“Sur le problème de Monge,” Bull. Soc. Math. **33** (1905), 201-210.

On the Monge problem [1]

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The publication of a recent note by Zervos (*Comptes rendus*, 10 April 1905) has attracted my attention to certain results that relate to the Monge problem, which although quite incomplete, can possibly offer some interest to the mathematicians that address that type of question. I shall indicate them rapidly.

1. – In order to facilitate the arguments and statements, it is convenient to employ the language of n-dimensional geometry. When we are given $(n + 1)$ independent variables $x_1, x_2, \ldots, x_{n+1}$, we say that any particular system of values for those variables represents the *coordinates of a point* in $(n+1)$-dimensional space. We continue to call any n-dimensional manifold that is situated in that $(n+1)$-dimensional space a *surface* and any *one*-dimensional manifold a *curve*. Any surface is represented by just one relation between the coordinates of its points. If that relation is linear then the surface will be called a *planar surface* or a *plane*. Similarly, we say *line* to mean a one-dimensional linear manifold, i.e., the set of points whose coordinates X_i satisfy n relations of the form:

$$\frac{X_1 - x_1}{a_1} = \frac{X_2 - x_2}{a_2} = \ldots = \frac{X_{n+1} - x_{n+1}}{a_{n+1}} . \tag{1}$$

$x_1, x_2, \ldots, x_{n+1}$ are the coordinates of a particular point, while $a_1, a_2, \ldots, a_{n+1}$ are the *direction parameters* of the line.

A plane P that passes through a point with coordinates $(x_1, x_2, \ldots, x_{n+1})$:

$$A_1 (X_1 - x_1) + A_2 (X_2 - x_2) + \ldots + A_{n+1} (X_{n+1} - x_{n+1}) = 0 \tag{2}$$

is, in general, determined when it contains n lines that pass through that point $(x_1, x_2, \ldots, x_{n+1})$, because the coefficients A_i must verify a system of n linear and homogeneous equations.

A line D that passes through a fixed point M whose coordinates are $(x_1, x_2, \ldots, x_{n+1})$ and whose direction parameters depend upon p arbitrary variables α_i $(p < n)$ generates a *conical manifold*

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[1] This note was presented to the Société Mathématique at the session on 15 June 1905. A note by Bottasso on the same subject was presented to the Académie des Sciences at the session on 13 June 1905.

whose summit is M. If $p = 1$ then we will say that this two-dimensional conical manifold is a *cone* whose summit is M. From that, any cone with summit M is represented by a system of $(n - 1)$ distinct equations:

$$(3) \qquad f_i\,(x_1, x_2, \ldots, x_{n+1}\,;\, X_1 - x_1, X_2 - x_2, \ldots, X_{n+1} - x_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n - 1)\,,$$

in which the f_i are homogeneous functions of the differences $X_i - x_i$. If the parameters $a_1, a_2, \ldots, a_{n+1}$ in equations (1) are functions of just one independent variable α then those lines will be the *generators* of a cone whose summit is M $(x_1, x_2, \ldots, x_{n+1})$, and each value of α will correspond to a well-defined generator.

If one is given a cone whose summit is M then any plane that passes through M will include a certain number of generators. If, for example, the generators of the cone are represented by equations (1), while $a_1, a_2, \ldots, a_{n+1}$ are functions of one independent variable α, then the values of α that correspond to the generators that are situated in the plane (2) will be roots of the equation:

$$(4) \qquad U\,(\alpha) = A_1\ \alpha_1 + A_2\ \alpha_2 + \ldots + A_{n+1}\ \alpha_{n+1} = 0\,.$$

One can assign the coefficients $A_1, A_2, \ldots, A_{n+1}$ in such a fashion that this plane has n generators in common with the cone that coincide with a well-defined generator. If α_0 is the corresponding value of the variable α then for that to be true, it is necessary and sufficient that α_0 is a multiple root of order n of equation $U\,(\alpha) = 0$. One will then have n condition equations:

$$(5) \qquad U\,(\alpha_0) = 0\,, \qquad U'(\alpha_0) = 0\,, \quad \ldots, \qquad U^{(n+1)}(\alpha_0) = 0\,,$$

which will, in general, determine the ratios of the coefficients $A_1, A_2, \ldots, A_{n+1}$. We say that the plane, thus-obtained, *osculates* the cone (T) along the generator (α_0). Upon eliminating α_0 from the n equation (5), one will generally arrive at $(n - 1)$ condition equations that are homogeneous in the $A_1, A_2, \ldots, A_{n+1}$:

$$(6) \qquad F_i\,(x_1, x_2, \ldots, x_{n+1}\,;\, A_1, A_2, \ldots, A_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n - 1)\,.$$

In addition, they depend upon the variables $x_1, x_2, \ldots, x_{n+1}$ if the direction parameters $a_1, a_2, \ldots, a_{n+1}$ depend upon, not only α, but also the coordinates $x_1, x_2, \ldots, x_{n+1}$ of the summit. We say, to abbreviate, that the relations (6) are the *tangential equations* of the cone considered at the summit $(x_1, x_2, \ldots, x_{n+1})$.

