# On some points regarding the theory of integral invariants 

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1.     - I propose to complete the results of my previous article "Sur les invariants intégraux" [Journal de Mathématiques (6) 4 (1908), 331-365] concerning a few points. I shall employ the same notations as in that work, except that in order to abbreviate, I shall write just one $\int$ sign in order to denote a multiple integral, which one can infer with no ambiguity.

Let $x_{1}, x_{2}, \ldots, x_{n}$ be a system of $n$ independent variables, and let $A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}(p \leq n)$ be a system of functions in those $n$ variables, each of which is affected with $p$ different indices $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}$ that are taken from the first $n$ numbers. The functions whose indices differ by only their order are equal up to sign. If $\left(\alpha_{1}^{\prime}, \alpha_{2}^{\prime}, \ldots, \alpha_{p}^{\prime}\right)$ is a new permutation of the indices $\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}\right)$ then one will have:

$$
A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}= \pm A_{\alpha_{1}^{\prime}, \alpha_{2}^{\prime}, \ldots, \alpha_{p}^{\prime}},
$$

in which the + sign pertains to the case in which the two permutations belong to the same class, and the - sign, to the case in which those permutations belong to different classes. The expression:

$$
I_{p}=\int \sum A_{\alpha_{1}, \alpha_{2}, \ldots \alpha_{p}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p}},
$$

in which the $\sum$ sign extends over all arrangements of the first $n$ numbers taken $p$ at a time, represents a multiple integral of order $p$. In order to have the value of that integral, when extended over a multiplicity $\left(E_{p}\right)$ in the $n$-dimensional space $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, we suppose that the coordinates of a point on that multiplicity are expressed by means of $p$ parameters $u_{1}, u_{2}, \ldots, u_{p}$, in such a fashion that the multiplicity $\left(E_{p}\right)$ will correspond point-by-point with another multiplicity $\left(e_{p}\right)$ in the $p$-dimensional space $\left(u_{1}, u_{2}, \ldots, u_{p}\right)$. The value of the integral $I_{p}$, when extended over the multiplicity $\left(E_{p}\right)$, can then be written in one or the other of the two forms:

$$
I_{p}=\int\left(\sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \frac{\partial x_{\alpha_{2}}}{\partial u_{2}} \cdots \frac{\partial x_{\alpha_{p}}}{\partial u_{p}}\right) d u_{1} d u_{2} \cdots d u_{p}
$$

in which the $\sum$ sign extends over all arrangements of the indices, taken $p$ at a time, or:

$$
I_{p}=\int\left[\sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} \frac{D\left(x_{\alpha_{1}}, x_{\alpha_{2}}, \ldots, x_{\alpha_{p}}\right)}{D\left(u_{1}, u_{2}, \ldots, u_{p}\right)}\right] d u_{1} d u_{2} \cdots d u_{p},
$$

in which the $\sum$ sign extends over all combinations of the indices, taken $p$ at a time, and the two integrals are extended over the multiplicity $\left(e_{p}\right)$. When one changes the order in which one writes the variables $u_{1}, u_{2}, \ldots, u_{p}$, the value of $I_{p}$ might change sign, just as a surface integral changes sign when one changes the side of the surface on which it is taken. In what follows, we will have to consider integrals that are extended over a multiplicity $\left(E_{p}\right)$ that varies in a continuous manner with a parameter $t$. The coordinates of a point of that multiplicity are functions of $p$ parameters $u_{1}, u_{2}$, $\ldots, u_{p}$, and $t$. Once one has chosen the order in which one writes the variables $x_{i}$ and the parameters $\left(u_{1}, u_{2}, \ldots, u_{p}\right)$ for a particular value of $t$, one will keep the same order for all values of $t$. Ir is clear that the value of the multiple integral is a continuous function of $t$.

## 2. - Let:

$$
\begin{equation*}
I_{p}=\int \sum A_{\alpha_{1}, \alpha_{2} \ldots, \alpha_{p}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p}} \tag{1}
\end{equation*}
$$

be an integral invariant of order $p$ of the system:

$$
\begin{equation*}
\frac{d x_{1}}{X_{1}}=\frac{d x_{2}}{X_{2}}=\ldots=\frac{d x_{n}}{X_{n}}=d t . \tag{2}
\end{equation*}
$$

I suppose, to fix ideas, that I have written all of the coefficients $A_{\alpha_{1}, \alpha_{2}, \ldots \alpha_{\rho}}$ that correspond to all arrangements of the first $n$ numbers taken $p$ at a time. Let $E_{p-1}$ be a ( $p-1$ )-dimensional multiplicity in the $n$-dimensional space $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ that is not composed of characteristics, i.e., of integral curves of equations (2). When one varies $t$ from 0 to $T$, the point ( $x_{1}, x_{2}, \ldots, x_{n}$ ), which coincides with a point $x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}$ in $E_{p-1}$ for $t=0$, will describe a segment of the characteristic $C$ that goes from the point $\left(x_{1}^{0}, \ldots, x_{n}^{0}\right)$ to a point $\left(x_{1}^{T}, x_{2}^{T}, \ldots, x_{n}^{T}\right)$, and those segments of the characteristics will generate a multiplicity $E_{p}$ that is bounded by $E_{p-1}$, the multiplicity $E_{p-1}^{\prime}$ that is described by the point $\left(x_{1}^{T}, x_{2}^{T}, \ldots, x_{n}^{T}\right)$, and another multiplicity $E_{p-1}^{\prime}$ that is generated by the segments of the characteristics that issue from the various points of the multiplicity $E_{p-2}$ that bounds $E_{p-1}$. We shall calculate the value of the integral $I_{p}$ when it is extended over that multiplicity $E_{p}$. In order to do that, suppose that the coordinates of a point of $\left(x_{1}^{0}, \ldots, x_{n}^{0}\right)$ are expressed by means of $p-1$ independent variables $u_{1}, u_{2}, \ldots, u_{p-1}$. The coordinates of a point on the multiplicity $E_{p}$ will then be functions of $u_{1}, u_{2}, \ldots, u_{p-1}$, and the variable $t$, for which one will have:

$$
\frac{d x_{i}}{d t}=X_{i} \quad(i=1,2, \ldots, n)
$$

The desired integral can be written:

$$
\begin{aligned}
I_{p} & =\int_{E_{p}} \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p}} \\
& =\int_{E_{p}} \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \frac{\partial x_{\alpha_{2}}}{\partial u_{2}} \cdots \frac{\partial x_{\alpha_{p}}}{\partial t} d u_{1} \cdots d u_{p-1} d t
\end{aligned}
$$

in which the summation indicated by the $\sum$ sign extends over all arrangement of indices taken $p$ at a time. From the relations ( $2^{\prime}$ ), one can again write $I_{p}$ :

$$
\begin{equation*}
I_{p}=\int_{0}^{T} d t \int_{E_{p-1}} \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \frac{\partial x_{\alpha_{2}}}{\partial u_{2}} \cdots \frac{\partial x_{\alpha_{p-1}}}{\partial u_{p-1}} d u_{1} \cdots d u_{p-1} \tag{3}
\end{equation*}
$$

upon setting:

$$
\begin{equation*}
C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}}=\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}, i} X_{i} . \tag{4}
\end{equation*}
$$

That expression for $I_{p}$ also takes the equivalent abbreviated form:

$$
I_{p}=\int_{0}^{T} d t \int_{E_{p-1}} \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p-1}} .
$$

The value of $I_{p}$ is a function of the variable $T$ whose derivative at $T=0$ has the value:

$$
\begin{equation*}
\left(\frac{d I_{p}}{d T}\right)_{0}=\int_{E_{p-1}} \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p-1}} \tag{5}
\end{equation*}
$$

In the second place, consider a multiplicity $E_{p}$ that is defined by starting from $E_{p-1}$ in a more general manner. Make each point $\left(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right)$ of $E_{p-1}$ correspond to a value of $\theta$ that varies continuously with the position of that point. The segment of the characteristic that issues from the point $\left(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right)$ and is described by the point $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ when $t$ varies between 0 and $\theta$ generates a multiplicity $E_{p}$ that is bounded by $E_{p-1}$ and another multiplicity $E_{p-1}$ that is the locus of the extremity of the variable segment of the characteristic. If one varies $t$ from 0 to $T$ then the point $m$ will occupy a position $m^{\prime}$ whose locus is another multiplicity that one can obviously deduce from $E_{p}$ by adding to it the multiplicity $\mu_{p}^{\prime}$ that is described by $E_{p-1}^{\prime}$ when one varies $t$ from 0 to $T$. Since $I_{p}$ is an integral invariant, the integral $I_{p}$ will have the same value for the
multiplicities $E_{p}$ and $E_{p}^{\prime}$, and as a result, for the multiplicities $\mu_{p}$ and $\mu_{p}^{\prime}$. The derivative $\left(\frac{d I_{p}}{d T}\right)_{0}$ will then have the same value for the two multiplicities $E_{p-1}$ and $E_{p-1}^{\prime}$ :

$$
\begin{equation*}
\int_{E_{p-1}} \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p-1}}=\int_{E_{p-1}^{\prime}} \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p-1}}, \tag{6}
\end{equation*}
$$

no matter how one varies $\theta$ with the position of the point $\left(x_{1}^{0}, \ldots, x_{n}^{0}\right)$ on $E_{p-1}$. In particular, if one supposes that $\theta$ has a constant value then one will see that:

$$
\begin{equation*}
I_{p-1}=\int \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{p-1}} \tag{7}
\end{equation*}
$$

is an integral invariant of order $p-1$ for equations (2). However, it is an integral invariant of a particular type because it preserves the same value for all multiplicities that one deduces from $E_{p-1}$ by making each point of $E_{p-1}$ describe the characteristic that issues from that point according to an arbitrary law. Upon continuing to employ the geometric language, if we say tube of characteristics to mean the multiplicity of order $p$ that is generated by the characteristics that issue from the various points of a multiplicity of order $p-1$ that is not composed of characteristics then we can say that the integral $I_{p-1}$ has the same value for all sections of a characteristic tube. We say, to abbreviate, that any integral invariant that enjoys that property is attached to the characteristics. From that, any invariant that is attached to the characteristics will also be an integral invariant for the system of differential equations that are obtained by multiplying $X_{1}, X_{2}, \ldots, X_{n}$ by an arbitrary function $\varphi\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ since the characteristics will remain the same.

The foregoing is basically just the argument that Poincaré (Acta mathematica, t. XIII, pp. 66) presented in its most-general form. The result can be stated as follows:

One can deduce (in general) an invariant $I_{p-1}$ that is attached the characteristics of the system (2) from any absolute integral invariant $I_{p}$ of that system.

One will then be led to pose the following two questions:

1. Can one obtain all of the integral invariants that are attached to the characteristics by that process?
2. If an integral invariant is given then how does one recognize whether it is attached to the characteristics?

