# On the relativity of five dimensions and an interpretation of the Schrödinger equation 

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Introduction. - This article is the development of four notes that were published last Summer in the Comptes-rendus de l'Académie des Sciences $\left({ }^{1}\right)$. Our purpose is to show, on the one hand, that the expression for the Lorentz force will necessarily lead to the consideration of a parallel displacement in a five-dimensional space whose metric is, admittedly, degenerate, like the one in Galileo-Newtonian space. On the other hand, we shall look for some consequences that one can infer from the study of that space when we have specified its nature by some hypotheses. We will recover some results that are due to Kaluza and relate to the Maxwell equations ( ${ }^{2}$ ), and we will obtain a new interpretation of the equation that Schrödinger placed at the root of his wave theory of mechanics.

It should be said that it was by meditating on the fundamental paper by E. Cartan on manifolds with affine connection $\left({ }^{3}\right)$, those of $\mathbf{E}$. Vessiot on the propagation of waves $\left({ }^{4}\right)$, the chapters that Hadamard dedicated to the theory of characteristics and bicharacteristics in his two great works $\left({ }^{5}\right)$, the beautiful thesis $\left({ }^{6}\right)$ and suggestive book $\left({ }^{7}\right)$ of L. de Broglie, as well as the brilliant papers of Schrödinger $\left({ }^{8}\right)$, which led us to the synthesis that we propose. Neither would we like to overlook Kaluza $\left({ }^{9}\right)$, who was the first to have interpreted electromagnetism in a five-dimensional space, or O. Klein $\left({ }^{10}\right)$ and Fock $\left({ }^{11}\right)$, whose work on those questions was rife with interest. Meanwhile, we shall depart from those three authors at some points. Our method is different, and we seem to conform more to the nature of things, and furthermore our results are more complete and lead to wave mechanics, as well.

[^0]Without pretending to give an exhaustive bibliography, we can further cite some works of $\mathbf{L}$. de Broglie ( ${ }^{1}$ ), Schidlof ( ${ }^{2}$ ), and Rosenfeld ( ${ }^{3}$ ) that appeared while our notes in the Comptes-rendu were being written and sent, or as they just appeared. In any event, we could not have been inspired by them; their viewpoints differ from ours, moreover. Finally, J. Struik has advised one of us that a paper that was written in collaboration with $\mathbf{N}$. Wiener will appear and that it will treat a fivedimensional universe. We have seen that paper $\left({ }^{4}\right)$ while our article was being written, but the methods are quite different.

One then sees that the consideration of a five-dimensional space has been the subject of several works. It would seem that what some believe to be merely an artifice that was created by mathematicians is, on the contrary, imposed or at least suggested by the nature of things.

## I. - On the equations of electromagnetism.

1.     - Consider a material point in a Minkowski space $E_{4}$ whose rest mass is $m$, and whose charge is $e$. Let $u^{0}, u^{1}, u^{2}, u^{3}$ be the components of the world-velocity of that point, and let $\xi^{0}$, $\xi^{1}, \xi^{2}, \xi^{3}$, with $\xi^{i}=m u^{i}$ denote the components of the world-impulse of that same point. Imagine that there is an electromagnetic field in $E_{4}$ whose tensorial components are the functions $F^{i k}$. That field exerts a force on the charged point, namely, the Lorentz force, whose components are calculated by means of the $F^{i k}$ and the components $s^{0}, s^{1}, s^{2}, s^{3}$ of the current that the moving point creates. The $s^{i}$ are given by the formulas:

$$
s^{i}=e u^{i}
$$

The Lorentz force $p^{0}, p^{1}, p^{2}, p^{3}$ is , moreover, defined by the equations:

$$
p^{i}=-F^{i k} s_{k} .
$$

The charged material point has a world-line whose differential equations are:

$$
d \xi^{i}=p^{i} d s
$$

in which $d s$ is the element of arc-length of that line. One knows that this element is defined by its square, which is a quadratic form in $d x^{0}, d x^{1}, d x^{2}, d x^{3}$ that is reducible to a sum of four squares of conveniently-chosen differentials. We suppose that they are the same ones that we have chosen. We further remark that:

[^1]$$
u^{i}=\frac{d x^{i}}{d s} .
$$

As a consequence, one will have:

$$
\begin{equation*}
d \xi^{i}=-\frac{e}{m} F_{\cdot k}^{i \cdot} \xi^{k} d s \quad(i=0,1,2,3) \tag{1}
\end{equation*}
$$

Those equations express the idea that one passes from the impulse vector $\boldsymbol{\Xi}\left(\xi^{0}, \xi^{1}, \xi^{2}, \xi^{3}\right)$ in $E_{4}$, which is located at the point $P\left(x^{0}, x^{1}, x^{2}, x^{3}\right)$ of the world-line to which it is tangent, to the vector $\boldsymbol{\Xi}$ that is located at the neighboring point $P\left(x^{i}+d x^{i}\right)$ along that line by adding the small vector $d \boldsymbol{\Xi}$ whose components are $d \xi^{i}$ to $\boldsymbol{\Xi}$ when it is transported parallel to itself from $P$ to $P^{\prime}$. That small vector is not zero, in general, since the world-line is curved. It is not a straight line, i.e., a geodesic, of $E_{4}$, and as a result, the Lorentz force has a significance that is not purely geometric with respect to Minkowski's $E_{4}$.