When that cone is represented by a system of $n - 1$ equations of the form (3), the n equations (2) and (3) determine the generators of a cone that is situated in the plane P. One will again get the tangential equations of the cone by expressing the idea that n of those generators coincide. For example, upon eliminating $n - 1$ of the ratios $\frac{X_2 - x_2}{X_1 - x_1}, \ldots, \frac{X_{n+1} - x_{n+1}}{X_1 - x_1}$ from the n relations (2) and (3), one will arrive at an equation for determining the last ratio, and that equation must have a multiple root of order n.

2. – Conversely, any system of n – 1 equations of the form (6) that is homogeneous in the A_1, A_2, …, A_{n+1} can be considered to represent the tangential equations of a cone whose summit is (x_1, x_2, …, x_{n+1}).

Indeed, imagine that one has inferred the values of (n – 1) of the ratios $\frac{A_2}{A_1}$, …, $\frac{A_{n+1}}{A_1}$ as functions of one of them α from the (n – 1) equations (6). The plane P that is represented by equation (2):

$$U(\alpha) = A_1 (X_1 - x_1) + A_2 (X_2 - x_2) + \ldots + A_{n+1} (X_{n+1} - x_{n+1}) = 0$$

depends upon only one variable parameter α. The n equations:

$$(7) \qquad U(\alpha) = 0 , \qquad U'(\alpha) = 0 , \qquad \ldots, \qquad U^{(n+1)}(\alpha) = 0$$

will represent a line D that passes through the point M (x_1, x_2, …, x_{n+1}) when one gives a particular value to α. When α varies, that line D will generate a cone with its summit at M, and from the way that it was obtained, it is clear that the plane P has n common generators with that cone that coincide with the generator (α). Upon eliminating α from the (n – 1) equations (7), one will then have the equations of the cone with its summit at M whose tangential equations are equations (6).

3. – Now consider a system of k Monge equations ($k \leq n - 1$):

$$(8) \qquad f_i (x_1, x_2, \ldots, x_{n+1} ; dx_1, dx_2, \ldots, dx_{n+1}) = 0 \qquad (i = 1, 2, \ldots, k),$$

in which f_i are functions that are homogeneous in the dx_1, dx_2, …, dx_{n+1} . The problem of integrating that system consists of expressing x_1, x_2, …, x_{n+1} explicitly as functions of one auxiliary variable t, $n - k$ arbitrary functions of that parameter, and their successive derivatives up to some well-defined order. That problem was solved by Monge in the particular case where he had n = 2, k = 1. We shall examine the case in which we have $k = n - 1$, where n is arbitrary.

If one replaces dx_i with $X_i - x_i$ in the (n – 1) equations (8) then one will have the equations of a cone whose summit is (x_1, x_2, …, x_{n+1}). At each point of the (n + 1)-dimensional space, the proposed equations will make a cone (T) whose summit at that point correspond to that point, and Monge problem can be formulated in these terms:

Determine the curves in (n + 1)*-dimensional space that are tangent at each of their points to one of the generators of the cone* (T) *that has that point for its summit.*

Write the equation of a plane that passes through the point (x_1, x_2, …, x_{n+1}) in the form:

$$(9) \qquad X_{n+1} - x_{n+1} = p_1 (X_1 - x_1) + \ldots + p_{n+1} (X_{n+1} - x_{n+1}) .$$

The tangential equations for the cone (T) whose summit is (x_1, x_2, …, x_{n+1}) then have the form:

$$(10) \qquad F_i (x_1, x_2, \ldots, x_{n+1} ; p_1, p_2, \ldots, p_n) = 0 \qquad (i = 1, 2, \ldots, n - 1) .$$

If one considers $x_1, x_2, \ldots, x_n$ to be n independent variables, x_{n+1} to be a function of those n variables, and $p_1, p_2, \ldots, p_n$ to be the partial derivatives of x_{n+1} $\left(p_i = \frac{\partial x_{n+1}}{\partial x_i} \right)$ then equations (10) will form a system of $(n - 1)$ first-order partial differential equations for just one unknown function. Any system of $(n - 1)$ Monge equations:

$$(9') \qquad f_i (x_1, x_2, \ldots, x_{n+1} ; dx_1, dx_2, \ldots, dx_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n - 1)$$

will then correspond to a system (10) of $(n - 1)$ simultaneous first-order partial differential equations that we shall call the *associated system* to the first system (9).

