q^i}{\partial x^i} - \sum_i \zeta_i Q^i = \sqrt{-g}\,J\,,$$

$$U_{ik} = \left(\frac{\partial P_i}{\partial x^k} - \frac{\partial P_k}{\partial x^i}\right) + \sum_l \sum_k \frac{g_{il}\,g_{km}}{\sqrt{-g}} \left(\frac{\partial Q_{\bar{l}}}{\partial x_{\bar{m}}} - \frac{\partial Q_{\bar{m}}}{\partial x_{\bar{l}}}\right). \qquad (c)$$

These sixteen equations are linear in the sixteen complex functions:

$$U_{ik}\,, \qquad P_i\,, \qquad Q_i\,, \qquad I, \qquad J, \tag{78}$$

and their first derivatives. We remark that the first eight (77.a) have the Maxwellian form that is characterized by the divergences that appear in the left-hand sides. The two equations (73.b) are the two complementary equations that relate to the potentials P^i and Q^i (i = 1, …, 4). Finally, the six equations (73.c) are the defining equations of the fields U^{ik} by means of the potentials.

One has set:

$$\zeta_i = \frac{2i\pi e}{hc}\,\Phi_i\,, \tag{79}$$

in which the Φ^i are the four components of the electromagnetic potential into which the electron in question is embedded. m_0, e are the rest mass and electron charge, respectively. J. M. Whittaker showed that some particular solutions of his equations were solutions of the Dirac equations.

On the other hand, Th. De Donder, who had already sketched out a theory of the photonic field ([1]), considered J. M. Whittaker's equations (73) to be the photonic equations of wave mechanics. We note that in that way of looking at things, equations (73) apply to not just the electron, but to any corpuscle that is or is not electrically-charged, as well as the photon.

([1]) *Loc. cit.* See also Bull. Acad. Roy. Belg. **15** (1929), pp. 116 and *ibid.* **16** (1930), pp. 3. The author reprised and developed that study in Annales de l'Institut H. Poincaré **1**, fasc. 2, pp. 77.

It is easy to show that L. de Broglie's fundamental equations (I.17), (I.18) can be recovered by starting with equations (73) in the particular case in which:

$$I = 0 \quad \text{and} \quad \left(\frac{2\pi\mu_0 c}{h}\right)^2 Q_i = \frac{\partial J}{\partial x_i}. \tag{80}$$

One must obviously set $e = 0$ in (73) and replace m_0 with the mass μ_0 of the photon.

In addition, the current-density vector that was introduced by J. M. Whittaker is written:

$$-\frac{2i\pi}{4h}\left\{\sum_l U^{kl} P_l^{\times} + \sum_l U_{\overline{kl}}\, Q_l^{\times} / \sqrt{-g} + J\, Q^k\right\} + \text{conj.}$$

That vector differs from the current-density vector (26) only by the term:

$$\sum_l U_{\overline{kl}}\, Q_l^{\times} / \sqrt{-g} \;+ \text{conj.},$$

which is a divergence; that term will be zero when J is constant. Finally, the impulse-energy tensor that is defined by J. M. Whittaker will become the electromagnetic tensor (36). That tensor is coupled to the asymmetric impulse-energy tensor of the theory of the photon by the relation (42).

CHAPTER III

APPLICATIONS

PRELIMINARIES

In a vacuum, the four components of the electromagnetic potential will satisfy the equation:

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \left(\frac{2\pi\mu_0 c}{h}\right)^2 U. \tag{1}$$

For a monochromatic wave of frequency $\nu = 2\pi W / h$, one can replace (1) with:

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0, \tag{2}$$

in which:

$$v^2 = c^2 / [1 - (\mu_0 \, c^2 / W)^2].$$

(2) is the classical equations, but with c replaced by v. Meanwhile, to simplify, we shall suppose in what follows that $(\mu_0 \, c^2 / W)^2$ is negligible in comparison to 1, in such a way that one can write c in place of v everywhere. That approximation is certainly valid for the applications that are of interest to us here. In this case, the indeterminacy in the potentials will persist in the form (II.28). It will make a difference which form one takes for (II.28), since the physical quantities of the theory of the photon will depend upon the potentials explicitly. In the examples that will be treated later, it will always be possible to annul the scalar potential V in a special reference system, because we will consider only monochromatic waves. The calculations above show that with V equal to zero, the results will be entirely satisfactory, whereas with non-zero V, one will be led to results that seem to be unacceptable.

