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Theory of adiabatic invariants of general differential systems

By Harald Geppert in Giessen, currently in Rome ([1])

Translated by D. H. Delphenich

I.

Like so many mathematical problems, the question of adiabatic invariants of mechanical systems can thank quantum theory for its origin, and the first theorem that was proved in regard to them, namely, Sommerfeld's theorem on the adiabatic invariance of the phase integral ([2]), was proved for their sake. The fact that this question is also important for the problems of classical mechanics, independently of the currents of modern atomic theory, was first recognized only recently, and one can thank Levi-Civita ([3]) for a very elegant and thorough theory of the adiabatic invariants of mechanical systems that defines the basis for an incisive mathematical treatment of the entire complex. Levi-Civita's theory is subject to two essential restrictions: First of all, it is true only for Hamiltonian differential systems, and secondly, it assumes that the known integrals of the latter are in involution (in the Lie sense), and that naturally raises the question *a posteriori* of the extent to which the results obtained are independent of the stated conditions. An examination of them will show that the concept of adiabatic invariants goes much further than it did up to now, that it must be adapted to general differential systems, and that a far-reaching mathematical theory of it that refers to functional analysis can be developed ([4]) that is rich in significant results. The following article shall be devoted to that theory.

According to its nature, the theory to be developed has two facets: viz., a mathematical one and a physical one. In our treatment, we will place most of the weight on the former without making contact with the physically-important problems. That will become especially noticeable in the questions that relate to taking the mean and the classification of systems by order of imprimitivity. At first, we shall deal with the general concept of adiabatic invariant for an arbitrary differential system. These invariants will themselves be determined by a new differential system whose integrability conditions, which are not satisfied identically, define the necessary and sufficient assumptions for the existence of adiabatic invariants. It is a noteworthy fact that the invariants can be given directly for an extensive class of cases, and in that way, the

([1]) Fellow of the International Education Board.
([2]) Cf., Burgers 1, 2, and the proof that Born gave in 1.
([3]) Cf., Levi-Civita 1, 2.
([4]) Cf., the survey notes of Geppert 1, 2, 3.

Gibbs-Hertz theorem will be generalized. The theory that will be developed is rich in applications to the mechanics of non-Hamiltonian systems, as well.

§ 1. The concept of adiabatic invariants

We start with a first-order differential system of the form:

$$\frac{dx_i}{dt} = X_i\,(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) \qquad (i = 1, \ldots, n) \tag{1.1}$$

in the variables $x_1, \ldots, x_n$. Let the X_i be continuously-differentiable functions (in the domain in question) that include certain parameters $a_1, \ldots, a_\rho$, in addition to the quoted variables. We will assume (although it will not be essential in what follows) that they are independent of t. If the parameters $a_1, \ldots, a_\rho$ are constant then the system will have solutions that one can write out formally as:

$$x_i = x_i\,(t - t_0, c_1, \ldots, c_{n-1} \mid a_1, \ldots, a_\rho) \qquad (i = 1, \ldots, n), \tag{1.2}$$

in which $t_0, c_1, \ldots, c_{n-1}$ mean the n integration constants. They are single-valued in the small and therefore soluble in a certain region in the form:

$$\begin{aligned} f_i(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho) &= c_i, \\ f_n(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho) &= t - t_0, \end{aligned} \qquad (i = 1, \ldots, n-1). \tag{1.3}$$

It would be preferable to interpret these relations in the space R_n of $x_1, \ldots, x_n$. If we interpret t as time then the equations (1.2) will represent the path of a point P that moves in R_n whose velocity components are given by (1.1) as functions of the position. We would like to denote that trajectory, *or integral curve*, into the future by $\mathfrak{T}$. The first $n - 1$ of equations (1.3) represent $n - 1$ hypersurfaces in R_n of dimensions $n - 1$, and $\mathfrak{T}$ appears as their one-dimensional curve of intersection. All of that is true only in a region of R_n in which equations (1.2) are regular and uniquely-soluble.

Now, if the parameters $a_1, \ldots, a_\rho$ are no longer constant, but functions of t, as well, then one can indeed leave the solutions to the system in the form (1.2) [(1.3), resp.], in general, but the $t_0, c_1, \ldots, c_{n-1}$ must also be regarded as functions of t then, and indeed the latter are determined from differential equations:

$$\frac{dc_i}{dt} = \sum_{\nu=1}^{\rho} \frac{\partial f_i}{\partial a_\nu}\frac{da_\nu}{dt}, \qquad \frac{dt_0}{dt} = -\sum_{\nu=1}^{\rho} \frac{\partial f_n}{\partial a_\nu}\frac{da_\nu}{dt} \qquad (i = 1, \ldots, n-1), \tag{1.4}$$

which are based in perturbation theory. In order to integrate them, it would be preferable to substitute the relations (1.2) in place of the x_i in $\partial f_i / \partial a_\nu$ ($i = 1, \ldots, n$), and in that way, we might find perhaps the equations:

$$(1.5)\qquad \frac{\partial f_i}{\partial a_\nu} = \varphi_{i\nu}\,(t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho) \qquad (i = 1,\, \ldots,\, n\,;\; \nu = 1,\, \ldots,\, \rho).$$

The system (1.4) will then go to:

$$(1.6)\qquad dc_i = \sum_{\nu=1}^{\rho} \varphi_{i\nu}\, da_\nu\,, \qquad dt_0 = -\sum_{\nu=1}^{\rho} \varphi_{i\nu}\, da_\nu \qquad (i = 1,\, \ldots,\, n-1),$$

in which only the variables $t,\, t_0,\, c_1,\, \ldots,\, c_n\,,\, a_1,\, \ldots,\, a_\rho$ still occur. As is clear from (1.4), these equations are integrable for given $a_\nu\,(t)$, and the $c_\nu\,(t)$ then prove to be *fonctions de ligne* (i.e., functions along a curve) relative to the $a_\nu\,(t)$. In order for the latter to be ordinary functions of the a_ν , it is, on the other hand, necessary that equations (1.6) should be integrable without restriction for arbitrary a_ν, which will imply further integrability conditions. The relations (1.6), together with (1.2), yield the trajectories of the "perturbed" motion. If one starts from the case of constant parameters as one's foundation then one can see that they can be regarded as another form of equations (1.1).

In place of this differential system (1.6), whose consideration would give us nothing new, we choose another differential system as our basis in what follows that will emerge from (1.6) when one subjects the coefficients $\varphi_{i\nu}$ to a functional operation that relates to t. Here, we would like to draw upon those operations, in particular, that have the character of taking a mean, but a treatment of the more general problem of an arbitrary operation is also practicable. Hence, let $T = T_0,\, \ldots,\, T_1$ be an arbitrary, but fixed, interval of the variables t over which we take the mean, and let $F\,(t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho)$ be the distribution function of the latter, so in general we will define the *mean value* of a quantity α that depends upon $t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho$ to be:

$$(1.7)\qquad \bar{\alpha} = \int_{t_0+T_0}^{t_0+T_1} \alpha\,(t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho)\, F\,(t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho)\, dt$$
$$: \int_{t_0+T_0}^{t_0+T_1} F\,(t - t_0,\, c_1,\, \ldots,\, c_{n-1} \mid a_1,\, \ldots,\, a_\rho)\, dt,$$

naturally, under the assumption that the integrals that appear are meaningful ([5]). We understand the integration in (1.7) to mean that it refers to only the argument $t - t_0$ that appears explicitly in α and F, and thus treats the quantities $t_0,\, c_1,\, \ldots,\, c_{n-1}\,,\, a_1,\, \ldots,\, a_\rho$ as constants.

An application of the operation (1.7) to the coefficients of (1.6) will produce the new n-term differential system:

$$(1.8)\qquad dc_i = \sum_{\nu=1}^{\rho} \bar{\varphi}_{i\nu}\, da_\nu\,, \qquad dt_0 = -\sum_{\nu=1}^{\rho} \bar{\varphi}_{i\nu}\, da_\nu \qquad (i = 1,\, \ldots,\, n-1)$$

([5]) Up to now, in physics, one has considered only the simplest case of defining a mean – namely, $F \equiv 1$ – but our viewpoint will not imply any complication of that.

that contain only the variables t_0, c_1, …, c_{n-1} , a_1, …, a_ρ, and in which the a_1, …, a_ρ play the role of independent variables. That will define the essential foundation for the theory that follows. In the realm of physics, we pose the question of the conditions under which the system (1.6) can be replaced with (1.8) *in practice*. That will always be the case when the parameters change "infinitely-slowly" or "adiabatically" in comparison to the motion of the point P in a way that shall be made more precise ([6]). Naturally, we assume the existence of the $\overline{\varphi}_{i\nu}$, which will make it necessary for T to lie in the interval of regularity of (1.2). We shall refer to (1.8) briefly as the *mean system.*

The mean system is not always completely-integrable, and we will have to discuss its integrability conditions, which express restrictions on the original system (1.1). We now define:

Definition 1. – *Any invariant of the mean system will be called an adiabatic invariant.*

The simultaneous existence of the following ρ differential equations:

$$\sum_{i=1}^{n-1} \frac{\partial J}{\partial c_i}\overline{\varphi}_{i\nu} - \frac{\partial J}{\partial c_n}\overline{\varphi}_{n\nu} + \frac{\partial J}{\partial a_\nu} = 0 \qquad (\nu = 1, \ldots, \rho) \tag{1.9}$$

is characteristic of an adiabatic invariant J (t_0, c_1, …, c_{n-1} | a_1, …, a_ρ). The systems (1.8) and (1.9) say the same thing and possess the same integrability conditions.

If we call a quantity *invariant in the mean* with respect to the system (1.1) when the total derivative of the mean value that is defined by (1.7) with respect to t vanishes due to (1.1) then we can also replace the definition above with the following one:

Definition 2. – A function of the integration constants and the parameters of the system (1.1) that is invariant in the mean for all values of the da_ν / dt ($\nu = 1, \ldots, \rho$) is called an *adiabatic invariant* of that system.

Namely, one needs only to observe that, from (1.6):

$$\frac{dJ}{dt} = \sum_{\nu=1}^{\rho} \left\{ \sum_{i=1}^{n-1} \frac{\partial J}{\partial c_i}\varphi_{i\nu} - \frac{\partial J}{\partial t_0}\varphi_{n\nu} + \frac{\partial J}{\partial a_\nu} \right\} \frac{da_\nu}{dt},$$

and since J should not depend upon t explicitly, Definition 2 demands the validity of the system of equations (1.9).

This second definition is the original conceptualization that the physicists assumed. They then dealt with the process of finding quantities that would remain invariant under arbitrary adiabatic changes of the parameters, in which the "infinitely-weak" character of the changes is made more precise by saying that invariance in the mean must enter in place of ordinary invariance. That has the practical advantage that with system (1.9), one will arrive at equations that are much easier to integrate than the original relations (1.6).

([6]) Cf., the note in Levi-Civita 3.

As we will see, we will then be in a position to actually give their integrals in a large class of cases.

The mean values that are defined by (1.7) still depend upon t_0 in general. However, in what follows, we would like to confine ourselves exclusively to those cases in which that is not true, namely, the ones in which the interval T does not depend upon t_0, and our interest will be mainly concentrated upon the mean values that are taken over infinite intervals, so one must set:

$$(1.10)\qquad \bar{\alpha} = \lim_{T\to\infty} \int_{t_0}^{t_0+T} \alpha\,(t - t_0, c_1, \ldots, c_{n-1} \mid a_1, \ldots, a_\rho)\, F\,(t - t_0, c_1, \ldots, c_{n-1} \mid a_1, \ldots, a_\rho)\, dt$$

$$: \int_{t_0}^{t_0+T} F\ \ (t - t_0, c_1, \ldots, c_{n-1} \mid a_1, \ldots, a_\rho)\, dt,$$

because our theory will yield especially beautiful results in that case. One can then address the problem of adiabatic invariants more closely and restrict oneself to the search for those invariants that are independent of t_0 ; i.e., to those functions J (c_1, …, c_{n-1} | a_1, …, a_ρ) for which the equations:

$$(1.11)\qquad \sum_{i=1}^{n-1} \frac{\partial J}{\partial c_i}\,\bar{\varphi}_{i\nu} + \frac{\partial J}{\partial a_\nu} = 0 \qquad\qquad (\nu = 1, \ldots, \rho)$$

are satisfied identically.

Since any function of one or more adiabatic invariants is also an adiabatic invariant, in order to find the independent particular solutions of (1.11), it will suffice to find what the largest possible number for $n - 1$ would be. It will relate the quantities c_1, …, c_{n-1} to the parameters. Equations (1.11) assume the existence of the $\bar{\varphi}_{i\nu}$, and that, in turn, assumes the unique solubility of (1.2) in the domain of T. Should that condition not be fulfilled for a sequence of quantities $\bar{\varphi}_{i\nu}$, the problem could nonetheless remain meaningful when one assumes that J is independent of the corresponding c_i . Ultimately, one infers the following trivial theorem from (1.11):

Theorem 1. – *Every integral f_i (x_1 , …, x_n) = c_i of the original system that is free of the parameters is an adiabatic invariant.*

§ 2. – Two examples

Before we turn to the general theory, two examples might find their places here that we will treat later on from a different viewpoint.

Example 1. The one-dimensional elastic motion can be represented as follows:

$$\frac{dx_1}{dt} = x_2\,, \qquad \frac{dx_2}{dt} = -\,a_1\,x_1\,,$$

if a_1 means the elasticity constant. The integrals are:

$$f_1 \equiv \sqrt{x_1^2 + \frac{1}{a_1}x_2^2}\; = c_1, \qquad f_2 \equiv \arctan\,\sqrt{a_1}\,\frac{x_2}{x_1} = t - t_0\,.$$

If one takes $F \equiv 1$, $T = \infty$, or, what amounts to the same thing, one takes T to be equal to the period $2\pi/\sqrt{a_1}$ of the motion, then one will have:

$$\overline{\varphi}_{11} = -\frac{c_1}{4a_1},$$

and (1.11) will read:

$$\frac{\partial J}{\partial a_1} + \frac{c_1}{4a_1}\frac{\partial J}{\partial c_1}\; = 0,$$

whose general solution is:

(2.1) $$J = \; f\,(a_1\,c_1^4)\,;$$

i.e., under adiabatic variation, the elasticity constants change the amplitude of the oscillation in inverse proportion to the fourth root of the elasticity constants.

Example 2. The differential system:

$$x_1\frac{dx_1}{dt} = \; a_1^2\,x_2\,, \qquad \frac{dx_2}{dt} = -\;a_2^2\,,$$

possesses the integrals:

$$f_1 \equiv \; x_1^2 + \frac{a_1^2}{a_2^2}x_2^2\; = c_1\,, \qquad f_2 \equiv -\frac{x_2}{a_2^2}\; = t - t_0\,,$$

and the (elliptical) trajectory will be real only when $t - t_0$ lies between the limits $-\;\dfrac{\sqrt{c_1}}{a_1\,a_2}$ and $\dfrac{\sqrt{c_1}}{a_1\,a_2}$. If we take that interval to be T and take $F \equiv 1$ then the system (1.9) will read as follows:

$$\frac{\partial J}{\partial a_1} + \frac{2}{3}\frac{c_1}{a_1}\frac{\partial J}{\partial c_1}\; = 0, \qquad \frac{\partial J}{\partial a_2} - \frac{2}{3}\frac{c_1}{a_2}\frac{\partial J}{\partial c_1}\; = 0.$$

It is completely-integrable and possesses the solution:

(2.2) $$J = f\left(\frac{a_2}{a_1}c_1^{3/2}; t_0\right)$$

as an adiabatic invariant.

At this point, we would like to only refer to the example of the damped oscillator, which we will deal with later.

§ 3. – Two-dimensional systems.

Before we turn to the general problem, we will first deal with the case of two-dimensional systems (n = 2), because they can be treated completely and will provide us with a guide in the following considerations. We shall then address the system:

(3.1) $$\frac{dx_1}{dt} = X_1\,(x_1, x_2 \mid a_2, \ldots, a_\rho), \qquad \frac{dx_2}{dt} = X_2\,(x_1, x_2 \mid a_2, \ldots, a_\rho),$$

whose integral reads (we omit the index 1):

(3.2) $$f\,(x_1, x_2 \mid a_2, \ldots, a_\rho) = c,$$

which must then satisfy the equation:

(3.3) $$\frac{\partial f}{\partial x_1}X_1 + \frac{\partial f}{\partial x_2}X_2 \equiv 0\ .$$

One infers from this relation that a quantity μ must exist such that:

(3.4) $$\frac{\partial f}{\partial x_1} = -\,\mu X_2\,, \qquad \frac{\partial f}{\partial x_2} = \mu X_1\,.$$

The integrability condition for (3.4) requires that m must be a solution to the equation:

(3.5) $$\frac{\partial f}{\partial x_1}(\mu X_1) + \frac{\partial f}{\partial x_2}(\mu X_2) \equiv X_1\frac{\partial \mu}{\partial x_1} + X_2\frac{\partial \mu}{\partial x_2} + \mu\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) = 0;$$

i.e., that μ must be a so-called *Jacobi multiplier* of the system (3.1). However, first of all, μ is not an arbitrary multiplier, but only one of the multipliers that are established uniquely by equations (3.4); μ is a function of $x_1, x_2 \mid a_2, \ldots, a_\rho$.

We would like to assume that the trajectory $\mathfrak{T}$ is free of all singularities, in such a way that, in particular, the two derivatives $\partial f / \partial x_1$ and $\partial f / \partial x_2$ do not vanish simultaneously. That says that as a result of (3.4), μ will vanish nowhere along $\mathfrak{T}$, and therefore also in a certain neighborhood of $\mathfrak{T}$, due to continuity, if we assume that X_1, X_2 themselves behave

regularly. μ will then have a well-defined sign in its domain of regularity along $\mathfrak{T}$. Due to (3.1) and (3.4), the arc length element of $\mathfrak{T}$ can be brought into the form:

$$(3.6) \qquad ds = \sqrt{dx_1^2 + dx_2^2} = dt \cdot \sqrt{X_1^2 + X_2^2} = \frac{dt}{\mu} \cdot \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} = dt \cdot \frac{\Gamma_{12}}{\mu},$$

in which we have set:

$$(3.7) \qquad \Gamma_{12}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2.$$

According our assumptions, Γ_{12} also vanishes nowhere along $\mathfrak{T}$ and can be assumed to have the same sign as μ when the sense of the direction of s is oriented towards advancing t.