One knows that Weyl gave a new extension of differential geometry by introducing the notion of gauge in order to succeed in geometrizing electromagnetism. We shall proceed differently and preserve classical differential geometry but introduce one more dimension.
2. - Indeed, introduce a coordinate $x^{4}$ without, nonetheless, specifying the metric on the fivedimensional universe $E_{5}$ that is composed of the set of points $\left(x^{0}, x^{1}, x^{2}, x^{3}, x^{4}\right)$. We shall simply define the differential $d x^{4}$ of the fifth coordinate of the material point in question, and in that way, we can interpret equations (1) as being those of the parallel displacement, or if one prefers, LeviCivita transport, of a certain vector in the space $E_{5}$.

Those desiderata will be realized if one sets:

$$
d x^{4}=\frac{e}{m} d s
$$

in which $d s$ is the arc-length element of the world-line in $E_{4}$, and one chooses the vector whose components are $\xi^{0}, \xi^{1}, \xi^{2}, \xi^{3}$, which was defined before, and:

$$
\xi^{4}=m \frac{d x^{4}}{d s}=e
$$

If that is the case then equations (1) can be written like the equations of parallel transport in $E_{5}$ :

$$
\begin{equation*}
d \xi^{\alpha}=-G_{\beta \gamma}^{\alpha} \xi^{\beta} d x^{\gamma} \quad(\alpha, \beta, \gamma=0,1,2,3,4) \tag{2}
\end{equation*}
$$

in which $G_{\beta \gamma}^{\alpha}$ are the components of the affine connection at each point of $E_{5}$. One will have:

$$
\left.\begin{array}{l}
G_{i k}^{l}=0, \\
G_{4 i}^{l}=G_{i 4}^{l}=F_{\cdot i}^{l}, \\
G_{44}^{l}=G_{i l}^{4}=G_{i 4}^{4}=G_{44}^{4}=0
\end{array}\right\} \quad(i, k, l=0,1,2,3) .
$$

The first four equations (2) are nothing but equations (1), and the fifth one can be written:

$$
d e=0
$$

which expresses the conservation of charge of the moving point.
3. - One can see that this geometrization extends immediately to the case in which $E_{4}$ is no longer Minkowskian, but Einsteinian, i.e., the case in which it is pervaded by a gravitational field.

Let $\Gamma_{i k}^{l}$ denote the components of that field, i.e., the components of the affine connection on the Einsteinian $E_{4}$.

The electromagnetic field that is embedded in $E_{4}$ will have the functions $F^{i k}$ for its components, although it is not necessary to specify them, and we will see that the:

$$
p^{i}=-F^{i k} s_{k}
$$

are the components of the Lorentz force that act upon the charged point.
Furthermore, the equations of motion of that are:

$$
\begin{equation*}
m\left(\frac{d^{2} x^{i}}{d s^{2}}+\Gamma_{k l}^{i} \frac{d x^{k}}{d s} \frac{d x^{l}}{d s}\right)=p^{i} \quad(i=0,1,2,3) \tag{3}
\end{equation*}
$$

One transforms them into the following ones:

$$
\begin{equation*}
d \xi^{i}+\Gamma_{k l}^{i} \xi^{k} d x^{l}+\frac{e}{m} F_{\cdot k}^{l \cdot} \xi^{k} d s=0 \tag{4}
\end{equation*}
$$

by setting:

$$
\xi^{i}=m u^{i}=m \frac{d x^{i}}{d s}
$$

If one agrees that:

$$
\begin{equation*}
d x^{4}=\frac{e}{m} d s \tag{5}
\end{equation*}
$$

and one defines the components $G_{\beta \gamma}^{\alpha}$ of the affine connection on a space $E_{5}$ by the equations:

$$
\left\{\begin{array}{l}
G_{i k}^{l}=\Gamma_{i k}^{l},  \tag{6}\\
G_{4 i}^{l}=G_{i 4}^{l}=F_{\cdot i}^{l}, \\
G_{44}^{l}=G_{i l}^{4}=G_{i 4}^{4}=G_{44}^{4}=0
\end{array} \quad(i, k, l=0,1,2,3)\right.
$$

then equations (4) can be written:

$$
d \xi^{i}+G_{\alpha \beta}^{i} \xi^{\alpha} d x^{\beta}=0
$$

and the equation:

$$
d \xi^{4}+G_{\alpha \beta}^{4} \xi^{\alpha} d x^{\beta}=0
$$

will be satisfied identically if one assumes the invariability of the charge. One can then state the following theorem:

If one associates a material point of mass $m$ and charge $e$ that moves in an electromagnetic field and a gravitational field with the vector whose components in five-dimensional space are:

$$
m \frac{d x^{0}}{d s}, \quad m \frac{d x^{1}}{d s}, \quad m \frac{d x^{2}}{d s}, \quad m \frac{d x^{3}}{d s}, \quad e
$$

then that vector will displace parallel to itself, Here, ds is the element of the projection of the world-line in $E_{5}$ onto $E_{4}$.