Therefore, set:

$$V = 0; \tag{3}$$

for a monochromatic wave of frequency $\nu = 2\pi W / h$, one will then have:

$$\mathbf{E} = - ik\,\mathbf{A}, \qquad \text{in which} \qquad k = 2\pi\nu fc. \tag{4}$$

In order to define the wave Φ completely then the electromagnetic potentials are known, one must again give the values of I, J, $\mathbf{Q}$, and Q_4 . Now, in the present theory of the photon, those non-Maxwellian terms play a special role that is not actually linked with the Maxwellian quantities. In the absence of any indication on the values of those terms, we shall take arbitrarily:

$$I = J = Q_1 = \ldots = Q_4 = 0 \tag{5}$$

in the vacuum.

Upon taking (3), (4), and (5), formulas (II.25) and (II.27) will become:

$$\begin{cases} \rho W = \frac{1}{2} |\mathbf{E}|^2, & (a) \\ W \rho \mathbf{u} = \dfrac{c}{4} [\mathbf{E}^\times \cdot \mathbf{H}] + \text{conj.} & (b) \end{cases} \tag{6}$$

$$\begin{cases} \mathbf{N} = \dfrac{i}{2kc} [\mathbf{E}^\times \mathbf{E}], & (a) \\ N_4 = \dfrac{c}{4} (\mathbf{E}^\times \mathbf{H}) + \text{conj.} & (b) \end{cases} \tag{7}$$

Here, we remark that:

$$c\, t^{4a} = \rho\, W\, u^a \qquad \text{and} \qquad t^{44} = W\, \rho. \tag{6′}$$

One recognizes the classical expression for the Poynting vector in (6.*b*), while (6.*a*) will be equal to the Maxwellian energy density when $|\mathbf{E}|^2 = |\mathbf{H}|^2$. Nonetheless, there is a difference: For a monochromatic wave, they will depend upon time by way of the exponential $e^{2i\pi\nu t}$, whereas in the classical theory, one takes real expressions for the fields that are functions that are sinusoidal in time; the Maxwellian energy density I will also depend upon time then. Meanwhile, the measurable quantity is not I, but $\bar{I}$; i.e., the mean value of I in time. The essential quantity is then $\bar{I}$, and that is what one finds here.

Finally, thanks to (3), (4), and (5), one will get:

$$\pi_x = \frac{i}{kc}\frac{1}{2}\left(\mathbf{E}^\times \frac{\partial \mathbf{E}}{\partial x}\right), \qquad \pi_y = \frac{i}{kc}\frac{1}{2}\left(\mathbf{E}^\times \frac{\partial \mathbf{E}}{\partial y}\right), \qquad \pi_z = \frac{i}{kc}\frac{1}{2}\left(\mathbf{E}^\times \frac{\partial \mathbf{E}}{\partial z}\right), \tag{8}$$

and

$$t^{ab} = \frac{i}{k}\frac{1}{2}\left\{\left[\mathbf{E}^\times \frac{\partial \mathbf{H}}{\partial x^a}\right]_b - \left[\mathbf{H}^\times \frac{\partial \mathbf{E}}{\partial x^a}\right]_b\right\} \qquad a, b = 1, 2, 3 \tag{9}$$

for the impulse density and impulse flux, resp.