The relation (3.6) puts us in a position to replace the mean value over t in our definitions with spatial integral constructions that are extended over $\mathfrak{T}$. Under this transition, by means of (1.2), a spatial interval S on $\mathfrak{T}$ for the variable s will enter in place of the interval T for the variable $t - t_0$. Furthermore, the distribution function $F(t - t_0, c_1 \mid a_1, \ldots, a_\rho)$ will be replaced with a single-valued, regular, distribution function $F(x_1, x_2 \mid a_1, \ldots, a_\rho)$ by means of (1.3), such that the fundamental operation (1.7) will go to:

$$(3.8) \qquad \bar{\alpha} = \int_S \alpha(x_1, x_2 \mid a_1, \ldots, a_\rho) F(x_1, x_2 \mid a_1, \ldots, a_\rho) \frac{\mu}{\Gamma_{12}} ds : \int_S F(x_1, x_2 \mid a_1, \ldots, a_\rho) \frac{\mu}{\Gamma_{12}} ds .$$

Here, we must add a remark about μ. Previously, μ was a multiplier of (3.1) that was determined uniquely by (3.4). Now, however, one has the easily-proved theorem that the quotient of two different multipliers is an invariant of the original system; in other words: The multipliers differ by only a constant factor along the trajectory $\mathfrak{T}$. Since the latter cancel out in (3.8), μ must then mean any arbitrary multiplier; i.e., any solution of (3.5).

If the system (3.1) is a *Liouville system* – i.e., its "divergence" vanishes:

$$(3.9) \qquad \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 0,$$

then (3.5) will show that one has set μ = const., and therefore μ can be dropped from (3.8) and following relations. A special case of the Liouville systems are, in turn, the *Hamiltonian systems* ([7]).

As a result of (3.8), the defining equations (1.11) for the adiabatic invariants will assume the form:

$$(3.10) \qquad \frac{\partial J}{\partial c} \cdot \int_S \frac{\partial f}{\partial a_\nu} F \frac{\mu}{\Gamma_{12}} ds + \frac{\partial J}{\partial a_\nu} \cdot \int_S F \frac{\mu}{\Gamma_{12}} ds = 0 \qquad (\nu = 1, \ldots, \rho).$$

([7]) Another derivative of (3.8) for the latter can be found in Levi-Civita, 1, pp. 335-337.

The most important case to be considered is that of simple averaging: $F \equiv 1$. Before we go on to the solution of those equations, we must investigate their integrability conditions.

§ 4. – The necessary integrability conditions.

The conditions for the complete integrability of the system (1.11), or what amounts to the same thing, the first $n - 1$ equations (1.8) read:

$$\frac{d}{da_\lambda}\overline{\varphi}_{1\kappa} - \frac{d}{da_\kappa}\overline{\varphi}_{1\lambda} \equiv \frac{\partial}{\partial a_\lambda}\overline{\varphi}_{1\kappa} - \frac{\partial}{\partial a_\kappa}\overline{\varphi}_{1\lambda} + \overline{\varphi}_{1l}\frac{\partial}{\partial c}\overline{\varphi}_{1\kappa} - \overline{\varphi}_{1\kappa}\frac{\partial}{\partial c}\overline{\varphi}_{1\lambda} = 0 \tag{4.1}$$

$$(\kappa, \lambda = 1, \ldots, \rho)$$

in our two-dimensional case. If we temporarily introduce the notation:

$$g = \int_S F\frac{\mu}{\Gamma_{12}}ds, \qquad g_\nu = \int_S \frac{\partial f}{\partial a_\nu}F\frac{\mu}{\Gamma_{12}}ds \qquad (\nu = 1, \ldots, \rho) \tag{4.2}$$

then we can put (4.1) into the form:

$$g\left(\frac{\partial g_\kappa}{\partial a_\lambda} - \frac{\partial g_\lambda}{\partial a_\kappa}\right) + g_\lambda\left(\frac{\partial g_\kappa}{\partial c} + \frac{\partial g}{\partial a_\kappa}\right) - g_\kappa\left(\frac{\partial g_\lambda}{\partial c} + \frac{\partial g}{\partial a_\lambda}\right) = 0 \qquad (\kappa, \lambda = 1, \ldots, \rho), \tag{4.3}$$

and our problem will now be to actually calculate the derivatives that appear in this.

The piece S of the trajectory $\mathfrak{T}$ to be considered might extend from the point P_0, which corresponds to the value $t - t_0 = T_0$, up to P_1, which is assumed when $t - t_0 = T_1$, in such a way that $T_1 - T_0 = T$ means the averaging interval. If one takes the mean over the entire, infinitely-long trajectory then the consideration of the boundary points P_0, P_1 will become superfluous in what follows. T_0, as well as T_1, are functions of $c \mid a_1, \ldots, a_\rho$. It will then be preferable to introduce the so-called *synchronous trajectories* [cf., (1.3)]:

$$f_2(x_1, x_2 \mid a_1, \ldots, a_\rho) = t - t_0 = \text{const.}, \tag{4.4}$$

which we would like to generally denote by $\mathfrak{S}$, along with the family of curves (3.2) that is defined by the trajectories $\mathfrak{T}$, and we would like to regard each of the points under consideration as the intersection of a $\mathfrak{T}$ with (at least) one $\mathfrak{S}$. The boundary points will be cut out of $\mathfrak{T}$ by the synchronous trajectories:

$$f_2(x_1, x_2 \mid a_1, \ldots, a_\rho) = T_0 \ (T_1, \text{ resp.}). \tag{4.5}$$

If we now change only c by dc and leave $a_1, \ldots, a_\rho$ fixed then $\mathfrak{T}$ will go over to a new trajectory $\mathfrak{T}'$ with the equation:

$$f(x_1, x_2 \mid a_1, \ldots, a_\rho) = c + dc, \tag{4.6}$$

and its boundary points P_0', P_1' will be cut out by the synchronous trajectories:

$$f(x_1, x_2 \mid a_1, \ldots, a_\rho) = T_0 + \frac{\partial T_0}{\partial c}\,dc \qquad (T_1 + \frac{\partial T_1}{\partial c}\,dc, \text{ resp.}). \tag{4.7}$$

We associate each point P of $\mathfrak{T}$ with the point on its normal n that is the point of intersection of $\mathfrak{T}$ with $\mathfrak{T}'$, and call that point P'. This association makes the points P_0 (P_1, resp.) correspond to two points $\bar{P}_0'$ ($\bar{P}_1'$, resp.), which will generally be different from P_0' P_1'. In order to calculate the derivatives with respect to c that appear in (4.3), we must extend the integral g_κ along $\mathfrak{T}'$ from P_0' to P_1' and then subtract the value that it assumes along $\mathfrak{T}$. We can then write:

$$\frac{\partial g_\kappa}{\partial c} = \lim_{dc \to 0} \frac{1}{dc} \left\{ \int_{\bar{P}_0'}^{\bar{P}_1'} + \int_{P_0'}^{P_1'} + \left(\int_{\bar{P}_0'}^{\bar{P}_1'} - \int_{P_0}^{P_1} \right) \right\}. \tag{4.8}$$

We call the first summands the (infinitesimal) *boundary terms*, while the third one can be easily calculated by means of our normal association $P \to P'$. Namely, let dn be the normal displacement $P' \to P$, and let its components be $d'x_1$, $d'x_2$, so:

$$d'x_1 = \frac{1}{\Gamma_{12}} \frac{\partial f}{\partial x_1} d'n, \qquad d'x_2 = \frac{1}{\Gamma_{12}} \frac{\partial f}{\partial x_2} d'n, \tag{4.9}$$

and all quantities that relate to $\mathfrak{T}'$ might be indicated by a prime. One will then have, up to second-order quantities:

$$d's = \sqrt{d(x_1 + d'x_1)^2 + d(x_2 + d'x_2)^2} = ds \left\{ 1 + \frac{1}{\Gamma_{12}} \frac{\partial f}{\partial x_2} \frac{d}{ds}(d'x_1) - \frac{1}{\Gamma_{12}} \frac{\partial f}{\partial x_1} \frac{d}{ds}(d'x_2) \right\}$$

$$= ds \left\{ 1 + \frac{d'n}{\Gamma_{12}^3} \left[\frac{\partial^2 f}{\partial x_1^2} \left(\frac{\partial f}{\partial x_2} \right)^2 - 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} + \frac{\partial^2 f}{\partial x_2^2} \left(\frac{\partial f}{\partial x_1} \right)^2 \right] \right\}, \tag{4.10}$$

$$\frac{1}{\Gamma_{12}'} = \frac{1}{\Gamma_{12}} \left\{ 1 - \frac{d'n}{\Gamma_{12}^3} \left[\frac{\partial^2 f}{\partial x_1^2} \left(\frac{\partial f}{\partial x_1} \right)^2 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} + \frac{\partial^2 f}{\partial x_2^2} \left(\frac{\partial f}{\partial x_2} \right)^2 \right] \right\}.$$

On the other hand, it follows immediately from (4.6) and (4.9) that:

$$\frac{\partial f}{\partial x_1} d'x_1 + \frac{\partial f}{\partial x_2} d'x_2 \equiv \Gamma_{12}\, d'n = dc,$$

so:

$$(4.11) \qquad d'n = \frac{dc}{\Gamma_{12}}, \qquad d'x_1 = \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1} dc, \qquad d'x_2 = \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2} dc,$$

and we will then be in a position to write down the third summand in (4.8).

We shall now calculate the boundary terms. From (4.11), the coordinates of $\bar{P}_0'$ are:

$$\bar{P}_0': \qquad x_1 + \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1} dc \Big|_0, \qquad x_2 + \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2} dc \Big|_0,$$

while we can find those of P_0' from the two equations (4.6) and (4.7). Namely, if ($\delta' x_1$, $\delta' x_2$) means the vector $P_0 P_0'$ then we must have:

$$\frac{\partial f}{\partial x_1}\delta' x_1 + \frac{\partial f}{\partial x_2}\delta' x_2 = dc,$$

$$\frac{\partial f_2}{\partial x_1}\delta' x_1 + \frac{\partial f_2}{\partial x_2}\delta' x_2 = \frac{\partial T_0}{\partial c} dc,$$

so when we temporarily set:

$$\mathfrak{F} = \frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_1}$$

we will have:

$$(4.12) \qquad \delta' x_1 = \frac{dc}{\mathfrak{F}}\left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial T_0}{\partial c}\right)\Big|_0, \qquad \delta' x_2 = \frac{dc}{\mathfrak{F}}\left(-\frac{\partial f_2}{\partial x_1} + \frac{\partial f}{\partial x_1}\frac{\partial T_0}{\partial c}\right)\Big|_0.$$

From the equation that follows from (3.1):

$$(4.13) \qquad \frac{\partial f}{\partial x_1} X_1 + \frac{\partial f}{\partial x_2} X_2 = 0, \quad \frac{\partial f_2}{\partial x_1} X_1 + \frac{\partial f_2}{\partial x_2} X_2 = 1,$$

it will follow that $\mathfrak{F} \neq 0$. The components of $P_0' \bar{P}_0'$ will then be:

$$dc\left\{\frac{1}{\mathfrak{F}}\left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial T_0}{\partial c}\right) - \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1}\right\}\Big|_0, \qquad dc\left\{\frac{1}{\mathfrak{F}}\left(-\frac{\partial f_2}{\partial x_1} + \frac{\partial f}{\partial x_1}\frac{\partial T_0}{\partial c}\right) - \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2}\right\}\Big|_0,$$

and therefore, the length of that segment will be:

$$\overline{P_0' \bar{P}_0'} = \frac{dc}{\mathfrak{F}\,\Gamma_{12}}\left[\Gamma_{12}^2\frac{\partial T_0}{\partial c} - \left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_0$$

$$= dc\cdot\sqrt{X_1^2 + X_2^2}\left[\frac{\partial T_0}{\partial c} - \frac{1}{\Gamma_{12}^2}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_0 .$$

One finds a corresponding expression for the other boundary point, so the boundary terms in the sum (4.8) will ultimately have the value:

$$\frac{\partial f}{\partial a_\kappa} F\mu\frac{\sqrt{X_1^2 + X_2^2}}{\Gamma_{12}}\left[\frac{\partial T_{0,1}}{\partial c} - \frac{1}{\Gamma_{12}^2}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_{P_0}^{P_1} .$$

We are finally in a position to summarize the expression for $\partial g_\kappa / \partial c$ then; from the foregoing, we have:

(4.14)
$$\begin{cases} \dfrac{\partial g_\kappa}{\partial c} = -\displaystyle\int_S \frac{\partial f}{\partial a_\kappa} F\frac{\mu}{\Gamma_{12}^2}\left\{\left(\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2}\right)\left(\left(\frac{\partial f}{\partial x_1}\right)^2 - \left(\frac{\partial f}{\partial x_2}\right)^2\right) + 4\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\right\} ds \\ \quad + \displaystyle\int_S F\frac{\mu}{\Gamma_{12}^2}\left(\frac{\partial^2 f}{\partial a_\kappa \partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial^2 f}{\partial a_\kappa \partial x_2}\frac{\partial f}{\partial x_2}\right) ds \\ \quad + \displaystyle\int_S \frac{\partial f}{\partial a_\kappa}\frac{1}{\Gamma_{12}^2}\left(\frac{\partial \mu}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial \mu}{\partial x_2}\frac{\partial f}{\partial x_2}\right) ds \\ \quad + \displaystyle\int_S \frac{\partial f}{\partial a_\kappa}\frac{\mu}{\Gamma_{12}^2}\left(\frac{\partial F}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial F}{\partial x_2}\frac{\partial f}{\partial x_2}\right) ds \\ \quad + \dfrac{\partial f}{\partial a_\kappa} F\mu\dfrac{\sqrt{X_1^2 + X_2^2}}{\Gamma_{12}}\left[\dfrac{\partial T_{0,1}}{\partial c} - \dfrac{1}{\Gamma_{12}^2}\left(\dfrac{\partial f}{\partial x_1}\dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_{P_1}^{P_0} . \end{cases}$$

We shall now carry out the calculation of the derivatives of g and g_κ with respect to a_λ in the same way. If a_λ goes to $a_\lambda + da_\lambda$, while the remaining quantities remain fixed, then the curve $\mathfrak{T}^*$ with the equation:

(4.15) $$f(x_1, x_2 \mid a_1, \ldots, a_\lambda + da_\lambda, \ldots, a_\rho) = c$$

will enter in place of $\mathfrak{T}$, and the boundary points will be cut out of its by the synchronous trajectories:

(4.16) $$f_2(x_1, x_2 \mid a_1, \ldots, a_\lambda + da_\lambda, \ldots, a_\rho) = T_0 + \frac{\partial T_0}{\partial a_\lambda} da_\lambda \qquad [T_1 + \frac{\partial T_1}{\partial a_\lambda} da_\lambda \text{, resp.}] .$$

We denote them by P_0^*, P_1^*, while $\bar{P}_0^*$, $\bar{P}_1^*$ shall denote the points that lie normally to $\mathfrak{T}^*$ over P_0, P_1, resp. We associate every point of $\mathfrak{T}$ with its point P^* that lies along the normal and let d^*n denote the vector PP^* whose components are:

$$d^*x_1 = \frac{1}{\Gamma_{12}}\frac{\partial f}{\partial x_1}\, d^*n, \qquad d^*x_2 = \frac{1}{\Gamma_{12}}\frac{\partial f}{\partial x_2}\, d^*n\,. \tag{4.17}$$

A decomposition that corresponds to (4.8) will also lead to our goal here. However, it follows from (4.15) and (4.17) that:

$$\frac{\partial f}{\partial x_1} d^*x_1 + \frac{\partial f}{\partial x_2} d^*x_2 \equiv \Gamma_{12}\, d^*n = -\frac{\partial f}{\partial a_\lambda} da_\lambda\,,$$

so

$$d^*n = -\frac{\partial f}{\partial a_\lambda}\frac{da_\lambda}{\Gamma_{12}}, \qquad d^*x_1 = -\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial a_\lambda} da_\lambda\,, \qquad d^*x_2 = -\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial a_\lambda} da_\lambda\,. \tag{4.18}$$

The quantities that refer to $\mathfrak{T}^*$ shall be given an asterisk in what follows. From (4.10), one will then have:

$$d^*n = ds\left\{1 - \frac{da_\lambda}{\Gamma_{12}^4}\frac{\partial f}{\partial a_\lambda}\left[\frac{\partial^2 f}{\partial x_1^2}\left(\frac{\partial f}{\partial x_2}\right)^2 - 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} + \frac{\partial^2 f}{\partial x_2^2}\left(\frac{\partial f}{\partial x_1}\right)^2\right]\right. \tag{4.19}$$

$$\left. - \frac{da_\lambda}{\Gamma_{12}^2}\left[\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial a_\lambda} + \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_2 \partial a_\lambda}\right]\right\},$$

and by means of these formulas, we will be in a position to write down terms that correspond to the third summand of (4.8).

It once more remains for us to determine the boundary terms. Due to (4.18), $\bar{P}_0^*$ has the coordinates:

$$\bar{P}_0^*: \qquad x_1 - \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial a_\lambda} da_\lambda\bigg|_0, \qquad x_2 - \frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial a_\lambda} da_\lambda\bigg|_0\,.$$

By contrast, P_0^* might have the coordinates $x_i + d^*x_i\,|_0$, so it will follow from (4.15) and (4.16) that:

$$\frac{\partial f}{\partial x_1}\delta^* x_1 + \frac{\partial f}{\partial x_2}\delta^* x_2 = -\frac{\partial f}{\partial a_\lambda} da_\lambda\,,$$

$$\frac{\partial f_2}{\partial x_1}\delta^* x_1 + \frac{\partial f_2}{\partial x_2}\delta^* x_2 = \frac{\partial (T_0 - f_2)}{\partial a_\lambda} da_\lambda\,,$$

are valid at P_0, and therefore:

$$\delta^* x_1 = \frac{da_\lambda}{\mathfrak{F}}\left(-\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial a_\lambda}-\frac{\partial f}{\partial x_2}\frac{\partial (T_0-T)}{\partial a_\lambda}\right),$$

(4.21)

$$\delta^* x_2 = \frac{da_\lambda}{\mathfrak{F}}\left(+\frac{\partial f_2}{\partial x_1}\frac{\partial f}{\partial a_\lambda}+\frac{\partial f}{\partial x_1}\frac{\partial (T-f_2)}{\partial a_\lambda}\right).$$

The components of the vector $P_0^* \bar{P}_0^*$ will then be:

$$da_\lambda\left[-\frac{1}{\mathfrak{F}}\left(-\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial a_\lambda}-\frac{\partial f}{\partial x_2}\frac{\partial (T_0-T)}{\partial a_\lambda}\right)+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial a_\lambda}\right],$$

$$da_\lambda\left[\frac{1}{\mathfrak{F}}\left(\frac{\partial f_2}{\partial x_1}\frac{\partial f}{\partial a_\lambda}+\frac{\partial f}{\partial x_1}\frac{\partial (T_0-f_2)}{\partial a_\lambda}\right)+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial a_\lambda}\right],$$

and therefore the length of that vector will be:

$$\overline{P_0^* \bar{P}_0^*} = \frac{da_\lambda}{\mathfrak{F}\,\Gamma_{12}}\left[\frac{\partial (T_0-T)}{\partial a_\lambda}\Gamma_{12}^2+\frac{\partial f}{\partial a_\lambda}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1}+\frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_0$$

$$= da_\lambda \cdot \sqrt{X_1^2+X_2^2}\left[\frac{\partial (T_0-T)}{\partial a_\lambda}+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial a_\lambda}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1}+\frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_0,$$

and a corresponding expression will be true at the other end point. Hence, for example, the boundary term in the decomposition of $\partial g / \partial a_\lambda$ that corresponds to the first two summands in (4.8) be:

$$F\,\mu\frac{\sqrt{X_1^2+X_2^2}}{\Gamma_{12}}\left[\frac{\partial (T_0-T)}{\partial a_\lambda}+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial a_\lambda}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1}+\frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_{P_1}^{P_0},$$

and analogously for $\partial g_\kappa / \partial a_\lambda$, where only the factor $\partial f / \partial a_\lambda$ enters in.