It is therefore possible to define inertial systems for both gravitation and electromagnetism in that way.
4. - However, we have not constructed a five-dimensional relativity yet, because the preceding considerations and the fact that we do not know the phenomena in which a variation of charge would be involved compel us to consider only the changes of variables:

$$
\left(x^{0}, x^{1}, x^{2}, x^{3}, x^{4}\right) \quad \text { into } \quad\left(\bar{x}^{0}, \bar{x}^{1}, \bar{x}^{2}, \bar{x}^{3}, \bar{x}^{4}\right)
$$

for which:

$$
\begin{equation*}
\left(\bar{x}^{0}, \bar{x}^{1}, \bar{x}^{2}, \bar{x}^{3}\right) \text { are functions of only }\left(x^{0}, x^{1}, x^{2}, x^{3}\right) \text {, and } \bar{x}^{4} \text { is a function of only } x^{4}, \tag{7}
\end{equation*}
$$

and we assume that the functions that enter into our reasoning do not depend upon $x^{4}$. Moreover, it is only under those conditions that equations (5) will present any character of invariance.

Indeed, calculation will easily show that if one makes a change of variables (7) then the components of the affine connection $\bar{G}_{\beta \gamma}^{\alpha}$ on $E_{5}$ with respect to the $x^{\alpha}$ can be expressed in terms of the components of the affine connection $\bar{\Gamma}_{k l}^{i}$ and the field $\bar{F}^{i k}$ by the following equations:

$$
\left\{\begin{array}{l}
\bar{G}_{i k}^{l}=\bar{\Gamma}_{i k}^{l}, \\
\bar{G}_{4 i}^{l}=\bar{G}_{i 4}^{l}=\bar{F}_{i}^{l}, \\
\bar{G}_{44}^{l}=\bar{G}_{i l}^{4}=\bar{G}_{i 4}^{4}=\bar{G}_{44}^{4}=0,
\end{array}\right.
$$

which are identical to equations (5), except for the overbar, which indicates that the overbarred functions relate to the new coordinates.

That calculation is based upon the transformation formulas for the components of the affine connection. They do not constitute a tensor. On the contrary, one has:

$$
\bar{G}_{\beta \gamma}^{\alpha}=G_{\tau \sigma}^{\rho} \frac{\partial x^{\tau}}{\partial \bar{x}_{\beta}} \frac{\partial x^{\sigma}}{\partial \bar{x}_{\gamma}} \frac{\partial \bar{x}^{\alpha}}{\partial x_{\rho}}+\frac{\partial^{2} x^{\rho}}{\partial \bar{x}_{\beta} \partial \bar{x}_{\gamma}} \frac{\partial \bar{x}^{\alpha}}{\partial x_{\rho}},
$$

and the last group of terms on the right-hand side exhibits that fact quite well. Now, if one introduces the hypotheses (7) and the values (6) into that right-hand side and takes the formulas:

$$
\bar{\Gamma}_{i k}^{l}=\Gamma_{r s}^{t} \frac{\partial x^{r}}{\partial \bar{x}_{i}} \frac{\partial x^{s}}{\partial \bar{x}_{k}} \frac{\partial \bar{x}^{l}}{\partial x_{t}}+\frac{\partial^{2} x^{s}}{\partial \bar{x}_{i} \partial \bar{x}_{k}} \frac{\partial \bar{x}^{l}}{\partial x_{i}}
$$

and

$$
\bar{F}_{\cdot k}^{i \cdot}=F_{\cdot s}^{r \cdot} \frac{\partial \bar{x}^{i}}{\partial x_{r}} \frac{\partial x^{s}}{\partial \bar{x}_{k}}
$$

into account then one will obtain the stated result.

## II. - On the metric on $E_{5}$.

5.     - We have therefore seen that in the universe $E_{5}$ that we have defined, the coordinates $\left(x^{0}, x^{1}, x^{2}, x^{3}\right)$ are still separate from $x^{4}$, and in particular, we have:

$$
G_{i k}^{l}=\Gamma_{i k}^{l} \quad(i, k, l=0,1,2,3),
$$

in which the $\Gamma_{i k}^{l}$ are the Christoffel symbols of the second type for the $d s^{2}$ of the $E_{4}$ of the ( $x^{0}, x^{1}, x^{2}, x^{3}$ ). Under those conditions, the transformation law for the $G_{\alpha \beta}^{\gamma}$ in $E_{5}$ will demand simply that the $\Gamma_{i k}^{l}$ must transform like the components of an affine connection on an $E_{4}$, and the $F_{\cdot k}^{i .}$ transform like those of a tensor in that same $E_{4}$.

Such a connection possesses a fundamental group with 15 parameters. That group plays the same role vis- $\grave{a}$-vis the Lorentz group of displacements in $E_{4}$ that the Galilean kinematic group of classical mechanics plays vis- $\grave{a}$-vis the group of displacements in $E_{3}$. Now, the Galilean kinematic
group cannot be characterized by a non-degenerate four-dimensional $d s^{2}$. The fusion takes place only thanks to the Lorentz group. The same thing will be true in our $E_{5}$. The 15 -parameter group cannot be characterized by a non-degenerate $d s^{2}$ on $E_{5}$.