THE ELECTRIC DIPOLE WAVE

As one knows, the problem of electric dipole radiation plays an important role in the theory of atomic spectra. The classical argument is the following one: Let ([1]) $\mathfrak{Re}(\rho)$, in which $\rho = \rho_0(P)\, e^{i\omega t}$, be the value of the potential of a continuous electric distribution at a point P. Let Q be a potential point with coordinates x, y, z in the tri-rectangular trihedron $Oxyz$. One assumes that $\rho_0(P)$ can take on appreciable values only at points P such that the distance OP is very small in comparison to OQ. Finally, let $\mathfrak{Re}(\mathbf{D})$, with $\mathbf{D} = \mathbf{D}_0\, e^{i\omega t}$ and $\mathbf{D}_0 = \int \rho_0 \mathbf{r}\, d\sigma$, be the electric moment of that distribution $R\rho$. The electric and

([1]) The symbol $\mathfrak{Re}$ signifies the real part of F.

magnetic fields at the point Q that correspond to the dipole part of the radiation are then $\mathfrak{Re}(\mathbf{E})$ and $\mathfrak{Re}(\mathbf{H})$, with:

$$\mathbf{E} = \frac{e^{i\omega\left(t-\frac{r}{c}\right)}}{r} k^2 [\mathbf{D}_0 - \mathbf{n}\,(n\,D_0)] + \ldots,$$
$$\mathbf{H} = \frac{e^{i\omega\left(t-\frac{r}{c}\right)}}{r} k^2 [\mathbf{n}\,\mathbf{D}_0] + \ldots, \tag{10}$$

in which:

$$k = \omega / c, \qquad r = \sqrt{x^2 + y^2 + z^2}\,, \tag{11}$$

and $\mathbf{n}$ is the unit vector that is placed on the vector OQ. The complete expressions for the fields involve terms in r^{-2}, in addition to terms in r^{-1} that were written explicitly in (10); however, it is only the knowledge of the terms in r^{-1} that will be necessary for us in what follows. The energy flux per unit area and time is calculated with the aid of the Poynting vector:

$$\mathfrak{S} = c\,[\mathfrak{Re}(\mathbf{E}) \cdot \mathfrak{Re}(\mathbf{H})] \tag{12}$$

(in Heaviside-Lorentz units). Meanwhile, as for the plane wave, it is the mean value $\bar{\mathfrak{S}}$ of $\mathfrak{S}$ in time that is the essential quantity. Now:

$$\bar{\mathfrak{S}} = \frac{c}{4}[\mathbf{E}^{\times}\,\mathbf{H}] + \text{conj.} \tag{13}$$

By a classical argument, we thus arrive at the vector $\bar{\mathfrak{S}}$, which can be interpreted in the theory of the photon. Indeed, by virtue of (10) and (6), $\bar{\mathfrak{S}}$ will be the mean-value density of the energy flux per unit time and for a surface that is normal to the vector $\bar{\mathfrak{S}}$, which corresponds to a wave Φ whose sixteen components are given by (10), (3), (4), (5). By hypothesis, that wave Φ will define the electric dipole wave in the theory of the photon. The functions Φ have a pole at the origin here that will prevent us from calculating the mean values by formulas (II.8, 9). However, we can calculate the flux of energy and moment of impulse that cross a spherical surface σ with its center at O and a radius of R that tends to infinity. We will then have to take the integrals over a surface of very large radius and keep only the terms that are annulled when one lets R tend to infinity.

For the energy flux:

$$\mathbf{\Sigma} = \lim_{R\to\infty} \int_\sigma (\rho\mathbf{u}\cdot\mathbf{n})\,dS\,, \tag{14}$$

one will easily get:

$$\mathbf{\Sigma} = \lim_{R\to\infty} \int_\sigma \frac{c}{2}\frac{k^2}{R^2}(|\,\mathbf{D}\,|^2 - |\,n\mathbf{D}\,|^2)\,dS\,.$$

Thanks to the relations:

$$\int_\sigma n_a^2 \, dS = \frac{4\pi}{3} R^2 \qquad \text{and} \qquad \int_\sigma n_a n_b \, dS = 0 \ \text{ for } a \neq b,$$

one will find the known result:

$$\boldsymbol{\Sigma} = c\,k^4\,|\,\mathbf{D}\,|^2 \frac{4\pi}{3}. \tag{15}$$

We remark that $\rho\,\mathbf{u}$ is normal to the surface σ and:

$$(\rho\,\mathbf{u}\cdot\mathbf{n}) \approx \rho\,c, \tag{16}$$

up to order R^{-3}.