If one combines everything then one will ultimately find that:

$$
\begin{aligned}
\frac{\partial g}{\partial a_\lambda} &= \int_S \frac{\partial f}{\partial a_\lambda} F \frac{\mu}{\Gamma_{12}^5}\left\{\left(\frac{\partial^2 f}{\partial x_1^2}-\frac{\partial^2 f}{\partial x_2^2}\right)\left(\left(\frac{\partial f}{\partial x_1}\right)^2-\left(\frac{\partial f}{\partial x_2}\right)^2\right)+4\frac{\partial^2 f}{\partial x_1\partial x_2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\right\} \\
&\quad -\int_S F\frac{\mu}{\Gamma_{12}^3}\left(\frac{\partial^2 f}{\partial a_\lambda \partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial^2 f}{\partial a_\lambda \partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\lambda} F\frac{1}{\Gamma_{12}^3}\left(\frac{\partial \mu}{\partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial \mu}{\partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\lambda}\frac{\mu}{\Gamma_{12}^3}\left(\frac{\partial F}{\partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial f}{\partial x_2}\right)ds+\int_S\left(F\frac{\partial \mu}{\partial a_\lambda}+\mu\frac{\partial F}{\partial a_\lambda}\right)\frac{1}{\Gamma_{12}}ds \\
&\quad +F\mu\frac{\sqrt{X_1^2+X_2^2}}{\Gamma_{12}}\left[\frac{\partial(T_{0,1}-f_2)}{\partial a_\lambda}+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial a_\lambda}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1}+\frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_{P_1}^{P_0},
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
\frac{\partial g_\kappa}{\partial a_\lambda} &= \int_S \frac{\partial f}{\partial a_\kappa}\frac{\partial f}{\partial a_\lambda} F \frac{\mu}{\Gamma_{12}^5}\left\{\left(\frac{\partial^2 f}{\partial x_1^2}-\frac{\partial^2 f}{\partial x_2^2}\right)\left(\left(\frac{\partial f}{\partial x_1}\right)^2-\left(\frac{\partial f}{\partial x_2}\right)^2\right)+4\frac{\partial^2 f}{\partial x_1\partial x_2}\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\right\}ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\kappa} F\frac{\mu}{\Gamma_{12}^3}\left(\frac{\partial^2 f}{\partial a_\lambda \partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial^2 f}{\partial a_\lambda \partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\lambda} F\frac{1}{\Gamma_{12}^3}\left(\frac{\partial^2 f}{\partial a_\kappa \partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial^2 f}{\partial a_\kappa \partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\kappa}\frac{\partial f}{\partial a_\lambda} F\frac{1}{\Gamma_{12}^3}\left(\frac{\partial \mu}{\partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial \mu}{\partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad -\int_S \frac{\partial f}{\partial a_\kappa}\frac{\partial f}{\partial a_\lambda}\frac{\mu}{\Gamma_{12}^3}\left(\frac{\partial F}{\partial x_1}\frac{\partial f}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial f}{\partial x_2}\right)ds \\
&\quad +\int_S \frac{\partial f}{\partial a_\lambda}\left(F\frac{\partial \mu}{\partial a_\lambda}+\mu\frac{\partial F}{\partial a_\lambda}\right)\frac{1}{\Gamma_{12}}ds+\int_S \frac{\partial^2 f}{\partial a_\kappa \partial a_\lambda}F\frac{\mu}{\Gamma_{12}}ds \\
&\quad +\frac{\partial f}{\partial a_\kappa}F\mu\frac{\sqrt{X_1^2+X_2^2}}{\Gamma_{12}}\left[\frac{\partial(T_{0,1}-f_2)}{\partial a_\lambda}+\frac{1}{\Gamma_{12}^2}\frac{\partial f}{\partial a_\lambda}\left(\frac{\partial f}{\partial x_1}\frac{\partial f_2}{\partial x_1}+\frac{\partial f}{\partial x_2}\frac{\partial f_2}{\partial x_2}\right)\right]\Bigg|_{P_1}^{P_0}.
\end{aligned}
\tag{4.23}
$$

This somewhat-lengthy formula puts us in a position to write down the integrability conditions (4.3) *in extenso*. One first finds the simple equations from (4.14), (4.22), and (4.23):

$$
\frac{\partial g_\kappa}{\partial a_\lambda}-\frac{\partial g_\lambda}{\partial a_\kappa}=\int_S\left\{\frac{\partial f}{\partial a_\kappa}\frac{\partial(\mu F)}{\partial a_\lambda}-\frac{\partial f}{\partial a_\lambda}\frac{\partial(\mu F)}{\partial a_\kappa}\right\}\frac{1}{\Gamma_{12}}ds \tag{4.24}
$$

$$+ F\mu \frac{\sqrt{X_1^2 + X_2^2}}{\Gamma_{12}} \left[\frac{\partial f}{\partial a_\kappa} \frac{\partial (T_{0,1} - f_2)}{\partial a_\lambda} - \frac{\partial f}{\partial a_\lambda} \frac{\partial (T_{0,1} - f_2)}{\partial a_\kappa} \right] \Bigg|_{P_1}^{P_0},$$

(4.25)
$$\frac{\partial g_\kappa}{\partial c} + \frac{\partial g}{\partial a_\kappa} = \int_S \frac{\partial (\mu F)}{\partial a_\kappa} \frac{1}{\Gamma_{12}} ds + F\mu \frac{\sqrt{X_1^2 + X_2^2}}{\Gamma_{12}} \left[\frac{\partial (T_{0,1} - f_2)}{\partial a_\kappa} + \frac{\partial f}{\partial a_\kappa} \frac{\partial T_{0,1}}{\partial c} \right] \Bigg|_{P_1}^{P_0}.$$

In the following, we would like to restrict ourselves exclusively to the case in which the boundary terms in these two equations that relate to P_0 and P_1 vanish in their own right, which represents a condition for T_0, T_1 that must now be interpreted. The vanishing that we require is obviously identical to the condition that the relations:

(4.26)
$$\frac{\partial (f_2 - T_{0,1})}{\partial a_\kappa} = \frac{\partial f}{\partial a_\kappa} \frac{\partial T_{0,1}}{\partial c} \qquad (\kappa = 1, \ldots, \rho)$$

must be true at the end points P_0, P_1. One then infers from (4.12) and (4.21) that:

(4.27)
$$\frac{\delta' x_1}{\delta' x_2} = \frac{\delta^* x_1}{\delta^* x_2};$$

i.e., that the points P_0, P_0', and P_0^* (P_1, P_1', P_1^*, resp.) lie along a line. Conversely, if that point-triple lies along a line then (4.27) must be true, and one will infer the validity of (4.26) from (4.12), (4.21), as we have already remarked above, $\mathfrak{F} \neq 0$. One can also express that differently: Namely, if one assumes that the boundary points lie along a curve $\mathfrak{C}$ that is independent of $a_1, \ldots, a_\rho$ and has the equation:

(4.28)
$$\Pi (x_1, x_2) = 0$$

then all boundary points with the same index must lie on the tangent to $\mathfrak{C}$ at P_0 (P_1, resp.), while it will, in fact, follow from (4.28) that:

$$\frac{\delta' x_1}{\delta' x_2} = \frac{\delta^* x_1}{\delta^* x_2} = - \frac{\partial \Pi}{\partial x_2} : \frac{\partial \Pi}{\partial x_1} \Bigg|_{0,1},$$

so (4.26) must be true. Conversely, if the boundary points lie along a line then they will be cut out of a curve whose equation does not include $a_1, \ldots, a_\rho$ (e.g., just that line). Our demand in regard to the vanishing of the boundary terms is then equivalent to the assumption:

Requirement: *The averaging interval S either extends over the infinite length of the trajectories or it will be cut out of it by a fixed curve that does not include the parameters.*

We have then combined *both* curves that cut out P_0 and P_1 into a single one, which is always allowed.

If that postulate is fulfilled then (4.3), (4.24), and (4.25) will ultimately give the integrability conditions:

$$\int_S F\frac{\mu}{\Gamma_{12}}ds\cdot\int_S\left\{\frac{\partial f}{\partial a_\kappa}\frac{\partial(\mu F)}{\partial a_\lambda}-\frac{\partial f}{\partial a_\lambda}\frac{\partial(\mu F)}{\partial a_\kappa}\right\}\frac{1}{\Gamma_{12}}ds+\int_S\frac{\partial f}{\partial a_\lambda}F\frac{\mu}{\Gamma_{12}}ds\cdot\int_S\frac{\partial(\mu F)}{\partial a_\kappa}\frac{\mu}{\Gamma_{12}}ds$$

(4.29)

$$+\int_S\frac{\partial f}{\partial a_\kappa}F\frac{\mu}{\Gamma_{12}}ds\cdot\int_S\frac{\partial(\mu F)}{\partial a_\lambda}\frac{\mu}{\Gamma_{12}}ds=0\qquad(\kappa,\lambda=1,\ldots,\rho).$$

If we employ the notation (3.8) then we can give it the following, very transparent, form:

$$\overline{\frac{\partial f}{\partial a_\kappa}\frac{\partial\log\mu F}{\partial a_\lambda}}-\overline{\frac{\partial f}{\partial a_\kappa}}\cdot\overline{\frac{\partial\log\mu F}{\partial a_\lambda}}=\overline{\frac{\partial f}{\partial a_\lambda}\frac{\partial\log\mu F}{\partial a_\kappa}}-\overline{\frac{\partial f}{\partial a_\lambda}}\cdot\overline{\frac{\partial\log\mu F}{\partial a_\kappa}}\quad(\kappa,\lambda=1,\ldots,\rho),\tag{4.30}$$

which requires a certain permutability of taking the mean and multiplication.

We conclude this paragraph with the:

Theorem 2. – *The necessary and sufficient conditions for the existence of the adiabatic invariants of a two-dimensional system are expressed by* (4.30).

§ 5. – Sufficient conditions for existence.

We shall now turn to the interpretation and evaluation of what we found. Along $\mathfrak{T}$, we can express the quantities $\frac{\partial f}{\partial a_\kappa}$, $\frac{\partial\log\mu F}{\partial a_\kappa}$ as functions of the parameter $t-t_0$. If we temporarily set:

$$\overline{\frac{\partial f}{\partial a_\kappa}}=\alpha_\kappa,\qquad\overline{\frac{\partial\log\mu F}{\partial a_\kappa}}=\beta_\kappa\qquad(\kappa=1,\ldots,\rho)$$

then we can perform a decomposition:

$$\frac{\partial f}{\partial a_\kappa}=\alpha_\kappa+h_\kappa(t-t_0),\qquad\frac{\partial\log\mu F}{\partial a_\kappa}=\beta_\kappa+j_\kappa(t-t_0),\tag{5.1}$$

in which h_κ, j_κ denote functions of their arguments whose mean values, when taken over the interval T, vanish. The condition (4.30) will then take on the form:

$$\overline{h_\kappa\,j_\lambda}=\overline{h_\lambda\,j_\kappa}\qquad(\kappa,\lambda=1,\ldots,\rho).\tag{5.2}$$

One can clarify what that means in the case of a periodic solution in which h_κ and j_κ will be Fourier series. However, we would not like to treat that in full generality, but choose three special cases that are amenable to deeper analysis:

Case I. All:

$$h_\kappa (t - t_0) = 0 \qquad (\kappa = 1, \ldots, \rho),$$

i.e., $\partial f / \partial a_\kappa$ are constant on $\mathfrak{T}$, so one must have:

$$\frac{\partial f}{\partial a_\kappa} = \Phi_\kappa (f \mid a_1, \ldots, a_\rho) \qquad (\kappa = 1, \ldots, \rho)\,. \tag{5.3}$$

Differentiating (3.3) with respect to a_κ will then imply the equations:

$$\frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial a_\kappa} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial a_\kappa} = 0 \qquad (\kappa = 1, \ldots, \rho); \tag{5.4}$$

i.e., f is not only an integral of original system (3.1), but also of the ρ following ones:

$$\frac{dx_1}{dt} = \frac{\partial X_1}{\partial a_\kappa}, \qquad \frac{dx_2}{dt} = \frac{\partial X_2}{\partial a_\kappa} \qquad (\kappa = 1, \ldots, \rho), \tag{5.5}$$

which says that f is a stationary integral with respect to the parameters. The functional operation vanished in equations (1.8), but here it will be identical with (1.6); i.e., with:

$$dc = \sum_{\nu=1}^{\rho} \frac{\partial f}{\partial a_\nu} da_\nu\,.$$

Furthermore, (3.3) and (5.4) will imply the proportions:

$$X_1 : X_2 = \frac{\partial X_1}{\partial a_\kappa} : \frac{\partial X_2}{\partial a_\kappa} \qquad (\kappa = 1, \ldots, \rho),$$

$$\frac{\partial}{\partial a_\kappa}\left(\frac{X_1}{X_2}\right) = 0,$$

from which, it will follow that along the trajectory $\mathfrak{T}$, the direction factor:

$$\frac{dx_1}{dx_2} = \frac{X_1}{X_2}$$

is a function of only x_1, x_2 . One can then choose the family of curves $\mathfrak{T}$ to be independent of the parameters and find an integral $f(x_1, x_2)$ of (3.1) that does not include the parameters at all and for which c will then be an adiabatic invariant, from Theorem 1. That case has then been dealt with.

Case II. For a well-defined multiplier μ, one will have:

$$j_\kappa(t - t_0) = 0 \qquad (\kappa = 1, \ldots, \rho)$$

in (5.1), so $\dfrac{\partial \log \mu F}{\partial a_\kappa}$ will be constant along $\mathfrak{T}$, and therefore:

(5.6) $$\frac{\partial \log \mu F}{\partial a_\kappa} = \Psi_\kappa(f \mid a_1, \ldots, a_\rho) \qquad (\kappa = 1, \ldots, \rho).$$

This condition is fulfilled, in particular, when the ρ equations are true:

(5.7) $$\frac{\partial \log \mu F}{\partial a_\kappa} = 0 \qquad (\kappa = 1, \ldots, \rho),$$

so μF does not depend upon the parameters at all, and we shall address that case in detail shortly. The relations (5.6) admit a simple interpretation when the distribution function is $F \equiv 1$. Namely, if:

(5.8) $$\Delta_0 = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}$$

denotes the divergence of the system (3.1) then one can write (3.5) as:

(5.9) $$\frac{\partial \log \mu}{\partial x_1} X_1 + \frac{\partial \log \mu}{\partial x_2} X_2 + \Delta_0 = 0,$$

and when (5.6) is true with $F \equiv 1$, differentiation with respect to a_κ will give:

(5.10) $$\frac{\partial \log \mu}{\partial x_1}\frac{\partial X_1}{\partial a_\kappa} + \frac{\partial \log \mu}{\partial x_2}\frac{\partial X_2}{\partial a_\kappa} + \frac{\partial \Delta_0}{\partial a_\kappa} = 0 \qquad (\kappa = 1, \ldots, \rho),$$

which says that μ is also a multiplier of the system (5.5), or in other words, a stationary multiplier with respect to the parameters. It follows from (5.9) and (5.10) that all three-rowed determinants in the matrix:

$$\begin{pmatrix} X_1 & X_2 & \Delta_0 \\ \dfrac{\partial X_1}{\partial a_\kappa} & \dfrac{\partial X_2}{\partial a_\kappa} & \dfrac{\partial \Delta_0}{\partial a_\kappa} \end{pmatrix} \qquad (\kappa = 1, \ldots, \rho)$$

must vanish. (5.7) is fulfilled, in particular, when $F \equiv 1$ and μ = const. so the system (3.1) is a Liouville system. With that, the following theorem, which goes back to Gibbs and Hertz ([8]), is proved as a special case of our investigations:

Theorem 3. – *A two-dimensional Hamiltonian differential system possesses an adiabatic invariant under simple averaging.*

Case III.

$$j_\kappa (t - t_0) = \text{const.}\ h_\kappa (t - t_0) \qquad (\kappa = 1, \ldots, \rho),$$

in which the constant is the same for all κ. It will then follow from (5.1) that one must have:

$$\frac{\partial \log \mu F}{\partial a_\kappa} = p\,(f \mid a_1, \ldots, a_\rho) + q_\kappa (f \mid a_1, \ldots, a_\rho) \qquad (\kappa = 1, \ldots, \rho). \tag{5.11}$$

If one then determines a function $\varphi (f \mid a_1, \ldots, a_\rho)$ in such a way that one has:

$$\frac{\partial \log \varphi (f \mid a_1, \ldots, a_\rho)}{\partial f} = p\,(f \mid a_1, \ldots, a_\rho)$$

then (5.11) will give:

$$\frac{\partial \log (\mu F : \varphi)}{\partial a_\kappa} = -\frac{1}{\varphi}\frac{\partial \varphi}{\partial a_k} + q_\kappa = \Psi_\kappa (f \mid a_1, \ldots, a_\rho) \qquad (\kappa = 1, \ldots, \rho),$$

and since $\mu : \varphi$ is also a multiplier of (3.1), along with μ, we come back to Case II.

We summarize our results in:

Theorem 4. – *Sufficient conditions for the existence of adiabatic invariants of a two-dimensional system are:*

a) *The stationarity of an integral,*

b) *The stationarity of a multiplier with respect to the parameters under simple averaging.*

([8]) Cf., Gibbs 1, Hertz 1, pp. 534-535.