One will see that those two questions can both be answered at the same time in a very simple manner.
5. - In the previous article that was cited above, I let $(E)$ denote the operation by which one passes from the invariant (1) $I_{p}$ to the invariant (7) $I_{p-1}$. The link between those two invariants can be summarized by the relation:

$$
\begin{equation*}
\left(\frac{d I_{p}}{d T}\right)_{T=0}=I_{p-1} \tag{8}
\end{equation*}
$$

in which the invariant $I_{p-1}$ extends over an arbitrary multiplicity $E_{p-1}$, and the invariant $I_{p}$ extends over the multiplicity $E_{p}$ that one deduces from $E_{p-1}$ by the process that was explained above. It is clear that this relation preserves when one changes the unknown functions $x_{1}, x_{2}, \ldots, x_{n}$ arbitrarily without changing the variable $t$. In other words, the operation $(E)$ is covariant for any transformation of the form:

$$
x_{i}=\varphi_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \quad(i=1,2, \ldots, n)
$$

From that transformation, equations (2) are replaced by a new system:

$$
\frac{d y_{1}}{Y_{1}}=\frac{d y_{2}}{Y_{2}}=\ldots=\frac{d y_{n}}{Y_{n}}=d t
$$

whereas the invariants $I_{p}$ and $I_{p-1}$ will become:

$$
I_{p}^{\prime}=\int \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}^{\prime} d y_{\alpha_{1}} \cdots d y_{\alpha_{p}}, \quad \quad I_{p-1}^{\prime}=\int \sum C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}}^{\prime} d y_{\alpha_{1}}^{\prime} \cdots d y_{\alpha_{p-1}}
$$

respectively. The invariant $I_{p-1}^{\prime}$ is deduced from $I_{p}^{\prime}$ in the same way that $I_{p-1}$ is deduced from $I_{p}$, i.e., one will have:

$$
C_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}}^{\prime}=\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}^{\prime} Y_{i},
$$

as one can verify directly.
When the operation $(E)$ is applied to an invariant $I_{p}$, that will lead to an invariant $I_{p-1}$ that is identically zero if the coefficients $A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}$ verify the relations:

$$
\begin{equation*}
\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} X_{i}=0 \tag{9}
\end{equation*}
$$

for all arrangements of the indices taken $p-1$ at a time. I shall denote the invariants that enjoy that property by $I_{p}^{(e)}$. One sees immediately that any invariant $I_{p}$ that is deduced from an invariant $I_{p+1}$ by the operation $(E)$ is an $I_{p}^{(e)}$. The converse will be established a little later.

One can characterize the invariants $I_{p}^{(e)}$ by the following property:

The value of an integral invariant $I_{p}^{(e)}$, when it is extended over an arbitrary multiplicity of order $p$ that is generated by characteristic curves, is always zero. Conversely, any integral invariant that enjoys that property will be an integral invariant $I_{p}^{(e)}$.

Any multiplicity of order $p$ that is composed of characteristics can be defined to be the multiplicity $E_{p}$ that was considered above. The coordinates $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ of a point in that multiplicity are functions of $p$ independent variables $u_{1}, u_{2}, \ldots, u_{p-1}, t$ whose partial derivatives $\partial x_{i} / \partial t$ are equal to functions $X_{t}$. The value of the integral invariant $I_{p}$, when extended over a multiplicity of that type, will then have the expression:

$$
I_{p}=\int_{E_{p}} \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}, i} \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \cdots \frac{\partial x_{\alpha_{p-1}}}{\partial u_{p-1}} X_{i} d u_{1} d u_{2} \cdots d u_{p-1} d t
$$

which can be further written:

$$
I_{p}=\int_{E_{p}} \sum \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \cdots \frac{\partial x_{\alpha_{p-1}}}{\partial u_{p-1}}\left(\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}, i} X_{i}\right) d u_{1} d u_{2} \cdots d u_{p-1} d t
$$

That integral can be zero for any multiplicity that one might consider only if all of the elements are zero, i.e., if one has:

$$
\sum \frac{\partial x_{\alpha_{1}}}{\partial u_{1}} \cdots \frac{\partial x_{\alpha_{p-1}}}{\partial u_{p-1}}\left(\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}, i} X_{i}\right)=0
$$

identically.
That will obviously be true if the coefficients $A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}$ of the integral invariant $I_{p}$ verify the relations (9). Conversely, those conditions are necessary because for a given system of values of $x_{1}, x_{2}, \ldots, x_{n}$, one can choose the values of all the partial derivatives $\partial x_{k} / \partial u_{i}$ arbitrarily. The invariant considered is then an invariant $I_{p}^{(e)}$.

It is obvious that this property is independent of the choice of unknowns. If a change in variables $x_{i}=\varphi_{i}\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ reduces the system (2) to the system ( $2^{\prime}$ ), and the integral invariant (1), to the form:

$$
I_{p}^{\prime}=\int \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}^{\prime} d y_{\alpha_{1}} \cdots d y_{\alpha_{p}}
$$

then the coefficients of the new invariant $I_{p}^{\prime}$ will verify the relations:

$$
\sum_{i=1}^{n} A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p-1}, i}^{\prime} Y_{i}=0,
$$

if the relations (9) are verified.
4. - In particular, suppose that one has reduced the system (2) to the reduced form that Poincaré often employed in his arguments by a change of the unknown functions:

$$
\begin{equation*}
\frac{d y_{1}}{0}=\frac{d y_{2}}{0}=\ldots=\frac{d y_{n-1}}{0}=\frac{d y_{n}}{0}=d t . \tag{10}
\end{equation*}
$$

Any integral invariant of order $p$ has the form:

$$
\int B_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} d y_{\alpha_{1}} d y_{\alpha_{2}} \cdots d y_{\alpha_{p}},
$$

in which the coefficients $B_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}$ do not depend upon $y_{n}$ and can be arbitrary functions of $y_{1}, y_{2}$, $\ldots, y_{n-1}$. With that system of variables, the conditions (9) will become:

$$
B_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}}=0 .
$$

Therefore, the differential $d y_{n}$ must not appear in the expression for the invariant if it is to be an invariant $I_{p}^{(e)}$. An invariant of that type will therefore contain neither $y_{n}$ nor $d y_{n}$, while an arbitrary integral invariant will not contain $y_{n}$, but it might contain $d y_{n}$. In order to apply the operation $(E)$ to an invariant $I_{p}$, we suppose that we have written it out while including the combinations of $n$ indices taken $p$ at a time, and always putting the factor $d y_{n}$ last in the terms where it appears. In order to pass from $I_{p}$ to the invariant $I_{p-1}^{(e)}$ that is deduced from it, it will then suffice to suppress the factor $d y_{n}$ in every term where it appears and suppress all of the terms that do not contain that factor.