4. – Having said that, it is easy to show that the Monge method can be effortlessly extended to the case in which the associated system of partial differential equations (10) is *in involution*.

Indeed, let:

$$(11) \qquad V (x_1, x_2, \ldots, x_{n+1} ; a, b) = 0$$

be a complete integral of that system. The tangent plane to one of the integral surfaces S that passes through a given point $(x_1, x_2, \ldots, x_{n+1})$ has the equation:

$$(12) \qquad \frac{\partial V}{\partial x_1}(X_1 - x_1) + \cdots + \frac{\partial V}{\partial x_{n+1}}(X_{n+1} - x_{n+1}) = 0 ,$$

in which the two parameters a and b are coupled by the relation (11). Therefore, in reality, that plane depends upon only one variable parameter, and from the way that the adjoint system (10) to the system (9′) was deduced, the plane that is represented by equation (12) will continue to osculate the cone (T) whose summit is $(x_1, x_2, \ldots, x_{n+1})$ and is represented by the equations:

$$(13) \qquad f_i (x_1, x_2, \ldots, x_{n+1} ; X_1 - x_1, X_2 - x_2, \ldots, X_{n+1} - x_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n - 1).$$

In order to deduce the equations (13) of the complete integral (11), one can then proceed as follows: To fix ideas, regard a as a variable parameter and b as a function of that parameter that is defined by the relation (11). The successive derivatives:

$$b' = \frac{db}{da}, \qquad b'' = \frac{d^2 b}{da^2}, \qquad \ldots, \qquad b^{(n-1)} = \frac{d^{n-1} b}{da^{n-1}}, \qquad b^{(n)} = \frac{d^n b}{da^n}$$

are determined by the relations:

$$(14)\quad \begin{cases} V_1 = \dfrac{\partial V}{\partial a} + \dfrac{\partial V}{\partial b} b' = 0\,, \\ V_2 = \dfrac{\partial^2 V}{\partial a^2} + 2\dfrac{\partial^2 V}{\partial a\,\partial b} b' + \dfrac{\partial^2 V}{\partial b^2} b'^2 + \dfrac{\partial V}{\partial b} b'' = 0\,, \\ \dotfill, \\ V_n = \dfrac{\partial^n V}{\partial a^n} + \cdots + \dfrac{\partial V}{\partial b} b^{(n)} = 0\,. \end{cases}$$

The equations that are obtained by differentiating equation (12) with respect to the parameter a (n – 1) times in succession, while b is supposed to have been replaced by its expression that one infers from the relation (11), can then be written:

$$(15)\quad \begin{cases} \dfrac{\partial V_1}{\partial x_1}(X_1 - x_1) + \cdots + \dfrac{\partial V_1}{\partial x_{n+1}}(X_{n+1} - x_{n+1}) = 0\,, \\ \dotfill, \\ \dfrac{\partial V_{n-1}}{\partial x_1}(X_1 - x_1) + \cdots + \dfrac{\partial V_{n-1}}{\partial x_{n+1}}(X_{n+1} - x_{n+1}) = 0\,, \end{cases}$$

and one will then get equations (13) upon eliminating a, b, b', b'', …, $b^{(n-1)}$ from the $2n$ equations (11), (12), (14) and (15).