§ 6. – The invariants of the generalized Liouville system.

In (5.7), we have already singled out the case in which a multiplier can be given, in such a way that μF is independent of the parameters $a_1, \ldots, a_\rho$. We will then refer to the system (3.1) as a *generalized Liouville system*, on grounds that will become clear later on. An adiabatic invariant must exist for it. Equations (4.24), (4.25) simplify (while preserving the demand of § **4**) to:

$$\frac{\partial g_\kappa}{\partial a_\lambda} = \frac{\partial g_\lambda}{\partial a_\kappa}, \qquad \frac{\partial g}{\partial a_\kappa} + \frac{\partial g_\kappa}{\partial c} = 0 \qquad (\kappa, \lambda = 1, \ldots, \rho), \tag{6.1}$$

and one must then be capable of finding a function $V(c \mid a_1, \ldots, a_\rho)$ such that:

$$\frac{\partial V}{\partial c} = g, \qquad \frac{\partial V}{\partial a_\kappa} = -g_\kappa . \tag{6.2}$$

Equations (3.10) then show that V is the desired adiabatic invariant of our two-dimensional system.

We can easily give V explicitly. Namely, if $\mathfrak{C}$ denotes a fixed curve that is independent of $a_1, \ldots, a_\rho$, and which, from our requirement, cuts out the averaging interval S along $\mathfrak{T}$ (an arbitrary fixed curve, resp., when $T = \infty$), and if B is the region in the x_1, x_2-plane that is bounded by $\mathfrak{C}$ and S then one will have:

$$J = V = \iint_S \mu F \, dx_1 \, dx_2 , \tag{6.3}$$

assuming that this integral is meaningful. The proof that (6.2) is fulfilled is easy to complete by means of the developments in § **4**.

The variation of only c will, in fact, leave μF and $\mathfrak{C}$ unchanged, while only the trajectory $\mathfrak{T}$ goes to $\mathfrak{T}'$. The normal displacement of the point P to P' will then be characterized by (4.11), and one will then have:

$$\frac{\partial V}{\partial c} = \int_S \mu F \, ds \, d'n \; : dc = \int_S F \frac{\mu}{\Gamma_{12}} ds = g.$$

One easily convinces oneself that the difference between P_0', $\bar{P}_0'$ ($P_1', \bar{P}_1'$, resp.) for the calculation of these derivatives is irrelevant when the triangle $P_0 P_0' \bar{P}_0'$ ($P_1 P_1' \bar{P}_1'$, resp.) is infinitely-small of order two. [Indeed, it is obvious that $\mathfrak{C}$ cannot contact $\mathfrak{T}$ at P_0, P_1, since otherwise, from (4.12), one would need to have $\Phi = 0$.] When only a_κ is varied, from the assumption (5.7), $\mathfrak{C}$ will likewise remain unperturbed, and $\mathfrak{T}$ will go to $\mathfrak{T}^*$ by the normal displacement $d^* n$, such that due to (4.18), it will follow that:

$$\frac{\partial V}{\partial a_\kappa} = \int_S \mu F\, ds\, d^* n \;:\; da_\kappa = -\int_S \frac{\partial f}{\partial a_\kappa} F \frac{\mu}{\Gamma_{12}} ds = -g_\kappa ,$$

as we have stated. We have then found the following theorem:

Theorem 5. – *The adiabatic invariant of a two-dimensional generalized Liouville system is* $V = \iint_B \mu F\, dx_1\, dx_2$.

If $\mathfrak{T}$ is periodic, in particular, and T is equal to infinity or the period then one will be able to take B to be the region that is included inside of $\mathfrak{T}$. If one is dealing with a simple averaging $F \equiv 1$ then the Liouville system ($\Delta_0 = 0$, μ = const.) will fall within the class considered. In the two-dimensional case (but only in that case), that class can be transformed into the class of Hamiltonian systems ([9]), and Theorem 5 will then coincide with the statement of the Gibbs-Hertz theorem ([10]), namely, that the *phase volume:*

$$V = \iint dx_1\, dx_2$$

is an adiabatic invariant. Equation (5.7) then demands that μ must be independent of the parameters, and a theorem is true for such systems that we will prove for n dimensions later on, but in the meantime, we would only like to state:

Theorem 6. – *If* μ *is independent of the parameters then there will be a coordinate transformation that is free of the parameters and that takes the original system to a Liouville system, and conversely.*

Such transformations that do not include the parameters are naturally permissible, and will have no effect on the calculation of the adiabatic invariants and one will then see that the aforementioned class of systems essentially coincides with the class of Liouville systems, and that will explain the terminology that was introduced at the beginning of this paragraph.

§ 7. – Examples. The damped oscillator.

The two examples of § **2** can be dealt with directly by means of the foregoing theorems.

1. Example. – One has $\Delta_0 = 0$, μ = const. = 1. Due to periodicity, we take B to be the interior of the ellipse:

([9]) Cf., Levi-Civita 1, pp. 336.
([10]) Cf., Gibbs 1, Hertz 1, pp. 534-535, Levi-Civita 1, pp. 339-342.

$$x_1^2 + \frac{1}{a_1} x_2^2 = c_1^2 ,$$

such that:

$$J = V = \iint_B dx_1\, dx_2 = \pi \sqrt{a_1}\, c^2 ,$$

which agrees with (2.1).

2. Example. – One has:

$$\Delta_0 = - a_1^2 \frac{x_2}{x_1^2} , \quad \mu = x_1 ,$$

so we have the case of § **6** before us. Therefore, an adiabatic invariant should exist. We take the axis $x_1 = 0$ to be the curve $\mathfrak{C}$ that determines the end points, and then take B to be the right (or left) part of the ellipse:

$$x_1^2 + \frac{a_1^2}{a_2^2} x_2^2 = c_1$$

that is bounded by the x_2-axis, such that it will follow from Theorem 5 that:

$$J = \iint_B x_1\, dx_1\, dx_2 = \frac{2}{3} \frac{a_2}{a_1} c_1^{3/2} ,$$

which is in harmony with (2.2).

3. Example. – Finally, we would like to address the damped oscillator, due to its physical interest; the harmonic oscillator falls within the classical examples ([11]). Let a_1 and a_2 denote the frequency and damping constant, resp., of the oscillator. Its motion will then be given by the differential equations:

$$\frac{dx_1}{dt} = x_2 , \qquad \frac{dx_2}{dt} = - 2a_2\, x_2 - a_1^2\, x_1 , \tag{7.1}$$

whose integrals read:

$$x_1 = c\ e^{-a_2 (t - t_0)} \cos \gamma (t - t_0),$$

$$x_2 = - c\ e^{-a_2 (t - t_0)} \{ a_2 \cos \gamma (t - t_0) + \gamma \sin \gamma (t - t_0) \}, \tag{7.2}$$

in which we have set:

$$\gamma = \sqrt{a_1^2 - a_2^2} .$$

When these equations are resolved, they will read:

([11]) Cf., Born 1, Levi-Civita 1, pp. 345; 2, pp. 13-14. We can thank Levi-Civita for the suggestion to treat this example.

$$f(x_1, x_2 \mid a_1, a_2) \equiv \frac{1}{\gamma}\sqrt{\gamma^2 x_1^2 + (a_2\, x_1 + x_2)^2}\; e^{-\frac{a_1}{\gamma}\arctan\frac{a_2 x_1 + x_2}{\gamma x_1}} = c,$$

$$f_2(x_1, x_2 \mid a_1, a_2) \equiv -\frac{1}{\gamma}\arctan\frac{a_2\, x_1 + x_2}{\gamma\, x_1} = t - t_0\,,$$

such that one finds that:

$$(7.3)\qquad \begin{cases} \dfrac{\partial f}{\partial a_1} = \varphi_{11} = -\dfrac{a_1 c}{\gamma^2}\left\{a_2(t-t_0) + \sin^2\gamma(t-t_0) + \dfrac{a_2}{2\gamma}\sin 2\gamma(t-t_0)\right\}, \\ \dfrac{\partial f}{\partial a_2} = \varphi_{12} = -\dfrac{c}{2\gamma^3}\left\{2a_1^2\gamma(t-t_0) + (2a_2^2 - a_1^2)\sin 2\gamma(t-t_0) + 2a_2\,\gamma\cos 2\gamma(t-t_0)\right\}, \\ \dfrac{\partial f_2}{\partial a_1} = \varphi_{21} = -\dfrac{a_1}{\gamma^2}\left\{(t-t_0) + \dfrac{1}{2\gamma}\sin 2\gamma(t-t_0)\right\}, \\ \dfrac{\partial f_2}{\partial a_2} = \varphi_{22} = \quad \dfrac{1}{\gamma^2}\left\{a_2(t-t_0) - \cos^2\gamma(t-t_0) + \dfrac{a_2}{2\gamma}\sin 2\gamma(t-t_0)\right\}. \end{cases}$$

The trajectory is a spiral that winds ever tighter around the origin. We take the x_2-axis to be the end point of the curve $\mathfrak{C}$ that determines the averaging interval, choose:

$$T_0 = \frac{\pi}{2\gamma}, \qquad T_1 = \frac{(2n+1)\pi}{2\gamma}, \qquad T = \frac{n\pi}{\gamma},$$

and then find the mean value (assuming that $F \equiv 1$):

$$\overline{\varphi}_{11} = -\frac{a_1 c}{2\gamma^3}\{a_2\,\pi(n+1) + \gamma\}, \qquad \overline{\varphi}_{12} = -\frac{a_1^2 c\,(n+1)\pi}{2\gamma^3},$$

such that the adiabatic invariant must be determined from the two equations (1.11), i.e.:

$$\frac{\partial J}{\partial c}\cdot a_2\, c\{a_2\,\pi(n+1) + \gamma\} - \frac{\partial J}{\partial a_1}\cdot 2\gamma^3 = 0,$$

$$\frac{\partial J}{\partial c}\cdot a_1^2\, c\,(n+1)\pi \qquad + \frac{\partial J}{\partial a_2}\cdot 2\gamma^3 = 0.$$

However, one sees directly that the integrability condition (4.1) is not fulfilled here, since its left-hand side assumes the value:

$$-\frac{a_1 a_2 c}{\gamma^4}.$$

The left-hand side of (4.29) will yield the same value when one observes that the quantity:

$$\mu = e^{-\frac{2a_2}{\gamma}\arctan\frac{a_2 x_1 + x_2}{\gamma x_1}}$$

is a multiplier of the system (7.1).

There is therefore no adiabatic invariant in the absolute sense; i.e., no quantity that will remain invariant under an *arbitrary* adiabatic change of a_1, a_2 . By the way, that serves as the proof that our existence conditions (4.30) are not trivial. The same thing will be true *a fortiori* for $T = \infty$, since the mean value that pertains to that case does not even exist.

However, another question that one can pose will lead to some remarkable results. We ask: Can one give a function $\lambda\,(a_1, a_2)$ such that an adiabatic invariant exists *for an absolutely constant* λ? The adiabatic changes in a_1, a_2 will no longer be arbitrary then, but they will be restricted by the constancy of λ, and such an invariant will have the character of a *relative adiabatic invariant.* We will introduce two functions $\kappa(a_1, a_2)$ and $\lambda\,(a_1, a_2)$, in place of a_1, a_2, and observe that:

$$\frac{\partial f}{\partial \kappa} = \frac{\partial f}{\partial a_1}\frac{\partial a_1}{\partial \kappa} + \frac{\partial f}{\partial a_2}\frac{\partial a_2}{\partial \kappa}, \qquad \frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial a_1}\frac{\partial a_1}{\partial \lambda} + \frac{\partial f}{\partial a_2}\frac{\partial a_2}{\partial \lambda}.$$

There is only one equation for the adiabatic invariant [cf. (1.9)]:

$$\frac{\partial J}{\partial c}\cdot\overline{\frac{\partial f}{\partial \kappa}} - \frac{\partial J}{\partial t_0}\cdot\overline{\frac{\partial f_2}{\partial \kappa}} + \frac{\partial J}{\partial \kappa} = 0, \tag{7.4}$$

from which, the equation that relates to λ drops out, by our assumption. For the sake of simplicity, we choose $T = \infty$. κ, λ is then chosen in such that at least one of the mean values in (7.4) exists – i.e., the term in $t - t_0$ drops out of either $\partial f / \partial \kappa$ or $\partial f_2 / \partial \kappa$ – so we will have:

$$-\,a_2\frac{\partial a_1}{\partial \kappa} + a_1\frac{\partial a_2}{\partial \kappa} = 0 \qquad \text{or} \qquad -\,a_1\frac{\partial a_1}{\partial \kappa} + a_2\frac{\partial a_2}{\partial \kappa} = 0. \tag{7.5}$$

It we establish that perhaps one should have:

$$\frac{\partial(\kappa, \lambda)}{\partial(a_1, a_2)} = 1$$

then the first equation (7.5) will give the value:

(7.6) $$\kappa = \tfrac{1}{2} a_1^2, \qquad \lambda = \frac{a_2}{a_1},$$

and the second one will give the transformation:

(7.7) $$\kappa = \frac{1}{4} \ln \frac{a_1 - a_2}{a_1 + a_2}, \quad \lambda = a_1^2 - a_2^2 = \gamma^2.$$

In the case of (7.6), we will get the equation:

$$- c \cdot \frac{\partial J}{\partial c} + 4\kappa (1 - \lambda^2) \cdot \frac{\partial J}{\partial \kappa} = 0,$$

$$J = f(a_1 c^{2(1-\lambda^2)}) = f\left(a_1 c_1^2 c^{-2\left(\frac{a_2}{a_1}\right)^2} \right)$$

from (7.4). *That will then be the invariant of the damped oscillator when the ratio a_2 / a_1 remains absolutely constant.* In the case of (7.7), the equation:

$$\mathfrak{Cos}\, 2\kappa \cdot \frac{\partial J}{\partial t_0} - \sqrt{\lambda} \frac{\partial J}{\partial \kappa} = 0$$

will then lead to the invariant:

$$J = f\left(t_0 + \frac{1}{2\sqrt{\lambda}} \mathfrak{Sin}\, 2\kappa \right) = f(2\gamma^2 t_0 - a_2),$$

which is true as such when the reduced frequency γ remains constant.

II.

§ 8. – Classification of the *n*-dimensional systems.

We now turn from the arguments that were constructed in the foregoing to the treatment of *n*-dimensional differential systems of type (1.1), and we will first carry out a classification of the problems that present themselves. An initial subdivision will come from the fact that the averaging interval T is either finite or infinitely large, so one can use the definition (1.7) [(1.10), resp.] as a basis. Namely, the essential component of our considerations will be the integrals (1.3) of those systems that arise from solving the single-valued relations (1.2). However, single-valuedness, as well as solubility, are properties in the small that are each true for only a restricted interval of the variables $t - t_0$, and therefore for the trajectory. However, the definitions of the mean that were

established in (1.7) [(1.10), resp.] demand more; they require knowledge of the integral f_i in the large, namely, in the entire interval T or ∞. As long as T is finite and the X_i are regular in that interval, that distinction is inessential, since it corresponds to a finite piece S of the trajectory $\mathfrak{T}$ that we would like to assume to be singularity-free and that will be covered by a finite number of the aforementioned domains of single-valuedness. Along that one-dimensional path S, the f_i are either single-valued or at most finitely multi-valued, and one can interpret S as a finite piece of the line of intersection of the $n - 1$ hypersurfaces in R_n :

$$f_i\,(x_1\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho) = c_i \qquad (i = 1,\,\ldots,\,n-1). \tag{8.1}$$

Under projection onto the $(x_i\,,\,x_k)$ coordinate plane, S will go to a finite path segment in the $x_i\,x_k$ -plane. We will have to consider precisely that projection later on, so it would be natural for us to adapt the demand that was made in § **4** in regard to the two-dimensional case that the end points of the averaging interval are cut out of a fixed curve that does not include the parameters to the new case. When back-projected to R_n , that will say that the end points of S must lie on a parameter-free hypersurface:

$$\Phi\,(x_1\,,\,\ldots,\,x_n) = 0.$$

We then summarize the first case as follows:

A. *T is finite, and likewise S, and the f_i are finitely multi-valued along S. The endpoints of S lie on a parameter-free hypersurface.*

That case then corresponds to the considerations up to now, on the whole. We shall assume, secondly, that $T = \infty$. As was pointed out above, we come to the behavior of the f_i in the large, and that is not very well-known up to the present day. Our knowledge of it is essentially restricted to the Poincaré-Carathéodory recurrence theorem, which is stated as follows: If the trajectory of (1.1) lies in a connected region G of R_n of finite measure (at least, for the values of the constants and parameters that come under consideration), and if one can find a multiplier μ that is positive in G and vanishes at most in a null set then almost all trajectories will return arbitrarily close to each of their points infinitely often ([12]). That is closely related to the assumption that in this case either the trajectories are closed, and thus periodic, or almost all trajectories fill up a certain manifold Φ densely. The integrals (1.3) that determine those trajectories then split into two irreducible groups (possibly after being previously combined), the first of which encompasses m of them that are single-valued or finitely multi-valued along the entire trajectory $\mathfrak{T}$:

$$f_\lambda\,(x_1\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho) = c_\lambda \qquad (\lambda = 1,\,\ldots,\,m), \tag{8.2}$$

and the second of which includes the infinitely multi-valued integrals along $\mathfrak{T}$, which are then irrelevant in practice. Equations (8.2) determine an $n - m$-dimensional manifold Φ

([12]) Cf., Poincaré 1, chap. 26. Carathéodory 1. What little that is known about the subject has been organized by Smekal 1, pp. 179-181; Levi-Civita 1, pp. 331.

in R_n, and we will address only those systems that satisfy the so-called *quasi-ergodic hypothesis: Almost all trajectories fill up the manifold* Φ *densely*. Quasi-ergodic systems exist ([13]). We would like to assume that Φ defines a closed manifold in R_n .

We shall refer to the number m, which defines a characteristic constant of the system, as its *order of imprimitivity* ([14]). The trajectories of a primitive system then fill up part of R_n densely and possess no single-valued integrals at all, so for them the question of adiabatic invariants will obviously make no sense. For m-fold imprimitive systems, one will seek those invariants that define a relationship between the m constant $c_1, \ldots, c_m$ and the parameters $a_1, \ldots, a_\rho$, and therefore, from a remark at the conclusion of § **1**, consider only those terms in equations (1.11) that relate to that. The largest value $m = n - 1$ will be assumed when all integrals of the system are finitely multi-valued. Φ will then reduce to the one-dimensional section of the hypersurfaces (8.1) – i.e., to $\mathfrak{T}$ – and thus the quasi-ergodic hypothesis will be trivial in that case. We summarize:

B. *T is finite. The system has order of imprimitivity m and fills up a closed n – m-dimensional manifold quasi-ergodically.*

We will break this up into three sub-cases:

B1. $m = 1$, so the system is simply-imprimitive.
B2. $1 < m < n - 1$, so the system is m-fold imprimitive.
B3. $m = n - 1$, so all integrals f_i are finitely multi-valued, and $\Phi \equiv \mathfrak{T}$ is one-dimensional.

