SCREW THEORY AND NULL SYSTEMS

INVITED PAPER

WITH THE

APPROVAL OF THE HIGHER PHILOSOPHICAL FACULTY

AT

UNITED FRIEDRICH UNIVERSITY HALLE-WITTENBERG

AN

INAUGURAL LECTURE:

ON THE

CENTRAL AXIS OF A FORCE SYSTEM

TO BE HELD ON

WEDNESDAY, 2 AUGUST, 12 A.M.

IN THE GREAT AUDITORIUM OF THE UNIVERSITY

BY THE HUMBLE INVITATION OF

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Translated by D. H. Delphenich

HALLE A. S.
BOOK PRINTERY OF THE ORPHANAGE
1899.

Section One

The reduction and decomposition of geometric structures into two components.

The method of reduction that was represented in my father's *Ausdehnungslehre* in the year 1862 ([1]) yields an analytical expression for the decomposition of a point and a plate, a rod and a screw into *two* components with certain properties, and thus a representation that is more preferable for many purposes than the decomposition of that structure into *four* or *six* components that is usually given in the applications of point, plane, and line coordinates. In particular, the decomposition of the geometric structure into two components is of use for the calculations with screws, and thus, likewise for the treatment of null systems, whose properties can be developed in a simpler way by means of the theory of screws.

If A and B are rods ([2]) in space whose lines do not intersect – i.e., two external products of any two points of those lines – *then every multiple point x in space can be represented as the sum of two multiple points y and z that belong to the lines of the rods A and B, respectively* (cf., Fig. 1).

In fact, if one sets:

(1) $x = y + z$

and progressively multiplies this equation by B then, since z lies on B, the product $[zB]$ will vanish, and that will yield the equation:

(2) $[A \cdot xB] = [A \cdot yB]$.

However, from A_2, no. 108, the right-hand side here can be replaced by the product:

$$y\,[A\,B]$$

when the point y lies on the line of the rod A, and the sum of the ranks of A and B is equal to 4. Equation (2) is then converted into:

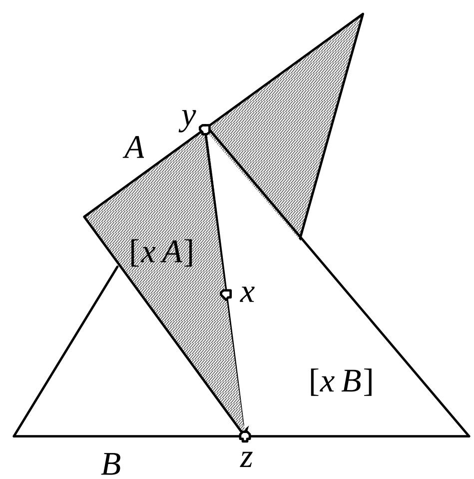

Figure 1.

([1]) Cf., *Hermann Grassmann's* Gesammelte mathematische und physikalische Werke. Volume One, part two. In conjunction with *H. Grassmann, Jr.*, and issued by Fr. *Engel*. Leipzig, Teubner, 1896, no. 127-133, and my remarks on pp. 419-424. In the sequel, that book might be cited briefly by "A_2," as usual.

([2]) On the meaning of the new terminology, one must confer A_2, pp. 437, *et seq.*, rem. on no. 346 and 347; on the introduction of the concepts of rod, field, and plate, see also the author's book: *Punktrechnung und projektive Geometrie. Erster Teil: Punktrechnung.* Contribution to the Festschrift of the Latin High School on the two-hundred year Jubilee of the University Halle-Wittenberg (Halle, 1894), pp. 81, *et seq.*, separate copy, pp. 7, *et seq.*

$$[A \cdot xB] = y\,[A\,B],$$

and since the product $[A\,B] \neq 0$ (because by assumption, the lines of the rods A and B do not intersect), that will yield the following representation for the point y:

$$y = \frac{[A \cdot xB]}{[AB]}. \tag{3}$$

One likewise proves that:

$$z = \frac{[B \cdot xB]}{[BA]}. \tag{4}$$

With that, we have expressed the two summands into the desired decomposition (1) in terms of the point x and the two rods A and B. Moreover, if one introduces these values (3) and (4) into equation (1) and considers the fact that $[B\,A] = [A\,B]$ then one will obtain the *decomposition formula:*

$$x = \frac{[A \cdot xB] + [B \cdot xA]}{[AB]} \tag{5}$$

or also:

$$[A\,B] = [A \cdot xB] + [B \cdot xA]. \tag{6}$$

The two points y and z that are represented by the expressions (3) and (4), which simultaneously define the components of the point x in formulas (1) and (5), are called the *reduction of the point x to the rods A and B, to the exclusion of the rods B and A.*

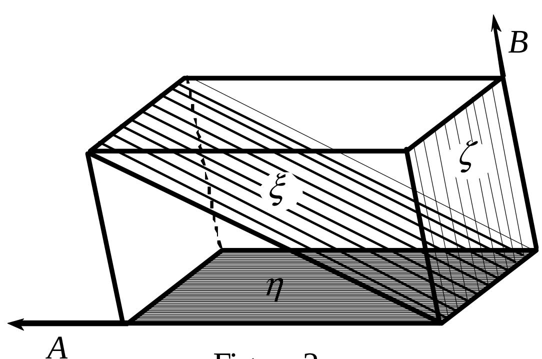

Figure 2.

If one *secondly* poses the dualistically-corresponding problem of *decomposing a plate ξ into two components η and ζ, the first of which – i.e., the plane η – contains a given rod A, while the second one – namely, the plate ζ – goes through a rod B that crosses A* (cf., Fig. 2), then one can proceed in a precisely corresponding way, except that one must replace progressive multiplication with regressive multiplication, and vice versa.