At each point of σ, the photon will then have a velocity that is normal to σ, up to order R^{-3}. The energy flux is then likewise given by:

$$\boldsymbol{\Sigma} = \lim_{R\to\infty} cW \int_\sigma \rho \, dS\,. \tag{17}$$

In order to calculate the flux of moment of impulse, the method that is used here is entirely different from that of the classical Maxwellian theory. Indeed, the flux of the total moment will be equal to the sum:

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^0 + \boldsymbol{\Phi}^s \tag{18}$$

of the flux of orbital moment and the flux of spin.

We first address the flux of orbital moment. Here again, we will verify that, for example:

$$\sum_a M_0^{1a2}\, n_a \approx [r\ \pi]_z \cdot c;$$

recall that π is defined by (8) and that from (II.60):

$$M_0^{1a2} = x\,t^{2a} - y\,t^{1a}\,.$$

We will then have, as we did for energy:

$$\Phi_z^0 = \lim_{R\to\infty} \int_\sigma \left(\sum M_0^{1a2} n_a\right) dS = \lim_{R\to\infty} c \int_\sigma [r\,\pi]_z \, dS\,. \tag{19}$$

On first glance, the calculation of the flux of orbital moment seems very complicated, because it seems that one must take terms in r^{-2} into account in the definition of the field (10) here; we will see that nothing of the sort is true. One has:

$$c\,[r\ \pi]_z = \frac{i}{2k}\left\{ x\left(\mathbf{E}^\times \frac{\partial \mathbf{E}}{\partial y}\right) - y\left(\mathbf{E}^\times \frac{\partial \mathbf{E}}{\partial x}\right)\right\}.$$

However:

$$\frac{\partial \mathbf{E}}{\partial y} = -\,ik\,n_2\,\mathbf{E} - \frac{n_2}{r}\mathbf{E} + \frac{e^{i\omega\left(t-\frac{r}{c}\right)}}{r}k^2\left\{-\frac{\partial \mathbf{n}}{\partial y}(n\mathbf{D}_0) - \mathbf{n}\left(\frac{\partial \mathbf{n}}{\partial y}\mathbf{D}_0\right)\right\} + \ldots$$

The unwritten terms are in r^{-3}, and the term $-\,ik\,n_2\,\mathbf{E}$ yields a zero contribution; that proves that the terms in r^{-2} in $\mathbf{E}$ do not need to be considered. The sequence of calculations is easy to perform and gives:

$$c\,[r\,\pi]_z = -\frac{ik^3}{2r^2}(n\,D_0)[n\,D_0^\times]_z + \ldots;$$

hence:

$$\Phi_z^0 = k^3\frac{i}{2}[\mathbf{D}^\times\,\mathbf{D}]_z\frac{4\pi}{3}.$$

The components x and y are obtained in the same manner. One has:

$$\boldsymbol{\Phi}^0 = k^3\frac{i}{2}[\mathbf{D}^\times\,\mathbf{D}]\frac{4\pi}{3}. \tag{20}$$

We finally calculate the spin flux. By virtue of (16), that flux will be equal to ([1]):

$$\boldsymbol{\Phi}^z = \lim_{R\to\infty} c\int_\sigma \mathbf{N}\,dS\,. \tag{21}$$

It suffices to replace $\mathbf{E}$ and $\mathbf{H}$ in (7) with their values that one infers from (10) in order to immediately deduce:

$$\boldsymbol{\Phi}^z = k^3\frac{i}{2}[\mathbf{D}^\times\,\mathbf{D}]\frac{4\pi}{3}. \tag{21$'$}$$

The flux $\mathbf{F}$ is then equal to:

$$\boldsymbol{\Phi} = k^3\frac{i}{2}[\mathbf{D}^\times\,\mathbf{D}]\frac{4\pi}{3}. \tag{22}$$

Thanks to (22) and (15), that will finally give:

$$\frac{\Phi}{\Sigma} = \frac{1}{\omega}\frac{i\,[\mathbf{D}^\times\mathbf{D}]}{|\,\mathbf{D}\,|^2}, \tag{23}$$

which is nothing but the important formula of Max Abraham and Arnold Sommerfeld.

