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On electrodynamics

(By Herrn **Hermann Grassmann** in Stettin)

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The law of mutual action of two current segments that I presented in 1845 in **Poggendorf**'s Annalen, Bd. 64, pp. 1, *et seq.*, as the putative correct one, in contrast to **Ampére**'s law, has found not only new support, but, one can say, a firm foundation, in the trailblazing work of Herrn **Clausius**, namely, his treatise in this journal, Bd. 82, pp. 85, *et seq.* In fact, the force law for current elements that **Clausius** derived from his general theory on pp. 130 of his aforementioned treatise agrees with my own law that was presented in *loc. cit.* precisely. Since, **Clausius**, who obviously missed out on my aforementioned treatise, did not mention that agreement in his works (cf., **Poggendorff**'s Annalen, Bd. 156, pp. 657, Bd. 157, pp. 489, and Verhandlungen des naturhist. Vereins der preuss. Rheinlande und Westfalens, Bd. 33), I will briefly discuss it here and connect it with some consequences that I believe to be not unimportant.

It is extremely easy to verify the agreement of both laws by the analysis that was treated in my *Ausdehnungslehren* (of 1844 and 1862). However, since I cannot assume that the reader is familiar with the laws of that analysis, I shall initially base that verification upon the laws of ordinary analysis, namely, upon the theorem that when a_1, a_2, a_3 are the perpendicular coordinates of a line segment, and a is its length, and b_1, b_2, b_3 are the corresponding coordinates of a second segment, while b is its length, the cosine of the angle between the directions of both segments [which I shall denote by cos $(a\,b)$ in the customary way] then:

$$\cos\,(a\,b) = \frac{a_1\,b_1 + a_2\,b_2 + a_3\,b_3}{ab}\,.$$

Now, if dx', dy', dz' are the coordinates of a current element in a perpendicular coordinate system, and ds' is it length, while dx, dy, dz are the coordinates of a second current element, and ds is its length, while X, Y, Z are the coordinates of the force with which the former element acts upon the latter, and furthermore, x', y', z' are the coordinates of the starting point of the first element, while x, y, z are the coordinates of the starting point of the second one, so $x-x'$, $y-y'$, $z-z'$ will be the coordinates of the segment that points from the former starting point to the latter one, and r is the length of that segment, so $(x-x')^2+(y-y')^2+(z-z')^2 = r^2$, and if i and i' are the two current intensities, moreover, while ε is the angle between the two current elements, and k is a constant numerical factor then, according to **Clausius** (*loc. cit.*, pp. 130):

(1) $$X = k\,i\,i'\,ds\,ds'\left(\frac{d(1/r)}{dx}\cos\varepsilon - \frac{d(1/r)}{ds}\frac{dx'}{ds'}\right).$$

Now:

$$d\,\frac{1}{r} = -\,\frac{r\,dr}{r^3} = -\frac{d(r^2)}{2r^3} = -\,\frac{d\,[(x-x')^2+(y-y')^2+(z-z')^2]}{2r^3}.$$

Hence:

$$\frac{d(1/r)}{dx} = -\,\frac{x-x'}{r^3}, \qquad \frac{d(1/r)}{dx}ds = -\,[(x-x')\,dx+(y-y')\,dy+(z-z')\,dz]:r^3.$$

Furthermore, one has:

$$\cos\,\varepsilon = (dx\,dx'+dy\,dy'+dz\,dz'):ds\cdot ds'.$$

When that value is substituted in (1), one will get:

$$X = -\,\frac{k\,i\,i'}{r^3}\{(x-x')(dx\,dx'+dy\,dy'+dz\,dz')-[(x-x')\,dx+(y-y')\,dy+(z-z')\,dz]\,dx'\},$$

and corresponding expressions for Y and Z. If one assumes that the z-axis is perpendicular to the plane in which r and ds' lies then one will have $z-z' = 0$ and $dz' = 0$, so one will also have $Z = 0$. We can then call the aforementioned plane the *plane of action*. The formula for X will then become:

$$X = -\,\frac{k\,i\,i'}{r^3}\{(x-x')(dx\,dx'+dy\,dy')-[(x-x')\,dx+(y-y')\,dy]\,dx'\},$$