Conversely, if one is given an invariant $I_{p}^{(e)}$ then one can deduce it from an invariant $I_{p+1}$ by the operation $(E)$ (and generally in an infinitude of ways). Indeed, if that invariant $I_{p}^{(e)}$ is written by selecting only the combinations of $n$ indices taken $p$ at a time then it will suffice to add the factor $d y_{n}$ to each term in $I_{p}^{(e)}$ and then add an arbitrary invariant $I_{p+1}^{(e)}$ in order to get an invariant of order $p+1$ whose invariant $I_{p}^{(e)}$ is deduced by the operation $(E)$. It is clear that one will then have all invariants of order $p+1$ whose given invariant $I_{p}^{(e)}$ is deduced by the operation $(E)$. There is only one way of applying that procedure when $p=n-1$ because there obviously exists no invariant $I_{n}^{(e)}$ except zero. Any invariant $I_{n-1}^{(e)}$ will then correspond to a well-defined system of differential equations, and conversely. It was, moreover, by starting from a multiplier that Poincaré gave the general construction of an invariant $I_{n-1}$ that is attached to the characteristics in the particular case where $n=3$ (see also no. 16 in my first article). The other theorems that were proved in that work (nos. 6-9) concerning the invariants $I_{p}^{(e)}$ can also be easily established in the reduced form. I shall not stop to do that here. I remark only that it is obvious from the expressions for those
invariants that they are attached to characteristics. Suppose, to simplify, that $n=3$. An invariant $I_{2}^{(e)}$ will then have the form:

$$
I_{2}^{(e)}=\iint B\left(y_{1}, y_{2}\right) d y_{1} d y_{2} .
$$

The characteristics are lines parallel to the $O y_{3}$ axis. If one is given a tube of characteristics then it is clear that the value of the integral $I_{2}^{(e)}$, when extended over an arbitrary section of that tube, will be the same for all of those sections.
5. - It remains for us to prove that the invariants $I_{p}^{(e)}$ are the only invariants that are attached to the characteristics. The following direct proof applies to both the proposition itself and its converse. As in no. 2, take a multiplicity $E_{p}(p<n)$ that is not composed of characteristics and is bounded by a multiplicity $E_{p-1}$ that can vanish if $E_{p}$ is a closed multiplicity. A characteristic starts from any point $m$ of $E_{p}$, and along that characteristic, we can take a point $m^{\prime}$ arbitrarily in such a fashion as to respect its continuity. When the point $m$ describes $E_{p}$, the segment $\mathrm{mm}^{\prime}$ of the characteristic will generate a multiplicity $E_{p+1}$ whose boundary is composed of:

1. The given multiplicity $E_{p}$.
2. The multiplicity $E_{p}^{\prime}$ that is described by the extremity $m^{\prime}$ of the segment of the characteristic considered when $m$ describes $E_{p}$.
3. The multiplicity $E_{p}^{\prime \prime}$ that is generated by the segments of the characteristics that issue from the various points of $E_{p-1}$.

That being the case, let $I_{p}$ be an integral invariant. From the generalized Stokes theorem, the difference between the values of the multiple integral $I_{p}$, when it is extended over two multiplicities $E_{p}$ and $E_{p}^{\prime}$ in the corresponding sense (no. 1), will be equal to the integral $I_{p}$, extended over $E_{p}^{\prime \prime}$, plus an integral $I_{p+1}^{(d)}$ that is extended over the multiplicity $E_{p+1}$, where $I_{p+1}^{(d)}$ is an integral invariant of order $p+1$ that is deduced form $I_{p}$ by the operation $(D)$ (see the cited article, no. 2). In order for that difference to always be zero, it is necessary and sufficient that the integral $I_{p}$, when extended over $E_{p}^{\prime \prime}$, and the integral $I_{p+1}^{(d)}$, when extended over $E_{p+1}$, should be separately zero.

That condition is necessary. Indeed, one can suppose that the multiplicity $E_{p}^{\prime \prime}$ vanishes. In order for that to happen, it is sufficient to bound $E_{p}$ and $E_{p}^{\prime \prime}$ with the same multiplicity $E_{p-1}$. The integral $I_{p+1}^{(d)}$, when extended over the multiplicity $E_{p+1}$, which is composed of characteristics, must then be zero, which demands that the integral invariant $I_{p+1}^{(d)}$ must be an invariant $I_{p+1}^{(d, e)}$ (no. 3). If that is true then the integral $I_{p}$, when extended over the multiplicity $E_{p}^{\prime \prime}$, which is an arbitrary multiplicity
that is composed of characteristics, must also be zero. Therefore, the invariant $I_{p}$ must be itself an invariant $I_{p}^{(e)}$. The latter condition implies the former, as I proved in my first article (no. 8).

In summary, the only invariants that are attached to the characteristics are the invariants $I_{p}^{(e)}$ whose coefficients verify the relations (9).

