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FASCICLE VI.

The Bäcklund problem

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THE

BÄCKLUND PROBLEM

By **E. GOURSAT**

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1. Statement of the problem. Generalities. – Upon studying certain transformations of surfaces with constant total curvature, A.-V. Bäcklund was led to pose the following problem [**1**], which I call, to abbreviate, the *Bäcklund problem*, or (B) problem:

Find two multiplicities M_2 and M_2' of contact elements in three-dimensional space that correspond element-by-element in such a fashion that the coordinates of two corresponding elements (x, y, z, p, q), (x', y', z', p', q') verify four relations that are given in advance:

(1) $$F_i(x, y, z, p, q; x', y', z', p', q') = 0 \qquad (i = 1, 2, 3, 4).$$

I will recall that a multiplicity M_k of contact elements (k = 1, 2) in three-dimensional space is a set of contact elements whose coordinates x, y, z, p, q are functions of k independent variables that verify the relation:

(2) $$dz = p\,dx + q\,dy.$$

When $k = 2$, the point (x, y, z), where the coordinates are functions of two variable parameters, generally describes a surface S, and the elements of M_2 are composed of points of S, each of which is associated with the tangent plane to S at that point. One says that the multiplicity M_2 has the surface S for its *point-wise support*. However, it might happen that the point (x, y, z) describes a curve C (or even remains fixed). In the former case, one obtains an element of M_2 by associating an arbitrary point of the curve C with a plane that passes through the tangent to C at that point; that set depends upon two parameters, and the point-wise support of M_2 is the curve C. If the point (x, y, z) remains fixed then the point-wise support of M_2 reduces to a point, and one obtains an element of M_2 by associating that fixed point with an arbitrary plane that passes through that point. One likewise obtains elements of a multiplicity M_1 by associating a point of a curve C with a plane that passes through the tangent to C at that point, or by associating a fixed point P with a tangent plane to a cone that has its summit at that point ([1]).

Bäcklund has studied only the case where the multiplicities M_2, M_2' have two surfaces S, S' for their point-wise supports. The problem then amounts to finding two surfaces S, S' such that it is possible to make them correspond point-by-point in such a fashion that the corresponding contact elements of these two surfaces verify relations (1). This is what we call the (B) problem *in the strict sense*. However, there is good reason to

([1]) In the pages that follow, we consider only analytic relations and analytic multiplicities.

pose the problem in the more general form that was stated in the beginning; when there is cause to make that distinction one will say that the new problem is (B) problem *in the broad sense*. That extension presents the same advantages as the generalized definition of S. Lie for the integral of a partial differential equation. One knows, moreover, that a multiplicity M_2 that has a curve or a point for its point-wise support gets converted into a multiplicity M_2 that has a surface for point-wise support by means of a Legendre transformation. In a general manner, if one subjects the multiplicities M_2, M_2' to two arbitrary contact transformations (T), (T') then they change into two new multiplicities of the same type, and the relations (1) are replaced by four new relations that are deduced from the original ones by performing a system of two transformations (T), (T') on the elements (x, y, z, p, q), (x', y', z', p', q'). We will not regard two problems (B) as distinct when they go to each other under a system of two transformations (T), (T').

If one takes the (B) problem in the strict sense then one must regard z and z' as two unknown functions in equations (1), the former, of the variables x, y, and the latter of the variables x', y', while the letters p, q, p', q' have the usual sense. It may happen that the elimination of the primed variables leads to just one partial differential equation of second order (E) for the function $z(x, y)$, while the elimination of the unprimed variables also leads to just one partial differential equation of second order (E') for the function $z'(x', y')$. Equations (1) then establish a correspondence between the integrals of the two equations (E), (E') that is different from the transformation (T). These new transformations are the *Bäcklund transformations*, or (B) transformations. The essential properties of these transformations are deduced very easily from the general study of (B) problem.

A particular case of (B) problem in the strict sense that is far-reaching has already given rise to a great number of papers. When the first two equations (1) are $x' = x$, $y' = y$, the system (1) reduces to a system of two first-order partial differential equations in the two unknown functions:

$$(3) \qquad F(x, y, z, z', p, p', q, q') = 0, \qquad \Phi(x, y, z, z', p, p', q, q') = 0.$$

For a long time, systems of this type have been known that lead to a second-order partial differential equation for each of the unknown functions.

For example, the system of two equations:

$$z' = f(x, y, z, p, q), \qquad z = \varphi(x, y, z', p', q')$$

leads, upon eliminating z', to second-order partial differential equations (E) in z, while the elimination of z leads to a second-order equation (E') in z'. The integrals of these two equations correspond in a ont-to-one fashion. These transformations, in particular, comprise the celebrated Laplace transformation. Likewise, the elimination of z' between the two equations:

$$p = f(x, y, z', p', q'), \qquad q = z'$$

leads to a second-order equation (E) in z, while the elimination of z leads to another equation (E') in z'; each integral of E corresponds to just one integral of (E'), while each integral of (E') corresponds to an infinitude of integrals of (E) that depend upon an arbitrary constant.

We again take the system $p' = a(x, y)\, p$, $q' = b(x, y)\, q$. The elimination of one of the unknowns z or z' leads to a second-order linear equation for the determination of the other unknown, and any integral of one of these equations corresponds to an infinitude of integrals of the other one that depends upon an arbitrary constant. If $a + b = 0$ then one recovers a well-known transformation of Moutard [**41**].

When the two functions F and Φ are linear in z, z', p, p', q, q', one may obtain numerous transformations that are analogous to the preceding ones, which permits us to pass from a second-order linear equation to another equation of the same type. The study of these transformations has been carried a long way [**22**, **26′**], but it is outside the scope of our subject.

2. Associated Pfaff system. – Any solution of (B) problem is represented by a system of ten functions $(x, y, \ldots, q')$ of two independent variables that satisfy equations (1) and two relations:

$$dz = p\,dx + q\,dy, \qquad dz' = p'\,dx' + q'\,dy'. \tag{4}$$

The four equations (1), which are assumed to be distinct and compatible, permit one to express $x, y, \ldots, p', q'$ by means of six parameters $x_1, x_2, \ldots, x_6$, in such a fashion that a system of values of $x, y, \ldots, q'$ corresponds to just one system of values of x_i and conversely, at least, in sufficiently restricted domains. When one makes this substitution in equations (4), they change into a system S of two Pfaff equations in six variables:

$$\omega_1 = \sum a_i\,dx_i = 0, \qquad \omega_2 = \sum b_i\,dx_i = 0 \qquad (i = 1, 2, \ldots, 6), \tag{5}$$

that we call the *associated system* to (B) problem. Any solution to (B) problem corresponds to a two-dimensional integral $\mathfrak{M}_2$ of the associated system. Conversely, any integral multiplicity $\mathfrak{M}_2$ of S corresponds to a two-dimensional multiplicity N_2 that is described by the point with coordinates $(x, y, \ldots, q')$ in six-dimensional space, since there is a bijective analytic correspondence between these two multiplicities. The two elements (x, y, z, p, q), (x', y', z', p', q') describe multiplicities M and M' that are generally two-dimensional. However, it might happen that M, for example, is only one-dimensional, while the element (x', y', z', p', q') describes an M_2'. Each element of M_1 then corresponds to ∞^1 elements of M_2'. It might likewise happen that these two elements each describe a one-dimensional multiplicity. This is what happens, for example, if the two equations $F_1 = 0$, $F_2 = 0$ refer only to x, y, z, p, q and the last two primed variables. If these two systems do not admit two-dimensional integral multiplicities (which is the general case) then (B) problem correspondingly does not admit solutions, even in the broad sense. Meanwhile, the associated Pfaff system has integrals $\mathfrak{M}_2$. Indeed, let M_1 be an integral of the system $F_1 = F_2 = 0$, and let M_1' be an integral of $F_3 = F_4 = 0$. Along M_1, x, y, z, p, q are functions of a parameter u; along M_1',

x', y', z', p', q' are functions of another parameter v. The point with coordinates $(x, \dots, q')$ thus describes a multiplicity N_2 in six-dimensional space, which corresponds to an integral $\mathfrak{M}_2$ of the Pfaff system. In this case, two arbitrary elements that are taken on M_1 and M_1' will correspond. One might also arrange that the first multiplicity M reduces to just one element, while the second one M_2' is two-dimensional. The corresponding multiplicity $\mathfrak{M}_2$ again possesses two dimensions. One then sees that one further generalizes the problem by replacing (B) problem, likewise in the broad sense, with the search for integrals $\mathfrak{M}_2$ of the associated Pfaff system. In particular, we see that the formation of the system S demands only that the four equations (1) be distinct and compatible, while the (B) problem, likewise in the broad sense, might have no meaning for certain systems of relations (1), like the ones that we just cited.

Equations (1) permit us to express the ten variables $(x, \dots, q')$ by means of six parameters in an infinitude of ways. If one expresses them by means of six parameters y_i that are different from the x_i then one is led to another Pfaff system in which the six variables y_i appear. However, the x_i are also expressed by means of y_i and, as a result, the new Pfaff system reduces to the first one by a change of variables. The associated Pfaff system to a (B) problem is therefore defined up to a change of variables.

For example, if the equations (1) may be solved with respect to the x', y', p', q' then one may take x, y, z; p, q, z' for parameters. The associated system will be composed of equation (2), and a second equation in which the differentials dx, dy, dp, dq, dz' appear.

Conversely, any system S of two Pfaff equations in six variables may be associated with an infinitude of problems (B), provided that they are not completely integrable. Indeed, let $\Omega_1 = 0$, $\Omega_2 = 0$ be two distinct linear combinations of two equations $\omega_1 = 0$, $\omega_2 = 0$ of S. If these equations are of the fifth class (which is the general case) then one may convert them into the canonical form (4); the variables (x, y, z, p, q), (x', y', z', p', q') that figure in these two forms are functions of the six variables x_i, and, in turn, are coupled by just four relations $F_i = 0$, in general. The system S is associated with (B) problem, corresponding to that system of relations. There thus exists an infinitude of problems (B) that have the same associated Pfaff system. We say, to abbreviate, that they belong to the same class. There are then some problems (B), in particular, that are converted into each other by two transformations (T), (T').

3. Singular elements of the associated system. – We first recall some definitions and some properties of Pfaff systems [**10**, **11**]. Any system of values $(dx_1, dx_2, \dots, dx_6)$ that are not all zero and verify equations (5) is a *linear integral element* of that system that issues from a point $(x_1, x_2, \dots, x_6)$ in six-dimensional space. An element will be represented by e or by (dx_i). Two elements (dx_i) and $(k\, dx_i)$ are not considered to be distinct, in such a way that any point of six-dimensional space is the origin of ∞^3 linear integral elements. Two linear integral elements (dx_i) and (δx_i) are said to be *in involution* when one has the two relations:

$$(6)\qquad \begin{cases} \omega_1' = \sum a_{ik}(dx_i\delta x_k - dx_k\delta x_i) = 0, & a_{ik} = \dfrac{\partial a_i}{\partial x_k} - \dfrac{\partial a_k}{\partial x_i}, \\ \omega_2' = \sum b_{ik}(dx_i\delta x_k - dx_k\delta x_i) = 0, & b_{ik} = \dfrac{\partial b_i}{\partial x_k} - \dfrac{\partial b_k}{\partial x_i}, \end{cases}$$

between the coordinates of these elements, the summation being extended over all combinations of the indices i and k. The left-hand sides of these relations ω_1', ω_2' are the *bilinear covariants* of the Pfaff forms ω_1, ω_2. In a general fashion, two elements (dx_i), (δx_i) are in involution *relative to a Pfaff equation* $\Omega = 0$ if they annul the bilinear covariant Ω'. This property is invariant with respect to an arbitrary change of variables.

We have already observed that the system S may be written in an infinitude of manners by replacing the variables x_i with a new arbitrary system of variables y_i that are distinct functions of the former ones. It might happen that by suitably choosing the variables y_i the system may be written in a form in which less than six variables appear. Let r be the *minimum* number of variables that appear in a system that is deduced from S by an arbitrary choice of variables; the system S is said to be of *class r*. In general, a system of two equations in six variables is of class 6, but it might be of class 5, 4, or 2 ([1]).

The class of a system is determined by looking for *characteristic elements* – i.e., elements (dx_i) that are in involution with all of the other linear integral elements (δx_i). In order for an element (dx_i) to be characteristic, it is necessary and sufficient that the equations $\omega_1' = 0$, $\omega_2' = 0$ be verified by all the integral elements (δx_i). Upon writing down these conditions, one obtains a certain number of linear relations in $dx_1, \dots, dx_6$, which, when combined with the equations $\omega_1 = 0$, $\omega_2 = 0$, determine the characteristic elements. If this system admits no other solutions than $dx_i = 0$ (which is the general case) then there are no characteristic elements, and the system S has class 6. If the equations that determine the characteristic elements admit other solutions than $dx_i = 0$ then they reduce to r distinct equations ($r < 6$); this system of r equations is completely integrable, and may be converted into the form $df_1 = 0, \dots, df_r = 0$. If one takes a system of six variables y_i such that $y_1 = f_1, \dots, y_r = f_r$ then these variables $y_1, y_2, \dots, y_r$ and their differentials appear in the equations of the system only after the transformations; *S is of class r*. One generally denotes a system of class p by S_p.