Recall that one infers some interesting properties from (22) and (23) that are concerned with the conservation of the moments of impulse in the phenomena of atomic radiation. In the classical theory, formulas (22) and (23) are obtained by using the

([1]) Cf., the remark (1) at the end of the paragraph.

Maxwell tensor. The right-hand side of (22) will then express the flux of the moment of electromagnetic impulse that crosses σ in a unit of time. The identity of the results that are obtained in the case of the dipole wave in the theory of the photon, on the one hand, and Maxwell's theory, on the other, can be understood as a consequence of the relations (II.74). We shall not insist upon that point. However, the following remark undoubtedly deserves to be pointed out:

a) We first argue in the context of Maxwell's theory. Decompose the electric vector field **E** into two components, one of which $\mathfrak{Re}(\mathbf{E}_p)$ is situated in the plane of the wave, while the other one $\mathfrak{Re}(\mathbf{E}_n)$ is normal to that plane. The component $\mathfrak{Re}(\mathbf{E}_p)$ gives zero in the calculation of the flux of the electromagnetic moment of impulse; it is the consideration of $\mathfrak{Re}(\mathbf{E}_n)$ that leads to the good values (22) for the flux of electromagnetic moment of impulse.

b) In the theory of the photon, one likewise decomposes the vector $\mathfrak{Re}(\mathbf{E})$ into its two components $\mathfrak{Re}(\mathbf{E}_p)$ and $\mathfrak{Re}(\mathbf{E}_n)$. Here, as the preceding calculation showed, it is the terms in r^{-1} in the fields (10) that yield (22). The terms in r^{-2} – in particular, $\mathfrak{Re}(\mathbf{E}_n)$ – then give zero for $\mathbf{\Phi}$.

Remarks:

1. As in Maxwell's theory, the fluxes of orbital moment are represented by the components (*adb*) of a tensor M_0^{ijk}. As far as the densities and flux of spin are concerned, we have only a completely-antisymmetric tensor S^{ijk} at our disposal. Now, we have already pointed out (II, pp. 23) that the functions S^{ijk} cannot be interpreted as the spin flux, in full generality.

However, one verifies that on σ:

$$(\mathbf{N}\,\mathbf{n}) \approx -N_4 \quad \text{and} \quad \mathbf{N} \approx -N_4\,\mathbf{n},$$

up to terms of order R^{-3}. It will then result that:

$$\mathbf{\Phi}^s = \lim_{R\to\infty} c\int_\sigma \mathbf{N}\,dS = \lim_{R\to\infty} \int_\sigma (-cN_4)\,\mathbf{n}\,dS\,;$$

hence:

$$\Phi_z^s = \lim_{R\to\infty} \int_\sigma \left(\sum_a c\,S^{1a2} n_a\right) dS\,.$$

The flux of total moment about the *z*-axis through a closed surface σ that tends to infinity will then be equal to:

$$\Phi_z = \lim_{R\to\infty} \int_\sigma \left(\sum_a M^{1a2} n_a\right) dS\,,$$

with

$$M^{1a2} = M_0^{1a2} + c\, S^{1a2}.$$

That is what we said in the preceding chapter (pp. 23). The same situation will come about for quadrupole radiation.

2. Based upon the example of electric dipole radiation, we shall show the importance of the choice of potential in the theory of the photon. Thanks to (I.17 *e*, *f*), the potentials:

$$\begin{aligned} \mathbf{A} &= \frac{e^-}{r} ik\mathbf{D}_0 + \cdots \\ \varphi &= \frac{e^-}{r} ik(nD_0) + \cdots \end{aligned} \qquad \left[e^- = e^{i\omega\left(t - \frac{c}{r}\right)} \right] \tag{24}$$

define the fields (10) to all be like the potentials that are used in the calculations above. They yield the same values for the fluxes of energy and spin, but with (24), the flux of orbital moment will be zero! It does not seem possible to us to find an explanation for that fact in the present theory, which does not account for the phenomena of interaction between matter and radiation.