Meanwhile, we shall determine a $d s^{2}$ that will produce almost the same connection, i.e., we shall work with a union of the fifth coordinate with the other ones that is analogous to the way that one works when one bases space and time in the Minkowskian synthesis, or if one prefers, similar to the one that allows us to pass from the classical Galilean kinematic group to the Lorentz group.

However, just as one abandons mechanics and classical kinematics when one passes from the absolute space and time to the Minkowski universe, we shall abandon Minkowski's electromagnetism (and as a result, we shall modify the expression for the force that was given by Lorentz) upon passing from the Minkowski-Einstein universe to the universe of $E_{5}$.
6. - If one first supposes that the $g_{i k}$ will be those of Einsteinian $E_{4}$ for $i, k=0,1,2,3,4$ then one will have:

$$
G_{i k}^{l}=\Gamma_{i k}^{l} .
$$

The equations:

$$
G_{4 i}^{l}=G_{i 4}^{l}=F_{\cdot i}^{l .}
$$

will then give:

$$
\begin{equation*}
G_{4 l, i}=G_{l 4, i}=F_{i l}, \tag{8}
\end{equation*}
$$

because one indeed has:

$$
G_{4 l, i}=g_{i r} G_{4 l}^{r}+g_{i 4} G_{4 l}^{4}=g_{i r} F_{\cdot l}^{r \cdot}=F_{i l},
$$

since $F_{i l}=g_{i r} F_{. l}^{r .}+g_{i 4} F_{. l}^{4 .}$, and $F_{. l}^{4 .}$ has no meaning in its own right.
As for the functions that we consider to not depend upon $x^{4}$ (one sees the analogy with the static case in relativity), upon recalling that:

$$
F_{i l}=\frac{\partial \varphi^{i}}{\partial x^{l}}-\frac{\partial \varphi^{l}}{\partial x^{i}},
$$

they can be written:

$$
\frac{\partial g_{4 i}}{\partial x^{l}}-\frac{\partial g_{4 l}}{\partial x^{i}}=2\left(\frac{\partial \varphi^{i}}{\partial x^{l}}-\frac{\partial \varphi^{l}}{\partial x^{i}}\right)
$$

We solve them by setting $\left({ }^{1}\right)$ :

$$
g_{4 l}=2 \varphi_{l} .
$$

One will have, moreover:

$$
G_{i 4,4}=g_{4 l} G_{i 4}^{l}+g_{44} G_{i 4}^{4}=2 \varphi_{l} F_{\cdot i}^{l},
$$

( ${ }^{1}$ ) More generally, one can set:

$$
g_{4 l}=2 \varphi_{l}+\frac{\partial \lambda}{\partial x_{l}},
$$

in which $\lambda$ is a function of $x^{0}, x^{1}, x^{2}, x^{3}$.

$$
G_{44, \mathrm{i}}=g_{i l} G_{44}^{l}+g_{44} G_{44}^{4}=0,
$$

so on the one hand:

$$
\frac{\partial g_{44}}{\partial x_{i}}=4 \varphi_{l} F_{\cdot i}^{l \cdot}
$$

and on the other hand:

$$
\frac{\partial g_{44}}{\partial x_{i}}=0 .
$$

It is then impossible to preserve the connection that was first introduced on the basis of classical electromagnetism.
7. - That is why we define a five-dimensional $d s^{2}$ arbitrarily. Moreover, that will be the $d s^{2}$ that determines the affine connection. Meanwhile, the arbitrariness that we shall use will be moderated by the preceding considerations. We set:

$$
g_{44}=\psi^{2},
$$

in which $\psi$ is a function of $x^{0}, x^{1}, x^{2}, x^{3}$, and from now on, the $d s^{2}$ whose square root we propose to take and which has a new electromagnetic form is:

$$
\begin{equation*}
d s^{2}=g_{i k} d x^{i} d x^{k}+4 \varphi_{i} d x^{i} d x^{4}+\psi^{2} d x^{4} d x^{4} . \tag{9}
\end{equation*}
$$

The $g_{i k}(i, k=0,1,2,3)$ are functions of $x^{0}, x^{1}, x^{2}, x^{3}$ that will reduce to the coefficients of the Einsteinian $d s^{2}$ that relates to the given gravitational field in the absence of an electromagnetic field.

The $g_{i 4}(i, k=0,1,2,3)$ are the components of the electromagnetic potential, up to the factor 2.

The coefficient $\psi^{2}$ is the square of a function of the $x^{0}, x^{1}, x^{2}, x^{3}$.
The world-lines (in $E_{5}$ ) of a charged material points are geodesics of that $d s^{2}$.
We can postulate the equation:

$$
\frac{d x^{4}}{d s}=\frac{e}{m},
$$

or the equations:

$$
\frac{d x^{4}}{d \sigma}=\frac{e}{m}
$$

in which $d \sigma$ is the line element in the $E_{4}$ that is defined by $x^{4}=$ const. They both define a variable ratio $e / m$. We might ask how that variability can be divided between $e$ and $m$, respectively.

## III. - On wave mechanics.

8.     - If one studies the papers of Vessiot $\left({ }^{1}\right)$, particularly the one in the Bulletin de la Société mathématique de France ( t . XXXIV) on the mechanical interpretation of contact transformations, and most especially the conclusion of that paper, and some of Hadamard's analyses of the bicharacteristics that are attached to an equation of propagation $\left({ }^{2}\right)$, then it will be possible to express the principles of wave mechanics in a very simple manner.