Having recalled these properties, let (dx_i) be an arbitrary linear integral element of S. The coordinates of dx_i of another element in involution with the first one must verify the two equations (5), where d is replaced with δ, and the two equations $\omega_1' = 0$, $\omega_2' = 0$. These four equations are generally distinct if the element (dx_i) is not chosen in any particular fashion, and consequently, there are ∞^1 linear integral elements in involution with the first one.

However, there may be an exception if the coordinates dx_i of the element e has been chosen in such a fashion that the four linear equations that determine the elements in

([1]) It cannot be of class 3. Indeed, a system of class 3 will be of the form $dy_2 + A\,dy_1 = 0$, $dy_3 + B\,dy_1 = 0$, A and B being functions of y_1, y_2, y_3. This system of differential equations is equivalent to two equations $df_1 = 0$, $df_2 = 0$.

involution with e are not distinct. Such elements are the *singular elements* of S. It is easy to prove that there are, in general, two distinct families of singular elements.

One may always suppose that the equations of S are solved with respect to two of the differentials – dx_5 and dx_6, for example – which amounts to writing the equations of S as:

$$
(7) \qquad \begin{cases} \omega_1 = dx_5 + a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + a_4 dx_4 = 0, \\ \omega_2 = dx_6 + b_1 dx_1 + b_2 dx_2 + b_3 dx_3 + b_4 dx_4 = 0. \end{cases}
$$

Any system of values for dx_1, dx_2, dx_3, dx_4 that are not all zero determines a linear integral element e, which we will make correspond to the point m of the three-dimensional space whose homogeneous coordinates are dx_1, dx_2, dx_3, dx_4. If one replaces dx_5, dx_6, δx_5, δx_6 in equations (6) with their values that are derived from equations (7) and analogous equations in which d is replaced by d then it is easy to verify that these two equations take the form:

$$
(8) \qquad \omega_1' = \sum A_{ik} (dx_i \, \delta x_k - dx_k \, \delta x_i) = 0 \qquad (i, k = 1, 2, 3, 4),
$$

$$
(9) \qquad \omega_2' = \sum B_{ik} (dx_i \, \delta x_k - dx_k \, \delta x_i) = 0
$$

in which the coefficients A_{ik}, B_{ik} are expressed by means of functions a_i, b_i, and their partial derivatives. Let m, m' be the image points of the two elements in involutions (dx_i), (δx_i). The conditions (8) and (9) express the idea that the line m, m' belongs to two linear complexes C_1 and C_2. If these two complexes C_1 and C_2 are distinct then *line m m' belongs to a linear congruence.* The integral element (dx_1, dx_2, dx_3, dx_4) being given, the elements (δx_i) in involution with it are replaced with the points of a lines that issues from m; that element is therefore in involution with ∞^1 linear integral elements that issue from the same point.

Things are no longer the same if the point m is situated on the one of the rectilinear directrices Δ_1, Δ_2 of the linear congruence. Any element (dx_i) that is represented by a point m of Δ_1, for example, is in involution with another element that is represented by a point m' of the plane that passes through m and Δ_2 ; that element (dx_i) is in involution with ∞^2 other integral elements. There are thus *two distinct families of singular elements, which are represented by the points of the two lines* Δ_1, Δ_2 .

This intuitive result is easily verified by means of the following calculation, which permits one to form the equations that determine the singular elements. In order for the two equations (8) and (9), which determine δx_1, δx_2, δx_3, δx_4, to not be distinct, it is necessary and sufficient that there exist two coefficients λ, μ such that one has $\lambda\omega_1' + \mu\omega_2'$ = 0 identically, for any δx_1, δx_2, δx_3, δx_4, which demands that dx_1, dx_2, dx_3, dx_4 verify the four equations:

$$
(10) \qquad \begin{cases} (\lambda A_{i1} + \mu B_{i1})\, dx_1 + (\lambda A_{i2} + \mu B_{i2})\, dx_2 + \cdots + (\lambda A_{i4} + \mu B_{i4})\, dx_4 = 0, \\ \qquad A_{ik} + A_{ki} = 0, \qquad B_{ik} + B_{ki} = 0 \qquad (i = 1, 2, 3, 4). \end{cases}
$$

In order for these equations to be verified by values of the dx_i that are not all zero, it is necessary and sufficient that the determinant $D(\lambda, \mu)$ of the coefficients be zero:

(11) $$D(\lambda, \mu) = \| \lambda A_{ik} + \mu B_{ik} \| = 0.$$

This skew-symmetric determinant is equal to the square of a quadratic form $[F(\lambda, \mu)]^2$, and the ratio λ / μ must be the root of a second-degree equation:

(12) $$F(\lambda, \mu) = 0.$$

Let $\lambda = \lambda_1$, $\mu = \mu_1$ be a system of solutions to this equation. Since all of the first-order minors of the determinant $D(\lambda_1, \mu_1)$ are zero, the four equations (10), where one has $\lambda = \lambda_1$, $\mu = \mu_1$ reduce to just two equations, and that solution of equation (12) indeed corresponds to a family of ∞^1 singular elements.

The same interpretation permits one to find the case where the determinant $D(\lambda, \mu)$ is identically zero.

For this, we remark that the relation $\| A_{ik} \| = 0$ is the necessary and sufficient condition for the complex C_1 to be a singular complex that is formed of lines that meet a fixed line Δ_1, because one obtains that condition by expressing the idea that there exist points $(dx_1, \ldots, dx_4)$ such that any line that passes through one of these points belongs to the complex.

In order for the determinant $D(\lambda, \mu)$ to be zero identically, it is therefore necessary that the two complexes C_1 and C_2 be singular complexes and that the same thing is true for all the complexes of the sheaf that is determined by these two complexes C_1 and C_2; this will be true if the axes Δ_1 and Δ_2 of the two singular complexes C_1 and C_2 have a common point P, and only in this case. The point P then represents an integral element that is in involution with all of the other integral elements of S – i.e., a characteristic element – and *the system S has a class that is less than six.*

Conversely, if the system S has class less than six then a characteristic element $(dx_1, \ldots, dx_4)$ is in involution with any other linear integral element, and the line that joins the two image points of these elements belongs to all of the complexes of the sheaf that is determined by C_1 and C_2 ; $dx_1, \ldots, dx_4$ thus verify equations (10) for any λ and μ, and, in turn, the determinant $D(\lambda, \mu)$ is identically zero.

When the two equations (8) and (9) are not distinct then two complexes C_1 and C_2 are identical, and the argument no longer applies. One may then find two coefficients λ, μ such that at least one of them is non-zero and $\lambda\omega_1' + \mu\omega_2'$ is identically zero for arbitrary integral elements. The bilinear covariant Ω_1' of the equation $\Omega_1 = \lambda\omega_1 + \mu\omega_2$ is zero for any two integral elements. We take that equation $\Omega_1 = 0$ to be one of the equations of the system, and suppose that it has class five and reduces to the canonical form:

$$\Omega_1 = dy_3 + y_2\, dy_1 + y_4\, dy_3 = 0.$$

One may take the second equation of the system to be an equation that does not refer to dy_5 :

$$\Omega_2 = Y_1\, dy_1 + Y_2\, dy_2 + Y_3\, dy_3 + Y_4\, dy_4 + Y_6\, dy_6 = 0.$$

The covariant $\Omega'_1 = dy_1\ \delta y_2 - dy_2\ \delta y_1 + dy_3\ \delta y_4 - dy_4\ \delta y_3$ might not be zero for two arbitrary integral elements if Y_6 is non-zero, since $dy_1, \ldots, dy_4, \delta y_1, \ldots, \delta y_4$ might then be chosen arbitrarily. If $Y_6 = 0$ then the equation $\Omega_2 = 0$ represents a plane P, upon adopting the same geometric interpretation, while the equation $\Omega'_1 = 0$ represents a non-singular complex C. In order for Ω'_1 to be zero for any arbitrary integral elements, one must therefore have that any of the lines of the plane P must belong to the complex C, which is impossible. The equation $\Omega_1 = \lambda\omega_1 + \mu\omega_2$ must therefore not be of class five. One confirms in the same fashion that it is of class *three*, so the system S is of class five. If S is of class six then one must thus have that Ω_1 is of class one, and this system admits an integrable combination $\Omega_1 = dy_5 = 0$.

The converse is immediate. If a system S of class six admits an integrable combination $dy_5 = 0$ then it is composed of that equation, combined with another equation of class five. Any arbitrary integral element is in involution with ∞^2 integral elements, and there are no singular elements.

In summary, any system S_6 for which there exists no integrable combination admits two families, which are distinct, in general, of ∞^1 singular elements, each of which is in involution with ∞^2 integral elements. The singular elements of each family are determined by a system of four distinct Pfaff equations; one may obviously take two of them to be the two equations $\omega_1 = 0$, $\omega_2 = 0$ of the system S_6. Let:

$$(13) \qquad \omega_1 = 0, \qquad \omega_2 = 0, \qquad \omega_3 = 0, \qquad \omega_4 = 0$$

be the equations that define one of these families of singular elements. There exists a family of one-dimensional integrals of that system that depend upon an arbitrary function, because if one establishes an arbitrary relation between two of the variables, such as $x_2 = f(x_1)$, then what remains is a system of four differential equations between five variables. These one-dimensional integrals of the system (13) are the *Monge characteristics* of the system S_6. There are thus, in general, two distinct families of Monge characteristics for the system S_6. These multiplicities enjoy properties that are analogous to those of the characteristics of a second-order partial differential equation.

The ∞^2 integral elements of a multiplicity $\mathfrak{M}_2$ that issue from a point of that multiplicity, being pair-wise in involution, are represented by the points of a line of the linear congruence that is represented by the relations (8) and (9), and the two points where that line encounters the directrices Δ_1, Δ_2 represent two singular elements. Any point of $\mathfrak{M}_2$ is therefore the origin of two tangent singular elements to $\mathfrak{M}_2$, and one easily concludes that $\mathfrak{M}_2$ may be generated by the Monge characteristics of each of the two families ([1]).

([1]) In this discussion, one always supposes that the elements (dx_i), (δx_i) issue from a point (x_i) of the *general situation* in six-dimensional space. For certain systems, it might happen that there exists a hypersurface H_k ($k < 6$) such that the two equations (8) and (9) reduce to just one when the point (x_i) is situated on H_k. All of the linear integral elements that have their origin at a point of H_k may thus be considered to be singular elements. Any integral multiplicity of S that belongs to H_k is a *singular integral*. The coordinates of a point of H_k may be expressed by means of k variables, so the search for these singular integrals may be reduced to the integration of a system of less than six variables.

4. Reduced forms for a system *S*. – Let S_6 be a system of class six that admits no integrable combination. Let (λ, μ) be a system of solutions of equation (12) that are not all zero. From the same way that one has obtained that equation, there exists a family of singular elements $(dx_1, \ldots, dx_4)$ that are in involution with any other integral element relative to the equation:

$$\Omega = \lambda\omega_1 + \mu\omega_2 = 0.$$

We say that this equation $\Omega = 0$ is a *singular equation* of the system S_6 ; the properties that define it are independent of the choice of variables. Any singular equation thus changes into a singular equation when one performs an arbitrary change of variables. First, suppose that the singular equation has class five. One may then choose a system of six variables x, y, z, p, q, u in such a fashion that the singular equation is put into the canonical form, and the equations of the system S_6 become:

$$\text{(14)} \qquad \begin{cases} \Omega_1 = dz - p\,dx - q\,dy = 0, \\ \Omega_2 = X\,dx + Y\,dy + P\,dp + Q\,dq + U\,du = 0. \end{cases}$$

If the second equation contains du then the condition:

$$\Omega_1' = dp\ \delta x - dx\ \delta p + dp\ \delta y - dy\ \delta q = 0$$

cannot be satisfied, no matter what the element $(\delta x, \delta y, \ldots, \delta u)$, only by supposing that $dx = dy = dp = dq = 0$, and, in turn, $dz = du = 0$, since $\delta x, \delta y, \delta p, \delta q$ may be chosen arbitrarily. If $\Omega_1 = 0$ is a singular equation then one necessarily has $U = 0$. If that condition is satisfied then the equation $\Omega_1' = 0$ will be identical to the second of equations (14), where d has been replaced with δ, provided that the integral element (dx, dy, dp, dq, dz) verifies the relations:

$$\text{(15)} \qquad \frac{dx}{P} = \frac{dy}{Q} = \frac{-dp}{X} = \frac{-dq}{Y} = \frac{dz}{Pp + Qq},$$

and one arrives at the following conclusion: *Any system of two Pfaff equations of class six may, in general, be converted in two different ways, and only two, to the reduced form:*

$$\text{(16)} \qquad \begin{cases} \Omega_1 = dz - p\,dx - q\,dy = 0, \\ \Omega_2 = X\,dx + Y\,dy + P\,dp + Q\,dq = 0. \end{cases}$$

Each reduced form corresponds to a family of singular elements that are defined by the four equations (15). These four equations determine the ratios of five variables $dx, \ldots, dq$, but du remains arbitrary. The proof shows, at the same time, what the operations are that must be performed in order to obtain that reduced form. If the equation $F(\lambda, \mu) = 0$ has been solved then one will have to convert the singular equation $\lambda\omega_1 + \mu\omega_2 = 0$ into a canonical form. The variables x, y, z, p, q that figure in this canonical form are determined up to a transformation (T). As for the sixth variable u, one may choose it at will, provided that it is distinct from the five variables $x, \ldots, q$.