and correspondingly for Y. Now let the product $i'\,ds'$ be denoted by a, as in my treatise (**Pogg. 64**, pp. 9), let the product $i\ ds$ be denoted by b, and let the (perpendicular) projection of b onto the plane of action be denoted by b_1. Furthermore, let the angles be set to:

$$\angle\, r\,a = \alpha, \qquad \angle\, r\,b_1 = \beta, \qquad \angle\, b_1\,a = \gamma,$$

such that one will then have:

$$\alpha = \angle\, r\,b_1 + \angle\, b_1\,a = \beta + \gamma.$$

One assumes that the y-axis is in the direction b_1 and that the x-axis lies in the direction c perpendicular to it in the plane of action, and indeed in such a way that $\angle\, b_1\,c = +90^\circ$. $i'\,dx'$ and $i'\,dy'$ are then the coordinates of a, while $i\ dx$ and $id\ y$ are those of b, so:

$$\cos\gamma = \cos\,(b_1\,a) = \frac{i\,i'\,(dx\,dx'+dy\,dy')}{a\,b_1}$$

and

$$\cos\beta = \cos(r\, b_1) = \frac{i\,(x-x')\,dx + i\,(y-y')\,dy}{r\, b_1} \ .$$

If we substitute those values then we will get:

$$X = -\frac{k}{r^3}[(x-x')\,a\,b_1\cos\gamma - i'\,dx'\,r\,b_1\cos\beta] \ ,$$
$$Y = -\frac{k}{r^3}[(y-y')\,a\,b_1\cos\gamma - i'\,dy'\,r\,b_1\cos\beta] \ .$$

Now if $y - y'$ is the projection of r onto b_1, so it is equal to r cos (r b_1) = r cos β, and $i'\,dy'$ is the projection of a onto b_1 , so it is equal to a cos (b_1 a) = a cos γ, then:

$$Y = -\frac{k\,a\,b_1}{r^2}(\cos\beta\cos\gamma - \cos\gamma\cos\beta) = 0 \ .$$

Furthermore, if $x - x'$ is the projection of r onto c, so it is equal to:

$$r\cos(r\,c) = r\cos(r\,b_1 + b_1\,c) = r\cos(90^{\circ} + \beta) = -\,r\sin\beta\,,$$

and $i'\,dx'$ is the projection of a into c, so it is equal to:

$$a\cos(a\,c) = a\cos(a\,b_1 + b_1\,c) = a\cos(90o - g) = a\sin\gamma\,,$$

so

$$X = -\frac{k\,a\,b_1}{r^2}(\sin\beta\cos\gamma + \cos\beta\sin\gamma) = \frac{k\,a\,b_1}{r^2}\sin(\beta+\gamma)\,,$$

then:

(2) $$X = \frac{k\,a\,b_1}{r^2}\sin\alpha\,.$$

Since Y and Z are zero, that expression represents the total force. It is identical to the expression that I published in **Pogg. 64**, pp. 9, formula (4), except that k, whose value depends upon which units one assumes, was set equal to unity in that paper.

However, formula (2) takes on an entirely new meaning with the basic law that **Clausius** proved. It no longer represents a hypothesis that can be posed with perhaps equal justification along with other hypotheses but proves to be one that is necessary in the **Clausius** representation. In order to show that and then connect it with other consequences, I would like to illustrate the process in the **Clausius** representation. **Clausius** proved that the basic law that the founder of a unified theory of electrodynamics – viz., Herr **W. Weber** – had exhibited agrees with experiment only under the assumption that in every galvanic current, the positive and negative electricity move in opposite directions with equal velocity. Namely, he showed that when the opposing electrical currents in a galvanic current move with unequal velocities (e.g., the negative one is at rest), based upon **Weber**'s law, the action of the constant current must be distributed over the electricity at rest, which contradicts the experiments. I point out that one can carry out that proof in a highly elementary way by means of a linear current that consists of two concentric circular arcs and two straight line segments, namely, when the common center of both circular arcs lies at the point at which the electricity is at rest, and the two segments are lengthened to go through that point. In that case, a mere glance at **Weber**'s formula:

$$\frac{e\,e'}{r^2}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2+\frac{2}{c^2}\,r\,\frac{d^2r}{dt^2}\right]$$