Finally, we offer as a last possibility that the system might posses no quasi-periodicity at all, which can be the case when singularities appear, in particular. Such systems will not enter into our treatment, to the extent that one cannot proceed in the same way that we did in cases A and B based upon one's knowledge of them. We must then exclude them from what follows.

C. *T is infinite. The system then possesses no quasi-ergodicity.*

Now that we have this classification, we will turn to the problem of adiabatic invariants.

§ 9. – Problem statement for the simply-imprimitive system.

The topological assumptions in regard to the trajectories that were discussed in the foregoing have only the one goal of converting the functional operation that was defined over t in (1.10) into a spatial operation that relates to Φ in a manner that is similar to how we could accomplish that for two-dimensional systems in (3.8). The true generalizations

([13]) Examples are included in Cherry 1, Levi-Civita 1, pp. 326-328.
([14]) We will then adopt the terminology of Levi-Civita 1, pp. 330, while in Levi-Civita 2, it is shifted by one unit.

of two-dimensional systems are the simply-imprimitive ones. We can ascribe all of the remaining cases to two-dimensional or simply-imprimitive systems by reducing by means of the known integrals.

Let the system (1.1) be simply-imprimitive, and let it possess the finitely multi-valued integral:

(9.1) $$f(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = c,$$

by means of which, the closed $n-1$-dimensional hypersurface Φ is characterized. For the moment, we would like to assume that the distribution function F is constant. We would then like to arrive at a point where an averaging (i.e., an integration) over Φ enters in place of the definition (1.10), which is possible as a result of the quasi-ergodic distribution of the trajectories over Φ. We must then ask only: What distribution function must be assigned to that averaging, or rather, what is the single-valued density κ that we have to multiply each surface element $d\Phi$ by?

κ must fulfill a certain invariance property, namely, it must be "carried by the trajectories"; i.e., if $d\Phi$, $d\Phi_1$ are two elements of Φ that emerge from each other as a result of equations (1.1) (in the sense that when all points of $d\Phi$ are assigned the value $t = t_0$, $d\Phi_1$ will represent the totality of points that belong to the corresponding trajectories for the value $t = t_1$) then one must have:

(9.2) $$\kappa\, d\Phi = \kappa_1\, d\Phi_1 .$$

That follows from the fact that the mean value (1.10) must be independent of t_0 – i.e., of the starting element $d\Phi$.

We now show that there is essentially *only one* function that satisfies that invariance. If the same thing were for a different single-valued density distribution λ then $\kappa\, d\Phi$, $\lambda\, d\Phi$, and therefore $\kappa : \lambda$, as well, would have to be invariant under (1.1), so the latter quantity would have to be an integral of the system (1.1). However, by assumption, it possesses only a finitely multi-valued integral (namely, f), so we must have:

$$\kappa : \lambda = \Psi\,(f(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho)),$$

so that ratio will be constant on Φ. However, there is no proportionality factor for the density, so there is essentially only one density function κ([15]).

One can easily extend the $(n-1)$-dimensional surface elements $d\Phi$, $d\Phi_1$ into space elements $d\tau$, $d\tau_1$ in R_n that emerge from each other as a result of (1.1). In order to do that, along with the surface (9.1), we consider the hypersurface Φ' with the equations:

(9.3) $$f(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = c + dc .$$

We associate each point P of Φ with the point of Φ' that lies on its normal and denote the vector PP' by $d'n$. Since we have assumed that Φ is closed and free of singularities, that association will be everywhere one-to-one. The direction cosines of the normal are:

([15]) Cf., Levi-Civita 1, pp. 337, *et seq.*

(9.4) $$\frac{1}{\Gamma_{1n}}\frac{\partial f}{\partial x_i} \qquad (i = 1, \ldots, n), \qquad \Gamma_{1n}^2 = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2,$$

and therefore, the components of $d'n$ will be:

$$d'x_i = \frac{1}{\Gamma_{1n}}\frac{\partial f}{\partial x_i}\, d'n.$$

On the other hand, it follows from (91.), (9.3), and (9.4) that:

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i} d'x_i \equiv \Gamma_{1n}\, d'n = dc,$$

so

(9.5) $$d'n = \frac{dc}{\Gamma_{1n}}; \qquad d'x_i = \frac{1}{\Gamma_{1n}^2}\frac{\partial f}{\partial x_i}\, dc.$$

We now let $d\tau = d\Phi\, d'n$ denote the space element of the layer that lies between Φ and Φ' over $d\Phi$, so as a result of equations (1.1), it will go to the layer element $d\tau_1 = d\Phi_1\, d'n_1$, since the trajectories that begin in the layer can never leave it and will fill it up everywhere densely. It will then follow from (9.2) that:

$$\kappa d\Phi\, dc = \kappa\Gamma_{1n}\, d\tau = \kappa_1\, d\Phi_1\, dc = (\kappa\Gamma_{1n})_1\, d\tau_1,$$

so the quantity:

(9.6) $$\mu = \kappa\Gamma_{1n}$$

must satisfy the relation:

$$\mu\, d\tau = \mu_1\, d\tau_1;$$

i.e., in modern terminology: $\int \mu\, d\tau$ *is an integral invariant of the system* (1.1). Now, one has the easily-proved theorem ([16]) that if that quantity is an integral invariant then the factor μ will be a Jacobi multiplier of (1.1); i.e., the differential equation:

(9.7) $$\sum_{i=1}^{n}\frac{\partial \mu}{\partial x_i} X_i + \mu\sum_{i=1}^{n}\frac{\partial X_i}{\partial x_i} = 0$$

must be fulfilled, and conversely, every multiplier must give an integral invariant. It then follows from (9.6):

(9.8) $$\kappa = \frac{\mu}{\Gamma_{1n}},$$

in which μ can mean any arbitrary single-valued (finitely multi-valued, resp.) multiplier of (1.1). There is essentially only one such multiplier μ, since the quotient of two such

([16]) Cf., Poincaré 1, pp. 41 *et seq.*

things will be a finitely multi-valued integral (1.1), so by assumption, it will be a function of just f, and therefore it will be constant on Φ.

In what follows, we will assume that there is a finitely multi-valued solution μ of (9.7) *and that μ does not vanish on Φ – except for finitely-many points* – and thus, it will have a definite sign, which we can choose to be positive. We shall see later (§ **11**) that this assumption is equivalent to the statement: The functions $f_1, \ldots, f_{n-1}$ that define the trajectory in the small shall possess no singularities along it. The averaging (1.10) will then be replaced by the definition (while still assuming that $F \equiv 1$):

$$\bar{\alpha} = \int_{\Phi} \alpha(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho)\frac{\mu}{\Gamma_{1n}}d\Phi : \int_{\Phi}\frac{\mu}{\Gamma_{1n}}d\Phi\,. \tag{9.9}$$

If $F \not\equiv 1$ then the finitely multi-valued distribution function $F(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho)$ that is transformed by (1.3) will enter inside the integral sign here such that:

$$\bar{\alpha} = \int_{\Phi} \alpha(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho)\,F(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho)\frac{\mu}{\Gamma_{1n}}d\Phi : \int_{\Phi} F(x_1,\ldots,x_n \mid a_1,\ldots,a_\rho)\frac{\mu}{\Gamma_{1n}}d\Phi\,.$$

With that, we shall now go into the differential equations (1.11) of adiabatic invariants and consider only the differentiations that relate to c and a_ν. This differential system then reads:

$$\frac{\partial J}{\partial c}\cdot\int_{\Phi}\frac{\partial f}{\partial a_\nu}F\frac{\mu}{\Gamma_{1n}}d\Phi + \frac{\partial J}{\partial a_\nu}\cdot\int_{\Phi}F\frac{\mu}{\Gamma_{1n}}d\Phi = 0 \qquad (\nu = 1, \ldots, \rho), \tag{9.11}$$

which defines the obvious generalization of (3.10). We will investigate its integrability conditions and its solution later.

§ 10. – Problem statement for m-fold imprimitive systems.

If the system (1.1) is m-fold imprimitive, i.e., it admits m finitely multiple-valued integrals:

$$f_\lambda(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = c_\lambda \qquad (\lambda = 1, \ldots, m) \tag{10.1}$$

that determine the closed $n - m$-dimensional manifold Φ, then arguments that are similar to the ones above will lead to our objective. The density function κ must again satisfy the invariance property (9.2) and will then be determined on Φ up to a constant factor, since the quotient of two such densities will be a finitely multi-valued integral of the system, and must then be a function $f_1, \ldots, f_m$.

We once more extend $d\Phi$, $d\Phi_1$ to space element of R_n and proceed from there as follows: Along with Φ, we introduce the manifold Φ' that is defined by the equations:

$$f_\lambda(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = c_\lambda + dc_\lambda \qquad (\lambda = 1, \ldots, m) \tag{10.2}$$

and together with Φ, it will define a layer that is filled up densely with trajectories that do not leave it. The normals to the hypersurfaces (10.1) have the direction cosines:

$$\frac{\partial f_\lambda}{\partial x_1} : \sqrt{\sum_{i=1}^{n}\left(\frac{\partial f_\lambda}{\partial x_i}\right)^2} \qquad (i = 1, \ldots, n\,;\ \lambda = 1, \ldots, m), \tag{10.3}$$

and they will naturally be perpendicular to any vector that is contained in Φ. We further determine $n - m$ more functions $u_j\ (x_1, \ldots, x_n \mid a_1, \ldots, a_\rho)$ in such a way that the hypersurfaces:

$$u_j\ (x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = \mathfrak{c}_j \qquad (j = 1, \ldots, n - m) \tag{10.4}$$

intersect all of the surfaces (10.1) orthogonally and are orthogonal to each other – i.e., that:

$$\sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i}\frac{\partial f_\lambda}{\partial x_i} = 0 \qquad (j = 1, \ldots, n - m,\ \lambda = 1, \ldots, m), \tag{10.5}$$

$$\sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_i} = 0 \qquad (j \neq k = 1, \ldots, n - m). \tag{10.6}$$

Such a choice of functions u_j is always possible. As a result of (10.5), the normal directions of (10.4) contact the manifold Φ and thus define the $n - m$-dimensional tangent manifold at a point. If we consider not only the surfaces (10.4), but also the following ones:

$$u_j\ (x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = \mathfrak{c}_j + d\mathfrak{c}_j \qquad (j = 1, \ldots, n - m), \tag{10.7}$$

then, from (9.5), for any pair of associated surfaces (10.4) and (10.7), the normal segment that lies between them will possess the length:

$$d'n_j = \left\{\sum_{i=1}^{n}\left(\frac{\partial u_j}{\partial x_i}\right)^2\right\}^{-1/2} d\mathfrak{c}_j\,.$$

By definition, the element $d\Phi$ is equal to that of its tangent manifold, and as a result of the orthogonality of the $d'n$, its magnitude will be, in turn, equal to:

$$d\Phi = \prod_{j=1}^{n-m} d'n_j = \left\{\prod_{j=1}^{n-m}\ \sum_{i=1}^{n}\left(\frac{\partial u_j}{\partial x_i}\right)^2\right\}^{-1/2} d\mathfrak{c}_1 \ldots d\mathfrak{c}_{n-m}\,. \tag{10.8}$$

The intersection of the $2n$ surfaces (10.1), (10.2), (10.4), and (10.7) defines a space element $d\tau$ in R_n. The linear independence of the functions f_λ, u_j allows one to introduce these new quantities as new coordinates on R_n and to then set:

(10.9) $$dt = dx_1 \ldots dx_n = \left[\frac{\partial(f_1,\ldots,f_m,u_1,\ldots,u_{n-m})}{\partial(x_1,\ldots,x_n)}\right]^{-1} dc_1 \ldots dc_m\, d\mathfrak{c}_1 \ldots d\mathfrak{c}_{n-m}\,.$$

On the other side, one infers from (10.5), (10.6) that:

$$\left[\frac{\partial(f_1,\ldots,f_m,u_1,\ldots,u_{n-m})}{\partial(x_1,\ldots,x_n)}\right]^2 = \prod_{j=1}^{n-m}\sum_{i=1}^{n}\left(\frac{\partial u_j}{\partial x_i}\right)^2 \cdot \Gamma_{m,n}^2\,,$$

in which:

$$\Gamma_{m,n}^2 = \operatorname*{Det}_{\kappa,\lambda}\sum_{i=1}^{n}\frac{\partial f_\kappa}{\partial x_i}\frac{\partial f_\lambda}{\partial x_i} = \left[\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}\right]^2$$

means the square of the matrix that was just written out. (10.9), in conjunction with (10.8), will then give:

(10.10) $$d\tau = d\Phi \cdot dc_1 \ldots dc_m$$

as the expression for the space element of the layer that lies between Φ and Φ'.

(9.2) then implies that:

$$\kappa\, d\Phi \cdot dc_1 \ldots dc_m = \kappa\Gamma_{m,n}\, d\tau = \kappa_1\, d\Phi_1\, dc_1 \ldots dc_m = (\kappa\Gamma_{m,n})_1\, d\tau_1\,,$$

from which, one concludes, as in § **9**, that the quantity:

(10.11) $$\mu = \Gamma_{m,n}\,\kappa$$

must be a multiplier of the given system, and conversely that any multiplier μ that is finitely multi-valued on Φ (which will then be established up to a constant factor by that demand) will give a valid density function:

$$\kappa = \frac{\mu}{\Gamma_{m,n}}\,.$$

The functional operation that one applies in place of (1.10) will then read:

(10.12) $$\bar{\alpha} = \int_\Phi \alpha F \frac{\mu}{\Gamma_{m,n}} d\Phi : \int_\Phi F \frac{\mu}{\Gamma_{m,n}} d\Phi$$

in that case, and the differential equations of the adiabatic invariants, which will be treated later, will be:

(10.13) $$\sum_{i=1}^{m}\frac{\partial J}{\partial c_\lambda}\cdot\int_\Phi \frac{\partial f_\lambda}{\partial a_\nu} F\frac{\mu}{\Gamma_{m,n}} d\Phi + \frac{\partial J}{\partial a_\nu}\cdot\int_\Phi F\frac{\mu}{\Gamma_{m,n}} d\Phi = 0 \qquad (\nu = 1, \ldots, \rho).$$

§ 11. – $(n-1)$-fold imprimitive systems.

The cases A and B3 of the classification in § **8** still remain to be discussed, and they are independent of all topological assumptions and the quasi-ergodic hypothesis. One knows the integral:

(11.1) $$f_i(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho) = c_i \qquad (i = 1, \ldots, n-1)$$

and the identities:

(11.2) $$\sum_{\kappa=1}^{n} \frac{\partial f_i}{\partial x_k} X_\kappa \equiv 0 \qquad (i = 1, \ldots, n-1)$$

imply the relations:

(11.3) $$X_i = \frac{(-1)^i}{\mu} \frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)} = \frac{(-1)^i}{\mu} \mathfrak{F}_i,$$

in which μ means a proportionality factor, and we have set:

$$\mathfrak{F}_i = \frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}.$$

Since the $\mathfrak{F}_i$ are not mutually independent, μ must satisfy an equation that is easy to find; namely, it is:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\mu X_i) = \sum_{\iota=1}^{n} (-1)^\iota \frac{\partial \mathfrak{F}_\iota}{\partial x_i}$$

$$= \sum_{i=1}^{n} \sum_{\kappa=1}^{n-1} \sum_{l=1}^{i-1} (-1)^{\kappa+l+i} \frac{\partial^2 f_\kappa}{\partial x_i \partial x_l} \frac{\partial(f_1, \ldots, f_{\kappa-1}, f_{\kappa+1}, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{l-1}, x_{l+1}, \ldots x_n)}$$

$$- \sum_{i=1}^{n} \sum_{\kappa=1}^{n-1} \sum_{l=i+1}^{n} (-1)^{\kappa+l+i} \frac{\partial^2 f_\kappa}{\partial x_i \partial x_l} \frac{\partial(f_1, \ldots, f_{\kappa-1}, f_{\kappa+1}, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{l-1}, x_{l+1}, \ldots x_n)}$$
$$\equiv 0,$$

as one will see upon switching i and l in the second sum. μ will then satisfy the differential equation:

(11.4) $$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\mu X_i) \equiv \sum_{i=1}^{n} \frac{\partial \mu}{\partial x_i} X_i + \mu \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = 0,$$

so it will then be a Jacobi multiplier of the system (1.1). If we introduce the divergence:

(11.5) $$\Delta_0 = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}$$

then we can also put (11.4) into the form:

(11.6) $$\sum_{i=1}^{n} X_i \frac{\partial \log \mu}{\partial x_i} + \Delta_0 = 0.$$

Here, as in the foregoing paragraphs, we would also like to pose the requirement that μ should not vanish on the trajectory (except for possibly a finite number of points). If μ were zero along $\mathfrak{T}$ then, from (11.3), since we have assumed that the X_i are each regular, that would say that all $\mathfrak{F}_i$ would vanish along $\mathfrak{T}$, and that, in turn, would say that equations (11.2) are linearly-dependent, so perhaps a relation:

$$\frac{\partial f_{n-1}}{\partial x_i} = \sum_{\kappa=1}^{n-2} \alpha_\kappa \frac{\partial f_\kappa}{\partial x_i} \qquad (i = 1, \ldots, n)$$

with constant α_κ exists. Geometrically, this says that the integral surface:

$$f_{n-1} - \sum_{\kappa=1}^{n-2} \alpha_\kappa f_\kappa = \text{const.}$$

of the system (1.1) is singular along $\mathfrak{T}$. Our assumption then demands *that no possible integral surface possesses the trajectories as singular lines.* The same argument is also valid in the context of the foregoing paragraphs, except that there it will be true *in the small.*

Moreover, one infers from (1.1) and (11.3) that:

(11.7) $$dt = \frac{dx_i}{X_i} = (-1)^\iota \mu \frac{dx_i}{\mathfrak{F}_i} = \mu \sqrt{\frac{\sum_{\iota=1}^{n} dx_\iota^2}{\sum_{\iota=1}^{n} \mathfrak{F}_\iota^2}} = \mu \frac{ds}{\Gamma_{n-1,n}},$$

in which *ds* denotes the arc length element of the trajectory $\mathfrak{T}$ in R_n and one has set:

(11.8) $$\Gamma_{n-1,n}^2 = \sum_{\iota=1}^{n} \mathfrak{F}_\iota^2 = \sum_{i=1}^{n} \left[\frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{i-1}, x_{i-1}, \ldots, x_n)} \right]^2.$$

As a result of a known identity, $\Gamma_{n-1,n}$ coincides with the expression that emerges from the quantity $\Gamma_{m,n}$ that was defined in the previous paragraph when $m = n - 1$. One can attribute a geometric meaning to it by way of the remark that:

$$(-1)^i \frac{\mathfrak{F}_i}{\Gamma_{n-1,n}} = \mu \frac{X_i}{\Gamma_{n-1,n}}$$

are the direction cosines of the tangent to $\mathfrak{T}$. On the basis of (11.7), the averaging process:

$$\bar{\alpha} = \int_S \alpha F \frac{\mu}{\Gamma_{n-1,n}} ds : \int_S F \frac{\mu}{\Gamma_{n-1,n}} ds , \tag{11.9}$$

enters in place of (1.7), in which S denotes the segment of the trajectory that corresponds to the interval T of t (the total length of the latter for $T = \infty$, resp.). The differential equations of the differential invariants will then read:

$$\sum_{i=1}^{n-1} \frac{\partial J}{\partial c} \cdot \int_S \frac{\partial J}{\partial a_\nu} F \frac{\mu}{\Gamma_{n-1,n}} ds + \frac{\partial J}{\partial a_\nu} \cdot \int_S F \frac{\mu}{\Gamma_{n-1,n}} ds = 0 \qquad (\nu = 1, \ldots, \rho) \tag{11.10}$$

in this case.