One may profit from this indeterminacy in u to further simplify the second of equations (16). Upon first performing, if necessary, a convenient transformation (T), one may suppose that the ratio Q / P contains the variable u, and take this ratio itself to be the last variable. Equations (16) then become:

$$\text{(I)} \qquad dz = p\,dx - q\,dy = 0, \qquad dp - u\,dq - a\,dx - b\,dy = 0,$$

in which a, b are functions of the six variables $x, y, \ldots, u$. Duport [**24**] was the first to prove, by a different method, that a system S in which six variables appear may generally be converted into the form (I) in two different ways. Two arbitrary functions of six variables appear in this reduced form. If the system S is *arbitrary* then one cannot obtain a reduced form in which less than two arbitrary functions appear. Indeed, if the system is assumed to have been solved for two of the differentials then it contains eight arbitrary coefficients. When one performs a change of variables, one disposes of six arbitrary functions that one may choose in such a fashion that six of the coefficients of the new system have expressions that are given in advance; there thus remain two indeterminate coefficients in the new system of equations.

Upon seeking the singular elements of the system (I) directly, one first obtains the system that is defined by the relations (15), which become:

$$(15)' \qquad \frac{dx}{1} = \frac{dy}{-u} = \frac{dp}{a} = \frac{dq}{b} = \frac{dz}{p-qu}$$

here, and a new family of singular elements that is determined by the four equations:

$$(17) \qquad \begin{cases} \Omega_1 = 0, \quad \Omega_2 = 0, \quad dq + \dfrac{\partial a}{\partial u}dx + \dfrac{\partial b}{\partial u}dy = 0, \\ \left(A + B\dfrac{\partial b}{\partial u} - C\dfrac{\partial a}{\partial u}\right)(u\,dx + dy) = \left(u\dfrac{\partial b}{\partial u} - b - \dfrac{\partial a}{\partial u}\right)(B\,dx + C\,dy - du), \end{cases}$$

in which we have set:

$$A = \frac{db}{dx} - \frac{da}{dy}, \qquad B = \frac{\partial a}{\partial q} + u\frac{\partial a}{\partial p}, \qquad C = \frac{\partial b}{\partial q} + u\frac{\partial b}{\partial p},$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z}, \qquad \frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z}.$$

The corresponding singular equation is:

$$(18) \qquad \left(u\frac{\partial b}{\partial u} - b - \frac{\partial a}{\partial u}\right)\Omega_2 - \left(A + B\frac{\partial b}{\partial u} - C\frac{\partial a}{\partial u}\right)\Omega_1 = 0.$$

If $u\frac{\partial b}{\partial u}-b-\frac{\partial a}{\partial u}$ is not zero, which is the general case, then the two families of singular elements are distinct. When $u\frac{\partial b}{\partial u}-b-\frac{\partial a}{\partial u}$ is zero, without $A+B\frac{\partial b}{\partial u}-C\frac{\partial a}{\partial u}$ being zero, the two families of singular elements coincide. Finally, if the coefficients of Ω_1 and Ω_2 in equation (18) are both zero then the system admits characteristic elements that are defined by the five relations:

$$u\,dx + dy = 0, \qquad dq + \frac{\partial a}{\partial u}dx+\frac{\partial b}{\partial u}dy = 0,$$

$$B\,dx + C\,dy - du = 0, \qquad \Omega_1 = 0, \qquad \Omega_2 = 0,$$

and the system is of class five.

Conversely, any system S_5 may be converted into the form (I) in an infinitude of ways, where the coefficients a and b verify the stated conditions. Let $\Omega_1 = 0$ be an equation of class five of the system S_5 ; if one assumes that it has been converted into the canonical form then the second equation of the system can contain the differential du of the sixth variable. Indeed, in order for an element (dx, dy, dp, dq) to be a characteristic element it is necessary that the relation:

$$dx\,\delta p - dp\,\delta x + dy\,\delta q - dq\,\delta y = 0$$

be verified for any integral elements (δx, δy, δp, δq), which is impossible if the second one contains du, since the values of δx, δy, δp, δq may then be taken arbitrarily.

A singular equation of a system S_6 may also be of class three, and conversely; if one may deduce a combination $\lambda\omega_1 + \mu\omega_2 = 0$ of class three from the equations $\omega_1 = 0$, $\omega_2 = 0$ of a system S_6 then that equation $\Omega_1 = 0$ is one of the singular equations of the system. Indeed, let $du - w\,dv = 0$ be a canonical form for that equation. Upon adding the second equation of the system S_6 to the three relations $du = 0$, $dv = 0$, $dw = 0$, one obtains a family of singular elements, each of which is in involution with any other integral element relative to the equation $\Omega_1 = 0$. There is no reason to return to the case where the system S_6 admits an integral combination, since there are no singular elements.

The discussion of all the possible singular cases is somewhat long, but presents no difficulties [**30**]. I will recall only the results.

1. *General case.* – Equation (12) has two distinct roots, and each of them corresponds to a singular equation of class five. The system S_6 may be converted into the form (I) in two different ways. There are two distinct families of singular elements, and the differential equations that define one family admit *at most two distinct integrable combinations*.

2. Equation (12) has a double root that corresponds to a singular equation of class five. The system S_6 may be converted into the form (I) in only one way, and one has:

$$\frac{\partial a}{\partial u} + b - u\frac{\partial b}{\partial u} = 0$$

for this reduced form.

3. Equation (12) has two distinct roots, one of which provides a singular equation of class five, and the second of which provides a singular equation of class three. The system S_6 may be converted into the form (I), and into another reduced form:

(II) $\Omega_1 = dy_3 - y_2\, dy_1 = 0, \qquad \Omega_2 = dy_5 - y_6\, dy_4 - a\, dy_1 - b\, dy_2 = 0,$

in which a and b are not both zero. The differential equations of the family of singular elements that correspond to the singular equation $\Omega_1 = 0$ are $dy_1 = 0$, $dy_2 = 0$, $dy_3 = 0$, $dy_3 - y_6\, dy_4 = 0$, and admit *three* distinct integrable combinations.

4. Equation (12) has two distinct roots, each of which corresponds to a singular equation of class three; S_6 may be converted into the *canonical form*:

(III) $\Omega_1 = dy_3 - y_2\, dy_1 = 0, \qquad \Omega_2 = dy_6 - y_5\, dy_4 = 0,$

and the differential equations of each family of singular elements admit *three* integrable combinations.

5. Equation (12) has a double root that gives a singular equation of class three; S_6 may be converted into a *canonical form:*

(IV) $\Omega_1 = dz - p\, dx - q\, dy = 0, \qquad \Omega_2 = du - q\, dp = 0,$

and the differential equations of the singular elements form a completely integrable system.

6. When the two equations (8) and (9) are not distinct, we have already remarked that the system S_6 admits an integrable combination; it may then be converted into the *canonical form:*

(V) $\Omega_1 = dz - p\, dx - q\, dy = 0, \qquad \Omega_2 = du = 0.$

In this case, where or not one has singular elements, equation (12) has a double root that corresponds to the equation $\Omega_2 = 0$.

In order to complete the enumeration of the reduced forms into which one may convert a system of two Pfaff equations in which six variables appear, it is necessary to add the forms that agree with the system S_5 and S_4 to the preceding types (26).

A system S_5 may generally be converted in an infinitude of ways into a reduced form [**26**]:

(VI) $\Omega_1 = dy_3 - y_2\, dy_1 = 0, \qquad \Omega_2 = dy_4 - f\, dy_1 - y_5\, dy_2 = 0,$

where f is not a linear function of y_5, and in certain cases, into the canonical form:

(VII) $dy_2 - y_4\, dy_1 = 0, \qquad \Omega_2 = dy_3 - y_5\, dy_1 = 0.$

A system S_4 may be likewise converted into one of the canonical forms:

(VIII) $\Omega_1 = dy_2 - y_3\, dy_1 = 0, \qquad \Omega_2 = dy_3 - y_4\, dy_1 = 0,$

(IX) $\Omega_1 = dy_2 = 0, \qquad \Omega_2 = dy_3 - y_4\, dy_1 = 0.$

A system S that is associated with a (B) problem cannot be completely integrable, because a linear combination of two equations $df_4 = 0$, $df_2 = 0$ cannot be of class five.

The reduction of a given system S to one of the forms that were just enumerated demands the integration of one or more systems of differential equations and changes of variables.

5. Search for integrals $\mathfrak{M}_2$. Resolvents of the first kind. – The determination of the integrals $\mathfrak{M}_i$ of the system S is simple when the system has been reduced to one of the canonical forms (III), (IV), (V), (VII), (VIII), (IX). For example, in the case of the form (IV), all of the integrals $\mathfrak{M}_2$ are given by a system of four equations:

(I) $u = f(p), \qquad q = f'(p), \qquad z - px - y\, f'(p) = \varphi(p), \qquad x + y\, f'(p) = -\varphi'(p),$

(II) $p = C_1, \qquad u = C_2, \qquad z - C_1\, x = \varphi(y), \qquad q = \varphi'(y),$

(III) $p = C_1, \qquad u = C_2, \qquad z - C_1\, x = C_3, \qquad y = C_4.$

We remark that when S may be reduced to one of the forms (VII), (VIII), (IX), that system admits integrals $\mathfrak{M}_3$. When the system is of class five and has been put into the reduced form (VI), all of the integrals $\mathfrak{M}_2$ are further defined by one of the systems of four relations:

$$(\alpha) \qquad \begin{cases} y_3 = F(y_1), \; y_4 = F'(y_1), \; y_4 = \Phi(y_1), \\ \qquad \Phi'(y_1) - f - y_5 F''(y_1) = 0, \end{cases}$$

$$(\beta) \qquad y_1 = C_1, \qquad y_3 = C_3, \qquad y_4 = \Phi(y_2), \qquad y_5 = \Phi'(y_2),$$

$$(\gamma) \qquad y_1 = C_1, \qquad y_3 = C_3, \qquad y_2 = C_2, \qquad y_4 = C_4.$$

In any case, the system S admits an explicit general integral that is represented by one or more systems of relations between the variables x_i ([1]).

([1]) In certain cases, there might also exist integrals that one calls *singular* that are not given by the application of general formulas. The transformations that permit one to convert the system into a canonical form do not apply to these integrals. This is a generalization of a well-known fact for first-order partial differential equations.

Here are some examples of problems (B) for which the associated system S falls into one of the preceding categories. The four equations $x' = x$, $y' = y$, $p' = -q$, $q' = p$ lead to the Pfaff system $dz = p\,dx + q\,dy$, $dz' = -q\,dx + p\,dy$, which is converted into the canonical form (III):

$$d(z + iz') = (p - iq)\,d(x + iy), \qquad d(z - iz') = (p + iq)\,d(x - iy);$$

this is, in another form, a classical result of the theory of analytic functions.

The (B) problem that is defined by the relations $p' = p$, $q' = q$, $x' = x$, $y' = y + p$ leads to the system in canonical form (IV):

$$dz - p\,dx - q\,dy, \qquad d(z - z') = q\,dp.$$

The solution is given by two developable surfaces with parallel generators that correspond point-by-point, from the given relations.

There also exists an infinitude of problems (B) whose associated system S_6 may be converted into the canonical form (V). Suppose that the equations $F_i = 0$ permit one to express x', y', z', p', q' by means of x, y, z, p, q, and a sixth variable u. If the associated system is reducible to the form (V) then one has an identity of the form:

$$dz' - p'\,dx' - q'\,dy' = \lambda\,dU + \mu\,(dz - p\,dx - q\,dy),$$

in which U, λ, μ are functions of the six variables that may be arbitrary *a priori*. If one adds the equation $U = C$ to the four relations (1), which determines u as a function of x, y, z, p, q, and the constant C then the five functions x', y', z', p', q' of the variables x, y, z, p, q thus obtained satisfy the identity:

$$dz' - p'\,dx' - q'\,dy' = \mu\,(dz - p\,dx - q\,dy);$$

these formulas thus define an infinitude of contact transformations that depend upon an arbitrary constant. One may choose the multiplicity M_2 arbitrarily, and it corresponds to ∞^1 multiplicities M_2'. For example, the (B) problem that is defined by the relations:

$$p' = p, \qquad q' = q, \qquad \frac{x' - x}{p} = \frac{y' - y}{q} = \frac{z' - z}{-1} = u$$

has the canonical system:

$$dz = p\,dx + q\,dy, \qquad d(u\sqrt{1 + p^2 + q^2}) = 0$$

for its associated system; the general property is verified immediately because the preceding formulas express the parallelism of two surfaces.

The four equations:

$$x' = q'\,y - \frac{x+y}{q}, \qquad y' = z - px, \qquad p' = p, \qquad z' = y + p'\,x'$$

have the system S_4:

$$d(z - px) = y\,dp, \qquad q\,dy = (x + y)\,dp$$

for their associated system, whose general integral is represented by a system of just three relations:

$$z = px + f(p), \qquad y = f'(p), \qquad x = q\,f''(p) - f'(p),$$

where the independent variables are p and q. On has, in turn:

$$x' = u\,f'(p) - f''(p), \quad y' = f'(p), \qquad z' = p\,x' + f'(p), \qquad p' = p, \qquad q' = u,$$

where u denotes a new independent variable. The two multiplicities M_2 and M_2' have their point-wise supports on two ruled surfaces whose generators (p = const.) correspond, but one may make the elements of these two multiplicities correspond in an infinitude of ways, because one may choose u to be an arbitrary function of q. This is attached to a general property of problems (B) whose associated systems admit three-dimensional integrals $\mathfrak{M}_3$. The point $(x, y, z, \ldots, q)$ then describes a multiplicity N_3 in ten-dimensional space, but the element (x, y, z, p, q) must generate a multiplicity M_i whose coordinates x, y, z, p, q depend upon at most two independent variables, and for the same reason, x', y', z', p', q' depend upon at most two independent variables. Suppose, to be specific, that these two elements describe two multiplicities M_2, M_2'. x, y, z, p, q are functions of two parameters u, v, and x', y', z', p', q' are functions of two other parameters u', v', but these four parameters are coupled by a relation $f_1 = 0$, since the multiplicity N_3 is three-dimensional. If one establishes another relation of the form $f_2 = 0$ between these four parameters then one establishes a correspondence between the elements M_2 and M_2'.