Remark I. - One can further prove that it follows that the conditions (9) are necessary for $I_{p}$ to be an invariant attached to the characteristics. Indeed, if that is true then the conditions that express the idea that $I_{p}$ is an integral invariant must once more be verified when one multiplies the functions $X_{i}$ by an arbitrary function $\lambda\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ since that will not change the characteristics. In order for that to be true, the function $\lambda$ must verify the relations:

$$
\frac{\partial \lambda}{\partial x_{\alpha_{1}}}\left(\sum_{h} A_{h \alpha_{1}, \ldots, \alpha_{p}} X_{h}\right)+\frac{\partial \lambda}{\partial x_{\alpha_{2}}}\left(\sum_{h} A_{h \alpha_{1}, \ldots, \alpha_{p}} X_{h}\right)+\cdots=0
$$

and since that function $\lambda$ is arbitrary, the coefficients $A_{\alpha_{1}}, \ldots, A_{\alpha_{p}}$ must satisfy the conditions (9).
Remark II. - It also results from the preceding proof that if an expression:

$$
U=\int \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} d x_{\alpha_{1}} \cdots d x_{\alpha_{p}}
$$

is an exact differential, and if the coefficients verify the conditions (9) then that expression will be an invariant $I_{p}^{(d, e)}$ of equations (2). It is easy to verify that result by combining equations (9) with the relations that express the idea that $U$ is an exact differential.
6. - Up to now, we have considered only absolute invariants. However, one can also define relative invariants that are attached to the characteristics. To fix ideas, let:

$$
J_{1}=\int A_{1} d x_{1}+\cdots+A_{n} d x_{1}
$$

be a first-order relative invariant of the system (2). One can then replace it with the absolute invariant:

$$
I_{2}^{(d)}=\int \sum\left(\frac{\partial A_{i}}{\partial x_{k}}-\frac{\partial A_{k}}{\partial x_{i}}\right) d x_{i} d x_{k}
$$

If that invariant $I_{2}^{(d)}$ is $I_{2}^{(d, e)}$ then the argument in the preceding section will prove that the integral $J_{1}$, when taken along an arbitrary closed curve $C_{0}$, is equal to the same integral, when taken along another closed curve $C_{1}$ that is obtained by taking a point $m^{\prime}$ on the characteristic that issues from a point $m$ on $C_{0}$. Indeed, from the generalized Stokes theorem, the difference between those two
integrals is equal to the double integral $I_{2}^{(d, e)}$, when it is extended over the multiplicity $E_{2}$ that is generated by characteristic segments $\mathrm{mm}^{\prime}$, which is an integral that will be zero, from the property that characterizes the invariants $I_{p}^{(e)}$.

One can likewise deduce an absolute or relative invariant of order $p$ that is attached to the characteristics from any invariant $I_{p+1}^{(d, e)}$.

It can happen that an integral invariant $I_{p}$ is an invariant $I_{p}^{(e)}$ for only a set of characteristics that satisfy certain conditions. For example, suppose that equations (2) admit a first integral $F=$ $C$. The characteristics for which the constant $C$ has a well-defined value form a system that depends upon only $(n-2)$ arbitrary constants, and can be considered to be the characteristics of a system of $(n-1)$ differential equations that one will obtain by solving the relation $F=C$ for one of the variables ( $x_{n}$, for example) and substituting that value in the equations:

$$
\frac{d x_{1}}{X_{1}}=\ldots=\frac{d x_{n-1}}{X_{n-1}}=d t
$$

If one makes the same substitution in the invariant $I_{p}$ then one will get the invariant $I_{p}^{\prime}$ of equations (2'), which might be an invariant $I_{p}^{(e)}$ for the system. However, one can dispense with that calculation by verifying that the integral $I_{p}$ is zero over any multiplicity $E_{p}$ that is composed of characteristics for which the constant $C$ has the same numerical value. For example, consider the canonical system:

$$
\begin{equation*}
\frac{d x_{i}}{d t}=\frac{\partial F}{\partial p_{i}}, \quad \frac{d p_{i}}{d t}=-\frac{\partial F}{\partial x_{i}} \quad(i=1,2, \ldots, n) \tag{11}
\end{equation*}
$$

which admits the integral invariant:

$$
\begin{equation*}
I_{2}=\iint d x_{1} d p_{1}+\cdots+d x_{n} d p_{n} \tag{12}
\end{equation*}
$$

As long as $F$ is not a constant, that invariant $I_{2}$ will not be $I_{2}^{(e)}$. On the other hand, the system (11) will admit the first integral $F=C$. We shall show that the integral $I_{2}$, when extended over any multiplicity $E_{2}$ that is generated by characteristics for which the constant $C$ has a well-defined numerical value will always be zero. Indeed, the coordinates $\left(x_{i}, p_{i}\right)$ of any point of a multiplicity of that type can be expressed by means of two independent variables $t$ and $u$, so the derivatives with respect to $t$ are given by the formulas:

$$
\frac{d x_{i}}{d t}=\frac{\partial F}{\partial p_{i}}, \quad \frac{d p_{i}}{d t}=-\frac{\partial F}{\partial x_{i}} .
$$

One will then have:

$$
\sum_{i=1}^{n} \frac{D\left(x_{i}, p_{i}\right)}{D(t, u)}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_{i}} \frac{\partial p_{i}}{\partial u}+\frac{\partial F}{\partial x_{i}} \frac{\partial x_{i}}{\partial u}\right)=\frac{\partial F}{\partial u} .
$$