If one is given a second-order partial differential equation (0) that is linear in the second derivatives then one can define the characteristic multiplicities that are attached to (0) by means of a first-order partial differential equation $(J)$ and curves, namely, the bicharacteristics of $(0)$, that are the characteristics of $(J)$.

If one identifies $(J)$ with the Jacobi equation of the motion of a material point then the trajectories of that material point will be the bicharacteristics of certain second-order partial differential equations (0), among which one finds the Schrödinger equation. One can single out some of them by way of their invariance in a certain number of cases.

We shall see that in five-dimensional relativity, it is easy to give a very precise sense to the Schrödinger equation, and furthermore, we can show that this equation is obtained very naturally.
9. - The five-dimensional theory of relativity that we propose here is a theory of invariance in $E_{5}$. However, like Einstein's gravity, it is, moreover, a physical theory that permits us to define the coefficients of $d s^{2}$ by means of masses and charges.

To begin with, we treat the motion of a material point by supposing that there is no electromagnetic field, so the potentials are zero.

From now on, the $d s^{2}$ of $E_{5}$ will have a coefficient matrix with the following appearance:

$$
\left|\begin{array}{ccccc}
g_{00} & g_{01} & g_{02} & g_{03} & 0  \tag{10}\\
g_{10} & g_{11} & g_{12} & g_{13} & 0 \\
g_{20} & g_{21} & g_{22} & g_{23} & 0 \\
g_{30} & g_{31} & g_{32} & g_{33} & 0 \\
0 & 0 & 0 & 0 & \psi^{2}
\end{array}\right| .
$$

The $g_{i k}(i, k=0,1,2,3)$ are determined in Einstein's theory by the equations:

$$
\begin{equation*}
R_{i k}=0 \quad(i, k=0,1,2,3) \tag{11}
\end{equation*}
$$

[^2]in the regions of $E_{4}$ where no mass is found. The $R_{i k}$ are the components of the contracted Riemann tensor relative to the $d s^{2}$ in $E_{4}$.

We suppose that the $g_{i k}$ and $\psi^{2}$ are determined outside of any masses by the equations:

$$
\begin{equation*}
R_{\alpha \beta}=0 \quad(\alpha, \beta=0,1,2,3,4) \tag{12}
\end{equation*}
$$

in which the $R_{\alpha \beta}$ are the components of the contracted Riemann tensor relative to the $d s^{2}$ in $E_{5}$ this time.

It is clear that equations (12) differ from equations (11) because in order to obtain the matrix (10), we had to modify the connection that was modified by equations (11). That modification will be very weak if one supposes that the derivatives $\psi_{i}=\partial \psi / \partial x_{i}$ are very small and negligible in comparison to the $\partial g_{i k} / \partial x_{l}$, and all of the functions $g_{i k}$ and $\psi$ have derivatives with respect to $x^{4}$ that are even smaller.

The calculation that allows one to find the $R_{\alpha \beta}$ will be very simple then. One finds that the equations:

$$
R_{i k}=0 \quad(i, k=0,1,2,3)
$$

are precisely the Einstein equations that determine the $g_{i k}$ of the Einsteinian $E_{4}$. The equations:

$$
R_{i 4}=0 \quad(i=0,1,2,3)
$$

are satisfied identically, and the equation:

$$
R_{44}=0
$$

is written:

$$
\psi \frac{\partial\left(g^{h i} \psi_{i}\right)}{\partial x^{h}}+\Gamma_{i h}^{h} \psi g^{i k} \psi_{k}=0
$$

or

$$
\begin{equation*}
g^{h i} \frac{\partial^{2} \psi}{\partial x^{i} \partial x^{h}}+\left(\Gamma_{i h}^{h} g^{i k}+\frac{\partial g^{h i}}{\partial x^{h}}\right) \frac{\partial \psi}{\partial x^{i}}=0 \tag{0}
\end{equation*}
$$

which is a second-order partial differential equation in $\psi$ that is linear in the second derivatives.
10. - Let us look for the characteristics of that equation. Since the coefficients $g_{i k}$ are functions of the $x^{0}, x^{1}, x^{2}, x^{3}$, and they do not contain either the unknown function or its first derivatives, the characteristics are not determined with the aide of a chosen integral of (0), but only by means of equation (0) $\left(^{1}\right.$ ).

Those characteristics are multiplicities:

$$
S\left(x^{0}, x^{1}, x^{2}, x^{3}\right)=0
$$

${ }^{1}$ ) Cf., Hadamard, Propagation, pp. 315.
that are defined by the first-order partial differential equation $\left({ }^{1}\right)$ :

$$
\begin{equation*}
g^{i k} \frac{\partial S}{\partial x^{i}} \frac{\partial S}{\partial x^{k}}=0 . \tag{J}
\end{equation*}
$$