One then regressively multiplies the equation:

$$\xi = \eta + \zeta \tag{7}$$

by B, and since the rod B lies in the plate ζ, so the product $[\zeta B]$ will vanish, one will get the equation:

$$[\xi B] = [\eta B],$$

and it will follow by progressively multiplying by A that:

(8) $$[A \cdot \xi B] = [A \cdot \eta B].$$

However (from A_2, no. 108 and the accompanying remark on pp. 416), the right-hand side of this can be replaced with the product:

$$\eta\, [A\, B].$$

The plane of the plate η then contains the rod A, and the sum of the ranks of A and B is equal to 4. Equation (8) thus converts into:

$$[A \cdot \xi B] = \eta\, [A \cdot B]$$

and since $[A\, B]$ is > 0 or < 0, it will give the following value for η:

(9) $$\eta = \frac{[A \cdot \xi B]}{[AB]}.$$

One likewise proves that:

(10) $$\zeta = \frac{[B \cdot \xi A]}{[BA]}.$$

This then yields the *decomposition formula* for the plate ξ:

(11) $$\xi = \frac{[A \cdot \xi B] + [B \cdot \xi A]}{[AB]},$$

in which, one can also write:

(12) $$[A\, B]\, \xi = [A \cdot \xi B] + [B \cdot \xi A].$$

Therefore, the plate η is again called *the reduction of the plate ξ to the rod A, to the exclusion of the rod B,* and the plate ζ *is the reduction of the plate ξ to the rod B, to the exclusion of the rod A.*

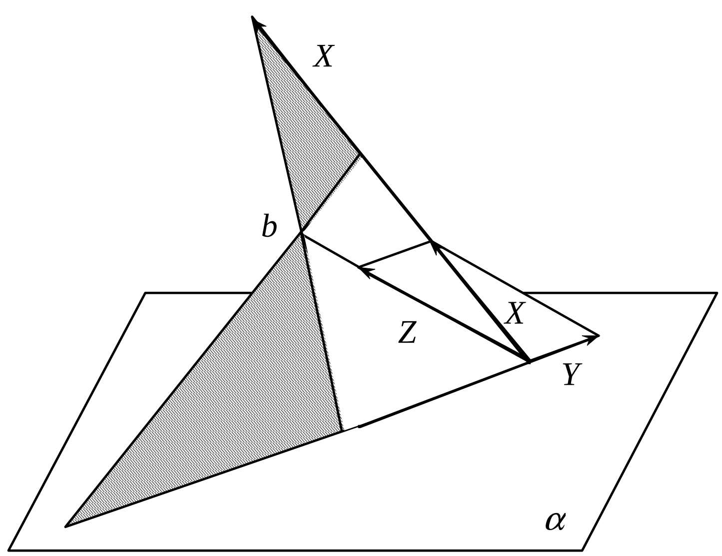

Figure 3.

Thirdly, should a rod X be *decomposed into two components Y and Z, the first of which – i.e., the rod Y – lies in a given plate α, while the second one – namely, the rod Z – lies on a line with a given point b* *that does not belong to the plane of the plate* α (Fig. 3) then one will again multiply the equation:

(13) $$X = Y + Z$$

progressively by b, and since b lies on the line of the rod Z, so the product $[Zb]$ will vanish, one will get the equation:

$$[X\,b] = [Y\,b],$$

and it will follow from this upon regressive multiplication by the plate α that:

(14) $$[\alpha \cdot Xb] = [\alpha \cdot Yb].$$

However (from A_2, no. 108), the right-hand side here can be replaced with the product:

$$Y\,[\alpha\, b].$$

The rod Y lies in the plane of the plate α, and the sum of the ranks of α and b is equal to 4. Equation (14) then converts to:

$$[\alpha \cdot Xb] = Y\,[\alpha \cdot b],$$

and since, by assumption, the point b does not belong to the plane of the plate α, so the product $[\alpha b] \neq 0$, this equation will yield the following representation for the rod Y:

(15) $$Y = \frac{[\alpha \cdot Xb]}{[\alpha b]}.$$

One likewise proves that:

(16) $$Z = \frac{[b \cdot X\alpha]}{[b\alpha]}.$$

The desired *decomposition formula* the reads:

(17) $$X = \frac{[\alpha \cdot Xb]}{[\alpha b]} + \frac{[b \cdot X\alpha]}{[b\alpha]}.$$

Here, the product $[b\alpha] = -[\alpha\, b]$, so the plate α is equal to the product of *three* points, and permuting it with the point b is then admissible only *with a change of sign.* Formula (17) then assumes the form:

(18) $$X = \frac{[\alpha \cdot Xb] - [b \cdot X\alpha]}{[\alpha b]}.$$

or finally:

(19) $$[\alpha\, b]\,X = [\alpha \cdot Xb] - [b \cdot X\alpha].$$

Section two

Application to the screw.

Since formulas (17), (18), and (19) are homogeneous and linear in X, one can also carry them over to a sum of rods. As is known, such a sum cannot be reduced to a single rod, in general, so it can, in the same way, be converted like a force system that acts upon a rigid body. In particular, it can be *represented as the sum of a rod and a field that is perpendicular to it*, assuming that one understands a field to mean the exterior product of two segments; that is, the geometric structure of a force-couple ([1]).

This is further connected with the fact that a sum of rods is especially suitable for the analytical treatment of a *screwing motion* ([2]), and *Hyde* has therefore proposed to follow a terminology of *Ball* by using the expression *screw* for a sum of rods ([3]).