However, since $F$ is constant along that multiplicity, one will also have:

$$
\frac{\partial F}{\partial u}=0
$$

and as a result, the value of $I_{2}$ will be zero (cf., Hadamard, Calcul des variations, pp. 154-155)
7. - The properties of the invariants $I_{2}^{(e)}$ give the simplest explanation for why knowing an invariant $I_{n-2}^{(e)}$ will permit one to find an equation $Y(f)=0$ that will define a complete system, together with the equation:

$$
\sum_{i=1}^{n} X_{i} \frac{\partial f}{\partial x_{i}}=0 .
$$

Suppose, first of all, that $n=3$, and let:

$$
\begin{equation*}
I_{1}^{(e)}=\int A_{1} d x_{1}+A_{2} d x_{2}+A_{3} d x_{3} \tag{13}
\end{equation*}
$$

be an integral invariant of the system:

$$
\begin{equation*}
\frac{d x_{1}}{X_{1}}=\frac{d x_{2}}{X_{2}}=\frac{d x_{3}}{X_{3}}=d t \tag{14}
\end{equation*}
$$

whose coefficients $A_{1}, A_{2}, A_{3}$ verify the relation:

$$
A_{1} X_{1}+A_{2} X_{2}+A_{3} X_{3}=0 .
$$

Let $\Gamma$ be a curve such that the coordinates $\left(x_{1}, x_{2}, x_{3}\right)$ of one of its points are functions of one parameter $u$ that verify the relation:

$$
A_{1} \frac{\partial x_{1}}{\partial u}+A_{2} \frac{\partial x_{2}}{\partial u}+A_{3} \frac{\partial x_{3}}{\partial u}=0 .
$$

There obviously exists an infinitude of curves of that type since one can choose two of the coordinates $x_{1}, x_{2}, x_{3}$ arbitrarily as functions of $u$. Suppose, moreover, that the curve $\Gamma$ is not a characteristic. In order for that to be true, it suffices to take $x_{1}$ and $x_{2}$ to be functions of $u$ such that $X_{2} \frac{\partial x_{1}}{\partial u}-X_{1} \frac{\partial x_{2}}{\partial u}$ is not zero. That being the case, the characteristics that issue from the various
points of $\Gamma$ will generate a surface, and the integral $\int A_{1} d x_{1}+A_{2} d x_{2}+A_{3} d x_{3}$, when taken along an arbitrary curve that is situated in $S$ will be zero since that integral is zero along $\Gamma$, and the invariant considered is $I_{1}^{(e)}$. Any linear element of $S$ will then satisfy the relation:

$$
\begin{equation*}
A_{1} d x_{1}+A_{2} d x_{2}+A_{3} d x_{3}=0 . \tag{15}
\end{equation*}
$$

Since a surface of that type obviously passes through any point in space, it will follow that the preceding equation is completely integrable, and integration will give a first integral of the system (14).

That result is verified immediately by means of the relations:

$$
\begin{gathered}
A_{1} X_{1}+A_{2} X_{2}+A_{3} X_{3}=0 \\
X\left(A_{i}\right)+A_{1} \frac{\partial X_{1}}{\partial x_{i}}+A_{2} \frac{\partial X_{2}}{\partial x_{i}}+A_{3} \frac{\partial X_{3}}{\partial x_{i}}=0 \quad(i=1,2,3),
\end{gathered}
$$

which express the idea that $I_{1}^{(e)}$ is an absolute invariant. Upon differentiating the first with respect to $x_{i}$ and comparing to the following ones, one will deduce that:

$$
\frac{X_{1}}{\frac{\partial A_{3}}{\partial x_{2}}-\frac{\partial A_{2}}{\partial x_{3}}}=\frac{X_{2}}{\frac{\partial A_{1}}{\partial x_{3}}-\frac{\partial A_{3}}{\partial x_{1}}}=\frac{X_{3}}{\frac{\partial A_{2}}{\partial x_{1}}-\frac{\partial A_{2}}{\partial x_{1}}}
$$

and as a result:

$$
A_{1}\left(\frac{\partial A_{3}}{\partial x_{2}}-\frac{\partial A_{2}}{\partial x_{3}}\right)+A_{2}\left(\frac{\partial A_{1}}{\partial x_{3}}-\frac{\partial A_{3}}{\partial x_{1}}\right)+A_{3}\left(\frac{\partial A_{2}}{\partial x_{1}}-\frac{\partial A_{1}}{\partial x_{2}}\right)=0,
$$

which expresses the idea that the equation $A_{1} d x_{1}+A_{2} d x_{2}+A_{3} d x_{3}=0$ is completely integrable. If the invariant $I_{1}^{(e)}$ is not an invariant $I_{1}^{(d, e)}$ then one can deduce an invariant $I_{2}^{(d, e)}$ by the operation $(D)$, and as a result, a multiplier. Integrating the system (14) will then be achieved by a quadrature.

More generally, consider an invariant $I_{n-2}^{(e)}$ :

$$
I_{n-2}^{(e)}=\int \sum A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-2}} d x_{\alpha_{1}} d x_{\alpha_{2}} \cdots d x_{\alpha_{n-2}} .
$$

One can determine an infinitude of multiplicities $E_{n-2}$ that are not composed of characteristics such that the integral $I_{n-2}^{(e)}$ will be zero when it is extended over those multiplicities. Indeed, suppose that the multiplicity $E_{n-2}$ is defined by the two equations:

$$
x_{n-1}=\varphi_{1}\left(x_{1}, x_{2}, \ldots, x_{n-2}\right), \quad x_{n}=\varphi_{2}\left(x_{1}, x_{2}, \ldots, x_{n-2}\right) .
$$

When the integral is extended over that multiplicity, it will have the expression:

$$
\int B d x_{1} d x_{2} \cdots d x_{n-2}
$$

in which $B$ depends upon the variables $x_{i}$ and the first-order partial derivatives of the functions $\varphi_{1}$ and $\varphi_{2}$. In the equation $B=0$, one can choose one of the functions $\varphi_{1}, \varphi_{2}$ arbitrarily, while the other one is determined by a first-order partial differential equation. One can even choose the multiplicity in such a fashion that it will pass through a multiplicity that is given in advance and is not composed of characteristic curves.