The equation $(J)$ will also have characteristics, which will be curves. Those curves will then be the bicharacteristics of (0). Their equations are:

$$
\begin{aligned}
& \frac{d x^{0}}{g^{0 k} \frac{\partial S}{\partial x^{k}}}=\frac{d x^{1}}{g^{1 k} \frac{\partial S}{\partial x^{k}}}=\frac{d x^{2}}{g^{2 k} \frac{\partial S}{\partial x^{k}}}=\frac{d x^{3}}{g^{3 k} \frac{\partial S}{\partial x^{k}}} \\
= & \frac{d\left(\frac{\partial S}{\partial x^{0}}\right)}{-\frac{1}{2} \frac{\partial g^{i h}}{\partial x^{0}}\left(\frac{\partial S}{\partial x^{i}}\right)\left(\frac{\partial S}{\partial x^{h}}\right)}=\frac{d\left(\frac{\partial S}{\partial x^{1}}\right)}{-\frac{1}{2} \frac{\partial g^{i h}}{\partial x^{1}}\left(\frac{\partial S}{\partial x^{i}}\right)\left(\frac{\partial S}{\partial x^{h}}\right)} \\
= & \frac{d\left(\frac{\partial S}{\partial x^{2}}\right)}{-\frac{1}{2} \frac{\partial g^{i h}}{\partial x^{2}}\left(\frac{\partial S}{\partial x^{i}}\right)\left(\frac{\partial S}{\partial x^{h}}\right)}=\frac{d\left(\frac{\partial S}{\partial x^{3}}\right)}{-\frac{1}{2} \frac{\partial g^{i h}}{\partial x^{3}}\left(\frac{\partial S}{\partial x^{i}}\right)\left(\frac{\partial S}{\partial x^{h}}\right)} .
\end{aligned}
$$

Now one knows that those equations define the geodesics of the $d s^{2}$ in $E_{4}$, and their common ratio will have a value equal to one-half the arc-length element along the aforementioned geodesics.

However, it is known that in the Einsteinian $E_{4}$, the trajectories of a material point are geodesics of $d s^{2}$ the in $E_{4}$.

One then sees that those trajectories are nothing but the bicharacteristics of (0).
11. - Meanwhile, under our hypotheses we have neglected the variability of the functions $g_{i k}$ and $y$ with respect to $x^{4}$, which amounts to considering the manifolds $x^{4}=$ const. in $E_{5}$. We then have the following result:

If one considers an Einsteinian universe $E_{4}$ to be a section $x^{4}=$ const. of a five-dimensional universe $E_{5}$ of $\left(x^{0}, x^{1}, x^{2}, x^{3}, x^{4}\right)$ whose $d s^{2}$ has the functions in the matrix (10) for its coefficients then the equations of gravitation will be the equations $R_{i k}=0(i, k=0,1,2,3)$ that relate to that $d s^{2}$, and the trajectories of a material point of mass that is small enough that it will not modify

[^3]thee gravitational field perceptibly will be the bicharacteristics of the equation $R_{44}=0$, which determines $\psi$ when the $g^{i k}$ are given.

The equation $R_{44}=0$ governs the propagation of waves. One can take it to be the Schrödinger equation of the wave mechanics of a material point.

If the gravitational field is zero then equation (0) is quite simply the d'Alembert equation, or if one prefers, the Laplace equation in $E_{4}$. Its bicharacteristics are the lines in $E_{4}$ that carry the time vectors, i.e., the world-lines of a free material point.

## IV. - Return to the equations of electromagnetism. The field equations and the Schrödinger equation.

12.     - We have obtained Einsteinian gravitation as an approximation to a five-dimensional theory of relativity by starting from a form $d s^{2}$ that is defined by equation (9), in which we suppose that the $\varphi^{i}$ are zero.

If one includes the $\varphi^{i}$, i.e., if one seeks to establish the field equations that define the $g_{i k}$, the $\varphi_{i}$, and $\psi$ then one will be led to some complicated equations that one can approximate by further supposing that the aforementioned functions have derivatives with respect to $x^{4}$ that are negligible. One assumes, moreover, that the $\varphi_{i}$ and the $F^{i k}$ are not considerable, so one can neglect them in comparison to $\psi$. One will then see that the electromagnetic potentials have been introduced with a calculated degree of prudence in order to not rush the process of approximation.

With those hypotheses, the components of the affine connection on $E_{5}$ will be:

$$
G_{i k}^{l}=\Gamma_{i k}^{l} \quad(i, k=0,1,2,3)
$$

in which the $\Gamma_{i k}^{l}$ are the Christoffel symbols of the second type that are attached to the $d s^{2}$ in Einsteinian $E_{4}$ of $\left(x^{0}, x^{1}, x^{2}, x^{3}\right)$, and:

$$
G_{4 i}^{l}=G_{i 4}^{l}=F_{\cdot i}^{l},
$$

in which the index $l$ was raised by means of the $g^{i k}$ in $E_{4}$ :

$$
\begin{aligned}
G_{i k}^{4} & =0, \\
G_{i 4}^{4} & =\frac{\psi_{i}}{\psi}, \\
G_{44}^{i} & =-\psi \psi^{i}, \\
G_{44}^{4} & =0 .
\end{aligned}
$$