3. Heitler ([1]) has defined multipole radiation by means of fields that are everywhere finite. He has permitted us to calculate the means (II.8, 9) with the aid of those fields. The relations (II.71) will then show that the results that are obtained by theory of the photon must be identified with those of Heitler, as far as mean values are concerned.

THE ELECTRIC QUADRUPOLE WAVE

In order to define the electric dipole wave Φ in the theory of the photon, it will suffice to argue as in the preceding paragraph. While always using the same notations, one will then obtain:

$$\begin{aligned} \mathbf{E} &= \frac{ik^3}{2} \frac{e^{i\omega\left(t - \frac{r}{c}\right)}}{r} [\mathbf{q} - \mathbf{n}\,(nq)] + \ldots, \\ \mathbf{H} &= \frac{ik^3}{2} \frac{e^{i\omega\left(t - \frac{r}{c}\right)}}{r} [nq] + \ldots, \end{aligned} \tag{25}$$

in which:

$$q_a = \sum_b \mathfrak{M}_{ab}\, n_b \qquad (a, b = 1, 2, 3), \tag{25'}$$

([1]) Proceedings of the Cambridge Philosophical Society **32** (1936), pp. 112, 126.

and the $\mathfrak{N}_{ab} = \mathfrak{N}_{ba}$ represent the "quadrupole moment." The vectors $\mathfrak{Re}(\mathbf{E})$ and $\mathfrak{Re}(\mathbf{H})$ are then the quadrupole fields of the classical theory ([1]).

As in the case of dipole waves, we shall study the flux of energy and the moment of impulse that cross the closed surface σ that tends to infinity. We will then once more have to calculate the values of (14), (19), and (21) in the case where **E** and **H** are defined by (25).

One immediately finds that the energy flux is:

$$\mathbf{\Sigma} = \lim_{R\to\infty} \int \frac{c}{8}\frac{k^6}{R^2}\{|\,q\,|^2 - |\,nq\,|^2\}\, dS\,;$$

hence, after some calculations and upon utilizing the identities:

$$\int_\sigma n_a^4\, dS = \frac{4\pi}{5} R^2 \qquad \text{and} \qquad \int_\sigma n_a^2\, n_b^2\, dS = \frac{4\pi}{15} R^2 \quad (a \neq b),$$

one will get:

$$\mathbf{\Sigma} = c\frac{k^6}{8}\frac{4\pi}{15}\left(6\sum_{a,b}|\,\mathfrak{N}_{ab}\,|^2 + \sum_{a,b}|\,\mathfrak{N}_{aa} - \mathfrak{N}_{bb}\,|^2\right). \tag{26}$$

The symbol $\sum_{a,b}$ indicates a sum over the simple combinations of the numbers 1, 2, 3, when taken two at a time. The energy flux will always be positive then.

On the hand, one will get:

$$\Phi_z^s = \frac{ik^5}{8}\frac{4\pi}{15}\sum_a(\mathfrak{N}_{a1}^\times\mathfrak{N}_{a2} - \mathfrak{N}_{a2}^\times\mathfrak{N}_{a1}) \tag{27}$$

for $\mathbf{\Phi}^s$.