Let $E_{n-2}$ be a multiplicity that is not composed of characteristics and satisfies the preceding condition. Let $E_{n-2}$ be the multiplicity that is generated by the characteristics that issue from the various points of $E_{n-2}$. We shall show that the integral $I_{n-2}^{(e)}$ will be zero when it is extended over an arbitrary multiplicity $E_{n-2}^{\prime}$ that is situated on $E_{n-1}$. Indeed, $E_{n-2}^{\prime}$ is deduced from $E_{n-2}$ by taking a point on each characteristic that issues from the various point of $E_{n-2}$, and from the essential property of the invariant $I_{n-2}^{(e)}$, the integral that is extended over $E_{n-2}^{\prime}$ will be equal to the integral that is extended over $E_{n-2}$, i.e., to zero.

Those multiplicities $E_{n-1}$, which are obviously integral multiplicities of the system (2), can be obtained directly. More generally, in order for the integral:

$$
I_{p}=\int \sum A_{\alpha_{1} \ldots, \alpha_{p}} d x_{\alpha_{1}} \cdots d x_{\alpha_{p}}
$$

to be zero when it is extended over any multiplicity $E_{p}$ that is included in the multiplicity $E_{n-1}$ that is defined by the equation $F=0$, one verifies effortlessly that the function $F$ must satisfy the following conditions, whose forms differ according to the parity of $p$ :

1. If $p$ is odd then one will have the conditions:

$$
A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} \frac{\partial F}{\partial x_{\alpha_{p+1}}}-A_{\alpha_{2}, \ldots, \alpha_{p}, \alpha_{p+1}} \frac{\partial F}{\partial x_{\alpha_{1}}}+\cdots+A_{\alpha_{p+1}, \alpha_{1}, \ldots, \alpha_{p-1}} \frac{\partial F}{\partial x_{\alpha_{p}}}=0
$$

with the signs + and - alternating.
2. If $p$ is even then one will have only + signs:

$$
A_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}} \frac{\partial F}{\partial x_{\alpha_{p+1}}}+A_{\alpha_{2}, \ldots, \alpha_{p}, \alpha_{p+1}} \frac{\partial F}{\partial x_{\alpha_{1}}}+\cdots=0 .
$$

In the present case, in which $p=n-2$, we know a priori that those equations must define a complete system. One immediately verifies that by supposing that the system (2) has been converted into the reduced form (10).
8. - In conclusion, I would like to show how one can complete a result that was proved in my previous article. Let:

$$
J=\int A_{1} d x_{1}+\cdots+A_{n} d x_{n}
$$

be a relative (or absolute) integral invariant of the system (2). The conditions for that to be true are then expressed by saying that the expression:

$$
\begin{equation*}
\left(\sum_{k=1}^{n} X_{k} B_{1 k}\right) d x_{1}+\left(\sum_{k=1}^{n} X_{k} B_{2 k}\right) d x_{2}+\cdots+\left(\sum_{k=1}^{n} X_{k} B_{n k}\right) d x_{n} \tag{16}
\end{equation*}
$$

in which:

$$
B_{i k}=\frac{\partial A_{i}}{\partial x_{k}}-\frac{\partial A_{k}}{\partial x_{i}}
$$

is an exact differential $d U$, and it is clear that $U=$ const. is a first integral of the system (2). Moreover, it can happen that the function $U$ reduces to a constant, and in that case, I have indicated what one can infer from knowing the invariant $J$ for the problem of integration.

Return to the case in which $U\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ does not reduce to a constant. Imagine that one makes a change of variables:

$$
x_{i}=\varphi_{i}\left(y_{1}, y_{2}, \ldots, y_{n}\right) \quad(i=1,2, \ldots, n),
$$

in such a fashion that the first integral will become $y_{n}=$ const. Equations (2) are replaced by a system:

$$
\frac{d y_{1}}{Y_{1}}=\frac{d y_{2}}{Y_{2}}=\ldots=\frac{d y_{n-1}}{Y_{n-1}}=\frac{d y_{n}}{0}=d t
$$

for which one has $Y_{n}=0$. As for the integral invariant $J$, it changes into an integral invariant $J^{\prime}$ of the system (2'):

$$
J^{\prime}=\int a_{1} d y_{1}+\cdots+a_{n-1} d y_{n-1}+a_{n} d y_{n}
$$

and it is clear that the first integral $y_{n}$ must be deduced from the invariant $J^{\prime}$ in the same way that $U\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is deduced from $J$. One will then have:

$$
d y_{n}=\left(\sum_{k=1}^{n} Y_{k} b_{1 k}\right) d y_{1}+\cdots+\left(\sum_{k=1}^{n} Y_{k} b_{n k}\right) d y_{n}, \quad b_{i k}=\frac{\partial a_{i}}{\partial y_{k}}-\frac{\partial a_{k}}{\partial y_{i}},
$$

and as a result, since $Y_{n}=0$ :

$$
\begin{equation*}
\sum_{k=1}^{n-1} Y_{k} b_{1 k}=0, \quad \ldots, \quad \sum_{k=1}^{n-1} Y_{k} b_{n-1, k}=0, \quad \sum_{k=1}^{n-1} Y_{k} b_{n k}=1 \tag{17}
\end{equation*}
$$

From the remark in no. $\mathbf{3}$, the first $n-1$ relations express the idea that the double integral:

$$
\begin{equation*}
\int \sum b_{i k} d y_{i} d y_{k} \quad(i, k=1,2, \ldots, n-1) \tag{18}
\end{equation*}
$$

is an integral invariant $I_{2}^{(e)}$ of the system:

$$
\frac{d y_{1}}{Y_{1}}=\frac{d y_{2}}{Y_{2}}=\ldots=\frac{d y_{n-1}}{Y_{n-1}}=d t
$$

in which one considered $y_{n}$ to be a constant. One saw the simplifications in the integration that might result in the first article.