Now suppose that one sets:

$$b = \varphi(a)\,,$$

in which the function $\varphi(a)$ is an arbitrary function of a, and consider the system of n – 1 equations:

$$(16)\qquad V = 0\,, \qquad \frac{dV}{da} = 0\,, \qquad \frac{d^2V}{da^2} = 0\,, \qquad \ldots, \qquad \frac{d^nV}{da^n} = 0\,,$$

in which one has set:

$$(17)\quad \begin{cases} \dfrac{dV}{da} = \dfrac{\partial V}{\partial a} + \dfrac{\partial V}{\partial b}\varphi'(a)\,, \\ \dfrac{d^2V}{da^2} = \dfrac{\partial^2 V}{\partial a^2} + 2\dfrac{\partial^2 V}{\partial a\,\partial b}\varphi'(a) + \dfrac{\partial^2 V}{\partial b^2}\varphi'^2(a) + \dfrac{\partial V}{\partial b}\varphi''(a)\,, \\ \dotfill, \\ \dfrac{d^nV}{da^n} = \dfrac{\partial^n V}{\partial a^n} + \cdots + \dfrac{\partial V}{\partial b}\varphi^{(n)}(a) = 0. \end{cases}$$

Those (n + 1) simultaneous equations generally permit one to express $x_1, x_2, \ldots, x_{n+1}$ as functions of the auxiliary variable a by way of $\varphi(a)$, $\varphi'(a)$, …, $\varphi^{(n)}(a)$. I say that the curve in the (n + 1)-

dimensional space that is determined in that way is an *integral* of the Monge system. Indeed, if we let $x_1', x_2', \ldots, x_{n+1}'$ denote the derivatives of those functions with respect to a then we will deduce from equations (16) that:

$$\text{(18)} \qquad \begin{cases} \dfrac{\partial V}{\partial x_1} x_1' + \cdots + \dfrac{\partial V}{\partial x_{n+1}} x_{n+1}' = 0\,, \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{dV}{da}\right) x_1' + \cdots + \dfrac{\partial}{\partial x_{n+1}}\left(\dfrac{dV}{da}\right) x_{n+1}' = 0\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{d^{n-1}V}{da^{n-1}}\right) x_1' + \cdots + \dfrac{\partial}{\partial x_{n+1}}\left(\dfrac{d^{n-1}V}{da^{n-1}}\right) x_{n+1}' = 0\,, \end{cases}$$

and the system of $2n + 1$ equations (16) and (18) will become identical to the system that is composed of equations (11), (12), (14), and (15), provided that one replaces $x_1', x_2', \ldots, x_{n+1}'$ with $X_1 - x_1, X_2 - x_2, \ldots, X_n - x_n$, respectively, and then replaces $\varphi(a)$ with b, $\varphi'(a)$ with b', …, $\varphi^{(n)}(a)$ with $b^{(n)}$. As a result, the elimination of a, $\varphi(a)$, $\varphi'(a)$, …, $\varphi^{(n)}(a)$ from equations (16) and (18) will lead to the relations:

$$f_i(x_1, x_2, \ldots, x_{n+1}\,;\; x_1', x_2', \ldots, x_{n+1}') = 0 \qquad (i = 1, 2, \ldots, n-1),$$

which are identical to the relations (9). Formulas (16) then represent the general integral of the proposed Monge system.

5. – The preceding systems are obviously very specialized. However, one can sometimes extend the method to more general Monge systems by imitating the Lagrange and Charpit method for integrating a partial differential equation. For example, if one is given a system of $n - k$ Monge equations ($k > 1$) then in order to apply the preceding method, it will suffice that one can append $k - 1$ new equations of the same form to it, in such a fashion that the associated system to the system thus-completed will be in involution.

For example, consider a Monge system of the form:

$$\text{(19)} \qquad f_i(dx_1, dx_2, \ldots, dx_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n-k),$$

in which the functions f_i do not include the variables $x_1, x_2, \ldots, x_{n+1}$. If $k = 1$ then the associated system will have the form:

$$\text{(20)} \qquad F_i(p_1, p_2, \ldots, p_n) = 0 \qquad (i = 1, 2, \ldots, n-1),$$

and is, consequently, in involution. If $k > 1$ then it will suffice to append $k - 1$ equations of the same form to equation (19), for example:

$$\frac{dx_3}{dx_1} = \psi_1\left(\frac{dx_2}{dx_1}\right), \qquad \ldots, \qquad \frac{dx_{k+1}}{dx_1} = \psi_{k-1}\left(\frac{dx_2}{dx_1}\right),$$

in which the functions $\psi_1, \psi_2, \ldots, \psi_{k-1}$ are arbitrary, in order to reduce it to the first case.

The systems (19) were considered by Darboux (*Journal de Mathématiques pures et appliquées*, 1887). The preceding method shows how one can attach a general theory to the integration of those systems that is completely analogous to the Monge theory for the equation with three variables.