It finally remains for us to examine the general case of a system S_6 that may be converted into the reduced form (16). Let $\mathfrak{M}_2$ be an integral of this system for which x and y are not related by any relation ([1]). If one takes x and y to be the independent variables then $\mathfrak{M}_2$ is represented by a system of relations:

([1]) If the system S_6 admits integrals $\mathfrak{M}_2$ for which x and y are not independent then the element (x, y, z, p, q) always describes a multiplicity M_2 or a multiplicity M_1. If the element describes a multiplicity M_2 then it will suffice to perform a transformation (T) that will convert it into the general case. If the element (x, y, z, p, q) that is described by a multiplicity M_2 is represented by the formulas:

$$x = f_1(\alpha), \qquad y = f_2(\alpha), \qquad z = f_3(\alpha), \qquad p = \varphi_1(\alpha), \qquad q = \varphi_2(\alpha),$$

α being a variable parameter, then it must be true that the second of equations (14) is verified identically for any u when one replaces x, y, z, p, q by their parametric expressions. The coordinates of a point $\mathfrak{M}_2$ then depend upon the two parameters α and u.

A system S_6 admits an infinitude of integrals of that type when the resolvent E_1 is a Monge-Ampère equation, and the corresponding multiplicities M_1 are the first-order characteristics of E_1.

(19) $$z = f(x, y), \qquad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}, \qquad u = \varphi(x, y).$$

The second of equations (15) proves that u must satisfy two conditions:

(20) $$X + Pr + Qs = 0, \qquad Y + Ps + Qt = 0,$$

in which r, s, t denote the second derivatives of $f(x, y)$. The elimination of u from these two relations leads to a second-order partial differential equation in z:

(21) $$F(x, y, z, p, q, r, s, t) = 0,$$

whose integration will make known all of the integrals $\mathfrak{M}_2$ of the system S_6 for which there is no relation between x and y. That second-order equation does not have an arbitrary form. Indeed, if one regards x, y, z, p, q in equations (20) as having given values and r, s, t as the Cartesian coordinates of a point then these equations represent a line that is parallel to a generator of the cone $rt - s^2 = 0$ that depends upon a parameter u, and the elimination of that parameter leads to an equation that, with the same conventions, represents a ruled surface whose generators are each parallel to a generator in a general variable of the cone $rt - s^2 = 0$. We say, to abbreviate, that equation (20) is a *resolvent of the first kind* of the system S_6 and represent it by E_1.

The equations of this type admit a family of *characteristics of the first kind* ([**26**], chap. IV). Upon eliminating the parameter u from the four equations (15), one obtains two homogeneous relations in dx, dy, dp, dq:

(22) $$\Phi_1(x, y, z, dx, dy, dp, dq) = 0, \qquad \Phi_2(x, y, z, dx, dy, dp, dq) = 0,$$

which, when combined with the equation $dz = p\,dx + q\,dy$, determines a family of first-order characteristics of equation (21).

Any system S_6 may generally be put into the form (16) in two different ways, so one concludes that *the search for integrals $\mathfrak{M}_2$ of the system S_6 may generally be converted in two different ways to the integration of a second-order partial differential equation that admits a family of first-order characteristics.*

In other words, any system S_6 generally possesses two distinct resolvents of the first type E_1, E_1', which are defined only up to a transformation T. There is only one resolvent of the first kind when equation (12) has a double root that corresponds to a singular equation of class 5, or when one of the singular equations is of class 5, while the other one is of class 3. There is no resolvent of the first kind when S_6 may be put into one of the canonical forms (III), (IV), (V).

Let E_1 be the resolvent of the first kind that is represented by equation (21). Other than the first-order system of characteristics (22), that equation admits another system of characteristics that are of second order, in general. Suppose that the system S_6 has been converted into the reduced form (I). Equation E_1 is obtained upon eliminating u from the two equations $r = us + a$, $s = ut + b$ that one might consider to define two functions r and

u of x, y, z, p, q, s, t. The usual rules of differential calculus easily give the following expressions for the partial derivatives:

$$\frac{\partial r}{\partial s} = u + \left(s+\frac{\partial a}{\partial u}\right):\left(t+\frac{\partial b}{\partial u}\right), \qquad \frac{\partial r}{\partial t} = u + \left(s+\frac{\partial a}{\partial u}\right):\left(t+\frac{\partial b}{\partial u}\right).$$

The differential equation in dy / dx that determines the two families of characteristics on an integral surface admits the root $dy / dx = - u$, which agrees with the first-order characteristics, and a second root $dy / dx = \left(s+\frac{da}{du}\right):\left(t+\frac{db}{du}\right)$.

In order for the two families of characteristics to coincide, it is necessary and sufficient that a and b verify a relation that was already obtained:

$$\frac{\partial a}{\partial u} + b - u\frac{\partial b}{\partial u} = 0$$

(page 12), which also expresses the idea that the two families of singular elements of S_6 coincide. Upon preserving the conventions that were already specified, equation E_1 then represents a developable surface whose tangent plane remains parallel to a plane tangent to the cone $rt - s^2 = 0$.

The systems S_6 are not the only ones that possess first-order resolvents. Indeed, we have seen that any system S_6 may be put into the form (14) in an infinitude of ways. If the ratios of the coefficients X,Y, P, Q are not independent of u then the system is generally of class 6, but it might be of class 5. Any system S_6 thus possesses an infinitude of first-order resolvents, but these resolvents for a *special class* that possesses very particular properties. The conditions obtained (pp. 12) that express the idea that the system (14) is of class 5 also express the idea that the corresponding resolvent E_1 has two families of characteristics that coincide, and furthermore, that the equations that determine the intermediate integrals $f(x, y, z, p, q) = C$ of equation E_1 form *a system in involution*. One may explicitly write the general integral of an equation of this class when one has integrated the system that determines the intermediary integrals of E_1, which is indeed in agreement with what was said above for the systems S_5 [**11**, **36**].

In summary, *the only systems* S_i *that possess resolvents of the first kind are the systems* S_6, *which cannot be converted into one of the canonical forms* (III), (IV), (V), *and the systems* S_5 . *A system* S_6 *has at most two resolvents of the first kind, while a system* S_5 *has an infinitude that belong to the special class.*

Conversely, *any second-order partial differential equation E that possesses a family of first-order characteristics is a resolvent of the first kind for a system* S_6 *if it does not belong to the special class, and for a system* S_3 *if it does not belong to the special class.*

Indeed, let:

(23) $$X + Pr + Qs = 0, \qquad Y + Ps + Qt = 0$$

be the equations that represent a rectilinear generator of the surface that is represented by the equation E in the space (r, s, t), where X, Y, P, Q are functions of x, y, z, p, q and a parameter u. The equation E is obtained by eliminating the parameter u from the relations (23); it is therefore a resolvent of the first kind for the system (14), where X, Y, P, Q are the same as in formulas (23). This system is of class 6, at least when E does not belong to the special class, and in the latter case it is of class 5. The equations of a generator of E may be written in an infinitude of ways in the form (23) by changing the parameter u, but the systems S thus obtained are not distinct, and can be converted into each other by a change of variables.

If equation E is a Monge-Ampère equation with two distinct families of characteristics then each of them corresponds to a system S_6 for which E is a resolvent of the first kind. For example, $s = 0$ is a resolvent of the first kind for the two systems ($dz = p\,dx + q\,dy$, $dp = u\,dx$), ($dz = p\,dx + q\,dy$, $dq = u\,dy$).

When a system S_6 possesses two distinct singular equations, one of class 5 and the other of class 3, it has only one resolvent of the first kind E_1, and *that resolvent admits an intermediate integral that depends upon an arbitrary function*. Indeed, suppose that one may deduce an equation of class 3 from the equations (16), namely, $dU = W\,dV$, where U, V, W are functions of x, y, z, p, q, u. For any integral of the system (16), one has two relations of the form $U = F(V)$, $W = F'(V)$, where F may be chosen arbitrarily. The elimination of u leads to a relation between x, y, z, p, q; i.e., to an intermediate integral of the resolvent that depends upon the arbitrary function F.

Conversely, if a second-oder equation E admits an intermediate integral that depends upon an arbitrary function, or, what amounts to the same thing, an intermediate integral that depends upon two arbitrary constants, such as $b = V(x, y, z, p, q, a)$, then that equation may be obtained by eliminating a from the two relations:

$$\frac{\partial V}{\partial x} + p\frac{\partial V}{\partial z} + \frac{\partial V}{\partial p}r + \frac{\partial V}{\partial q}s = 0, \qquad \frac{\partial V}{\partial y} + q\frac{\partial V}{\partial z} + \frac{\partial V}{\partial p}s + \frac{\partial V}{\partial q}t = 0.$$

It is therefore a resolvent of the first kind of the system:

$$dz = p\,dx - q\,dy = 0,$$

$$\left(\frac{\partial V}{\partial x} + p\frac{\partial V}{\partial z}\right)dx + \left(\frac{\partial V}{\partial y} + q\frac{\partial V}{\partial z}\right)dy + \frac{\partial V}{\partial p}dp + \frac{\partial V}{\partial q}dq = 0,$$

in which the six variables x, y, z, p, q, a appear, and one may immediately deduce an equation of class 3, $dV - (\partial V / \partial a)\,da = 0$ from them. This general fact gives the reason for a remark of Clairin [**13**]. Suppose that one deduces a relation that contains only x', y', z', p', q' from the four equations (1); indeed, by performing a suitable transformation (T), one may assume that this relation is $y' = 0$. Upon taking the variables to be x, y, z, p, q, and one of the primed variables, the associated system S is then:

$$dz = p\,dx + q\,dy, \qquad dz' = p'\,dx'.$$

This system therefore admits a singular equation of class 3, and consequently, if it is not reducible to one of the canonical forms (IV) or (V), so there is a resolvent of the first kind that possesses an intermediate integral that depends upon an arbitrary function. We further remark that in a system S_5 one may find an infinitude of equations of class 3, which is completely in agreement with the properties of the resolvents of that system.

6. The B_i transformations. – Let E_1, E_1' be two resolvents of the first kind of a system S_6 . This system may be written in the form (16) with a particular choice of the variables x, y, z, p, q, u, and in an analogous form with another system of variables x', y', z', p', q', u', where the letters $x, y, z, \ldots$ are replaced by the primed letters. The resolvents E_1, E_1' correspond to the two forms in which one may write the system S_6, respectively. *The integrals of the two equations E_1, E_1' correspond to each other in a one-to-one fashion*. Indeed, any integral M_2 of E_1 is contained in one and only one integral $\mathfrak{M}_2$ of S_6, and that integral $\mathfrak{M}_2$ itself contains one and only one integral of E_1'. In a more precise fashion, let $z = f(x, y)$ be an integral of E_1 ; one has:

$$p = \frac{df}{dx}, \qquad q = \frac{df}{dy},$$

and u is given by the two compatible equations (20). Since the variables $x', y', \ldots$ are expressed by means of the first ones, the formulas that give x', y', z', p', q' by means of two independent variables x and y define an integral of E_1'. In the same fashion, one may deduce one and only one integral of E_1 from any integral of E_1'. With the classification of Clairin [**13**], we say that one passes from one of the two equations E_1 , E_1' to the other by a *Bäcklund transformation B_1* .

The elimination of the parameter u from the five equations that permit one to express x', y', z', p', q' in terms of x, y, z, p, q, u will lead to a system of four relations between the coordinates of the two contact elements. Conversely, being given a system of four relations $F_i = 0$ between the coordinates of two elements, we seek the cases in which these relations define a B_1 transformation. We always set aside the case in which one may deduce an equation that contains only the coordinates of one of the elements from the relations $F_i = 0$. Indeed, we have remarked that the associated system cannot admit two resolvents of the first kind if it is of class 6 ([1]). Upon taking the variables to be x, y, z, p, q, and a sixth variable u that is distinct from them, in such a fashion that x, y, z, p, q are expressed by means of the x', y', z', p', q', u', the associated Pfaff system to the (B) problem that is defined by the relations $F_i = 0$ is:

$$dz = p\,dx + q\,dy, \qquad dz' = p'\,dx' + q'\,dy'. \tag{24}$$

In order for the elimination of the variable u to lead to a resolvent of the first kind for the determination of z as a function of x and y, it is necessary and sufficient that the

([1]) Any system S_6 may be converted into the form (16) in an infinitude of ways, so it possesses an infinitude of resolvents of the first kind that are of the special class that was defined above. Cartan has proved that all of these resolvents may be deduced from each other by contact transformations [**11**].

second of equations (24) does not refer to du, and contains only the differentials dx, dy, dz, dp, dq. If that is true then the relation $dz' = p'\,dx' + q'\,dy'$ is a consequence of the relations $x = x_0$, $y = y_0$, $z = z_0$, $p = p_0$, $q = q_0$, and consequently the relations $F_i = 0$ make an arbitrary element $(x_0, y_0, z_0, p_0, q_0)$ correspond to a multiplicity M_1' of elements (x', y', z', p', q'). One likewise verifies that the elimination of the unprimed variables will lead to the second resolvent of the first kind E_1' of the system S if the equations $F_i = 0$ make an arbitrary element $(x_0', y_0', z_0', p_0', q_0')$ correspond to a multiplicity M_1 of elements (x, y, z, p, q). Therefore, *in order for the relations $F_i = 0$ to define a B_1 transformation, it is necessary and sufficient that an arbitrary elements of each family correspond to ∞^1 elements of the other family that form a one-dimensional multiplicity M.*

J. Clairin, to whom one attributes this interpretation [**13**], has pointed out a very broad case in which these conditions are verified. Let:

$$\varphi_i(x, y, z, p, q) = C_i, \quad \varphi_i'(x', y', z', p', q') = C_i' \qquad (i = 1, 2, 3, 4)$$

be the equations of two families of multiplicities M_1, M_1' that depend upon four parameters C_i or C_i', respectively. It is obvious that the equations $\varphi_i = \varphi_i'$ indeed satisfy the conditions of the statement, and consequently define a transformation B_1 .