The conclusion will break down if the integral invariant (18) is identically zero, which would demand that $a_{1} d y_{1}+\cdots+a_{n-1} d y_{n-1}$ must be the total exact differential of a function:

$$
V=\int a_{1} d y_{1}+a_{2} d y_{2}+\cdots+a_{n-1} d y_{n-1}
$$

while always regarding $y_{n}$ as a constant. Set:

$$
W=a_{n}-\frac{\partial V}{\partial y_{n}}=a_{n}-\int \frac{\partial a_{1}}{\partial y_{n}} d y_{1}+\cdots+\frac{\partial a_{n-1}}{\partial y_{n}} d y_{n-1} .
$$

One has:

$$
\begin{aligned}
\frac{d W}{d t} & =\frac{\partial a_{n}}{\partial y_{1}} \frac{d y_{1}}{d t}+\cdots+\frac{\partial a_{n-1}}{\partial y_{n-1}} \frac{d y_{n-1}}{d t}-\left(\frac{\partial a_{1}}{\partial y_{n}} \frac{d y_{1}}{d t}+\cdots+\frac{\partial a_{n-1}}{\partial y_{n}} \frac{d y_{n-1}}{d t}\right) \\
& =\sum_{i=1}^{n-1} Y_{i}\left(\frac{\partial a_{n}}{\partial y_{i}}-\frac{\partial a_{i}}{\partial y_{n}}\right)
\end{aligned}
$$

and the right-hand side is equal to unity, from the last of the relations (17). In that case, one will then have a new first integral that includes $t$ :

$$
\begin{equation*}
a_{n}-\int \frac{\partial a_{1}}{\partial y_{n}} d y_{1}+\cdots+\frac{\partial a_{n-1}}{\partial y_{n}} d y_{n-1}=t+\text { const. } \tag{19}
\end{equation*}
$$

That will be true for $n=3$, in particular.
9. - Return to the case in which one knows an integral invariant $I_{2}^{(d, e)}$ of the system (2):

$$
\begin{equation*}
I_{2}^{(d, e)}=\int \sum A_{i k} d x_{i} d x_{k} \tag{20}
\end{equation*}
$$

The $n$ relations:
which reduce to $n-p$ distinct relations ( $p>0$ ), form a completely integrable system whose integrals are also integrals of the system (2). If $p>1$ then one can define a complete system that admits less than $n-1$ distinct integrals that belong to the system (2). If $p=1$ then the system (21) is equivalent to the system (2), but one can deduce a multiplier.

Suppose that $p>1$. The equations (21) admit $q=n-p$ distinct integrals that one will obtain by integrating a complete system of $p$ equations. Imagine that one has made a change of variables $x_{i}=\varphi_{i}\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ in such a way that $y_{1}, y_{2}, \ldots, y_{n}$ are precisely those $n$ integrals. The invariant (20) will change into a new invariant of the same type $I_{2}^{(d, e)}$ :

$$
I^{\prime}=\int \sum a_{i k} d y_{i} d y_{k}
$$

and the invariant system (21) will become:

$$
\left\{\begin{array}{c}
a_{11} d y_{1}+a_{12} d y_{2}+\cdots+a_{1 n} d y_{n}=0 \\
a_{21} d y_{1}+a_{22} d y_{2}+\cdots+a_{2 n} d y_{n}=0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
a_{n 1} d y_{1}+a_{n 2} d y_{2}+\cdots+a_{n n} d y_{n}=0
\end{array}\right.
$$

That system must be equivalent to the system $d y_{1}=0, \ldots, d y_{q}=0$. It would then be necessary for all of the coefficients $a_{i k}$ for which one of the indices $i$ or $k$ is greater than $q$ to be zero. Moreover, the system:

$$
\begin{gathered}
a_{11} d y_{1}+\ldots+a_{1 q} d y_{q}=0, \\
a_{21} d y_{1}+\ldots+a_{2 q} d y_{q}=0, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
a_{q 1} d y_{1}+\ldots+a_{q q} d y_{q}=0
\end{gathered}
$$

must imply the relations $d y_{1}=0, \ldots, d y_{q}=0$. The Pfaff determinant must be nonzero, and as a result $q=n-p$ will be necessarily an even number $2 r$, which can also be easily deduced from the theory of the Pfaff problem. The conditions:

$$
\frac{\partial a_{i k}}{\partial y_{l}}+\frac{\partial a_{k l}}{\partial y_{i}}+\frac{\partial a_{l i}}{\partial y_{k}}=0
$$

which express the idea that $\sum a_{i k} d y_{i} d y_{k}$ is an invariant $I_{2}^{d}$, show, in addition, that the coefficients $a_{i k}$, in which $i$ and $k$ are equal to at most $2 r$, depend upon only $y_{1}, y_{2}, \ldots, y_{n}$. It is clear that knowing one such invariant cannot be of any use in achieving the integration of the transformed system, which has the form:

$$
\frac{d y_{1}}{0}=\ldots=\frac{d y_{2 r}}{0}=\frac{d y_{n-1}}{Y_{n-1}}=\ldots=\frac{d y_{n}}{Y_{n}}=d t
$$