6. – For example, apply that method to the Serret equation:

(21) $$dx_1^2 + dx_2^2 + dx_3^2 = dx_4^2.$$

Combine that equation with a relation of arbitrary form:

(22) $$\frac{dx_3}{dx_1} = \varphi\left(\frac{dx_2}{dx_1}\right).$$

Equations (21) and (22) form a Monge system, and the equations of the cone (T) that corresponds to the summit (x_1, x_2, x_3, x_4) are:

(23) $$\begin{cases} (X_4 - x_4)^2 = (X_1 - x_1)^2 + (X_2 - x_2)^2 + (X_3 - x_3)^2, \\ \dfrac{X_3 - x_3}{X_1 - x_1} = \varphi\left(\dfrac{X_2 - x_2}{X_1 - x_1}\right). \end{cases}$$

In order to get the corresponding associated system, form the equation that gives the common generators to the cone (T) and the plane:

$$X_4 - x_4 = p_1 (X_1 - x_1) + p_2 (X_2 - x_2) + p_3 (X_3 - x_3).$$

One can set:

$$\frac{X_2 - x_2}{X_1 - x_1} = \alpha, \qquad \frac{X_3 - x_3}{X_1 - x_1} = \varphi(\alpha),$$

which gives:

$$\frac{X_2 - x_2}{X_1 - x_1} = p_1 + p_2\,\alpha + p_3\,\varphi(\alpha).$$

The equation α is then:

$$[p_1 + p_2\,\alpha + p_3\,\varphi(\alpha)]^2 = 1 + \alpha^2 + \varphi^2(\alpha),$$

or

(24) $$p_1 + p_2\,\alpha + p_3\,\varphi(\alpha) = \sqrt{1 + \alpha^2 + \varphi^2(\alpha)} = U(\alpha).$$

From the general theory, one will get the equations of the associated system of partial differential equations by eliminating α from the relation (24) and the relations (25):

$$(25) \qquad \begin{cases} p_2 + p_3\,\varphi'(\alpha) = U'(\alpha), \\ \qquad p_3\,\varphi''(\alpha) = U''(\alpha). \end{cases}$$

It is pointless to carry out that elimination, because one will get a complete integral of that system by taking the plane:

$$X_4 = p_1\,X_1 + p_2\,X_2 + p_3\,X_3 + b\,,$$

in which p_1, p_2, p_3 are replaced by their expressions that one infers from formulas (24) and (25), while α and b are arbitrary constants.

Now replace b with a second arbitrary function $\psi(\alpha)$. From the general theorem, the functions x_1, x_2, x_3, x_4 of the parameter α are defined by the four equations:

$$(26) \qquad \begin{cases} x_4 = p_1\,x_1 + p_2\,x_2 + p_3\,x_3 + \psi(\alpha), \\ \;0 = p_1'\,x_1 + p_2'\,x_2 + p_3'\,x_3 + \psi'(\alpha), \\ \;0 = p_1''\,x_1 + p_2''\,x_2 + p_3''\,x_3 + \psi''(\alpha), \\ \;0 = p_1'''\,x_1 + p_2'''\,x_2 + p_3'''\,x_3 + \psi'''(\alpha), \end{cases}$$

in which p_i', p_i'', p_i''' are the derivatives of p_i with respect to α, must satisfy the proposed relation (21).

That is easy to verify. Indeed, one deduces from the relations (24) and (25) that one has:

$$(27) \qquad \begin{cases} p_1 + p_2\,\alpha + p_3\,\varphi(\alpha) = U(\alpha), \\ p_1' + p_2'\,\alpha + p_3'\,\varphi(\alpha) = 0, \\ p_1'' + p_2''\,\alpha + p_3''\,\varphi(\alpha) = 0. \end{cases}$$

On the other hand, upon differentiating the relations (26), they will become:

$$(28) \qquad \begin{cases} dx_4 = p_1\,dx_1 + p_2\,dx_2 + p_3\,dx_3\,, \\ \;\;0 = p_1'\,dx_1 + p_2'\,dx_2 + p_3'\,dx_3\,, \\ \;\;0 = p_1''\,dx_1 + p_2''\,dx_2 + p_3''\,dx_3\,. \end{cases}$$

The relations (27) and (28) prove that one must have:

$$\frac{dx_1}{1} = \frac{dx_2}{\alpha} = \frac{dx_3}{\varphi(\alpha)} = \frac{dx_4}{U(\alpha)}\,,$$

and since one has:

$$U^2(\alpha) = 1 + \alpha^2 + \varphi^2(\alpha),$$

it will then follow that one also has:

$$dx_4^2 = dx_1^2 + dx_2^2 + dx_3^2 .$$

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