Finally, the calculation of the flux of orbital moment will give rise to the same remarks as in the case of a dipole wave. Here again, the terms in r^{-2} yield a zero contribution for $\mathbf{\Phi}^0$, and one will find:

$$\Phi_z^0 = \frac{ik^5}{8}\frac{4\pi}{15}\sum_a(\mathfrak{N}_{a1}^\times\mathfrak{N}_{a2} - \mathfrak{N}_{a2}^\times\mathfrak{N}_{a1})\cdot 5\,; \tag{28}$$

hence:

$$\Phi_z = \frac{ik^5}{8}\frac{4\pi}{15}\sum_a(\mathfrak{N}_{a1}^\times\mathfrak{N}_{a2} - \mathfrak{N}_{a2}^\times\mathfrak{N}_{a1})\cdot 6\,. \tag{29}$$

Finally:

$$\frac{\Phi_z}{\Sigma} = \frac{1}{\omega}\frac{6i\sum_a(\mathfrak{N}_{a1}^\times\mathfrak{N}_{a2} - \mathfrak{N}_{a2}^\times\mathfrak{N}_{a1})}{6\sum_{a,b}|\,\mathfrak{N}_{ab}\,|^2 + \sum_{a,b}|\,\mathfrak{N}_{aa} - \mathfrak{N}_{bb}\,|^2}\,, \tag{30}$$

([1]) See CONDON and SHORTLEY, *The Theory of Atomic Spectra*, Cambridge, 1935, pp. 93, *et seq.*

along with two others analogous expressions in Φ_x and Φ_y.

Formulas (26) to (30) can be utilized in the study of the conservation of moment of impulse in the phenomena of atomic electric quadrupole radiation, just as formulas (22), (23) were in the case of atomic (electric) dipole radiation ([1]). By way of example, suppose that:

$$\begin{aligned} &\mathfrak{N}_{11} = 1, \quad \mathfrak{N}_{12} = \pm i, \quad \mathfrak{N}_{22} = -1, \\ &\mathfrak{N}_{13} = \mathfrak{N}_{23} = \mathfrak{N}_{33} = 0. \end{aligned} \tag{31}$$

Thanks to (26),…(30) and (31), we immediately find:

$$\boldsymbol{\Sigma} = c\frac{k^6}{2}\frac{4\pi}{15}\cdot 3, \tag{32}$$

$$\Phi_z^s = \mp\frac{k^5}{2}\frac{4\pi}{15}, \qquad \Phi_x^s = \Phi_y^s = 0, \tag{33}$$

$$\Phi_z^0 = \mp\frac{k^5}{2}\frac{4\pi}{15}\cdot 5, \qquad \Phi_x^0 = \Phi_y^0 = 0; \tag{34}$$

hence:

$$\frac{\Phi_z}{\Sigma} = (\mp 2)\frac{1}{\omega}. \tag{35}$$

That formula shows that for a radiated photon of energy $h\,\nu$, the moment of impulse about the z-axis that is transported by the wave is equal to $\mp\ 2h / 2\pi$. On the other hand, the wave that is defined by (31) does not transport moment of impulse along the x-axis or the y-axis, in the mean. A deeper examination of the process of atomic quadrupole radiation shows that the relation (35) expresses the theorem of conservation of moment of impulse in the phenomenon of atomic quadrupole radiation. Indeed, one knows ([2]) that the values (31) correspond to the case in which, for the emitted photon, the projection along the z-axis of the moment of impulse of the electron that is responsible for the radiation increases by $\pm\ 2h / 2\pi$.

Remark: Remark (1) of the preceding paragraph is valid here, as well ([3]).

THE MONOCHROMATIC PLANE WAVE

The monochromatic plane solutions were studied in a detailed fashion by L. de Broglie. Here, we confine ourselves to the case in which the plane is defined by (II.30). The (complex) electromagnetic potentials and fields of that plane wave are:

([1]) For dipole radiations, see CONDON and SHORTLEY, *loc. cit.*, pp. 93.

([2]) CONDON and SHORTLEY, *loc. cit.*, pp. 23.