Here, it is intended that the $G_{\alpha \beta}^{\gamma}$ are the Christoffel symbols of the second type for the $d s^{2}$ of $E_{5}$, but our hypotheses permit us to single out the Christoffel symbols of the $d s^{2}$ of $E_{4}$, because in calculating the determinant of the $g_{\alpha \beta}(\alpha, \beta=0,1,2,3,4)$ and its minors, we neglected the $\varphi_{i}$ and the derivatives in comparison with $\psi$.
13. - That being the case, one easily calculates:

$$
R_{\alpha \beta}=\frac{\partial G_{\alpha \beta}^{\gamma}}{\partial x^{\gamma}}-\frac{\partial G_{\alpha \gamma}^{\gamma}}{\partial x^{\beta}}+G_{\alpha \beta}^{\gamma} G_{\alpha \delta}^{\delta}-G_{\alpha \gamma}^{\delta} G_{\alpha \delta}^{\gamma},
$$

and one has the following results:
The $R_{i k}(i, k=0,1,2,3)$ are those of Einstein.
The $R_{i 4}(i=0,1,2,3)$ are the divergences of the electromagnetic field when taken with respect to the $d s^{2}$ of the $E_{4}\left({ }^{1}\right)$ :

$$
R_{i 4}=\frac{\partial F_{\cdot i}^{h .}}{\partial x^{h}}-\Gamma_{i h}^{l} F_{. l}^{h .}+\Gamma_{h r}^{r} F_{. i}^{h .}=F_{\cdot i / h}^{h .} .
$$

Finally:

$$
R_{44}=-\psi\left[\frac{\partial\left(g^{r i} \psi_{i}\right)}{\partial x^{r}}+\Gamma_{i h}^{h} \psi^{i}\right] .
$$

There is nothing difficult about that; it is a very simple algebraic calculation.
14. - We can no longer write $R \alpha \beta=0$ in order to determine the coefficients of the $d s^{2}$ of the $E_{5}$ because the presence of the field forces us to consider an energy tensor. As one knows, in an Einstein $E_{4}$, one sets:

$$
R_{i k}=\kappa\left(T_{i k}-\frac{1}{2} g_{i k} T\right) \quad(i, k=0,1,2,3),
$$

in which $T_{i k}$ is the tensor of energy and quantity of motion, and $\kappa$ is a constant on the order of $10^{-47}$ CGS. Now:

$$
T_{i k}=m u_{i} u_{k} \quad \text { and } \quad T=m u_{i} u^{i} \quad\left(u^{i}=\frac{d x^{i}}{d \sigma}\right)
$$

We then take the field equations in $E_{5}$ to be the equations:

$$
\begin{equation*}
R_{\alpha \beta}=\mu\left(T_{\alpha \beta}-\frac{1}{2} g_{\alpha \beta} T\right), \tag{13}
\end{equation*}
$$

[^4]with $T_{\alpha \beta}=m u_{\alpha} u_{\beta}, T=m u_{\alpha} u^{\alpha}, \mu$ is a constant that must coincide with $\kappa$ when one does not take the variations of $x^{4}$ into account.

Having assumed that, for $\alpha, \beta=0,1,2,3$, equations (13) will give back the Einstein equations in the approximation that we have assumed.

For $\alpha=0,1,2,3$, and $\beta=4$, one will have:

$$
F_{. i / h}^{h \cdot}=\mu\left(m u_{i} \cdot u_{4}-\frac{1}{2} g_{4 i} T\right),
$$

or

$$
F_{. i / h}^{h \cdot}=\mu m\left(u_{i} u_{4}-\varphi_{i} u_{\alpha} u^{\alpha}\right) .
$$

Now:

$$
u_{4}=g_{4 \alpha} u^{\alpha}=2 \varphi_{i} \frac{d x^{i}}{d s}+\psi^{2} \frac{d x^{4}}{d s},
$$

and

$$
u_{\alpha} u^{\alpha}=g_{\alpha \beta} u^{\alpha} u^{\beta}=1 .
$$

Consequently, upon neglecting $\varphi_{i} \frac{d x^{i}}{d s}$ and $\varphi_{i}$ in comparison to $\psi^{2} \frac{d x^{4}}{d s}$ (which conforms to our hypotheses), we will have:

$$
F_{\cdot i / h}^{h .}=\mu m u_{i} \psi^{2} \frac{d x^{4}}{d s} .
$$

Now, we have set:

$$
d x^{4}=\frac{e}{m} d \sigma
$$

above, so we assume that in $E_{5}$, we must set:

$$
d x^{4}=\frac{e}{m} d s
$$

and if one further sets $e u_{\alpha}=s_{\alpha}$ then the $s_{i}$ will be the components of the current in $E_{5}$, and $s_{0}, s_{1}$, $s_{2}, s_{3}$ will reduce to the components of the current in $E_{4}$ if $x_{4}=$ const. One sees that:

$$
F_{. i / h}^{h .}=\mu \psi^{2} \cdot s_{i}
$$

Now, the Maxwell equations are written:

$$
F_{. i / h}^{h \cdot}=s_{i}
$$

in any Einstein universe. Therefore, assume that $\mu \psi^{2}$ is very close to unity, which gives the order of magnitude of $\psi$ since $\mu$ is the order of $\kappa$. Equations (13) will then reduce to one of the Maxwell equations for $\alpha=0,1,2,3, \beta=4$, with our approximations. Kaluza has shown that the other group results from an identity that the Christoffel symbols for $d s^{2}$ must satisfy $\left({ }^{1}\right)$.