With the help of formula (17), a sum of rods:

$$S = \sum_{i=1}^{n} X_i \,, \tag{20}$$

or as we, with Hyde, would now like to say, a *screw S*, can now be represented as a sum of *two* rods A and B, one of which – the rod A – belongs to an arbitrarily given plane α, while the other rod B goes through a fixed point b that cannot lie in the plane of the plate α.

In fact, if one forms the two reductions Y_i and Z_i that belong to α and b for any rod X_i in the sum (20) using the prescription of formulas (15) and (16), with which, one will get:

$$Y_i = \frac{[\alpha \cdot X_i b]}{[\alpha b]}, \qquad i = 1, 2, \ldots, n, \tag{21}$$

$$Z_i = \frac{[b \cdot X_i \alpha]}{[b \alpha]}, \qquad i = 1, 2, \ldots, n, \tag{22}$$

and then sums from 1 to n then, due to (20), that will produce the formulas:

([1]) On this, cf., A_2, no. 346 and 347, and furthermore, *E. W. Hyde*, “The directional theory of screws,” Annals of Mathematics, v. IV, no. 5, October 1888, pp. 137, *et seq.* is an important paper for, above all, the formulation of the theory of screws. There is also the self-sufficient book of the same author: *The directional calculus, based upon the methods of Hermann Grassmann*, Boston, 1890. Both works are also useful for the following presentation. The same is also true of the two papers of E. Müller: “Die Liniengeometrie nach der Prinzipien der Grassmannschen Ausdehnungslehre,” in the Monatsheften für Mathematik und Physik, II. Jahrg. Wien 1891, and “Neue Methode zur Ableitung der statischen Gesetze,” in the Mitteilungen des K. K. technologischen Gewerbemuseums in Wien, Neue Folge, III Jahrg., Wien 1893, and finally, the great work of Whitehead: *A treatise on Universal Algebra with applications*, vol. I, Cambridge, 1898.

([2]) Cf., J. Lüroth, “Die Bewegung eines starren Körpers,” Zeitschrift für Mathematik und Physik, 43 Jahrg., 1898.

([3]) Cf., the two aforementioned papers of Hyde.

(23) $$\sum_{i=1}^{n} Y_i = \frac{[\alpha \cdot Sb]}{[\alpha b]} \qquad \text{and} \qquad \sum_{i=1}^{n} Z_i = \frac{[b \cdot S\alpha]}{[b\alpha]}.$$

The sums on the left-hand side of them can be reduced to a single rod. The rods Y_i then all belong to the plane of α, so their sum $\sum_{i=1}^{n} Y_i$ will likewise be a rod in that plane. The lines of the rods Z_i all go through the point b, so their sum $\sum_{i=1}^{n} Z_i$ must likewise be a rod whose line contains the point b. If one sets these sums of rods equal to:

(24) $$\sum_{i=1}^{n} Y_i = A \qquad \text{and} \qquad \sum_{i=1}^{n} Z_i = B$$

the equations (23) will be converted into:

(25) $$A = \frac{[\alpha \cdot Sb]}{[\alpha b]} \qquad \text{and} \qquad B = \frac{[b \cdot S\alpha]}{[b\alpha]}.$$

However, the fact that the screw S is actually the *sum* of two rods A and B, thus obtained, is deduced immediately from equation (17). Namely, if one forms equation (17) for all rods X_i in the sum $S = \sum_{i=1}^{n} X_i$ and sums the resulting equations then one will find the corresponding equation for the sum $\sum_{i=1}^{n} X_i$:

(26) $$\sum_{i=1}^{n} X_i = \frac{\left[\alpha \cdot \left(\sum_{i=1}^{n} X_i\right) b\right]}{[\alpha b]} + \frac{\left[b \cdot \left(\sum_{i=1}^{n} X_i\right) \alpha\right]}{[b\alpha]},$$

or, from (20):

(27) $$S = \frac{[\alpha \cdot Sb]}{[\alpha b]} + \frac{[b \cdot S\alpha]}{[b\alpha]},$$

or finally, from (25):

(28) $$S = A + B.$$

One then has the theorem:

Any screw – that is, any arbitrary sum of rods in space – can be represented as the sum of two rods, one of which belongs to the plane of a given plate α, while the line of the other one will go through a given point b that does not lie in the plane of that plate ([1]).

([1]) One finds another proof of this theorem in A_2, no. 285.

Moreover, one can give equation (27), and also equations (18) and (19), the corresponding forms:

(29)
$$S = \frac{[\alpha \cdot Sb] - [b \cdot S\alpha]}{[\alpha b]}$$

and

(30)
$$[\alpha b]\, S = [\alpha \cdot Sb] - [b \cdot S\alpha].$$

The representation of a screw S as the sum of two rods A and B likewise produces an analytic criterion for deciding whether a screw (i.e., a force system) can or cannot be reduced to a single rod (i.e., a single force) or a field (i.e., a force-couple). Namely, while every rod and every field yields a vanishing product under exterior multiplication with itself, a non-reducible screw S always produces a non-zero product under exterior multiplication with itself. Since a rod A is always representable as the exterior product of two points a and b, and a field F, as the exterior product of two segments f and g, the products will be:

$$[A\, A] = [ab \cdot ab] = [abab] = 0$$

and

$$[F\, F] = [fg \cdot fg]\, [fgfg] = 0,$$

because a product of points or segments with two equal factors will vanish. By contrast, for a screw S, as a result of its representability by the sum of two rods A and B – that is, with hindsight of the equation:

(31)
$$S = A + B,$$

the product will be:

$$\begin{aligned}[S\, S] &= [(A + B)(A + B)] \\ &= [B\, A]\, [A\, B] \\ &= 2\, [A\, B],\end{aligned}$$

so the one-half the product will be:

(32)
$$\frac{[S\, S]}{2} = [A\, B].$$

However, the exterior product $[A\, B]$ will vanish if and only if the two rods A and B belong to *the same* plane. In this case, and only in this case, however, can the rod sum $S = A + B$ be reduced to a single rod or a single field. The exterior product of two rods A and B, as whose sum a screw can be represented – or what amounts to the same thing, from equation (32), one-half the exterior product of the screw S with itself – is a characteristic number for the nature of the screw S, and might be called *the characteristic of the screw*. If we denote it by $\mathfrak{c}$, and thus set:

(33)
$$\mathfrak{c} = \frac{[S\, S]}{2} = [A\, B],$$

then result that we obtained can be summarized in the theorem:

A screw S degenerates into a rod or a field if and only if its characteristic $\mathfrak{c} = [S\ S] / 2$ *vanishes.*

Section three

The point-plate conversion of the null system and its inversion.

A screw:

$$S = A + B \tag{34}$$

proves to be especially suitable for the analytical representation of a special kind of reciprocity, namely, for the treatment of the relationship that is defined by a *null system*.

One obtains the point-plate association of a null system when one associates every point p in space with the structure π that is represented by the exterior product $[p\ S]$. One likewise recognizes that this structure π is a plate in space whose plane goes through the point p. Namely, if one introduces the product:

$$\pi = [p\ S] \tag{35}$$

into the rod sum $A + B$, which is equal to S, then the product will decompose into the sum:

$$[p\ S] = [p\ A] + [p\ B], \tag{36}$$

whose summands $[p\ A]$ and $[p\ B]$ will be the exterior products of the point p and the rods A and B, resp., and thus represent two plates whose planes will contain the point p to be mapped and one of the rods A and B. However, the addition of two plates will again yield a plate whose plane goes through the edge of intersection of the two summand-planes, and since this edge of intersection also belongs to the point p that is common to the two planes, the product $[p\ S]$ will actually represent a plate π whose plane will contain the point p, as was asserted above.

Each point p will then be associated by the screw S with a certain plate π whose plane will go through the point p.

However, one also easily convinces oneself that *all plates in the same plane will also be associated with the same point.* Then, if:

$$\pi_1 = [p_1\ S] \qquad \text{and} \qquad \pi_2 = [p_2\ S] \tag{37}$$

are two plates that correspond to the points p_1 and p_2 and themselves *belong to the same plane*, so they will differ by at most a numerical factor $\mathfrak{g}$, such that one has:

$$\pi_1 = \mathfrak{g}\ \pi_2\ , \tag{38}$$

or, due to (37):

$$[p_1\ S] = \mathfrak{g}\ [p_2\ S], \tag{39}$$

then it can be shown that the numerical relationship that corresponds to equation (38), namely:

(40) $$p_2 = \mathfrak{g}\, p_1$$

must also exist between the points p_1 and p_2, such that the two points p_1 and p_2 must coincide in a single point.

Thus, one multiplies equation (39) regressively by S and obtains the new equation:

(41) $$[p_2\, S \cdot S] = \mathfrak{g}\, [p_1\, S \cdot S].$$

However, one has, in general:

$$[p\, S \cdot S] = [p\, (A + B)(A + B)] = [p\, B \cdot A] + [p\, A \cdot B],$$

and since the second-rank factors A and B can be permuted with the third-rank factors $[p\, B]$ and $[p\, A]$ with no change of sign, this will be:

$$[p\, S \cdot S] = [A \cdot pB] + [B \cdot pA].$$

However, from the decomposition formula (6), the right-hand side of the equation is precisely the expression for the product $[A\, B]\, p$, and one thus obtains the formula:

(42) $$[p\, S \cdot S] = [A\, B]\, p,$$

or, with hindsight of (33):

(43) $$[p\, S \cdot S] = \mathfrak{c}\, p.$$

As a result of this formula, equation (41) can then also be written in the form:

(44) $$\mathfrak{c}\, p_2 = \mathfrak{g}\ \mathfrak{c}\, p_1\, .$$

Therefore, if $\mathfrak{c} \neq 0$ – that is, if S is an actual (i.e., non-degenerate) screw – then one will also have:

(45) $$p_2 = \mathfrak{g}\, p_1\, ,$$

which was stated above. *Therefore, all plates in a plane will actually correspond to one and the same point in that plane under our relationship.*

One calls the plane of the plate $\pi = [p\, S]$ that is associated with the point p by the screw S the *null plane of the point p*, and the point p, the *null point of the plane* π, and the relationship between the points p and their null planes $[p\, S]$ is called *the null system of the screw S*, relative to that screw S.

One easily convinces oneself that the relationship of the null system is only *a special case of a general reciprocal relationship.*

Then, *first of all*, the null system will be explained by the *assignment of each point of space to a plane.*

However, *secondly, the points of a fixed plane will always correspond to planes that go through one and the same point.*

In fact, any point p of the plane of the plate:

$$\rho = [p_1\, p_2\, p_3] \tag{46}$$

can be represented as a sum of the form:

$$p = \mathfrak{g}_1\, p_1 + \mathfrak{g}_2\, p_2 + \mathfrak{g}_3\, p_3\,,$$

in which the $\mathfrak{g}_i$ are numerical quantities. However, the null plane $[p\ S]$ of this point p will be:

$$[p\ S] = \mathfrak{g}_1\, [p_1\ S] + \mathfrak{g}_2\, [p_2\ S] + \mathfrak{g}_3\, [p_3\ S],$$

and will then be the corresponding multiple sum of the three plates $[p_1\ S]$, $[p_2\ S]$, $[p_3\ S]$, which will be assigned to the three points p_1, p_2, p_3 of the plate ρ by the null system. They will therefore certainly go through the point of intersection:

$$t = [p_1\ S \cdot p_2\ S \cdot p_3\ S] \tag{47}$$

of the planes of those three plates.