In order for the four equations:

$$x' = x, \;\; y' = y, \;\; F_1(x, y, z', p, q, p', q') = 0, \qquad F_2(x, \ldots) = 0$$

define B_1 transformation, it is necessary and sufficient that $dz' = 0$ be a consequence of $dx = 0$, $dy = 0$, …, $dq = 0$, and likewise that $dz = 0$ be a consequence of $dx' = 0$, …, $dq' = 0$. One may thus conclude values of z and z' from the two equations $F_1 = 0$, $F_2 = 0$ such that $z' = f_1(x, y, z, p, q)$, $z = f_2(x', y', z', p', q')$. The converse is obvious. In the particular case where f_2 is deduced from f_1 by putting primes on the letters z, p, q, it is clear that E_1' is deduced from E_1 by putting primes on the letters z, p, q, r, s, t. The B_1 transformation thus permits one to deduce a new integral of E_1 from an integral of that equation.

Any second-order equation to which one may apply a transformation B_1 necessarily possesses a family of first-order characteristics. This condition is, in general, sufficient. Indeed, suppose that this equation possesses two distinct families of characteristics of first order and one of second order, and that it does not admit an intermediate integral that depends upon two arbitrary constants. This equation is then a resolvent of the first kind of a system S_6 that is completely determined and which possesses a second resolvent of the first kind E_1' that is itself completely determined up to a transformation (T). *From the given equation E, one may thus deduce another equation of second order E', and only one, up to a contact transformation, by a B_1 transformation* ([1]) (**13**. **30**). At the same time, the proof shows that this is the path to follow if one is to obtain this transformation. The second singular equation of S_6 is determined by linear calculations, and one must then convert this Pfaff equation to a canonical form. The latter problem indeed admits an

([1]) This result was obtained for the first time by J. Clairin [**15**] by a totally different method.

infinitude of solutions, but the second-order equations to which one is led can be deduced from each other by transformations (T).

The conclusion is incorrect if the two families of characteristics coincide, at least if the equation E does not belong to the special class (*see* the note on page 20). It is also incorrect if, the two families of characteristics being distinct – one of first order, one of second order – the equation E admits an intermediate integral that depends upon two arbitrary constants. Finally, *if the equation E is a Monge-Ampère equation that has two distinct families of characteristics then the equation E provides two distinct B_1 transformations, provided that it does not admit an intermediate integral that depends upon an arbitrary function for any system of characteristics.*

Let $z = f(x, y)$ be an integral of E_1 and let $z' = \varphi(x', y')$ be the integral of E_1' that it corresponds to by the B_1 transformation. *The characteristics correspond to these two integrals.* Indeed, let M_1 be a characteristic of the first integral – i.e., a multiplicity of ∞^1 first-order elements that also belongs to an infinitude of other integrals of E_1. In particular, there exist an infinitude of E_1 that have second-order contact with the first one at each element of M_1. Along M_1, x, y, z, p, q, r, s, t have the same values for all of these surfaces and are functions of one parameter α. The corresponding elements (x', y', z', p', q'), which are expressed by means of x, y, z, p, q, r, s, t, thus generates a multiplicity M_1' that belongs to an infinitude of integrals of E_1'. The point-wise support of M_1' is therefore a characteristic curve that is common to all of these integrals.

In order to specify the correspondence between the two families of characteristics, we remark that the equation E_1 that is from the reduced form (16) has a first system of first-order characteristics C_1 that are defined by the relations (15), and a second system of characteristics C_2 that are of second order, in general. To abbreviate, we say that the B_1 transformation by which one passes from E_1 to E_1' is deduced from the family C_1 of characteristics. The equation E_1' likewise admits a family of first-order characteristics C_1' whose B_1 transformation is deduced, relative to that equation, and a second family of characteristics C_2' that is of second order, in general. Now, the two families of characteristics C_1 and C_1' belong to two distinct families of singular elements of S_6. These two systems of characteristics cannot therefore correspond, and consequently, *the characteristics C_1' and C_2' of E_1' correspond to the characteristics C_2 and C_1 of E_1, respectively* (**13**).

Examples.

1. A second-order equation $s = f(x, y, z, p, q)$ is a resolvent of the first kind E_1 for the system S_6 :

(25) $$\omega_1 = dz - p\,dx - q\,dy = 0, \qquad \omega_2 = dp - u\,dx - f\,dy = 0.$$

Upon applying the general search method for singular elements (no. 3), one finds two families that are defined by the following equations:

$$dy = 0, \qquad dq = f\,dx, \qquad dz = p\,dx, \qquad dp = u\,dx,$$

$$dx = 0, \qquad dz = q\,dy, \qquad dp = f\,dy, \qquad du = \left[\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + u\frac{\partial f}{\partial p} + f\,\frac{\partial f}{\partial q}\right] dy,$$

in which the first one gives the singular equation $\omega_1 = 0$, while the second one provides the second singular equation:

$$\omega_3 = \omega_2 - \frac{\partial f}{\partial q}\,\omega_1 = dp - \frac{\partial f}{\partial q}\,dz - \left(u - p\frac{\partial f}{\partial p}\right)dx - \left(f - q\frac{\partial f}{\partial q}\right)dy = 0,$$

which must be converted into a canonical form in order to deduce the second resolvent of the first kind of the system (25).

If f does not contain q then the second singular equation is $\omega_2 = 0$, and it is has a canonical form. The (B) problem that leads to the system (25) belongs to a category that we have already pointed out.

In the case of the Laplace equation, one has $f = -ap - bq - cz$, where a, b, c are functions of x and y, so the equation $\omega_3 = 0$ is easily put into a canonical form:

$$d(p + bz) - \left[z\frac{\partial b}{\partial y} + ap - cz\right]dy - \left[z\frac{\partial b}{\partial y} - bp\right]dx = 0.$$

In order to achieve this calculation, it is sufficient to observe that the Laplace equation provided by (B) problem is defined by the formulas:

$$x' = x, \qquad y' = y, \qquad z' = p + bz, \qquad q' = z\,\frac{\partial b}{\partial y} + ap - cz,$$

from which, one conversely infers, if $\frac{\partial b}{\partial y} + ab - c = k$ is not zero, that:

$$z = \frac{q' + az'}{k}, \qquad p = z' - \frac{bq' + abz'}{k}.$$

The elimination of z leads to a new linear equation of the same form, and the B_1 transformation is identical to the Laplace transformation [**22**, **26′**].

2. The equation of Gomes Teixeira [**44**]:

$$s - A(x, y, z, p)\,q - B(x, y, p, r) = 0$$

is a resolvent of the first kind for the system:

$$dz - p\,dx - q\,dy = 0, \qquad dp - u\,dx - (Aq + B)\,dy = 0,$$

in which r is replaced by u in B. The second singular equation of the system is:

$$dp = A\,dz - (u - Ap)\,dx - B\,dy = 0.$$

In order to convert this equation into a canonical form, it suffices to find an integrating factor m for the Pfaff expression $dp - A\,dz$, where x and y are regarded as parameters. The product $\mu\,(dp - A\,dz)$ is indeed of the form $d\varphi - \frac{\partial\varphi}{\partial x}dx - \frac{\partial\varphi}{\partial y}dy$, and the preceding equation is then put into canonical form.

The calculations are easily carried out. Upon assuming that B is independent of r, one has the Imschenetsky transformation [**37**]. This example was further generalized by J. Clairin [**13**].

By starting with a Laplace equation, one may generally repeat the B_1 transformation indefinitely in the two senses of application; the operation terminates in one sense only if one arrives at a Laplace equation that admits an intermediate integral with an arbitrary function. It is clear that the same property belongs to any Monge-Ampère equation that reduces to a Laplace equation by a transformation (T). It will be interesting to examine whether these are the only ones that possess this property, and, more generally, form all of the Monge-Ampère equations that give another Monge-Ampère equation under a B_1 transformation.

This problem was studied by J. Clairin [**19**] by assuming that the sequence of B_1 transformations preserves the independent variables. With this restrictive condition, it is proved that any second-order equation that one may deduce, by a sequence of B_1 transformations of this type, more than three consecutive equations, can be converted into a Laplace equation by a transformation (T), or to one of the equations that were studied by Moutard [**41**], which also reduce to Laplace equations.

7. Resolvents of the second kind. – The integration of a system of class 6 may, in certain cases, be converted into the integration of one second-order equation in another fashion. Let $\omega_1 = 0$ be a non-singular equation of that system; it must necessarily be of class 5. If one has put it into a canonical form $dz = p\,dx + q\,dy$ then an equation of the system S that is distinct from it contains the differential du of the sixth variable u, since otherwise $\omega_1 = 0$ will be a singular equation of S (no. 4). This system is thus composed of two equations:

$$\left\{\begin{aligned} \omega_1 &= dz - p\,dx - q\,dy = 0, \\ \omega_2 &= du - Xdx - Ydy - Pdp - Qdq = 0, \end{aligned}\right. \tag{26}$$

X, Y, P, Q being functions of x, y, z, p, q, u. Any system of class 6 may be converted into the form (26) in an infinitude of ways, and we have remarked above (page 13) that any system of that form is of class 6.

Let $\mathfrak{M}_2$ be a integral of this system, such that x and y are not coupled by any relation ([1]). If one takes x or y to be independent variables then this multiplicity $\mathfrak{M}_2$ is represented by a system of four equations:

$$(27)\qquad z = f(x, y), \qquad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}, \qquad u = \varphi(x, y).$$

Upon replacing dp with $r\,dx + s\,dy$ and dq with $s\,dx + t\,dy$ in the second equation (26), it becomes:

$$(28)\qquad du = (X + P\,r + Qs)\,dx + (Y + Ps + Qt)\,dy.$$

By developing the integrability condition for this equation, one obtains a linear equation in $r, s, t, rt - s^2$:

$$(29)\qquad Hr + 2\,K\,s + L\,t\,M + N(rt - s^2) = 0,$$

when the coefficients H, K, L, M, N have the following values:

$$(30)\qquad \begin{cases} H = \dfrac{dP}{dy} - \dfrac{\partial Y}{\partial p} + Y\dfrac{\partial P}{\partial u} - P\dfrac{\partial Y}{\partial u}, \qquad L = \dfrac{\partial X}{\partial q} - \dfrac{dQ}{dx} + Q\dfrac{\partial X}{\partial u} - X\dfrac{\partial Q}{\partial u}, \\ 2K = \dfrac{\partial X}{\partial p} + P\dfrac{\partial X}{\partial u} + \dfrac{dQ}{dy} + Y\dfrac{\partial Q}{\partial u} - Q\dfrac{\partial Y}{\partial u} - \dfrac{\partial P}{\partial u} - X\dfrac{\partial P}{\partial u}, \\ M = \dfrac{dQ}{dy} - \dfrac{\partial Y}{\partial x} + Y\dfrac{\partial X}{\partial u} - X\dfrac{\partial Y}{\partial u}, \qquad N = \dfrac{\partial P}{\partial q} - \dfrac{\partial Q}{\partial p} + Q\dfrac{\partial P}{\partial u} - P\dfrac{\partial Q}{\partial u}. \end{cases}$$

In general, the ratios of these coefficients depend upon u, and equation (29) determines u as a function of x, y, z, p, q, r, s, t. Upon writing down that the function thus obtained is an integral of equation (28), one obtains two third-order partial differential equations that determine the function $f(x, y)$. These equations are not arbitrary, moreover, since we know *a priori* that they admit an infinitude of common integrals.

For certain functions X, Y, P, Q, it might happen that the ratios of the coefficients (30) do not depend upon u. Equation (29) is then a Monge-Ampère that determines the function $f(x, y)$. Any integral of that equation corresponds to an infinitude of integrals $\mathfrak{M}_2$ of the system S that depend upon an arbitrary constant, because u is determined by a completely integrable total differential equation.

In this case, equation (29) is a *resolvent of the second kind* E_2 of the system S, and it results from the form itself of system (26) that the systems that admit a resolvent of the second kind are of class 6. A system of class 6 might admit several resolvents of the

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([1]) If $\mathfrak{M}_2$ is an integral of S_6 on which x and y cannot be taken to be independent variables then the element (x, y, z, p, q) describes a multiplicity M_2 or M_1. In the first case, it is sufficient that a transformation (T) converted it into the case that was studied in the text. The second case must be rejected; indeed, if x, y, z, p, q are functions of one variable α then the second equation $\omega_2 = 0$ becomes $du = F(\alpha, u)\,d\alpha$, and in turn, u will also be a function of just the variable α.

second kind, and we remark that a system S_6 might have resolvents of the second kind without having resolvents of the first kind. For example, the canonical system:

$$dz = p\,dx + q\,dy, \qquad du = -\,q\,dx + p\,dy$$

admits the resolvent of the second kind $r + t = 0$. Likewise, the canonical system (IV) admits the resolvent $rt - s^2 = 0$.