§ 12. – The invariants of a simply-imprimitive system.

Now that the differential equations of adiabatic invariants have been presented in all cases, we would next like to investigate their integrability conditions and possible solutions for simply-imprimitive systems. If we introduce the notations:

$$g = \int_\Phi F \frac{\mu}{\Gamma_{1n}} d\Phi , \qquad g_\nu = \int_\Phi \frac{\partial f}{\partial a_\nu} F \frac{\mu}{\Gamma_{1n}} d\Phi , \tag{12.1}$$

in analogy to (4.2), then the integrability conditions for the system (9.11) will read just as they did in (4.3). In order to get them in an explicit form, we let §§ **4**, **6** lead us to a gimmick.

Let a continuous function of position $G(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho)$ be given in the interior of Φ. We then consider the integral:

$$H = \int G(x_1, \ldots, x_n \mid a_1, \ldots, a_\rho)\, dx_1 \ldots dx_n , \tag{12.2}$$

in which the integration should be extended over the entire interior of Φ, and we would like to calculate its derivatives. In order to find $\partial H / \partial c$, along with Φ, we must consider the hypersurface Φ' that is defined by (9.3), and we have, with the notations of that equation:

$$\frac{\partial H}{\partial c} = \int_\Phi G\, d\Phi\, d'n : dc = \int_\Phi G \frac{1}{\Gamma_{1n}} d\Phi , \tag{12.3}$$

as would follow from (9.5). By contrast, in order to ascertain $\partial H / \partial a_\nu$, along with Φ, we introduce the surface Φ^* whose equation reads:

$$f(x_1, \ldots, x_n \mid a_1, \ldots, a_\nu + d\,a_\nu, \ldots, a_\rho) = c, \tag{12.4}$$

and whose points P^* we think of as arising from those of P on Φ by the normal displacement d^*n whose direction cosines are given by (9.4), and whose components then read:

$$d^*x_i = \frac{1}{\Gamma_{1n}} \frac{\partial f}{\partial x_i} d^*n .$$

On the other hand, (12.4) gives us:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d^*x_i \equiv \Gamma_{1n}\, d^*n = -\frac{\partial f}{\partial a_\nu} da_\nu ,$$

so

$$d^*n = -\frac{\partial f}{\partial a_\nu} \frac{1}{\Gamma_{1n}}\, da_\nu , \qquad d^*x_i = -\frac{\partial f}{\partial a_\nu} \frac{1}{\Gamma_{1n}}\, da_\nu . \tag{12.5}$$

In the calculation of $\partial H / \partial a_\nu$, one should observe that not only will Φ vary when the a_ν are varied, but also the function G in the interior of Φ, such that one will have:

$$\begin{aligned} \frac{\partial H}{\partial a_\nu} &= \int_\Phi G\, d\Phi\, d^*n : da_\nu + \int \frac{\partial G}{\partial a_\nu} dx_1 \cdots dx_n \\ &= -\int_\Phi \frac{\partial f}{\partial a_\nu} G \frac{1}{\Gamma_{1n}} d\Phi + \int \frac{\partial G}{\partial a_\nu} dx_1 \cdots dx_n . \end{aligned} \tag{12.6}$$

If one compares these formulas with the integrals g, g_ν then that will suggest that one might introduce the integral:

$$V = \int F\mu\, dx_1 \ldots dx_n , \tag{12.7}$$

which is extended over the interior of Φ. Namely, from (12.3) and (12.6), one has:

$$g = \frac{\partial V}{\partial c}, \qquad g_\nu = -\frac{\partial V}{\partial a_\nu} - \int \frac{\partial (F\mu)}{\partial a_\nu}\, dx_1 \ldots dx_n , \tag{12.8}$$

from which, it will follow, when one appeals to (12.3) and (12.6) again, that:

$$\begin{aligned} \frac{\partial g_\kappa}{\partial a_\lambda} - \frac{\partial g_\lambda}{\partial a_\kappa} &= \int_\Phi \left\{ \frac{\partial f}{\partial a_\lambda} \frac{\partial (F\mu)}{\partial a_\kappa} - \frac{\partial f}{\partial a_\kappa} \frac{\partial (F\mu)}{\partial a_\lambda} \right\} \frac{1}{\Gamma_{1n}} d\Phi , \\ \frac{\partial g}{\partial a_\kappa} + \frac{\partial g_\kappa}{\partial c} &= -\int_\Phi \frac{\partial (F\mu)}{\partial a_\kappa} \frac{1}{\Gamma_{1n}} d\Phi . \end{aligned} \tag{12.9}$$

As a result, the integrability conditions (4.3) will then read explicitly:

$$\int_\Phi F\frac{\mu}{\Gamma_{1n}}d\Phi\cdot\int_\Phi\left\{\frac{\partial f}{\partial a_\lambda}\frac{\partial(F\mu)}{\partial a_\kappa}-\frac{\partial f}{\partial a_\kappa}\frac{\partial(F\mu)}{\partial a_\lambda}\right\}\frac{1}{\Gamma_{1n}}d\Phi$$

(12.10)
$$+\int_\Phi\frac{\partial f}{\partial a_\kappa}F\frac{\mu}{\Gamma_{1n}}d\Phi\cdot\int_\Phi\frac{\partial(F\mu)}{\partial a_\lambda}\frac{1}{\Gamma_{1n}}d\Phi$$

$$-\int_\Phi\frac{\partial f}{\partial a_\lambda}F\frac{\mu}{\Gamma_{1n}}d\Phi\cdot\int_\Phi\frac{\partial(F\mu)}{\partial a_\kappa}\frac{1}{\Gamma_{1n}}d\Phi\equiv 0\qquad(\kappa,\lambda=1,\ldots,\rho).$$

With the use of the notation (9.10), we can then formulate the theorem:

Theorem 7. – *The necessary and sufficient conditions for the existence of an adiabatic invariant for a simply-imprimitive system read:*

(12.11)
$$\overline{\frac{\partial f}{\partial a_\kappa}\frac{\partial\log\mu F}{\partial a_\lambda}}-\overline{\frac{\partial f}{\partial a_\kappa}}\cdot\overline{\frac{\partial\log\mu F}{\partial a_\lambda}}\equiv\overline{\frac{\partial f}{\partial a_\lambda}\frac{\partial\log\mu F}{\partial a_\kappa}}-\overline{\frac{\partial f}{\partial a_\lambda}}\cdot\overline{\frac{\partial\log\mu F}{\partial a_\kappa}}$$

$$(\kappa,\lambda=1,\ldots,\rho).$$

A comparison with (4.30) will show that this has the same form as in the two-dimensional case, and therefore one can link it with statements that correspond to the ones in that case.

The assumption that Φ is closed in R_n makes it possible for us to establish that on Φ, the coordinates $x_1, \ldots, x_{n-1}$, as well as all functions of them, as well – in particular, $\frac{\partial f}{\partial a_\kappa}$ and $\frac{\partial\log\mu F}{\partial a_\kappa}$ – can be developed into multiple Fourier series in $n-1$ real parameters. If we then set:

(12.12)
$$\frac{\partial f}{\partial a_\kappa}=\alpha_\kappa+h_\kappa(x_1,\ldots,x_n),\qquad\frac{\partial\log\mu F}{\partial a_\kappa}=\beta_\kappa+j_\kappa(x_1,\ldots,x_n)\qquad(\kappa=1,\ldots,\rho),$$

in which:

$$\alpha_\kappa=\overline{\frac{\partial f}{\partial a_\kappa}},\qquad\beta_\kappa=\overline{\frac{\partial\log\mu F}{\partial a_\kappa}}$$

mean the constant terms in the Fourier series for $F\frac{\partial f}{\partial a_\kappa}$ and $F\frac{\partial\log\mu F}{\partial a_\kappa}$, resp., then the h_κ, j_κ will be functions whose mean value over Φ vanishes, and one can give (12.11) the form:

(12.13)
$$\overline{h_\kappa\,j_\lambda}=\overline{h_\lambda\,j_\kappa},$$

which can also be written as a relation between Fourier constants.

Three special cases appear with those of § **5**. The developments are the same as before, so we shall only state the result:

Case I: $$h_\kappa(x_1, \ldots, x_n) = 0 \qquad (\kappa = 1, \ldots, \rho).$$

The functional operation vanishes from equations (1.8), so it will be identical with (1.6). f is an integral that is stationary with respect to the parameters – i.e., it is not only an integral of the system (1.1), but also the ρ systems:

(12.14) $$\frac{dx_i}{dt} = \frac{\partial X_i}{\partial a_\kappa} \qquad (\kappa = 1, \ldots, \rho),$$

for which it will be necessary that all n-rowed determinants in the matrix:

$$\begin{pmatrix} X_1 & \cdots & X_n \\ \dfrac{\partial X_1}{\partial a_\kappa} & \cdots & \dfrac{\partial X_n}{\partial a_\kappa} \end{pmatrix} \qquad (\kappa = 1, \ldots, \rho)$$

must vanish.

Case II: For a well-defined multiplier μ, one has:

$$j_\kappa(x_1, \ldots, x_n) = 0 \qquad (\kappa = 1, \ldots, \rho),$$

so

(12.15) $$\frac{\partial \log \mu F}{\partial a_\kappa} = \Psi_\kappa(f \mid a_1, \ldots, a_\rho) \qquad (\kappa = 1, \ldots, \rho).$$

For $F \equiv 1$, it will follow that μ is a stationary multiplier; i.e., it is also such a thing for the system (12.14). For that to be true, it is necessary that all $(n + 1)$-rowed determinants must vanish in the matrix:

$$\begin{pmatrix} X_1 & \cdots & X_n, & \Delta_0 \\ \dfrac{\partial X_1}{\partial a_\kappa} & \cdots & \dfrac{\partial X_n}{\partial a_\kappa}, & \dfrac{\partial \Delta_0}{\partial a_\kappa} \end{pmatrix} \qquad (\kappa = 1, \ldots, \rho).$$

That is the case for Liouville systems ($\Delta_0 = 0$), and especially Hamiltonian systems, in particular.

Case III: $$j_\kappa(x_1, \ldots, x_n) = \text{const. } h_\kappa(x_1, \ldots, x_n) \qquad (\kappa = 1, \ldots, \rho)$$

can lead back to the foregoing case, just as in § **5**.

In summary, along with Theorem 4, one also has the following one:

Theorem 8. – *Sufficient conditions for the existence of the adiabatic invariants of a simply-imprimitive system are:*

a) *The stationarity of the finitely multi-valued integrals.*

b) *The stationarity of a multiplier with respect to the parameters under a simple averaging process.*

§ 13. – The generalized Liouville system.

(12.15) includes the particular possibility that one has:

$$\frac{\partial \mu F}{\partial a_\kappa} = 0 \qquad (\kappa = 1, \ldots, \rho). \tag{13.1}$$

We would then like to refer to such a system as a *generalized Liouville system.*

It follows from (12.8), on the one hand, and (9.11), on the other, that the equations of the adiabatic invariants assume the form:

$$\frac{\partial J}{\partial c}\frac{\partial V}{\partial a_\kappa} - \frac{\partial J}{\partial a_\kappa}\frac{\partial V}{\partial c} = 0 \qquad (\nu = 1, \ldots, \rho),$$

i.e.:

$$J = V = \int F \mu \, dx_1 \ldots dx_n \tag{13.2}$$

is the adiabatic invariant of our system. Theorem 5 will then be related to the following one:

Theorem 9: – *The adiabatic invariant of a simply-primitive generalized Liouville system:*

$$V = \int F \mu \, dx_1 \ldots dx_n \, .$$

If one is dealing with a simple averaging ($F \equiv 1$) then the class of systems under consideration will include the Liouville systems, in particular, for which $\Delta_0 = 0$, $\mu =$ const. For them, the invariant will reduce on the volume in R_n that is enclosed by Φ. Once again those Hamiltonian systems that possess no other finitely multi-valued integral beyond the energy integral are included in that as a particular possibility, and for them, our theorem will coincide with the statement of the Gibbs-Hertz theorem ([17]) that the phase volume of the energy surface is an adiabatic invariant. More generally, under simple averaging, (13.1) will encompass the systems that admit a multiplier that is independent of the parameters, and as we stated in Theorem 6, they can be converted into Liouville systems by a transformation that is free of parameters, and therefore allowed.

We shall now prove Theorem 6. We first introduce new coordinates $\xi_1, \ldots, \xi_n$ in place of the $x_1, \ldots, x_n$, which are regular, invertible functions of $x_1, \ldots, x_n$ that do not

([17]) Cf., Levi-Civita 1, pp. 339-342.

include the parameters $a_1, \ldots, a_n$, so the system (1.1) will be transformed into a new system:

$$\frac{\partial \xi_i}{\partial t} = H_i = \sum_{\kappa=1}^{n} \frac{\partial \xi_i}{\partial x_\kappa} X_\kappa \qquad (i = 1, \ldots, n).$$

We demand that this transformed system should be a Liouville type, so its divergence must vanish:

$$\sum_{i=1}^{n} \frac{\partial H_i}{\partial \xi_i} = \sum_{i=1}^{n} \sum_{\lambda=1}^{n} \frac{\partial H_i}{\partial x_\lambda} \frac{\partial x_\lambda}{\partial \xi_i} \equiv 0$$

or

$$\sum_{i=1}^{n} \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{n} \frac{\partial^2 \xi_i}{\partial x_\kappa \partial x_\lambda} \frac{\partial x_\lambda}{\partial \xi_i} X_\lambda + \sum_{\kappa=1}^{n} \sum_{\lambda=1}^{n} \frac{\partial X_\kappa}{\partial x_\lambda} \sum_{i=1}^{n} \frac{\partial x_\lambda}{\partial \xi_i} \frac{\partial \xi_i}{\partial \xi_\kappa} \equiv 0. \tag{13.3}$$

On the other hand, one has the identities:

$$\frac{\partial x_\lambda}{\partial x_\kappa} = \sum_{i=1}^{n} \frac{\partial x_\lambda}{\partial \xi_i} \frac{\partial \xi_i}{\partial \xi_\kappa} = \delta_{\lambda\kappa},$$

from which, one will infer:

$$\frac{\partial x_\lambda}{\partial \xi_i} \cdot \frac{\partial(\xi_1, \ldots, \xi_n)}{\partial(x_1, \ldots, x_n)} = (-1)^{i-\lambda} \frac{\partial(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_n)}{\partial(x_1, \ldots, x_{\lambda-1}, x_{\lambda+1}, \ldots, x_n)},$$

and (13.3) will then take on the form:

$$\sum_{\kappa=1}^{n} X_\kappa \sum_{i=1}^{n} \sum_{\lambda=1}^{n} (-1)^{i+\lambda} \frac{\partial^2 \xi_i}{\partial x_\kappa \partial x_\lambda} \frac{\partial(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_n)}{\partial(x_1, \ldots, x_{\lambda-1}, x_{\lambda+1}, \ldots, x_n)} + \frac{\partial(\xi_1, \ldots, \xi_n)}{\partial(x_1, \ldots, x_n)} \sum_{\kappa=1}^{n} \frac{\partial X_\kappa}{\partial x_\kappa} \equiv 0,$$

or finally:

$$\sum_{\kappa=1}^{n} \frac{\partial}{\partial x_\kappa} \left\{ X_\kappa \cdot \frac{\partial(\xi_1, \ldots, \xi_n)}{\partial(x_1, \ldots, x_n)} \right\} \equiv 0, \tag{13.4}$$

from which one will find from a comparison with (11.4) that the functional determinant:

$$\frac{\partial(\xi_1, \ldots, \xi_n)}{\partial(x_1, \ldots, x_n)} = \mu \tag{13.5}$$

must be a multiplier of the system (1.1). By assumption, the left-hand side is independent of the parameters, and the same thing must then be true for one of the possible multipliers of (1.1).

Conversely, if there is a multiplier of (1.1) (we shall call it μ) that is independent of $a_1, \ldots, a_\rho$ then one can always find a transformation of the x into the ξ whose functional determinant (13.5) is equal to μ, and which is free of the parameters. As a result of the

equations above, that will then convert the system (1.1) into one with a Liouville character. Theorem 6 is then proved with that.

§ 14. – The reduction of imprimitive systems.

We shall now turn to a treatment of the general m-fold imprimitive systems, so when one uses the terminology of the classification in § **8**, the cases A, B2, and B3, for which, the differential systems (10.13) and (11.10) will characterize their invariants. We then know the m independent integrals of (1.1):

$$f_i\,(x_1\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho) = c_i\,, \qquad (i = 1,\,\ldots,\,m \leq n-1), \tag{14.1}$$

which determine the closed manifold Φ in R_n . The next thing to consider would then be to make use of one's knowledge of these integrals in order to lower the rank of the original differential system, so that it will go to a simply-imprimitive or two-dimensional system.