that the positive current (if one ignores the static effect, which always cancels the effect of the negative current) exerts an attraction on the positive electricity at rest that goes through the center of the current and is proportional to the square of the velocity $\left[\frac{dr}{dt}\right]$, while the negative electricity will be repelled just as strongly in such a way that the action will have the opposite sense to the negative current, but in in such a way that those actions will be proportional to the square of the velocity with which the negative electricity flows. The total effect is then zero only when both velocities are equally large, but in any other case, the electricity at rest must be distributed, which contradicts the experiments. **Clausius** proved the corresponding statement for **Riemann**'s basic law. Both were then consistent with experiments only when one could assume that positive and negative electricity flowed with *equal* velocity (in opposite directions) in any galvanic current. However, that assumption is not admissible, since, e.g., in electrolytes, the electrical currents move with the ions, and they will generally possess unequal velocities.

Now, **Clausius** started from the assumption, which also lies at the foundations of **Weber**'s and **Riemann**'s law, that the force with which a moving electrical particle e' acts upon another e depends upon only the mutual distance between the two particles, along with the directions and magnitudes of their velocities and accelerations. When he simply coupled the results of verified observations and the principle of the conservation of energy, he arrived at his fundamental formula (66), in which, however, an unknown function of r appeared. Nonetheless, that unknown function will drop out automatically when one determines the force with which a current element ds' acts upon another one ds, and one will then arrive at equation (1) and its equivalent one (2), which must then be regarded as being established completely, as long as one does not wish to go higher than perhaps the second time differential of the moving electrical currents.

It should be remarked in regard to the foregoing that formulas (1) and (2) also remain true when the two opposite electrical currents in the opposite directions, but one must then understand the intensity of the current to mean the sum of the positive and negative electricity that flows in the opposite direction that flows through a cross-section of the conductor per unit time.

I shall connect that with an entirely elementary derivation of the action of a constant closed current on a current element that leads to the same results, and which I believe is unknown up to now.

When one replaces a with its value $i' ds'$ in (2), upon integration, one will immediately find the force v that a current-carrying segment BC with the intensity i' exerts upon a current element whose starting point lies at A and whose projection onto the plane ABC is equal to b_1. Namely, when $AD = h$ is the height of the triangle ABC, and can be called the pieces of the triangle (*die Stücke des Dreiecks*) in the more traditional way, one will have:

$$v = \frac{k\, i' b_1}{h} (\cos \beta + \cos \gamma) .$$

In particular, if b_1 points in the same direction as BC then v will have the same direction as AD, and when b_1 is rotated through an arbitrary angle on the plane of action, the direction of v will rotate through the same angle, while the *value* of v will remain the same. If a is the third angle of the triangle and m is the centerline, which bisects that angle and reaches to the opposite side, then it is known that:

$$\frac{\cos \beta + \cos \gamma}{h} = \frac{2 \sin \alpha / 2}{m} ,$$

and the formula above will become:

(3) $$v = \frac{2k\, i' b_1 \sin \alpha / 2}{m} .$$

However, in order to be able to employ that formula directly, it is necessary to also represent the direction of the force v in it, as well. To that end, I must recall some concepts in geometric analysis that I presented in my *Ausdehnungslehren* of 1844 and 1862, namely, the concepts of segments, surface spaces, the addition of segments and surface spaces, and the inner product of the surface space with a segment. Namely, I say that two bounded straight lines are equal to each other as *segments* only when the have the same lengths and directions, and two planar regions are equal as *surface spaces* only when the planes are parallel, and the planar regions have equal areas that point in the same direction. Two segments are *added* when one attaches them continuously so the segment from the starting point of the first one to the endpoint of the last one will be sum of the two segments. Two surface spaces are added when one attaches them continuously as parallelograms, i.e., one attaches them in such a way that the base side of the second one coincides with the top side (i.e., the side that is opposite to base side) of the first one, so the parallelogram whose base side is the base side of the first one and whose top side is the top side of the second parallelogram will be the sum of the two surface spaces. Finally, I understand the *inner product* of a surface space F whose area is unity with a segment b, which I write as $[F \mid b]$, to mean a segment g that is just as long as the projection b_1 of b onto F, is perpendicular to b_1 in the plane F (so also to b), and which is continuously attached to b_1 and bent to the side in which the perimeter of F runs. If one replaces F with λF, where λ expresses the area, then one will have $[\lambda F \mid b] = \lambda g$.