When equation (29) is a resolvent E_2 of the system (26), each family of singular elements of S corresponds to a family of characteristics of E_2. In order to prove this, we may suppose that equation (29) contains a term in $rt - s^2$, since it suffices for it to be converted into this case by a transformation (T). In order for two integral linear elements (dx, dy, dp, dq), (δx, δy, δp, δq) of the system (26) to be in involution; these elements must verify the two relations:

$$dp\,\delta x + dq\,\delta y - dx\,\delta p - dy\,\delta q = 0,$$

$$\begin{aligned} L(dx\,\delta q - dq\,\delta x) + 2K(dx\,\delta p - dp\,\delta x) + M(dx\,\delta y - dy\,\delta x) \\ + H(dp\,\delta y - dy\,\delta p) + N(dp\,\delta q - dq\,\delta p) = 0. \end{aligned}$$

In order for an element (dx, dy, dp, dq) to be a singular element, it is necessary and sufficient that the coefficients of δx, δy, δp, δq in the two preceding equations be proportional. Upon writing down these conditions, one finds that dx, dy, dp, dq must satisfy one of the following relations:

$$(31)^1 \qquad N\,dp + L\,dx + \lambda_1\,dy = 0, \qquad N\,dq + \lambda_2\,dx + H\,dy = 0,$$
$$(31)^2 \qquad N\,dp + L\,dx + \lambda_2\,dy = 0, \qquad N\,dq + \lambda_1\,dx + H\,dy = 0,$$

λ_1 and λ_2 being the two roots of the equation:

$$(32) \qquad \lambda_2 + 2K\,\lambda + HL - MN = 0.$$

One obtains the equations that define a family of singular elements of the system by adjoining equations (26) to one of the systems (31). Now, upon adjoining only the first of equations (26) to one of the systems $(31)^1$ or $(31)^2$, one obtains the equations that define a family of characteristics of E_2, which proves the stated theorem.

Any characteristic M_1 of E_2 is formed of ∞^1 elements (x, y, z, p, q) that verify one of these systems. Upon replacing x, y, z, p, q with their expressions in terms of a parameter variable in the last of equations (26), one obtains a first-order differential equation to determine u, and consequently any first-order characteristic of E_2 belongs to ∞^1 Monge characteristics of S (no. 3), and conversely any Monge characteristic of S contains a characteristic of E_2. It also results from that study that if the characteristic equations of E_2 admit i integrable combinations (i = 1, 2, 3) then the differential equations of the corresponding system of singular elements of S admit at least i integrable combinations. If the two families of singular elements for a system S_6 coincide then any resolvent E_2 of this system also has two families of characteristics that coincide, and conversely.

The integrability condition (29) generally contains a term in $rt - s^2$. In order for this term to not exist, it is necessary that one have $N = 0$, a condition that expresses the idea that the equation $du = P\,dp + Q\,dq$, where one regards x, y, z as parameters, is completely integrable. Let $U(x, y, z, p, q) =$ const. be one of the forms into which one may put the general integral of that equation. The function U verifies the two relations:

$$\frac{\partial U}{\partial p} + P\frac{\partial U}{\partial u} = 0, \qquad \frac{\partial U}{\partial q} + Q\frac{\partial U}{\partial u} = 0,$$

and, if one takes $U(x, y, z, p, q, u)$ to be the variable in place of u then the second of equations (26) is replaced by:

$$dU = \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial z}p\right)dx + \left(\frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}q\right)dx + \frac{\partial U}{\partial u}(X\ dx + Y\,dy),$$

and the system takes the form:

$$(26) \qquad dz = p\,dx + q\,dy, \qquad du = f(x, y, z, p, q, u)\,dx + \varphi(x, y, z, p, q, u)\,dy,$$

the variable u no longer being the same as in the system (26). Conversely, for any functions f and φ, it is clear that the integrability condition of system (26)′ does not refer to the term in $rt - s^2$.

In particular, when equation (29) is independent of u, one may, by a transformation (T), convert it into one that does not refer to $rt - s^2$. Any resolvent of the second kind E_2 of a system S_6 that is linear in r, s, t is therefore identical to the integrability condition of an equation of the form (26)′, where one must have, moreover, that f and φ are such that this condition does not depend upon u.

Condition (29) refers to neither r nor t if s is independent of q and φ is independent of p. The system (26) takes the form:

$$(26)'' \qquad dz = p\,dx + q\,dy, \qquad du = f(x, y, z, p, u)\,dx + \varphi(x, y, z, p, u)\,dy.$$

The integrability condition is then:

$$(33) \qquad \left(\frac{\partial f}{\partial p} - \frac{\partial \varphi}{\partial q}\right)s = \frac{d\varphi}{dx} - \frac{df}{dy} + \frac{\partial \varphi}{\partial u}f - \frac{\partial f}{\partial u}\varphi\ .$$

This integrability condition contains no second-order derivative if one has:

$$f = A(x, y, z, u) + C(x, y, z, u)\,p, \qquad \varphi = B(x, y, z, u) + C(x, y, z, u)\,q.$$

The system (26) then contains an equation:

$$du = A\,dx + B\,dy + C\,dz$$

which refers to only x, y, z, u, and which is, consequently, of third class. If that integrability condition does not refer to u, but contains at least one of the derivatives p, q, then one has a resolvent of the second kind and first order. One may suppose that one has converted it into the form $p = 0$ by a transformation (T). The coefficients A, B, C must satisfy the two conditions:

$$\frac{\partial A}{\partial y} + \frac{\partial A}{\partial u} B = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial u} A, \qquad \frac{\partial A}{\partial z} + C \frac{\partial A}{\partial u} = \frac{\partial C}{\partial x} + A \frac{\partial C}{\partial u},$$

the first of which expresses the idea that the equation $du = A\,dx + B\,dy$, where one regards z as a parameter, is completely integrable. One sees, as we will quite soon, that by a change of the variable u the system (26) is converted into the form:

(34) $$dz = p\,dx + q\,dy, \qquad du = \varphi(x, y, z)\,dz,$$

which is reducible to the canonical form (IV), because the two families of singular elements coincide, and admit the four integrable combination $dx = 0$, $dy = 0$, $du = 0$ ([1]).

One sees, in the same fashion, that if the condition (29) does not refer to any derivative of z then the second equation may be converted into the form:

$$du = \varphi(x, y, u)\,dx.$$

Finally, it might happen that the integrability condition (29) is verified identically. In order for this to be true, it is necessary that the functions A, B, C verify three conditions that express the idea that the equation:

$$du = A\,dx + B\,dy + C dz$$

is completely integrable, and in this case, it suffices to replace u with a function $U(x, y, z, u)$ in order to convert the system (26) into the canonical form:

$$dz - p\,dx - q\,dy = 0, \quad dU = 0.$$

We recall a particular case that was examined above (pp. 15).

The problems that relate to the resolvents of the second kind are generally much more difficult than the analogous problems that concern the resolvents of the first kind. The principal questions that one poses are the following two:

([1]) The integrals that satisfy the relation $p = 0$ are *singular* integrals (*see* the note on page 9), and these integrals depend upon only an arbitrary function $z = f(y)$, u being given by the integration of a differential equation $du = \varphi[y, f(y), u], f'(y)\,dy$. The *general* integral is given by the formulas:

$$y = f(x), \qquad z = g(x), \qquad p = g'(x) - q\,f'(x),$$

u being determined by the differential equation:

$$du = \varphi[f(x), g(x), u]\,g'(x)\,dx.$$

1. *Being given a system S_6, find the resolvents of the second kind of the system, if they exist.*

2. *Being given a Monge-Ampère equation, find the systems S_6 for which it is a resolvent of the second kind.*

The results indicated above (pp. 25-26) permit us to state a *necessary* condition for a system S_6 to admit a resolvent of the second kind. Indeed, we have seen that in this case the differential equations that define the singular elements of each family involve three distinct equations in which only five variables appear.

Therefore: *In order for a system S_6 to admit a resolvent of the second kind, it is necessary that one may deduce three equations that form a system of class five from the four differential equations that define the singular elements of each family.*

This is a particular case of a very general problem that relates to Pfaff systems that does not seem to have been studied up to the present. We will confirm later on (no. 10) that there are systems S_6 that admit an infinitude of resolvents of the second kind.

In order to study the converse problem, one may limit oneself to the case of a Monge-Ampère equation E that is linear in r, s, t.

In order for E to be a resolvent of the second kind of a system S_6, it is necessary and sufficient that one may find two functions:

$$f(x, y, z, p, q, u), \qquad \varphi(x, y, z, p, q, u),$$

such that E is identical to the integrability condition of the equation $du = f\,dx + \varphi\,dy$. *This is not always possible.* For example, if E does not refer to the second-order derivative r then f must be independent of q and φ must be linear in q, and in this case *the integrability condition is bilinear in r and q.* On the other hand, a Monge-Ampère equation may a resolvent of the second kind for distinct systems S_6 .

Therefore, the canonical system (IV) admits a resolvent that one can convert into the form $s = 0$ (pp. 26). This equation is also a resolvent of the second kind for the system $dz = p\,dx + q\,dy$, $dz' = z\,dx + x\,q\,dy$ that is distinct from the first, since it admits the resolvent of the first kind $xs' - q' = 0$.

One has, above all, studied the systems S_6 that admit a resolvent E_2 that refers to only the second-order derivative s. J. Clairin [**14**, **17**, **18**, **20′**] has determined the systems that admit a resolvent of the first kind and a resolvent of the second kind of that form, with the same variables x and y, when one of these resolvents of a Laplace equation.

One has also determined [**31**, **33**] the systems S_6 that admit a resolvent of the second kind $s = \rho\,pq + ap + bq + c$, where a, b, c, ρ are functions of x, y, z.

If a system S_6 admits a resolvent E_2 that is reducible to the form $r = 0$ by a transformation T then this system may be converted into the canonical form (IV). Indeed, the two families of singular elements must coincide, and their differential equations admit at least *three* integrable combinations (pp. 26). Now, we have seen (no. 4) that if a system S_6 admits a resolvent E_1 then the differential equations of the singular elements cannot admit more than two integrable combinations. The system:

$$dz = p\,dx + q\,dy, \qquad dz' - \lambda\,dz = (p - \lambda q)^k\,(\lambda\,dx + dy),$$

belongs to that category, where λ and k are constants that were encountered by E. Picard [**42**, **43**] in the context of a question on partial differential equations.

8. The B_2 and B_3 transformations. – Let E_1 and E_2 be two resolvents of a system S_6, the one, of the first kind, and the other, of the second kind. An integral I_1 of E_1 corresponds to one and only one integral $\mathfrak{M}_2$ of S_6 (no. 5), and in turn, one and only one integral I_2 of E_2. Conversely, an integral I_2 of E_2 belongs to ∞^1 integrals $\mathfrak{M}_2$ of S_6, and in turn, one may deduce integrals of E_1 from it. The transformation by which one passes from E_1 to E_2, or vice versa, is a B_2 *transformation* (J. Clairin, [**13**]). One sees that the two equations E_1, E_2 do not play the same role in this transformation. If one may pass from an equation E_1 to an equation E_2 by a B_2 transformation then it is clear that the same is true for the equations that one may deduce from it by arbitrary (T) transformations.

Likewise, let E_2, E_2' be two resolvents of the second kind of a system S_6. Each integral of the one of the equations corresponds to ∞^1 integrals $\mathfrak{M}_2$ of S_6, and in turn, ∞^1 integrals of the second equation, and conversely. The transformation by which one passes from E_2 to E_2', or vice versa, is a B_3 *transformation* (**13**); the two equations play a symmetric role in this transformation. One proves, as in no. 6, that if two second-order equations can be deduced from each other by a B_2 or B_3 transformation then their characteristics correspond in the corresponding integrals of the two equations. If one of them is integrable by the method of Darboux then the same is true for the second one [**13**, **22**].

Let E_1 be a resolvent of the first kind of the system S_6, where E_2, E_2' are two resolvents of the second kind. The B_3 transformation by which one passes from E_2 to E_2' may obviously be replaced by the sequence of two transformations B_2, B_2' by which one passes from E_2 to E_1 and then from E_1 to E_2'. Since S_6 generally admits two resolvents of the first kind, one sees that *any B_3 transformation may, in general, be decomposed into a sequence of two B_2 transformations in two different fashions.* At the same time, the argument shows what the exceptional cases are.

When two B_2 transformations are applied to an equation that admits only one system of first-order characteristics, this leads to two resolvents of the second kind of the same system S_6, and consequently, may be replaced by a unique B_3 transformation.

When the two B_2 transformations are applied to a Monge-Ampère equation, this might lead to two equations E_2 that are resolvents of the second kind of the two distinct systems S_6, S_6'. It might happen that one cannot pass from E_2 to E_2' by a B_3 transformation; later on (no. 9), we shall discuss a case in which one passes from E_2 to E_2' by a B_1 transformation.