However, with that, we have actually proved that the null planes of the points of a fixed plane all go through a fixed point, and a relationship that assigns points of space with planes will be characterized as a reciprocal relationship by this property.

The special peculiarity of the null systems, as compared to other reciprocal relationships, then consists in just the fact that *any point p of space will itself belong to the plane of the plate that is associated with it*, which is a property that can be expressed analytically by the equation:

$$[p \cdot pS] = 0, \tag{48}$$

which will be satisfied by every point p in space.

This equation can thus serve to derive a further property of the null system, by which, it can be related to a polar system. Namely, if y and z are two entirely arbitrary points in space then the point $p = y + z$ that is represented by their sum will also satisfy equation (48); that is, one will have the equation:

$$[(y + z) \cdot (y + z)\ S] = 0,$$

or, from (48):

$$[z \cdot y\ S] + [y \cdot z\ S] = 0,$$

or finally, the equation:

$$[z \cdot y\ S] = -\ [y \cdot z\ S], \tag{49}$$

which can, moreover, also be developed very easily and directly from the properties of the exterior product.

If follows, in particular, from equation (49) that the equation:

(50) $[z \cdot y\, S] = 0$

will always imply the equation:

(51) $[y \cdot z\, S] = 0,$

and one will then have the theorem:

If z lies on the null plane of the point y then y will also lie on the null plane of the point z.

This theorem already shows that the null system has a certain connection with a *polar system*, which indeed likewise possess a corresponding property, and which is, moreover, just like the null system, a special case of a reciprocity.

In fact, the two relationships of the null system and the polar system have yet another important property in common with each other that can be derived from the first property: Namely, the two relationships are *involutory*.

As was shown above on pp. 10, by way of the null system that belongs to the screw S, the points that lie in a plane with the plate:

(46) $\rho = [p_1\, p_2\, p_3]$

will be assigned to the planes of a bundle of planes with the vertex:

(47) $t = [p_1\, S \cdot\ p_2\, S \cdot p_3\, S].$

However, that plate ρ will likewise be assigned to the point t in that way, if only indirectly. The relationship of the null system then contains, along with the originally developed *point-plate association*, likewise a *plate-point association*, and it shall be shown that:

If one subjects an arbitrary point p to first the point-plate association of the null system, and thus derives its null plane:

(52) $\pi = [p\, S]$

from it, then represents the plate π as the product of three points p_1, p_2, p_3 – that is, in the form:

(53) $\pi = [p\, S] = [p_1\, p_2\, p_3],$

and finally invokes the plate-point relationship of the null system on this plate $[p_1\, p_2\, p_3]$, when one again takes it to a point by means of the equation:

(47) $t = [p_1\, S \cdot\ p_2\, S \cdot p_3\, S],$

then the point t will differ from the original point p by at most a numerical factor.

Since the point t is the point of intersection of the three planes of $[p_1\, S]$, $[p_2\, S]$, $[p_3\, S]$, as a result of equation (47), it will satisfy the three equations:

(54) $[t \cdot p_1\, S] = 0, \quad [t \cdot p_2\, S] = 0, \quad [t \cdot p_3\, S] = 0.$

However, as was shown above, these three equations imply the equations:

(55) $$[p_1 \cdot t\,S] = 0, \quad [p_2 \cdot t\,S] = 0, \quad [p_3 \cdot t\,S] = 0,$$

which say that the plane of the plate $[t\,S]$ goes through the three points p_1, p_2, p_3, so the plate $[t\,S]$ will coincide with the plate $[p_1\,p_2\,p_3]$, up to a numerical factor $\mathfrak{g}$; that is, that one will have:

(56) $$[t\,S] = \mathfrak{g}\,[p_1\,p_2\,p_3]\,.$$

Due to (53), one will thus also have:

(57) $$[t\,S] = \mathfrak{g}\,[p\,S],$$

and it will again follow from this, as on pp. 8, *et seq.*, that the corresponding numerical relation will arise between the points t and p, so one will have:

(58) $$t = \mathfrak{g}\,p.$$

However, with that, we have actually proved that the plate-point relationship of a null system will once more assign the null plane of any point p with precisely its null point, so the relationship is *involutory*.

This reciprocity in the relations between null points and null planes, with hindsight of equation (45), makes it possible to give a simpler representation of the plate-point relationship of the null system. The equation:

(43) $$[pS \cdot S] = \mathfrak{c}\,p$$

shows, in fact, that one can obtain the null point p to the plane of the plate $[p\,S]$ by simple multiplication with the screw S, so in precisely the same way by which one derived the null plane from the point p.

One then gets the formula:

(59) $$r = [\rho\,S]\,.$$

If one recalls the values of S:

(34) $$S = A + B$$

then one will obtain the representation:

(60) $$r = [\rho\,A] + [\rho\,B]$$

for the point r, and since the products $[\rho\,A]$ and $[\rho\,B]$ are the points of intersection of the plane of ρ with the lines of the rods A and B, one will then have the theorem:

The null point r of a plate ρ relative to a screw S lies on the connecting line of the two points that the plane of the plate r cuts out of any two rods A and B whose sum expresses the screw S.

If one then knows *two* such representations of the screw S as a sum then one can construct the null point r of a plate ρ linearly.