If we apply that to the case above and agree that F denotes the same thing as the surface space of the triangle ABC then it will be clear that from the above that the direction of the force will be the opposite of $[F \mid b]$, and therefore when the direction of the force is, at the same time, expressed by – $[F \mid b]$ and substituted for b_1 in (3), that will give:

$$v = -\frac{2k\,i'\sin\alpha/2}{m}\,[F \mid b] \;. \tag{4}$$

The galvanic current now flows through an arbitrary spatial polygon, one of whose sides is *BC*, while *A* remains the starting point of the current element *b*. For the triangle that is connected to *ABC*, let α be replaced with α_1, let *m* be replaced with m_1, and let *F* be replaced with F_1, etc. The force *V* with which the entire polygon acts upon the current element $\underline{b}$ will then be:

$$V = -2k\,i'[Q \mid \underline{b}] \;, \qquad \text{where} \qquad Q = \frac{\sin\alpha/2}{m}F + \frac{\sin\alpha_1/2}{m_1}F_1 + \cdots \tag{5}$$

That equation includes the following important theorem:

If an arbitrary closed current in space is given then there is a well-defined plane at every point A that one can assume goes through A, and which can be called the plane of action of the current relative to the point A. It has the property that every current element (*b*) *that starts from A will, first of all, experience no effect when it is perpendicular to that plane, and secondly, experience the same effect when it is skew to that plane that its (perpendicular) projection* (b_1) *onto that plane would suffer. Thirdly, the force that it experiences lies in that plane and is perpendicular to the projection* (b_1) *of the current element, and therefore also to the current element itself. Fourthly, when g is the force that the current element* (*b*) (*which starts from A*) *experiences in any position and the projection* (b_1) *of the current element onto the plane of action is rotated through an angle in that plane, the force g will also be rotated through the same angle without changing its value.*

That plane of action is easiest to construct when the current is a polygonal current in space. That is because it is parallel to *Q*, and *Q* can be found directly from formula (5) by adding the surface spaces.

What is much more convenient than the path that was pursued here is the method that involves introducing geometric analysis right from the outset. However, one must then introduce the concept of the outer product of two segments. Namely, I understand the outer product $[a \cdot b]$ of two segments *a* and *b* to be the surface space of the parallelogram that has *a* for its base and *b* for the side that is attached to it. Formula (1) will then immediately imply (although I shall not prove this here) the formula:

$$P = \frac{k}{r^3}[\underline{r} \cdot \underline{a} \mid \underline{b}] \;, \tag{2*}$$

where $\underline{r}$, $\underline{a}$, $\underline{b}$ represent segments whose lengths we denoted by *r*, *a*, *b*, resp., above, and *P* is the force in magnitude and direction.

If we set $\underline{a} = i'\,\underline{ds'}$ here, where $\underline{ds'}$, at the same time, represents the direction of the current element ds', and take $\underline{ds'}$ to be the element of an arbitrary closed current then we will get:

$$(5^*) \qquad V = k\,i'[Q\,|\,b]\,, \qquad \text{where} \qquad Q = \int \frac{[\underline{r} \cdot \underline{ds'}]}{r^3}$$

when we extend the integration over the entire closed current.

Here, as everywhere, there is not the slightest difficulty in converting the formulas of geometric analysis into the (as a rule, very complicated) formulas of ordinary analysis. To that end, one must assume only three segments along the three mutually-perpendicular coordinate axes whose directions are those of the positive axes and whose lengths are unity; let them be e_1, e_2, e_3 . If a_1, a_2, a_3 are the coordinates of a segment a then one must only replace a with $a_1\,e_1 + a_2\,e_2 + a_3\,e_3$. If one applies this process to each segment and then performs the additions and multiplications according to the usual rules of algebra, except that one does not switch the factors in a product without further analysis and combines them then no other segments will remain the formula besides e_1, e_2, e_3, whose multiplication, whether outer or inner, is performed according to the definition the product. In our formula (5*), we will then ultimately get V is the form $V = V_1\,e_1 + V_2\,e_2 + V_3\,e_3$, in which V_1, V_2, V_3 are the desired algebraic expressions.

Stettin, 10 January 1877.