A sequence of two B_3 transformations may also sometimes be replaced by a unique transformation of the same kind. Let E_2, E_2', E_2'' be three resolvents of the second kind S_6. The B_3'' transformation by which one passes from E_2 to E_2'' may obviously be obtained by the succession of B_3 and B_3' transformations by which one passes from E_2 to E_2', and then from E_2' to E_2''. This is no longer true if E_2 and E_2' are resolvents of S_6, while E_2' and E_2'' are resolvents of a different system S_6'. The two equations E_2 and E_2'' are not necessarily resolvents of the same system.

The importance of resolvents of the second kind in the search for integrals $\mathfrak{M}_2$ of a system S_6 amounts to the following property, whose proof is immediate: *If one knows one resolvent of the second type E_2 of a system S_6 then one may deduce ∞^1 integrals $\mathfrak{M}_2$ of the system from any integral $\mathfrak{M}_2$ of that same system by the integration of a first-order differential equation.*

Indeed, let $\mathfrak{M}_2$ be an integral that is represented by the equations:

$$z = f(x, y), \qquad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}, \qquad u = \varphi(x, y)$$

of a system S_6 that has been put into the form (26), where the integrability condition of the second equation does not depend upon u. This integrability condition is a resolvent of the second kind E_2 of S_6, where $f(x, y)$ is a particular integral. That integral $f(x, y)$ of E_2 corresponds to an infinitude of functions $\varphi(x, y)$ that one obtains by the integration of a completely integrable total differential equation for which one already knows a particular integral. If one knows only one integral of E_2 then that integral belongs to ∞^1 integrals $\mathfrak{M}_2$ of S_6 that one determines by the integration of the same total differential equation, no integral of which is assumed to be known.

Now suppose that one knows two resolvents of the second kind E_2, E_2' of S_6. One may deduce ∞^1 integrals of S_6 from an integral I_2 of E_2 by the integration of a differential equation, and in turn, ∞^1 integrals of E_2'. One may then deduce ∞^1 integrals of S_6 and E_2 from each of these new integrals I_2' by the same process, and since these integrals I_1' themselves depend upon an arbitrary constant, one will thus have ∞^2 integrals of S_6, and in turn, ∞^2 integrals of E_2 . This alternating process may obviously be continued indefinitely, and one imagines that its application might lead, upon starting with just one integral of S_6, to an infinitude of integrals of the same system that depend upon as many arbitrary constants as one desires (*see* no. 10). However, it might also happen that the application of this method permits one to obtain only integrals that depend upon a definite number of arbitrary constants, no matter how far one prolongs it (no. 9).

All of these remarks are naturally extended to the case in which one knows more than two resolvents of the second kind.

Remark. – Being given a system of four equations $F_i = 0$ that may be solved for the x', y', z', p', q', we have seen above that that the elimination of the primed variables leads to a resolvent of the second kind E_2 if the integrability condition of the equation $dz' = p'\,dx' + q'\,dy'$ is independent of z'. This integrability condition is expressed by means of partial derivatives of x', y', z', p', q' with respect to x, y, z, p, q, z', derivatives that one always calculates by means of the classical rules that give the derivatives of implicit functions.

One may arrive at this integrability condition by a more elegant process ([**22**], chap. XII). From equations (1), one infers the relations:

$$\left(\frac{dF_i}{dx}\right)dx + \left(\frac{dF_i}{dy}\right)dy + \frac{dF_i}{dx'}dx' + \frac{dF_i}{dy'}dx' + \frac{dF_i}{dp'}dp' + \frac{dF_i}{dq'}dq' = 0, \qquad (i = 1, 2, 3, 4),$$

where one has set:

$$\left(\frac{dF_i}{dx}\right) = \frac{\partial F_i}{\partial x} + \frac{\partial F_i}{\partial z}p + \frac{\partial F_i}{\partial p}r + \frac{\partial F_i}{\partial q}s, \qquad \left(\frac{dF_i}{dy}\right) = \frac{\partial F_i}{\partial y} + \cdots + \frac{\partial F_i}{\partial q}t.$$

These four equations, when solved for dx, dy, dp', dq', give expressions of the following form:

$$N\,dp' = H\,dx' + K\,dy', \qquad N\,dq' = L\,dx' + M\,dy',$$

where H, K, L, M, N are linear functions of r, s, t, $rt - s^2$. In order for p' and q' to be partial derivatives of the same function with respect to x' and y', one must have $K = L$.

Upon carrying out the calculations, one arrives at the following condition, which was given by Bäcklund [**2**]:

$$\begin{aligned}(35)\qquad &(12)[F_3\,F_4] + (13)[F_4\,F_2] + (14)[F_2\,F_3] \\ &+ (34)[F_1\,F_2] + (42)[F_1\,F_3] + (23)[F_1\,F_4] = 0,\end{aligned}$$

where one has set:

$$(ik) = \left(\frac{dF_i}{dx}\right)\left(\frac{dF_k}{dy}\right) - \left(\frac{dF_k}{dx}\right)\left(\frac{dF_i}{dy}\right),$$

and where the bracket [ ] has its usual sense.

If the five equations (1) and (35) may be solved with respect to x', y', z', p', q', upon writing that the expressions obtained satisfy the relation $dz' = p'\,dx' + q'\,dy'$, then one is led to two third-order equations in z. If the elimination of x', y', z', p', q' from these five equations is possible then z is determined by a Monge-Ampère equation, which is a resolvent of the second kind.

9. Systems S_6 that admit a continuous group. – Let S_6 be a system that admits a continuous, one-parameter group of transformations g that are derived from an infinitesimal transformation ε. Choose the variables x_i in such a fashion that the symbol of that infinitesimal transformation is $\partial f / \partial x_i$. With this choice of variables the system S_6 is written:

$$(36)\qquad \omega_1 = dx_1 + \Omega_1 = 0, \qquad \omega_2 = \Omega_2 = 0,$$

Ω_1 and Ω_2 being two Pfaff forms in which only the five variables x_i ($i < 6$) appear, along with their differentials. In order to determine the singular elements, one must write down the idea that for certain values of the dx_i the equations $\omega_1' = \Omega_1' = 0$, $\omega_2' = \Omega_2' = 0$ reduce to just one. The coefficients A_{ik} of equations (8) and (9) do not depend upon x_6, and as a result the two roots of the equation in λ / μ are also independent of x_6. One may leave aside the case of a double root $\lambda = 0$, because the only singular equation of the system will be $\Omega_2 = 0$, and it cannot be of class five. The system will then be reducible to the canonical form (IV). If one sets aside this very special case then one sees that the system S_6 admits at least one singular equation of the form $dx_6 + \Omega_3 = 0$, where Ω_3 does not depend upon x_6. There is at least one of these singular equations for which Ω_3 is of class five or four; in other words, the system S_6 will be reducible to one of the canonical forms (III) or (IV). If one converts Ω_3 into a canonical form then the system S_6 will be written:

$$dx_6 + dy_5 - y_2\, dy_1 - y_4\, dy_2 = 0, \qquad \Omega_2 = 0,$$

in which Ω_2 does not contain x_6. In order for the first equation to be a singular equation, one sees, as in no. 4, that Ω_2 must not refer to dy_5. It will thus suffice to make a simple change of notations in order to be able to write the system S_6 in the form:

$$dz - p\, dx - q\, dy = 0, \qquad X\, dx + Y\, dy + P\, dp + Q\, dq = 0, \tag{37}$$

in which X, Y, P, Q *do not depend upon* z. The corresponding resolvent of the first kind will no longer depend upon z; it thus admits the infinitesimal transformation $\partial f / \partial x_i$.

Conversely, any equation that has a system of first-order characteristics and admits an infinitesimal transformation (T) is a resolvent of the first kind for a system S_6 that admits an infinitesimal transformation. Indeed, if one supposes that E_1 does not contain z then the equations of the generators of the surface in r, s, t (no. 5):

$$X + P\, r + Q\, s = 0, \qquad Y + P\, s + Q\, t = 0$$

no longer depend upon z, and the Pfaff system that admits E_1 as its resolvent of the first kind does not change when one changes z into $z + C$. Therefore, *when a system* S_6 *admits an infinitesimal transformation, a resolvent of the first kind of that system admits an infinitesimal contact transformation* (T), *and conversely*.

From any infinitesimal transformation of S_6, one may likewise deduce a resolvent of the second kind of that system. Suppose that the second of equations (37) is put into the canonical form $dz' = p'\, dx' + q'\, dy'$, where x', y', p', q' are functions of x, y, p, q, u, so the first equation takes the form:

$$dz = X'\, dx' + Y'\, dy' + P'\, dp' + Q'\, dq',$$

in which X', Y', P', Q' do not depend upon z, and the integrability condition for the latter equation is a resolvent of the second kind E_2' in z' of the system. The conclusion is not true if the second of equations (37) is of class three or one. In the latter case, the system is of the form (V), and admits an *infinite* group of transformations. In the other case, the resolvent of the first kind will admit an intermediate integral that depends upon an arbitrary function. By setting aside this exceptional case, one may thus say that *any* *infinitesimal transformation* (ε) *of a system* S_6 *corresponds to a resolvent of the second kind* E_2 *of that system*.

We say, to abbreviate, that E_2 is deduced from the infinitesimal transformation ε. One does not therefore obtain all of the resolvents of the second kind; indeed, we will study (no. 10) the systems S_6 that have resolvents of the second kind, but admit no continuous group. The resolvents E_2 that are deduced from a transformation ε may be characterized by the following property: *The integrals* $\mathfrak{M}_2$ *that correspond to a particular integral of* E_2 *are deduced from each other by the transformations of a one-parameter group* g.

A Monge-Ampère equation E_1 that admits a transformation ε is a resolvent of the first kind for two systems S_6 that admits two resolvents of the second kind E_2, E_2', respectively, that are deduced from the transformation ε. The integrals of these two equations correspond to each other in a one-to-one fashion, since each of them corresponds to ∞^1 integrals of S_6 that are deduced from each other by transformations of the group g that is deduced from ε. *One passes from* E_2 *to* E_2' *by a* B_1 *transformation*. Indeed, suppose that E_1 does not refer to z; the formulas for the B_2 transformation between E_1 and E_2 refer to only $x, y, z, p, q, x', y', z', p', q'$, and likewise the formulas for the B_2' transformation between E_1 and E_2' refers to only $x, y, z, p, q, x'', y'', z'', p'', q''$. The elimination of x, y, z, p, q will thus lead to four relations between the coordinates of the two elements $(x', y', \ldots, q')$. For example, the equation $s = 2\lambda(x, y)\sqrt{pq}$ is a resolvent of the first kind for each of the systems:

$$dz = p\,dx + q\,dy, \qquad dp = u\,dx + 2\lambda\sqrt{pq}\,dy,$$
$$dz = p\,dx + q\,dy, \qquad dq = 2\lambda\sqrt{pq}\,dx + u\,dy,$$

each of which admits a resolvent of the second kind that is deduced from the infinitesimal transformation $\partial f / \partial z$. These two resolvents are obtained by taking the unknowns to be $\sqrt{p}$ or $\sqrt{q}$, and are two Laplace equations that are deduced from each other by a Laplace transformation [**27**, **28**].

If the system S_6 admits a continuous group G_n with n parameters then any resolvent of the first kind E_1 also admits a continuous group G_n' with n parameters, and conversely. Each infinitesimal transformation of G_n corresponds to a resolvent of the second kind, and S_6 admits an infinitude of resolvents E_2 that might not all be different, moreover. Let $\mathfrak{M}_2$ be an integral of S_6; the knowledge of the group G_n permits one to deduce an infinitude of other integrals from that integral that depend upon m ($m \le n$) arbitrary constants, the set of which we denote by $\mathcal{E}_G$. Let ε and ε' be two infinitesimal transformations of G_n that give rise to two one-parameter groups g, g'. We likewise let $\mathcal{E}_g$, $\mathcal{E}_g'$ denote the two sets of integrals that are deduced from $\mathfrak{M}_2$ by means of the transformations of g and g', respectively. If the set $\mathcal{E}_G$ depends upon m parameters then it is composed of ∞^{m-1} sets $\mathcal{E}_g$ and ∞^{m-1} sets $\mathcal{E}_g'$. Having said this, let E_2, E_2' be resolvents of the second kind that provide transformations ε, ε'. From an integral I_2 of E_2, one may, by the integration of a differential equation, deduce a set $\mathcal{E}_g$ of integrals of S_6 that belong to a set $\mathcal{E}_G$ of ∞^m integrals of S_6 . Each integral of $\mathcal{E}_g$ corresponds to an integral I_2' of the second resolvent E_2', from which, one may further deduce a new set $\mathcal{E}_g'$ of integrals of S_6 by the integration of a differential equation. However, since all of these sets $\mathcal{E}_g'$ have a common integral $\mathfrak{M}_2$ with $\mathcal{E}_G$ they must be a subset of $\mathcal{E}_G$. The same thing is obviously true for the integrals of S_6 and of E_2 that one might obtain by pursuing the application of the same process. Consequently, if two resolvents of the second kind E_2, E_2' provide two infinitesimal transformations of a group G of S_6 then the repeated application of the B_3 transformation between these two equations, upon starting with an integral of one of them, cannot furnish other integrals of these two equations than the ones that one may deduce from the knowledge of the group G, which depends upon $m - 1$ arbitrary constants. This result is clear *a priori*, since the resolvents E_2, E_2' are themselves deduced from the group G.

Examples.