To complete the analogy between the analytic representation of the point-plate and plate-point relationship of the null system, one might ultimately develop the dualistic counterpart to formula (43) – that is, a formula for the product $[\rho S \cdot S]$. It is:

$$[\rho S \cdot S] = [\rho\,(A + B) \cdot (A + B)] = [\rho B \cdot A] + [\rho A \cdot B],$$

or since the point factors $[\rho B]$ and $[\rho A]$ can be placed after the rod factors A and B with no change of sign:

$$[\rho S \cdot S] = [A \cdot \rho B] + [B \cdot \rho A].$$

However, from the decomposition formula (12), the right-hand side of the equation is precisely the expression for the product $[A\,B]\,\rho$, and one then obtains the formula:

$$[\rho S \cdot S] = [A\,B]\,\rho,$$

or finally, recalling (33):

$$[\rho S \cdot S] = \mathfrak{c}\,r, \tag{61}$$

which is a formula that once more says that the null system is an involutory relationship.

Section four

The rod relationship of the null system.

It is clear that a null system will assign the points of a line to the planes of a pencil of planes. Every point p of the line of the rod:

$$X = [p_1\,p_2] \tag{62}$$

can be represented as a multiple sum of the points p_1 and p_2 that determine the line – that is, in the form:

$$p = \mathfrak{g}_1\,p_1 + \mathfrak{g}_2\,p_2\,. \tag{63}$$

The null plane $[p\,S]$ of this point will then be:

$$[p\,S] = \mathfrak{g}_1\,[p_1\,S] + \mathfrak{g}_2\,[p_2\,S], \tag{64}$$

and will thus be the *corresponding* multiple sum of the two plates $[p_1 S]$ and $[p_2 S]$ that is associated with the two points p_1 and p_2 that determine the rod X by the null system. In particular, it will then go to the edge of intersection:

(65) $$Y = [p_1 S \cdot p_2 S]$$

of the planes of each plate.

The point sequence of the line X will then be associated with the pencil of planes with the axis Y, and indeed this association will be *projective*, due to the equality of the coefficients of both multiple sums (63) and (64). Indeed, *it is perspective*, since every plane $[p S]$ of the pencil of planes will go through the corresponding point p of the point sequence, from the basic property of the null system.

In addition to this association between the *points of the line X* and the *planes of the pencil with the axis Y*, however, there is also a relation that is worthy of interest that the null system establishes *between the lines X and Y*.

Corresponding to the terminology above, the line Y might be referred to as the *null line* of the line of the rod X, the rod Y itself, as the *null rod* of the rod X, and finally, the relationship between the two rods X and Y, as the *rod relationship of the null system S*.

However, up to now, the relation between the two rods X and Y was mediated by the *two* equations (62) and (65) in a seemingly indirect way. In order to formulate the analytical relationship between the two rods more rigorously, one poses the problem of representing the rod Y directly as a function of the rod X, and – if possible – similar to what one did for the point-plate and plate-point relationship of the null system, to give a *transformation factor* $\mathfrak{S}$, by which, one must multiply the rod X in order to obtain its null rod, and thus satisfy the equation:

(66) $$Y = X\,\mathfrak{S}.$$

For such a factor $\mathfrak{S}$, its *dimension* will agree with that of a numerical quantity, insofar as it takes a rod to another rod under multiplication. However, *apart from that*, it seems to be an *extensive quantity* that is essentially different from a numerical quantity, since it converts a rod X into a rod Y that (generally) *lies on another line*.

Now, in order to ascertain an analytical expression for this conversion factor $\mathfrak{S}$, one introduces a brief symbol for the first factor, namely, the plate $p_1 S$, in the product $[p_1 S \cdot p_2 S]$, which, from (65), represents the expression for the null rod Y of the rod $X = [p_1 p_2]$, so one sets, say:

(67) $$[p_1 S] = \pi_1 .$$

Every product then assumes the form:

(68) $$[p_1 S \cdot p_2 S] = [\pi_1 \cdot p_2 S],$$

in which the main thing is that it agrees with the first term on the right-hand side of the decomposition formula (30). In order to complete this agreement, one changes the order of factors in both terms of the right-hand side of this formula, and thus obtains the equation:

$$[\alpha\, b]\, S = [\alpha \cdot bS] - [b \cdot \alpha S],$$

in which, the first term on the right-hand side will now correspond to the right-hand side of (68) precisely, and from which, the value of that term will follow:

$$[\alpha \cdot bS] = [\alpha\, b]\, S + [b \cdot \alpha S] .$$

Correspondingly, this will then yield the following expression for the product on the right-hand side of (68):

$$[\pi_1 \cdot p_2\, S] = [\pi_1\, p_2]\, S + [p_2 \cdot \pi_1\, S],$$

and if one again replaces p_1 with its value $[p_1 S]$ in (67) then one will get the following representation for the product $[p_1\, S \cdot p_2\, S]$, whose conversion we wish to arrive at:

$$[p_1\, S \cdot p_2\, S] = [p_1 S\, p_2]\, S + [p_2 \cdot p_1\, S\, S],$$

or, with hindsight of (43):

$$[p_1\, S \cdot p_2\, S] = [p_1 S\, p_2]\, S + [p_2 \cdot \mathfrak{c}\, p_1],$$

or also:

(69) $$[p_1\, S \cdot p_2\, S] = [p_1\, p_2\, S]\, S + \mathfrak{c}\, [p_1\, p_2].$$

Finally, if one replaces the products:

$$[p_1\, S \cdot p_2\, S] \qquad \text{and} \qquad [p_1\, p_2]$$

with their values Y and X, respectively, then that will yield the final formula:

(70) $$Y = [X\, S]\, S - \mathfrak{c}\, X.$$

Therefore, the *first* of the two demands that were posed above will be fulfilled when one, in fact, has represented the rod $Y = [p_1\, S \cdot p_2\, S]$ as a function of the rod $X = [p_1\, p_2]$.