That leads to the following process: We take the integrals:

$$f_1, f_2, \ldots, f_{\kappa-1}, f_{\kappa+1}, \ldots, f_m \tag{14.2}$$

and solve them for $m-1$ of the $x_1,\,\ldots,\,x_n$, which is always possible, due to the independence of the latter. Let those variables, whose choice can naturally depend upon κ, be denoted by $x_1,\,\ldots,\,x_{m-1}$. The solution in Φ will then imply the existence of $m-1$ relations of the type:

$$x_i = \psi_{i\kappa}\,(x_1\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho \mid c_1\,,\,\ldots,\,c_{\kappa-1}\,,\,c_{\kappa+1}\,,\,\ldots,\,c_m) \qquad (i = 1,\,\ldots,\,m-1), \tag{14.3}$$

which satisfy the following equations identically:

$$f_\lambda\,(\psi_{1\kappa},\,\ldots,\,\psi_{m-1,\kappa}\,,\,x_m\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho) \equiv c_\lambda\,, \tag{14.4}$$

$$(\lambda = 1,\,\ldots,\,\kappa-1,\,\kappa+1,\,\ldots,\,m).$$

Since the quantities (14.2) are integrals of (1.1), the further identities will follow:

$$\frac{d\psi_{i\kappa}}{dt} = \sum_{j=m}^{n} \frac{\partial \psi_{i\kappa}}{\partial x_j} X_j \equiv X_i \qquad (i = 1,\,\ldots,\,m-1), \tag{14.5}$$

which one must naturally write in terms of the variables $x_m\,,\,\ldots,\,x_n$, and on the other hand, a system of $n-m+1$ differential equations will remain:

$$\frac{dx_j}{dt} = X_j\,(\psi_{1\kappa},\,\ldots,\,\psi_{m-1,\kappa}\,,\,x_m\,,\,\ldots,\,x_n \mid a_1\,,\,\ldots,\,a_\rho) \qquad (j = m,\,\ldots,\,n), \tag{14.6}$$

which, from our assumptions, admits a single known finitely multi-valued integral, namely:

(14.7) $$f_\kappa(\psi_{1\kappa}, \ldots, \psi_{m-1,\kappa}, x_m, \ldots, x_n) = c_\kappa,$$

so it will be simply-imprimitive or two-dimensional, resp., when $m = n - 1$.

That integral is nothing but the relation that emerges from (14.1) upon eliminating the $x_1, \ldots, x_{m-1}$; i.e., geometrically-speaking, it represents the projection Φ^* of the manifold Φ in the R_n of $x_1, \ldots, x_n$ onto the $n - m + 1$-dimensional coordinate space R^* of the $x_m, \ldots, x_n$. That illuminates the fact that this closed manifold Φ^* is invariant under the choice of index κ; assuming that the variables that one solves for remain the same in that way. According to our topological assumptions, Φ^* will be covered quasi-ergodically by the trajectories of the system (14.6) [coincide with them in the $(x_{n-1}, \ldots, x_n)$-plane, resp.]. We will now examine this reduced system (14.6), instead of the original system (1.1), and we will first look for the adiabatic invariants of the latter and then look for the conditions under which they are also invariants of the original system.

It would be convenient for that to first present some relations that relate to the system (14.6). All quantities can appear in out calculations in two different forms: First, as functions of the $x_1, \ldots, x_n, a_1, \ldots, a_\rho$ that are free of the $c_1, \ldots, c_m$, and then as functions of the $x_1, \ldots, x_n, a_1, \ldots, a_\rho \,|\, c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_m$. When they are written in the latter form, in which they also represent the projection onto R^*, we would like to provide them with an asterisk. Moreover, the matrix:

(14.8) $$D \equiv \left(\frac{\partial f_i}{\partial x_1} \cdots \frac{\partial f_i}{\partial x_{m-1}} \right) \qquad (i = 1, \ldots, m)$$

will play a role in what we shall do. We would like to denote the determinants that emerge from deleting rows $p, q, r, \ldots$ and columns $\lambda, \mu, \ldots$ by:

$$D^{\lambda,\mu,\cdots}_{p,q,r,\cdots}$$

We shall next determine the divergence of (14.6); it is:

(14.9) $$\Delta_\kappa = \sum_{j=m}^{n} \frac{\partial X_j^*}{\partial x_j} = \sum_{i=1}^{m-1} \sum_{j=m}^{n} \frac{\partial X_j}{\partial x_i} \frac{\partial \psi_{i\kappa}}{\partial x_j} + \sum_{j=m}^{n} \frac{\partial X_j}{\partial x_j}.$$

On the other hand, (14.4) implies the equations:

$$\frac{\partial f_\lambda}{\partial x_1} \frac{\partial \psi_{1\kappa}}{\partial x_j} + \ldots + \frac{\partial f_\lambda}{\partial x_{m-1}} \frac{\partial \psi_{m-1,\kappa}}{\partial x_j} = - \frac{\partial f_\lambda}{\partial x_j} \qquad (j = m, \ldots, n;\ \lambda \neq \kappa = 1, \ldots, m),$$

whose solution will yield:

(14.10) $$\frac{\partial \psi_{i\kappa}}{\partial x_j} = \frac{(-1)^{i-1}}{D_\kappa}\left\{\sum_{\lambda=1}^{\kappa-1}(-1)^\lambda \frac{\partial f_\lambda}{\partial x_j} D^i_{\lambda\kappa} - \sum_{\lambda=\kappa+1}^{n}(-1)^\lambda \frac{\partial f_\lambda}{\partial x_j} D^i_{\kappa\lambda}\right\},$$

so

$$\Delta_\kappa = \frac{1}{D_\kappa}\sum_{i=1}^{m-1}\sum_{j=m}^{n}(-1)^{i-1}\frac{\partial X_j}{\partial x_i}\left[\sum_{\lambda=1}^{\kappa-1}(-1)^\lambda \frac{\partial f_\lambda}{\partial x_j} D^i_{\lambda\kappa} - \sum_{\lambda=\kappa+1}^{n}(-1)^\lambda \frac{\partial f_\lambda}{\partial x_j} D^i_{\kappa\lambda}\right] + \sum_{i=m}^{n}\frac{\partial X_i}{\partial x_i}.$$

However, the $f_1, \ldots, f_m$ are integrals of (1.1); i.e., one has the identities:

(14.11) $$\sum_{j=1}^{n}\frac{\partial f_\lambda}{\partial x_j}X_j \equiv 0 \qquad (l = 1, \ldots, m),$$

and upon differentiating this with respect to x_i, one will find that:

$$\sum_{j=1}^{n}\frac{\partial f_\lambda}{\partial x_j}\frac{\partial X_j}{\partial x_i} = -\sum_{j=1}^{m-1}\frac{\partial f_\lambda}{\partial x_j}\frac{\partial X_j}{\partial x_i} - \sum_{j=1}^{n}\frac{\partial^2 f_\lambda}{\partial x_i \partial x_j}X_j,$$

such that ultimately one will have:

$$\Delta_\kappa = \frac{1}{D_\kappa}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\frac{\partial X_j}{\partial x_i}\left[\sum_{\lambda=1}^{\kappa-1}(-1)^{i+\lambda}\frac{\partial f_\lambda}{\partial x_j}D^i_{\lambda\kappa} - \sum_{\lambda=\kappa+1}^{n}(-1)^{\lambda-i}\frac{\partial f_\lambda}{\partial x_j}D^i_{\kappa\lambda}\right]$$

$$+ \frac{1}{D_\kappa}\sum_{j=1}^{n}X_j\sum_{i=m}^{m-1}(-1)^{i+\lambda}\frac{\partial X_j}{\partial x_i}\left[\sum_{\lambda=1}^{\kappa-1}(-1)^{i+\lambda}\frac{\partial^2 f_\lambda}{\partial x_i \partial x_j}D^i_{\lambda\kappa} - \sum_{\lambda=\kappa+1}^{n}(-1)^{i+\lambda}\frac{\partial^2 f_\lambda}{\partial x_i \partial x_j}D^i_{\kappa\lambda}\right] + \sum_{i=m}^{n}\frac{\partial X_i}{\partial x_i}$$

$$= \sum_{i=1}^{n}\frac{\partial X_i}{\partial x_i} + \sum_{j=1}^{n}X_j\frac{\partial \log D_\kappa}{\partial x_j},$$

or, when Δ_0 means the divergence of the original system:

(14.12) $$\Delta_\kappa = \Delta_0 + \sum_{i=1}^{n}X_i\frac{\partial \log D_\kappa}{\partial x_i}.$$

It is important to remark that D_κ cannot vanish in this derivation, since we have assumed the solubility of the quantities (14.2) for the $x_1, \ldots, x_{m-1}$. In passing, we infer from the formula (14.2) the fact that the divergence of the reduced system will be identical to the original one when D_κ is an integral of (1.1), and in particular, we will have:

Theorem 10. – *If the original system has Liouville type then the same thing will be true for the reduced system if and only if D_κ is an integral of the first one.*

We are now in a position to find the multiplier μ_κ of the reduced system (14.6), whose differential equation reads:

$$\sum_{j=m}^{n} X_j^* \frac{\partial \mu_\kappa^*}{\partial x_j} + \mu_\kappa^* \Delta_\kappa^* = 0$$

or

$$\sum_{j=m}^{n} \sum_{i=1}^{m-1} \frac{\partial \mu_\kappa}{\partial x_i} \frac{\partial \psi_{i\kappa}}{\partial x_j} X_j + \sum_{j=m}^{n} \frac{\partial \mu_\kappa}{\partial x_j} X_j + \mu_\kappa \Delta_\kappa = 0,$$

and with the use of (14.10) and the identity:

$$\sum_{j=m}^{n} \frac{\partial f_\lambda}{\partial x_j} X_j = - \sum_{j=1}^{m-1} \frac{\partial f_\lambda}{\partial x_j} X_j \qquad (\lambda = 1, \ldots, m),$$

which follows from (14.11), that will assume the form:

$$\sum_{i=1}^{m-1} \frac{1}{D_\kappa} \frac{\partial \mu_\kappa}{\partial x_i} \sum_{j=1}^{m-1} X_j \left[\sum_{\lambda=1}^{\kappa-1} (-1)^{i+\lambda} \frac{\partial f_\lambda}{\partial x_j} D^i_{\lambda\kappa} - \sum_{\lambda=\kappa+1}^{n} (-1)^{i+\lambda} \frac{\partial f_\lambda}{\partial x_j} D^i_{\kappa\lambda} \right] + \sum_{j=m}^{n} \frac{\partial \mu_\kappa}{\partial x_j} X_j + \mu_\kappa \Delta_\kappa = 0,$$

or

$$\sum_{i=1}^{n} \frac{\partial \mu_\kappa}{\partial x_i} X_i + \mu_\kappa \Delta_\kappa = 0. \tag{14.13}$$

In conjunction with (14.12), it will follow from this that:

$$\sum_{i=1}^{n} \frac{\partial \log(\mu_\kappa D_\kappa)}{\partial x_i} X_i + \Delta_0 = 0 \ ;$$

i.e., if μ_0 denotes a multiplier of the original system then:

$$\mu_\kappa = \frac{\mu_0}{D_\kappa} \tag{14.14}$$

will be a multiplier of the reduced system (14.6).

When we completely overlook its origin and consider it for its own sake, the system (14.6) is a simply-imprimitive $n - m + 1$-dimensional system whose right-hand side includes the parameters $a_1, \ldots, a_\rho \mid c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_m$, and we can come to those parameters in two different ways: First of all, if the $a_1, \ldots, a_\rho$ were already included in the X_j then they would once more come to the fore in the same way that c did when we performed the substitution (14.3). We would like to examine their appearance in the integral f_κ^* and the multiplier μ_κ^* in more detail. First of all, it follows from the identities (14.4) by differentiation with respect to c_λ that:

$$\frac{\partial f_j}{\partial x_1}\frac{\partial \psi_{1\kappa}}{\partial c_\lambda} + \ldots + \frac{\partial f_j}{\partial x_{m-1}}\frac{\partial \psi_{m-1,\kappa}}{\partial c_\lambda} \equiv \delta_{j\lambda} \qquad (j, \lambda = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m),$$

from which, we will find by solution that:

$$\frac{\partial \psi_{i\kappa}}{\partial c_\lambda} = \begin{cases} (-1)^{i+\lambda}\dfrac{D^i_{\lambda\kappa}}{D_\kappa} & \text{when} \quad \lambda < \kappa, \\ (-1)^{i+\lambda+1}\dfrac{D^i_{\lambda\kappa}}{D_\kappa} & \text{when} \quad \lambda > \kappa. \end{cases} \tag{14.15}$$

It further follows that:

$$\frac{\partial f^*_\kappa}{\partial c_\lambda} = \sum_{i=1}^{m-1}\frac{\partial f_\kappa}{\partial x_i}\frac{\partial \psi_{i\kappa}}{\partial c_\lambda} = (-1)^{\kappa+\lambda+1}\frac{D_\lambda}{D_\kappa} \qquad (\lambda \neq \kappa) \tag{14.16}$$

and

$$\frac{\partial \mu^*_\kappa}{\partial c_\lambda} = \sum_{i=1}^{m-1}\frac{\partial \mu_\kappa}{\partial x_i}\frac{\partial \psi_{i\kappa}}{\partial c_\lambda} = \begin{cases} \dfrac{1}{D_\kappa}\displaystyle\sum_{i=1}^{m-1}(-1)^{i+\lambda}D^i_{\lambda\kappa}\frac{\partial \mu_\kappa}{\partial x_i} & \text{when} \quad \lambda < \kappa, \\ \dfrac{1}{D_\kappa}\displaystyle\sum_{i=1}^{m-1}(-1)^{i+\lambda+1}D^i_{\kappa\lambda}\frac{\partial \mu_\kappa}{\partial x_i} & \text{when} \quad \lambda > \kappa. \end{cases} \tag{14.17}$$

In a similar way, the differentiation of (14.4) with respect to a_ν will give the equations:

$$\frac{\partial f_j}{\partial x_1}\frac{\partial \psi_{1\kappa}}{\partial a_\nu} + \ldots + \frac{\partial f_j}{\partial x_{m-1}}\frac{\partial \psi_{m-1,\kappa}}{\partial a_\nu} = -\frac{\partial f_j}{\partial a_\nu}$$

$$(j = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m;\ \nu = 1, \ldots, \rho),$$

from which it will follow that:

$$\frac{\partial \psi_{i\kappa}}{\partial a_\nu} = \frac{1}{D_\kappa}\left[-\sum_{\lambda=1}^{\kappa-1}(-1)^{i+\lambda}\frac{\partial f_\lambda}{\partial a_\nu}D^i_{\lambda\kappa} + \sum_{\lambda=\kappa+1}^{n}(-1)^{i+\lambda}\frac{\partial f_\lambda}{\partial a_\nu}D^i_{\kappa\lambda}\right]. \tag{14.18}$$

With that, one will have:

$$\begin{aligned}\frac{\partial f^*_\kappa}{\partial a_\nu} &= \sum_{i=1}^{m-1}\frac{\partial f_\kappa}{\partial x_i}\frac{\partial \psi_{i\kappa}}{\partial a_\nu} + \frac{\partial f_\kappa}{\partial a_\nu} = \sum_{\lambda=1}^{\kappa-1}(-1)^{\kappa+\lambda}\frac{D_\lambda}{D_\kappa}\frac{\partial f_\lambda}{\partial a_\nu} + \sum_{\lambda=\kappa+1}^{m}(-1)^{\kappa+\lambda}\frac{D_\lambda}{D_\kappa}\frac{\partial f_\lambda}{\partial a_\nu} + \frac{\partial f_\kappa}{\partial a_\nu} \\ &= \sum_{\lambda=1}^{m}(-1)^{\kappa+\lambda}\frac{\partial f_\lambda}{\partial a_\nu}\frac{D_\lambda}{D_\kappa}.\end{aligned} \tag{14.19}$$

Incidentally, this, in conjunction with (14.16), will imply the formulas:

$$\frac{\partial f_\kappa^*}{\partial a_\nu} - \frac{\partial f_\kappa}{\partial a_\nu} = - \sum_{\lambda=1}^{m}{}' \frac{\partial f_\lambda}{\partial a_\nu} \frac{\partial f_\kappa^*}{\partial c_\lambda} \qquad (\nu = 1, \ldots, \rho), \tag{14.20}$$

in which the prime is intended to mean a sum that leaves out the terms with $\lambda = \kappa$. Ultimately, we have:

$$\begin{aligned} \frac{\partial \mu_\kappa^*}{\partial a_\nu} &= \sum_{i=1}^{m-1} \frac{\partial \mu_\kappa}{\partial x_i} \frac{\partial \psi_{i\kappa}}{\partial a_\nu} + \frac{\partial \mu_\kappa}{\partial a_\nu} \\ &= \frac{1}{D_\kappa} \left[\sum_{\lambda=1}^{m-1} \frac{\partial f_\lambda}{\partial a_\nu} \sum_{i=1}^{m-1} (-1)^{i+\lambda+1} \frac{\partial \mu_\kappa}{\partial x_i} D_{\lambda\kappa}^i + \sum_{\lambda=\kappa+1}^{m} \frac{\partial f_\lambda}{\partial a_\nu} \sum_{i=1}^{m-1} (-1)^{i+\lambda} \frac{\partial \mu_\kappa}{\partial x_i} D_{\kappa\lambda}^i \right] + \frac{\partial \mu_\kappa}{\partial a_\nu}, \end{aligned} \tag{14.21}$$

$$\frac{\partial \mu_\kappa^*}{\partial a_\nu} - \frac{\partial \mu_\kappa}{\partial a_\nu} = - \sum_{\lambda=1}^{m} \frac{\partial f_\lambda}{\partial a_\nu} \frac{\partial \mu_\lambda^*}{\partial c_\lambda}.$$

§ 15. – Existence conditions for the adiabatic invariants.