([3]) For the general case of spherical waves, see: J. GÉHÉNIAU, *Collection: La Chimie mathématiques*, 1938, Brussels.

$$A_x = (b - a)\,\frac{2hc}{2\pi\,\Delta}\,P, \qquad A_y = i\,(b + a)\,\frac{2hc}{2\pi\,\Delta}\,P, \qquad A_z = V = 0, \tag{36}$$

$$\begin{aligned} E_x &= -\,2i\,\frac{W}{\Delta}(b - a)\,P, \qquad E_y = 2\,\frac{W}{\Delta}(b + a)\,P, \qquad E_z = 0, \\ H_x &= -\,2\frac{cp}{\Delta}(b + a)\,P, \qquad H_y = -\,2i\,\frac{cp}{\Delta}(b - a)\,P, \qquad H_z = 0. \end{aligned} \tag{37}$$

Formulas (6.*a*, *b*) easily give:

$$\begin{aligned} \rho\,W &= (|\,a\,|^2 + |\,b\,|^2)\,\frac{4W^2}{\Delta^2}\,, \\ W\,\rho\,u_z &= W\,\rho\cdot\frac{c^2 p}{W}, \quad \rho\,u_z = \rho\,u_z = 0, \end{aligned} \tag{38}$$

which are the expected results. Similarly, formulas (7.*a*, *b*) give:

$$\begin{aligned} N_z &= \frac{h}{2\pi}\cdot\frac{4W^2}{\Delta^2}(|\,a\,|^2 - |\,b\,|^2), \qquad N_x = N_y = 0, \\ c\,S^{123} &= -\,c\,N_4 = N_z\,c^2\,p\,/\,W. \end{aligned} \tag{39}$$

Here again, one has (¹):

$$(\mathbf{N}\,\mathbf{u}) = -\,c\,N_4\,.$$

As for formulas (8) and (9), here they will become:

$$\begin{aligned} \pi_z &= \rho\,p, \qquad \pi_x = \pi_y = 0, \\ t^{33} &= p\,\rho\,u_z\,; \end{aligned} \tag{40}$$

the other components of t^{ab} are zero.

Calculate the density of the moment of total impulse (II.64) about the *z*-axis. By virtue of (40), $M_z^0 = 0$; hence:

$$M_z = N_z = \frac{h}{2\pi}\frac{4W}{\Delta^2}(|\,a\,|^2 - |\,b\,|^2). \tag{41}$$

Finally, since M_0^{123} is zero:

$$M^{123} = c\,S^{123} = N_z\,c^2\,p\,/\,W. \tag{42}$$

It results from (38) and (39) that a plane wave that propagates along *Oz* transports a moment of impulse about the *z*-axis, and the ratio of the moment flux to the energy flux (per unit time) is:

(¹) Cf., L. DE BROGLIE, *L'électron magnetique*, Paris, Hermann and Co., 1934. pp. 222.

$$\frac{M^{123}}{W\rho u_z} = \frac{N_z u_z}{\rho W u_z} = \frac{1}{2\pi\nu} \cdot \frac{|a|^2 - |b|^2}{|a|^2 + |b|^2}. \tag{43}$$

As E. Henriot remarked before ([1]), formula (43) is easily obtained Fresnel's theory of the elastic ether, in which the moment of impulse about the *z*-axis is provided by the motion of the ether particles in the plane of the wave. On the contrary, the result (43) seems to contradict the usual definition of the electromagnetic moments of impulse in Maxwell's theory. Indeed, the density of electromagnetic moment of impulse is defined by the vector:

$$\boldsymbol{\mu} = [\mathbf{r}\ \mathbf{G}], \tag{44}$$

in which **G** denotes the density of electromagnetic impulse. Now, **G** is normal to the plane wave. Hence, μ_z will be zero, and the same thing will be true for the electromagnetic flux of moment of impulse about the *z*-axis. By a deeper analysis of the question of the electromagnetic couples and moments, E. Henriot was led to introduce a moment of the second type into Maxwell's theory ([2]). He referred to that moment by the name of "momentor" and represented it by a tensor of order three. That seems quite natural in the expression for the theorem of the conservation of electromagnetic moment; E. Henriot obtained formula (43) in Maxwell's theory by the use of those momentors.

(September 1937)

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([1]) E. HENRIOT, "Les couples de radiation et les moments électromagnétiques," Mémorial des Sciences physiques, Paris, Gauthier-Villars, 1936, fasc. **30**.

([2]) *Loc. cit.* That tensor is nothing but the Maxwellian part of (41), with the difference that the fields are complex here.