However, *secondly*, in order to convert the expression thus-obtained for the rod Y as a *product*, one of whose factors is the rod X to be mapped, one extracts the factor X from the difference on the right-hand side of (70) and marks the place at which the factor X stood in the first term of the difference before the extraction (since it indeed does not mean the same thing as the original expression) with an arbitrary symbol – perhaps with the character L – which should be interpreted as saying that a *gap* has been created by the extraction of the factor X; likewise, the choice of an uppercase Latin character shall say that the factor that enters into that gap – viz., the *filling* in the gap – must be a rod (or also a sum of rods).

Equation (70) will assume the following form after the conversion that was described:

(71) $$Y = X\, \{[L\, S]\, S - \mathfrak{c}\}.$$

If one then lets $\mathfrak{S}$ denote the *expression for the filling*, by which the rod X will be multiplied, and thus sets:

(72) $$\mathfrak{S} = [L\, S]\, S - \mathfrak{c},$$

then one will actually have:

$$Y = X\,\mathfrak{S} \tag{73}$$

as required, and with that, the second of the demands that were imposed above can be fulfilled, as well. *One has then found a transformation factor $\mathfrak{S}$ that mediates the relationship between the rods X and Y – or, as we would like to say, the "rod relationship" of the null system of the screw S.*

However, one can give the relationship factor $\mathfrak{S}$ yet *another* form, when one performs a corresponding conversion on the difference representation (70) of the rod Y, whose original product representation was (65), and thus converts it into a product of the form $X\,\mathfrak{S}$, at the same time. For this, one needs a conceptual determination of the *combinatorial multiplication of a product of points or plates with a missing expression that contains just as many point or plate gaps as the number of factors that the product possesses.*

It is self-explanatory that the expressions:

$$p\,[l\,S] \qquad \text{and} \qquad \rho\,[\lambda\,S],$$

in which the symbols:

$$p \text{ and } \rho, \qquad l \text{ and } \lambda,$$

refer to a point and a plate and a point and the lack of a plate and a point, respectively, mean nothing else but the products:

$$[p\,S] \text{ and } [\rho\,S].$$

By contrast, expressions of the form:

$$[p_1\,p_2\,(l\,S_1 \cdot lS_2)], \qquad [\rho_1\,\rho_2\,(\lambda\,S_1 \cdot \lambda S_2)],$$

for example, or the general expressions:

$$[p_1\,p_2\,\ldots\,p_n\,(l\,S_1 \cdot lS_2\,\ldots\,lS_n)], \qquad [\rho_1\,\rho_2\,\ldots\,\rho_n\,(\lambda\,S_1 \cdot \lambda S_2\,\ldots\,\lambda S_n)],$$

require further clarification.

One understands the combinatorial product:

$$[p_1\,p_2\,\ldots\,p_n\,\mathbf{A}]$$

to mean a product of n quantities $p_1\,p_2\,\ldots\,p_n$ of arbitrary, but equal, rank with a missing expression **A** with just as many gaps of that rank as the arithmetic mean of all quantities that emerge when one lets the factors of the product $[p_1\,p_2\,\ldots\,p_n]$ enter into the gaps of the missing expression **A** in all possible sequences and prefixes the resulting expressions with a + or – sign according to whether the sequences of quantities p_i that enters into the gaps in the expression **A** experience an even or odd number of inversions relative to the original sequence of these quantities, respectively ([1]).

In the following development, this concept will generally be made use of only for the case in which the missing expression **A** has the form $[lS \cdot lS]$ or $[\lambda S \cdot \lambda S]$; that is, it has the form of a *combinatoric square*, for which we would also like to write the briefer symbols $(l\,S)^2$ and $(\lambda\,S)^2$. The formulas of the general explanation simplify appreciably for such combinatorial squares of expressions with gaps, and one gets:

$$[p_1\, p_2\, (l\,S)^2] = \frac{[p_1S \cdot p_2S] - [p_2S \cdot p_1S]}{2},$$

and since

$$[p_1\, S \cdot p_2\, S] = -\,[p_2\, S \cdot p_1\, S],$$

the above expression will simplify to:

(74) $$[p_1\, p_2\, (l\,S)^2] = [p_1\, S \cdot p_2\, S],$$

and one will likewise get:

(75) $$[\rho_1\, \rho_2\, (\lambda\, S)^2] = [\rho_1\, S \cdot \rho_2\, S]\,.$$

If one again sets:

$$[p_1\, p_2] = X \qquad \text{and} \qquad [p_1\, S \cdot p_2\, S] = Y$$

then equation (74) can also be written in the form:

(76) $$Y = [X\,(l\,S)^2],$$

and when one again excises the rod factor X and sets an equivalent gap L in its place, it can be written in the form:

(77) $$Y = X\,[L\,(l\,S)^2]\,.$$

If one compares this expression for Y with the expression (73) and considers that the two formulas (73) and (77) are true for an arbitrary rod X then it will follow that the factor of X in equation (77) represents a second form for the relationship factor $\mathfrak{S}$; that is, that one will have the equation:

(78) $$\mathfrak{S} = [L\,(l\,S)^2]\,.$$

Finally, one will find a *third* expression for the relationship factor $\mathfrak{S}$ that is dualistic to (78) when one starts with the plate-point association of the null system, not the point-plate one that we started with up to now. In fact, if one considers the rod X that will be converted by the null system, not as the progressive product of two points p_1 and p_2, as above, but as the regressive product of two plates ρ_1 and ρ_2, and thus sets:

([1]) Cf., A_2, no. 504, *et seq.* In particular, it is shown there what is necessary in order to complete the concept above, namely, that every alteration of the quantities p_i for which the product $[p_1\, p_2\, \dots\, p_n]$ remains the same will also leave the product $[p_1\, p_2\, \dots\, p_n\, \mathbf{A}]$ invariant.