1. A Laplace equation $s = ap + bq + cz$ is a resolvent of the first kind for a system S_6 (no. 6) that admits the transformation $z\frac{\partial f}{\partial z}+p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}$. The resolvent of the second kind E_2 provided by this transformation is obtained by setting $z = e^Z$ and then taking $\partial f / \partial x$ to be the unknown, which leads to a transformation that was known to Moutard (**41**). Likewise, if z_1 is a particular integral of the Laplace equation then that equation does not change when one changes z into $z + az_1$; the resolvent E_2 of S_6 that is deduced from this one-parameter group is obtained by taking the unknown to be $\frac{\partial}{\partial x}\left(\frac{z}{z_1}\right)$, and the B_2 transformation is identical to the transformation of Lucien Lévy [**38**].

2. A system S_6 of the form $dz + \Omega_1 = 0$, $dz' + \Omega_2 = 0$, where Ω_1 and Ω_2 are two Pfaff forms in four variables x_1, x_2, x_3, x_4 , admits two permutable infinitesimal transformations, each of which leads to a resolvent of the second kind. The integrals of E_2' that correspond to an integral of E_2 are obtained by adding an arbitrary constant to one of them, and conversely. The group G has two parameters, and the integrals of E_2 and E_2' correspond to each other by sets that depend upon one parameter. If, in particular, the system S_6 has the form (7):

$$dz = p\,dx + q\,dy, \qquad dz' = f(p, q)\,dx + \varphi(p, q)\,dy$$

then the resolvent E_2 has the form $H\,r + 2\,K\,s + L\,t = 0$, where H, K, L depend upon only p, q, and may be converted into a Laplace equation.

10. Examples. – It was the research of Bianchi [**7**] on surfaces of constant negative curvature that led A,-V. Bäcklund to pose the general problem that was studied here. Bianchi had proved that from any surface Σ of constant negative curvature $-1/a^2$, one may deduce an infinitude of other surfaces Σ′ that enjoy the same property. The points M and M' of Σ and a transformed Σ′ correspond to each other in such a fashion as to satisfy the following conditions: The distance MM' is constant and equal to a, while the tangent planes at M and M' contain the line MM' and are orthogonal. It is clear that these conditions translate into *four* relations between the coordinates of an element (x, y, z, p, q) of Σ and the coordinates of the corresponding element (x', y', z', p', q') of Σ′. Upon replacing the orthogonality condition for the tangent planes with the condition of making a constant angle, Bäcklund was led to a more general problem that gave a new method of transforming surfaces of constant total curvature.

G. Darboux further generalized the problem by replacing the Bäcklund conditions with the following ones: The system that is composed of two points M, M', and the tangent planes to the surfaces Σ, Σ′ at the points M and M', respectively, has an invariable form. Abstracting from parallel surfaces, one further finds that the surfaces Σ, Σ′ must be parallel to minimal surfaces or to surfaces of constant total curvature. Finally, J. Clairin [**13**] extended this result to non-Euclidian space.

The study of Bianchi and Bäcklund transformations led to a system of two simultaneous equations of a very simple form, and which possesses remarkable properties. The search for surfaces of total curvature – 1 [**22**] depends upon the integration of the second-order partial differential equation:

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \sin\theta\cos\theta. \tag{38}$$

One sees immediately that if $\theta = f(x, y)$ is a particular integral then $\theta = f\left(mx, \frac{y}{m}\right)$ is also an integral for any constant m; this is the Lie transformation. The study of the Bianchi transformation leads to the study of the system:

$$\begin{cases} \dfrac{\partial\theta}{\partial x} + \dfrac{\partial\varphi}{\partial x} = \sin(\theta - \varphi), \\ \dfrac{\partial\theta}{\partial x} - \dfrac{\partial\varphi}{\partial x} = \sin(\theta + \varphi), \end{cases} \tag{39}$$

which, along with the relations $x' = x$, $y' = y$, forms a Bäcklund system. The elimination of φ from the two equations (39) leads to equation (38), and, by reason of symmetry, the elimination of θ likewise leads to the equation:

$$\frac{\partial^2\varphi}{\partial x\,\partial y} = \sin\theta\cos\theta; \tag{40}$$

the two equations (38) and (40) are two resolvents of the second kind of the system (39). The knowledge of a particular integral $\theta(x, y)$ of the resolvent (38) permits one to obtain an infinitude of integrals of equation (40) that depend upon an arbitrary constant by the integration of the completely integrable system (39), which comes down to a Ricatti equation. Upon operating likewise on the integral $\varphi(x, y)$ of (40) thus obtained, one may deduce new integrals that depend upon another arbitrary constant, and so on. For the study of this sequence of operations and the integrations that it demands, I will refer the reader to Chapter XIII of the *Leçons sur la théorie générale des surfaces* of G. Darboux (tome 3, book VII).

The Bäcklund transformation leads to the more general system:

$$\text{(41)} \qquad \begin{cases} \dfrac{\partial\theta}{\partial x} + \dfrac{\partial\varphi}{\partial x} = m\sin(\theta - \varphi), \\[2mm] \dfrac{\partial\theta}{\partial x} - \dfrac{\partial\varphi}{\partial x} = \dfrac{1}{m}\sin(\theta + \varphi), \end{cases}$$

where m is an arbitrary constant. The elimination of φ further leads to equation (38) and that of θ, to equation (40), in such a way that these two equations are, moreover, two resolvents of the second kind for the more general system (41). However, this system may itself be converted into the simple form (39) by taking the new variables to be $x' = mx$, $y' = (1 / m)\, y$, in such a way that the Bäcklund transformation for the surfaces of constant curvature is a combination of the two transformations of Lie and Bianchi.

Let $\theta(x, y)$ be a particular integral of the resolvent (38), and let $\varphi = \varphi_1(x, y, m, C)$ be the general integral of the system (41) that depends upon the parameter m and the constant of integration C. If one replaces φ with φ_1 and m with a new constant m_1 then the system:

$$\text{(42)} \qquad \begin{cases} \dfrac{\partial\theta}{\partial x} + \dfrac{\partial\varphi_1}{\partial x} = m_1\sin(\theta - \varphi_1), \\[2mm] \dfrac{\partial\theta}{\partial x} - \dfrac{\partial\varphi_1}{\partial x} = \dfrac{1}{m_1}\sin(\theta + \varphi_1) \end{cases}$$

is again completely integrable, and it results from a beautiful theorem of Bianchi on *permutability* [**9**] that this system can be integrated by algebraic operations and differentiations if one has obtained the general integral of the first system for any m. In this case, one may thus deduce from the integral $\theta(x, y)$ of (38), an infinitude of other integrals that depend upon as many arbitrary constants as one desires, without any new integration.

An important theorem of Weingarten [**21**] on the deformation of surfaces may also be attached to the Bäcklund problem. Let S be a surface that admits the linear element:

$$\text{(43)} \qquad ds^2 = du^2 + 2\, dv\, d\psi,$$

where $\psi(u, v)$ is a given function of u, v. The rectangular coordinates of a point m of that surface are functions of the variable u, v that verify the classical equations:

(44) $$S\left(\frac{\partial x}{\partial u}\right)^2 = 1, \quad S\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial \psi}{\partial u}, \qquad S\left(\frac{\partial x}{\partial v}\right)^2 = 2\frac{\partial \psi}{\partial v}.$$

One makes the point m of S correspond to the point M with the coordinates:

(45) $$X = \frac{\partial x}{\partial v}, \qquad Y = \frac{\partial y}{\partial v}, \qquad Z = \frac{\partial z}{\partial v},$$

and one easily deduces from the relations (44) that one therefore has:

(46) $$\frac{\partial x}{\partial u}dX + \frac{\partial y}{\partial u}dY + \frac{\partial z}{\partial u}dZ = 0.$$

When the point m describes a surface S that admits the linear element (43), the point M describes a surface Σ whose normal has the direction cosines $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$. Let P, Q be the angular coefficients of the plane tangent to that surface; one has:

(47) $$\frac{P}{\frac{\partial x}{\partial u}} = \frac{Q}{\frac{\partial y}{\partial u}} = \frac{-1}{\frac{\partial z}{\partial u}},$$

and some simple combinations show that the common value of the ratios is equal to:

$$\frac{PX + QY - Z}{\frac{\partial \psi}{\partial u}} = \sqrt{1 + P^2 + Q^2}.$$

One thus has the following four relations:

(48) $$\begin{cases} Z = \dfrac{\partial z}{\partial v}, & X^2 + Y^2 + Z^2 = 2\dfrac{\partial \psi}{\partial v}, \\[2ex] \dfrac{\partial z}{\partial u} = \dfrac{-1}{\sqrt{1 + P^2 + Q^2}}, & \dfrac{\partial \psi}{\partial u} = \dfrac{PX + QY - Z}{\sqrt{1 + P^2 + Q^2}} \end{cases}$$

between $X, Y, Z, P, Q, u, v, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$. This is a Bäcklund system in which z does not appear; the Pfaff system thus admits an infinitesimal transformation that corresponds to a resolvent of the second kind (no. 9). In order to obtain it, it suffices to deduce $u, v, \frac{\partial z}{\partial u}$,

$\frac{\partial z}{\partial v}$ from the preceding formulas by means of *X*, *Y*, *Z*, *P*, *Q*, and to write the integrability condition for the equation:

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv .$$

The Monge-Ampère equation to which one is led is precisely the second-order equation into which Weingarten converted the determination of the surfaces that admit the linear element (43).

The system (48) admits another infinitesimal transformation. Indeed, if one sets $X = \rho \cos \omega$, $Y = \rho \sin \omega$ then these equations become:

$$Z = \frac{\partial z}{\partial v}, \qquad \rho^2 + Z^2 = 2\,\frac{\partial \psi}{\partial v};$$

$$\left(\frac{\partial z}{\partial u}\right)^2 \left\{1 + \left(\frac{\partial Z}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial Z}{\partial \omega}\right)^2\right\} = 1,$$

$$\frac{\partial \psi}{\partial u}\sqrt{1 + \left(\frac{\partial Z}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial Z}{\partial \omega}\right)^2} = \rho\,\frac{\partial Z}{\partial \rho} - Z,$$

and do not refer to ω. The corresponding resolvent of the second kind is identical with the classical equation that *z* must satisfy when considered as a function of the two parameters *u*, *v*.

The search for surfaces that are mappable to a surface of second degree that is tangent to the circle at infinity is thus converted into the determination of the surfaces of constant curvature [**23**].

11. Diverse generalizations. – The statement of the Bäcklund problem may be generalized in various ways. Indeed, one may augment the dimensions or the order of the contact elements of the two multiplicities that one makes correspond element-by-element, or the number of relations between these two elements. Cerf [**12**] has studied in detail the case where one establishes *four* relations between two elements of arbitrary order of two two-dimensional multiplicities, and showed that if certain conditions are satisfied then the solution of this new problem comes down to the integration of just one partial differential equation. Bäcklund himself has studied the correspondences between two multiplicities of first-order elements in spaces of dimension more than three, where the number of relations is augmented [**6**]. No matter what the manner by which one generalizes the problem, one always comes down to the search for integral multiplicities of a Pfaff system with a known number of dimensions. It results from the foregoing that the integration of such a system is, is in certain cases and in several ways, converted into the integration of just one partial differential equation, but one is still quite far from a general solution to the problem.

I shall point out only the various circumstances that one may expect in a particularly simple case [**35**]. The integration of the second-order equation:

$$r = f(x, y, z, p, q, s, t) \tag{49}$$

may be replaced with a slightly more general problem, viz., the search for the two-dimensional integrals of the system S_3 of three Pfaff equations in seven variables x, y, z, p, q, s, t:

$$dz = p\,dx + q\,dy, \qquad dp = f\,dx + s\,dy, \qquad dq = s\,dx + t\,dy, \tag{50}$$

which is not, moreover, the most general of this type. Equation (49) is obviously a resolvent of this system, but it might admit others. This is what happens, in particular, if one might find two equations of S_2 that form a system S_2 of class 6. The various resolvents of S_2 will also be resolvents of S_3 . The same is true, in particular, if equation (49) does not refer to z. The last two equations (50) then form a system with six variables x, y, z, p, q, s, t. Since any second-order equation that admits an infinitesimal contact transformation may be converted into an equation that does not refer to z, one concludes from this that the integration of a second-order equation that admits an infinitesimal contact transformation may be converted into the integration of a second-order equation that possesses at least one system of first-order characteristics (**11**, **35**).

The system S_3 may admit resolvents of another kind. Let X, Y, Z, P, Q, U, V be a new system of variables such that the equations S_3 are, with these variables:

$$\begin{cases} dZ = P\,dX + Q\,dY, \\ dU = A\,dX + B\,dY + C\,dP + E\,dQ, \\ dV = A_1dX + B_1dY + C_1dP + E_1dQ, \end{cases} \tag{51}$$

in which $A, B, C, E, A_1, B_1, C_1, E_1$ are functions of the new variables. If one takes X and Y to be the independent variables, and if one supposes that Z is replaced by a function $F(X, Y)$, and P, Q, by the partial derivatives of F then the integrability conditions of the last two equations furnish two linear equations in $R, S, T, RT - S^2$. It results from the special properties of the system (50) that these two conditions must reduce to just one that generally contains U and V. If it contains neither U nor V then it forms a resolvent of the system, such that to any integral of that equation there correspond ∞^2 integrals of S_3 .

The system (50) may be *prolonged* by introducing the derivatives of z up to an arbitrary order, and the properties of the system (50) may also be extended to these new systems. These considerations are attached to the general results that were due to Clairin [**14**, **17**] on the second-order equations that admit a group of transformations and some other transformations that were pointed by Gau [**25**].

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