Finally, equation (13) is written:

$$
\frac{\partial\left(g^{r i} \psi_{i}\right)}{\partial x^{r}}+\Gamma_{i h}^{h} \psi^{i}=-\mu \psi^{3} \cdot \frac{e^{2}}{m}
$$

for $\alpha=\beta=4$, with the permitted approximations, and after dividing by $\psi$.
Now, if $\mu \psi^{2}$ is close to unity then one will have:

$$
\begin{equation*}
\frac{\partial\left(g^{r i} \psi_{i}\right)}{\partial x^{r}}+\Gamma_{i h}^{h} \psi^{i}+\frac{e^{2}}{m} \psi=0 \tag{1}
\end{equation*}
$$

That is a second-order partial differential equation that will define $\psi$ when the $g_{i k}$ are known. The bicharacteristics of $\left(0_{1}\right)$ are once more geodesics of the $d s^{2}$ of $E_{4}$, and consequently, the trajectories of material point, if the electromagnetic field is negligible in comparison to the gravitational field. It is the Schrödinger equation.

If one neglects the gravitational field, or rather, if one neglects the curvature of space, then that equation will reduce to:

$$
\square \psi+\frac{e^{2}}{m} \psi=0 .
$$

The periodic solutions of the form:

$$
\psi=\varphi(x, y, z) e^{2 \pi i v t}
$$

are such that:

$$
\Delta \psi+\left(\frac{4 \pi^{2} v^{2}}{c^{2}}+\frac{e^{2}}{m}\right) \psi=0
$$

That is just the Schrödinger equation with the additional term $\left(e^{2} / m\right) \psi$. Now, $e^{2} / m$ will be negligible in comparison to $4 \pi^{2} v^{2} / c^{2}$ for the frequencies $v$ that are of interest in the physics of the atom. Our approximation is therefore good enough.
15. - The simplifying hypotheses that have permitted us to neglect the electromagnetic potential in comparison with the gravitational potential and $\psi^{2}$, which we can call the wave potential, give us Einsteinian mechanics as a first approximation and the equations of the

[^5]electromagnetic field. Moreover, upon generalizing the theory of relativity by extending to $E_{5}$, we have been able to give a new interpretation of the Schrödinger equation.

Dealing with equations (13) is much more complicated when one includes all of the factors. The relation between the bicharacteristics of equation (13), in which $\alpha=\beta=4$, and the geodesics of $E_{5}$ is less simple than what had believed to begin with ( ${ }^{1}$ ).

Meanwhile, the facts that we could recover the field equations and those of motion that are given by the theories in $E_{4}$ in the context of a new theory of relativity in $E_{5}$, and that we have obtained an interpretation of the Schrödinger equation, in addition, encourage us to pursue our attempt further.

If the electromagnetic field becomes dominant then the approximation methods must be transformed from one end to the other. What we have just said up to now concerns macroscopic wave mechanics. In order to recover the laws of atomic phenomena in their detailed form, one must first construct microscopic wave mechanics.

[^6]
[^0]:    ${ }^{1}$ ) C. R. Acad. Sci. Paris 185, pps. 341, 412, 448, 535.
    $\left(^{2}\right)$ Sitzungsber. Berlin (II) (1921), 966-972.
    ${ }^{(3}$ ) Ann. Ec. norm. sup. (3) 40, pp. 325, ibid., 41, pp. 1
    $\left({ }^{4}\right)$ "Essai sur la propagation par ondes," Ann. Ec. norm. sup. (3) 26, pp. 403. "Sur l'interprétation mécanique des transformations de contact," Bull. Soc. math. de France 34, pp. 265.
    $\left({ }^{5}\right)$ Leçons sur la propagation des ondes, Paris, 1903, and Lectures on Cauchy's Problem in linear partial Equations, New Haven, 1923.
    $\left({ }^{6}\right)$ Thesis, Paris, 1924.
    ${ }^{7}$ ) Ondes et Mouvements, Paris, 1927.
    $\left({ }^{8}\right)$ Ann. Phys. (Leipzig) (1926), passim.
    ${ }^{9}$ ) Loc. cit.
    $\left({ }^{10}\right)$ Zeit. Phys. 37, pp. 895.
    $\left({ }^{11}\right)$ Zeit. Phys. 39, pp. 226.

[^1]:    ( ${ }^{1}$ ) Journal de Physique 6, pp. 65-73, 225-241.
    $\left(^{2}\right)$ C. R. Acad. Sci. Paris, t. 185.
    $\left(^{3}\right)$ Bull. Acad. roy. de Belgique, Classe des Sciences (5) 13, no. 6.
    $\left(^{4}\right)$ Publ. from the Massachusetts Institute of Technology, 2, 133, Dec. 1927.

[^2]:    ${ }^{1}$ ) Loc. cit., note 4.
    $\left(^{2}\right)$ Loc. cit., note 5 .

[^3]:    (1) Hadamard, loc. cit., pp. 271.

[^4]:    $\left.{ }^{1}\right)$ Cf., Kaluza, loc. cit.

[^5]:    ( ${ }^{1}$ ) Loc.cit.

[^6]:    ( ${ }^{1}$ Cf., our fourth note, at the end.