The system (14.6) with the parameters $a_1, \ldots, a_\rho, c_1, \ldots, c_{\kappa-1}, c_{\kappa+1}, \ldots, c_m$, and the integral (14.7) can possess an adiabatic invariant J_κ^*, and according to (9.11), it will be defined by the system of equations:

$$\begin{cases} \dfrac{\partial J_\kappa^*}{\partial c_\kappa} \cdot \displaystyle\int_{\Phi^*} \frac{\partial f_\kappa^*}{\partial c_\kappa} F^* \frac{\mu_\kappa^* \, d\Phi^*}{\Gamma^*} + \frac{\partial J_\kappa^*}{\partial c_\lambda} \cdot \int_{\Phi^*} F^* \frac{\mu_\kappa^* \, d\Phi^*}{\Gamma^*} = 0, \quad (\lambda = 1, \ldots, \kappa-1, \kappa+1, \ldots, m), \\ \dfrac{\partial J_\kappa^*}{\partial c_\kappa} \cdot \displaystyle\int_{\Phi^*} \frac{\partial f_\kappa^*}{\partial c_\kappa} F^* \frac{\mu_\kappa^*}{\Gamma^*} d\Phi^* + \frac{\partial J_\kappa^*}{\partial a_\nu} \cdot \int_{\Phi^*} F^* \frac{\mu_\kappa^* \, d\Phi^*}{\Gamma^*} = 0 \quad (\nu = 1, \ldots, \rho), \end{cases} \tag{15.1}$$

in which Γ^* means the analogue for f_κ^* of the quantity Γ_{1n} that is defined by (9.4); J_κ^* is a function of only the $a_1, \ldots, a_\rho, c_1, \ldots, c_m$. The integrals that appear here over the projection Φ^* can be easily converted into ones over Φ. One observes that from (10.10), the space element of R_n will possess the magnitude:

$$d\tau = dx_1 \ldots dx_n = \frac{1}{\Gamma_{mn}} \, d\Phi \, df_1 \, df_2 \ldots df_m,$$

while on the other hand, from (9.5), the space element of R^* will possess the expression:

$$d\tau^* = dx_m \ldots dx_n = \frac{1}{\Gamma^*} \, d\Phi^* \, df_\kappa,$$

from which the relations will follow:

$$\frac{d\Phi}{\Gamma_{m,n}} = \frac{1}{D_\kappa}\frac{d\Phi^*}{\Gamma^*}, \qquad \mu_0\frac{d\Phi}{\Gamma_{m,n}} = \mu_\kappa\frac{d\Phi^*}{\Gamma^*} \tag{15.2}$$

[cf., (14.14)]. By projecting onto the manifold Φ, we can then give equations (15.1) the form:

$$\begin{cases} \dfrac{\partial J_\kappa^*}{\partial c_\kappa}\cdot\displaystyle\int_{\Phi^*}\frac{\partial f_\kappa^*}{\partial c_\kappa}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi + \frac{\partial J_\kappa^*}{\partial c_\lambda}\cdot\int_{\Phi^*}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi = 0, \quad (\lambda = 1,\ldots,\kappa-1,\kappa+1,\ldots,m), \\ \dfrac{\partial J_\kappa^*}{\partial c_\kappa}\cdot\displaystyle\int_{\Phi^*}\frac{\partial f_\kappa^*}{\partial c_\kappa}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi + \frac{\partial J_\kappa^*}{\partial a_\nu}\cdot\int_{\Phi^*}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi = 0 \quad (\nu = 1,\ldots,\rho). \end{cases} \tag{15.3}$$

We introduced the reduced system in order to make it possible for us to calculate the adiabatic invariants of the original system (1.1). We were inclined to say *a priori* that any adiabatic invariant of the reduced system would also be an adiabatic invariant of the original system. *However, we will see that this is not the case,* since certain conditions must be fulfilled, moreover. Our question then reads: When is a solution of (15.3) also a solution of (10.13)? Since one naturally has $\partial J_\kappa^*/\partial c_\kappa \neq 0$, the elimination of the derivatives of J_κ^* from (15.3) and (10.13) will give the conditions:

$$\begin{aligned} &\sum_{\lambda=1}^{m}\int_\Phi\frac{\partial f_\kappa^*}{\partial c_\lambda}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi\cdot\int_\Phi\frac{\partial f_\lambda}{\partial a_\nu}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi \\ &+\int_\Phi\left\{\frac{\partial f_\kappa^*}{\partial a_\nu}-\frac{\partial f}{\partial a_\nu}\right\}F\frac{\mu_0}{\Gamma_{m,n}}d\Phi\cdot\int_\Phi F\frac{\mu_0}{\Gamma_{m,n}}d\Phi = 0 \qquad (\nu = 1,\ldots,\rho), \end{aligned} \tag{15.4}$$

which will take on the simple form:

$$\sum_{\lambda=1}^{m}(-1)^\lambda\left\{\overline{\frac{D_\lambda}{D_\kappa}\frac{\partial f_\lambda}{\partial a_\nu}}-\overline{\frac{D_\lambda}{D_\kappa}}\,\overline{\frac{\partial f_\lambda}{\partial a_\nu}}\right\} = 0 \qquad (\nu = 1,\ldots,\rho) \tag{15.5}$$

by means of (10.12), (14.16), and (14.19).

Theorem 11. – *The necessary and sufficient conditions for an adiabatic invariant of the reduced system to also be an adiabatic invariant of the original system read:*

$$\sum_{\lambda=1}^{m}(-1)^\lambda\left\{\overline{\frac{D_\lambda}{D_\kappa}\frac{\partial f_\lambda}{\partial a_\nu}}-\overline{\frac{D_\lambda}{D_\kappa}}\,\overline{\frac{\partial f_\lambda}{\partial a_\nu}}\right\} = 0 \qquad (\nu = 1,\ldots,\rho).$$

For example, (15.5) is fulfilled when some of the f_λ are (in the terminology of § **12**) stationary integrals, and the extended D_λ / D_κ are constant on Φ, and therefore functions of $f_1, \ldots, f_m$. The case in which *all of the* $\partial f_\lambda / \partial a_\nu$ ($\lambda = 1, \ldots, m,\ \nu = 1, \ldots, \rho$) are constant on Φ, so all known integrals will be stationary with respect to the parameters, is trivial, so the mean in (10.13) [(1.8), resp.] will vanish, J will become the *ordinary* invariant of (1.6), and there will exist no analogue of the reduction conditions (15.5) for it. Another possibility for the fulfillment of (15.5) is that:

$$\text{(15.6)} \qquad \frac{D_\lambda}{D_\kappa} = \Psi\,(f_1, \ldots, f_m \mid a_1, \ldots, a_\rho) \qquad (\lambda = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m)\,;$$

this can also be interpreted. Namely, if one employs the following integrals $f_1, \ldots, f_{\lambda-1}, f_{\lambda+1}, \ldots, f_m$ for the reduction, instead of (14.2), and if those integrals can be solved for the same variables $x_1, \ldots, x_{m-1}$ then, from (14.14) and (15.6), the multipliers μ_λ ($\lambda = 1, \ldots, m$) of all m reduced systems will be identical to each other, up to a factor that is constant on Φ. In particular, from (14.13), this will happen *when the divergence* Δ_λ *is the same for all m reduced systems.*

If (15.5) is fulfilled then we know that the adiabatic invariants of the reduced systems will also be adiabatic invariants of the original one, and we therefore need only to write the existence conditions for the first one – i.e., the integrability conditions of (15.1). We then appeal to Theorem 7 and obtain three groups of conditions:

$$\text{(15.7)} \quad \left\{ \begin{array}{ll} (a) & \overline{\dfrac{\partial f_\kappa^*}{\partial a_\nu}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\mu}} - \overline{\dfrac{\partial f_\kappa^*}{\partial a_\nu}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\mu}} = \overline{\dfrac{\partial f_\kappa^*}{\partial a_\mu}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\nu}} - \overline{\dfrac{\partial f_\kappa^*}{\partial a_\mu}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\nu}} \\ & \qquad\qquad (\nu, \mu = 1, \ldots, \rho), \\ (b) & \overline{\dfrac{\partial f_\kappa^*}{\partial c_\lambda}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\nu}} - \overline{\dfrac{\partial f_\kappa^*}{\partial c_\lambda}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial a_\nu}} = \overline{\dfrac{\partial f_\kappa^*}{\partial a_\nu}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_\lambda}} - \overline{\dfrac{\partial f_\kappa^*}{\partial a_\nu}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_\lambda}} \\ & \qquad\qquad (\lambda = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m), \\ (c) & \overline{\dfrac{\partial f_\kappa^*}{\partial c_\lambda}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_j}} - \overline{\dfrac{\partial f_\kappa^*}{\partial c_\lambda}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_j}} = \overline{\dfrac{\partial f_\kappa^*}{\partial c_j}\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_\lambda}} - \overline{\dfrac{\partial f_\kappa^*}{\partial c_j}}\ \overline{\dfrac{\partial \log \mu_\kappa^* F^*}{\partial c_\lambda}} \\ & \qquad\qquad (j, \lambda = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m), \end{array} \right.$$

in which, from the foregoing remarks, we can think of the mean as being taken over Φ, in the sense of (10.12). By means of (14.16) to (14.20) and (15.5), it can be brought into a generally less obvious form in which the appearance of the starred quantities will be avoided. If the $\frac{1}{2}\rho\,(\rho - 1) + (m - 1)\,\rho + \frac{1}{2}(m - 1)(m - 2)$ conditions (15.7) are fulfilled, as well as ρ equations (15.5) then we will know from the foregoing that the reduced system (14.6) possesses an adiabatic invariant and that it is, at the same time, an adiabatic invariant of the original system. The totality of these conditions is stronger than the

integrability conditions of (10.13), which have a very complicated form *in extenso* ([18]), but offer the advantage of being clearer to deal with.

Two special cases in which the relations (15.7) are satisfied are worthy of note:

$$\text{(I)} \qquad \frac{\partial \log \mu_\kappa^*}{\partial c_j} = A_j(f_1^*, \ldots, f_m^*, a_1, \ldots, a_\rho); \quad \frac{\partial \log \mu_\kappa^* F^*}{\partial a_\nu} = B_\nu(f_1^*, \ldots, f_m^*, a_1, \ldots, a_\rho)$$

$$(j = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m;\ \nu = 1, \ldots, \rho),$$

i.e., the left-hand sides are constant on Φ:

$$\text{(II)} \qquad \frac{\partial f_\kappa^*}{\partial c_j} = \bar{A}_j(f_1^*, \ldots, f_m^*, a_1, \ldots, a_\rho); \quad \frac{\partial f_\kappa^*}{\partial a_\nu} = \bar{B}_\nu(f_1^*, \ldots, f_m^*, a_1, \ldots, a_\rho).$$

The case of the proportionality of the left-hand sides in (I), (II) can be reduced to the two possibilities that were just mentioned, as in § **12**.

Case. I. If $F \equiv 1$ then μ_κ will be a stationary multiplier of the system (14.6), when one uses the previous terminology. If all $A_j = 0$ then one will have:

$$\text{(15.8)} \qquad \frac{\partial \log \mu_\kappa^* F^*}{\partial c_j} = 0 \qquad (j = 1, \ldots, \kappa - 1, \kappa + 1, \ldots, m),$$

and from (14.17), if an equation that is similar to the one that is also true for $\mu_\kappa^* F^*$ is true then that will say that:

$$\text{(15.9)} \qquad \frac{\partial \mu_\kappa F}{\partial x_i} = 0 \qquad (i = 1, \ldots, m - 1)$$

must be true. (14.21) will then further give:

$$\frac{\partial \mu_\kappa^* F^*}{\partial a_\nu} = \frac{\partial \mu_\kappa F}{\partial a_\nu} \qquad (\nu = 1, \ldots, \rho)$$

such that one must have:

$$\text{(15.10)} \qquad \frac{\partial \log \mu_\kappa F}{\partial a_\nu} = B_\nu(f_1, \ldots, f_m) \qquad (\nu = 1, \ldots, \rho)$$

If one also has that all $B_\nu = 0$ then one will have:

$$\text{(15.11)} \qquad \frac{\partial \mu_\kappa F}{\partial a_\nu} = 0 \qquad (\nu = 1, \ldots, \rho),$$

([18]) In regard to this, it should be mentioned that Fermi 1 sought to present those conditions for Hamiltonian systems. He proved their existence only in an example.

and in conjunction with (15.9), that will say that *the quantities:*

$$\mu_\kappa F = \frac{\mu_0}{D_\kappa} F$$

cannot depend upon $x_1, \ldots, x_{m-1}, a_1, \ldots, a_\rho$. In the previous terminology, the system (14.6) will be of generalized Liouville type. One must still add the reduction condition (15.5). If it has the special form (15.6) then all possible reduced systems will have generalized Liouville type (when $D_\lambda \neq 0$).

Case II. – f_κ^* is a stationary integral of the differential systems (14.6), and from equations (15.1), the functional operation will vanish. This case will come up, in particular, when all $\overline{A}_j$, $\overline{B}_\nu$ vanish; i.e., from (14.16) and (14.20), when:

(15.12) $$D_\lambda = 0 \qquad (\lambda \neq \kappa = 1, \ldots, m)$$

and

(15.13) $$\frac{\partial f_\kappa^*}{\partial a_\nu} = \frac{\partial f_\kappa}{\partial a_\nu} = 0 \qquad (\nu = 1, \ldots, \rho);$$

i.e., *when* f_κ *depends upon either* $x_1, \ldots, x_{m-1}$ *or* $a_1, \ldots, a_\rho$. Due to (15.12), the reduction conditions (15.5) will be satisfied identically then.

If the aforementioned possibility of the simultaneous validity of (15.9) and (15.11) arises then Theorem 9 will give us the adiabatic invariants of the reduced system directly and therefore those of the original one:

(15.14)
$$\begin{aligned} J = J_\kappa^* &= \int_{f_\kappa^*} \mu_\kappa^* F^* dx_m \ldots dx_n = \int_{f_\kappa^*} \frac{\mu_0^*}{D_\kappa^*} F^* dx_m \ldots dx_n \\ &= \int_{f_\kappa^*} \frac{\mu_0}{D_\kappa} F \, dx_m \ldots dx_n . \end{aligned}$$

It is possible, at best, to calculate all m different invariants that couple the $c_1, \ldots, c_m$ with the $a_1, \ldots, a_\rho$ by performing the m existing reductions in that way. Namely, that will be the case *when all reduced systems have generalized Liouville type with essentially different multipliers.* In cases A and B3, the classification of § **8** is applied in accordance with Theorem 5. We summarize:

Theorem 12. – *Other than the necessary reduction condition* (15.15), *there are sufficient conditions for the existence of an adiabatic invariant of an m-fold imprimitive system:*

a) *That* $\dfrac{\mu_0 F}{D_\kappa}$ *or*

b) f_κ

does not depend upon $x_1, \ldots, x_{m-1}, a_1, \ldots, a_\rho$. *In the first case:*

$$J = \int_{\Phi^*} \frac{\mu_0 F}{D_\kappa}\, dx_m \ldots dx_n$$

will be an adiabatic invariant.

This theorem admits many applications; we would like to mention only one special case:

Theorem 13. – *If the original system, like all m reduced systems, has Liouville type, and* $D_\kappa \neq 0$ ($\kappa = 1, \ldots, m$) *then the spatial volume of the projection:*

$$J = \int_{\Phi^*} dx_m \ldots dx_n$$

will be adiabatically invariant under simple averaging.

All μ_κ will then be constant on Φ, and the same thing will be true for μ_0, and then for D_κ, and (15.5) will be fulfilled identically.

§ 16. – An example.

As an example that will illustrate the foregoing theory, we would like to treat planar elliptical oscillations by means of the system:

$$(16.1) \qquad \frac{dx_1}{dt} = a_2 x_2 - a_3 x_3\,, \quad \frac{dx_2}{dt} = a_3 x_3 - a_1 x_1\,, \quad \frac{dx_3}{dt} = a_1 x_1 - a_2 x_2\,.$$

The system admits the two integrals:

$$f_1 \equiv x_1 + x_2 + x_3 = c_1\,, \qquad f_2 \equiv a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = c_2\,,$$

so we find ourselves in case B3. We take the simple mean $F \equiv 1$ and $T = \infty$. There are two paths that we can pursue in order to calculate the adiabatic invariants:

1. The direct method in § **1**, with the use of the x_i as functions of $t - t_0$, and

2. The process of § **15** by means of reducing to a two-dimensional coordinate plane.

First path: If one sets:

$$h^2 = a_1 a_2 + a_2 a_3 + a_3 a_1\,, \qquad k^2 = c_2 h^2 - a_1 a_2 a_3 a_4\,,$$

to abbreviate, then the integration of (16.1) will give the relations:

$$x_1 = \frac{1}{h^2}\frac{k}{\sqrt{a_1 + a_2}}\{a_2 \cos h\,(t - t_0) + h \sin h\,(t - t_0)\} + \frac{a_2\, a_3\, c_1}{h^2},$$

$$x_2 = \frac{1}{h^2}\frac{k}{\sqrt{a_1 + a_2}}\{a_2 \cos h\,(t - t_0) - h \sin h\,(t - t_0)\} + \frac{a_1\, a_3\, c_1}{h^2},$$

$$x_3 = -\frac{1}{h^2} k\sqrt{a_1 + a_2} \cos h\,(t - t_0) + \frac{a_1\, a_2\, c_1}{h^2}.$$

In the notation of (1.5), one will then have:

$$\varphi_{11} = \varphi_{12} = \varphi_{13} = 0,$$

$$\bar{\varphi}_{21} = \frac{1}{2h^4}\{2a_2^2\, a_3^2\, c_1^2 + k^2(a_2 + a_3)\}, \qquad \bar{\varphi}_{32} = \frac{1}{2h^4}\{2a_1^2\, a_3^2\, c_1^2 + k^2(a_1 + a_3)\},$$

$$\bar{\varphi}_{23} = \frac{1}{2h^4}\{2a_1^2\, a_2^2\, c_1^2 + k^2(a_1 + a_2)\}.$$

The system of equations (1.11) must be constructed with that, and one easily verifies that its two integrals will read:

(16.2) $$J_1 = c_1, \qquad J_2 = \frac{k^2}{h^3}.$$

Second path: First of all, from Theorem 1, c_1 is an adiabatic invariant, so:

$$J_1 = c_1.$$

In order to reduce, we employ f_1; i.e., we set:

$$x_1 \equiv \psi_{12} = c_1 - x_2 - x_3,$$

such that (16.1) will reduce to the following system:

(16.3)
$$\frac{dx_2}{dt} = a_1 x_2 + (a_1 + a_3) x_3 - a_1 c_1,$$

$$\frac{dx_3}{dt} = -(a_1 + a_3) x_2 - a_1 x_3 + a_1 c_1,$$

with the integral:

$$\Phi^* = f_2^* \equiv (a_1 + a_2)\ x_2^2 + (a_1 + a_3)\ x_3^2 - a_1 c_1 - 2\ a_1 c_1 (x_2 + x_3) + a_1 c_1^2 - c_2 = 0. \tag{16.4}$$

If a_1, a_2, a_3 are positive then the trajectory in R_3 will be the intersection of the ellipsoid f_2 with the fixed plane f_1 , and therefore an ellipse. Φ^* is its projection onto the x_2, x_3-plane, and thus another ellipse. The reduced system (16.3), like the original one (16.1), has zero divergence, so:

$$\mu_0 = \mu_2 = 1.$$

Due to the fact that $\partial f_1 / \partial a_\nu = 0$, the reduction condition (15.5) is fulfilled, and likewise the integrability conditions, from Theorem 12. The invariant is the area of the ellipse Φ^*, which is calculated to be:

$$J_2 = \pi \frac{k^2}{h^3}.$$

By contrast, the reduction by means of f_2 cannot be applied.

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