(79) $$X = [\rho_1\ \rho_2],$$

and denotes the points that are associated with the plates ρ_1 and ρ_2 by the screw S by r_1 and r_2, resp., such that one has:

(80) $$r_1 = [\rho_1\ S], \qquad r_2 = [\rho_2\ S],$$

then it can be shows that the rod that is represented by the product of these two points:

(81) $$[r_1\ r_2] = [\rho_1\ S \cdot \rho_2\ S]$$

is *also identical in its length and sense* with the null rod Y that is associated with the rod X by each of formulas (65), (71), (73), (77).

One thus produces the formula:

(82) $$[\rho_1\ S \cdot \rho_2\ S] = [\rho_1 \rho_2\ S]\ S - \mathrm{c}\ [\rho_1 \rho_2]$$

that corresponds dualistically to formula (69) in precisely the same way as on pp. 14. One replaces the product $[\rho_1\rho_2]$ in its right-hand side with its value X from (79), and thus obtains the formula:

(83) $$[\rho_1\ S \cdot \rho_2\ S] = [X\ S]\ S - \mathfrak{c}\ X,$$

and in fact when one compares this with (70), it will also emerge that the length and sense of the product $[\rho_1\ S \cdot \rho_2\ S]$ is equal to that of the rod Y that is defined by equation (65); that is, that:

(84) $$Y = [\rho_1\ S \cdot \rho_2\ S],$$

or due to (75):

$$Y = [\rho_1\ \rho_2\ (\lambda\ S)^2]$$
$$= [X\ (\lambda\ S)^2]$$

or

(85) $$Y = X\ [L\ (\lambda\ S)^2],$$

from which it will, in fact, follow that there is a third value for $\mathfrak{S}$:

(86) $$\mathfrak{S} = [L\ (\lambda\ S)^2]\ .$$

On the basis of the three representations (72), (78), and (86) for the relationship factor $\mathfrak{S}$, the rod relationship of the null system that it mediates can now be examined more closely.

It next follows that, according to whether the rod X is:

a progressive product of two points p and x, and thus takes the form $X = [p\ x]$

or

a regressive product of two plates ρ and τ, and thus takes the form $X = [\rho\ \tau]$,

and whether one employs (78) or (86), resp., for the representation of $\mathfrak{S}$, one will have the equations:

$$X\ \mathfrak{S} = [p\ x]\ \mathfrak{S} = [p\ x]\ [L\ (l\ S)^2] = [p\ x\ (l\ S)^2] = [p\ S \cdot x\ S] = [\pi\ \xi] \tag{87}$$

or

$$X\ \mathfrak{S} = [\rho\ \tau]\ \mathfrak{S} = [\rho\ \tau]\ [L\ (\lambda\ S)^2] = [\rho\ \tau\,(\lambda\ S)^2] = [\rho\, S \cdot \tau\, S] = [r\ t], \tag{88}$$

resp., in which the null planes of p and x are denoted by π and ξ, resp., while the null points of ρ and τ are denoted by r and t, resp., so one sets:

$$[p\ S] = \pi, \qquad [x\ S] = \xi, \qquad \text{and} \qquad [\rho\, S] = r, \qquad [\tau\, S] = t\,. \tag{89}$$

Formulas (87) and (88), the first of which only summarizes the results that were scattered above, includes the theorem:

If one represents a rod X as a product of two points then its null rod X $\mathfrak{S}$ *will be the product of any points of the plates that are associated with those two points by the screw S; that is, a piece of their edge of intersection. Moreover, if one represents a rod X as a product of two plates then its null rod X* $\mathfrak{S}$ *will be the product of any points that are associated with those plates by the screw S; that is, a piece of their connecting line.*

Furthermore, if X and Y are two *intersecting* rods, and p is their point of intersection, while ρ is their plane, then the two rods can be represented in the forms:

(90) $\quad X = [p\ x], \qquad Y = [p\ y], \qquad\qquad$ (91) $\quad X = [\rho\ \tau], \qquad Y = [\rho\ \varphi]$.

Therefore, if one again sets:

(92) $\quad [p\ S] = \pi,\ [x\ S] = \xi,\ [y\ S] = \eta, \qquad\qquad$ (93) $\quad [\rho\, S] = r,\ [\tau\, S] = t,\ [\varphi\, S] = f$

then, from (87) and (88), one will have:

(94) $\quad X\ \mathfrak{S} = [\pi\ \xi], \quad Y\ \mathfrak{S} = [\pi\ \eta], \qquad\qquad$ (95) $\quad X\ \mathfrak{S} = [r\ t], \quad Y\ \mathfrak{S} = [r\ f]$.

These equations say that:

The two null rods $X\ \mathfrak{S}$ and $Y\ \mathfrak{S}$ of the intersecting rods X and Y both belong to the null plane π of the point of intersection p of X and Y.

and

The two null rods $X\,\mathfrak{S}$ and $Y\,\mathfrak{S}$ of the intersecting rods X and Y have the null point r of the plane ρ that connects the rods X and Y in common.

One then has the theorem:

If two rods X and Y intersect then their null rods $X\,\mathfrak{S}$ and $Y\,\mathfrak{S}$ will also intersect.

Furthermore, in order to answer the question of whether *the equation:*

$$[Z \cdot Y\mathfrak{S}] = 0 \tag{96}$$

is invariantly coupled with the equation:

$$[Y \cdot Z\mathfrak{S}] = 0 \tag{97}$$

by the rod relationship $\mathfrak{S}$ of the null system, similarly to its plate-point relationship [cf., eqs. (50) and (51)], when the line of the rod Z cuts the null line of Y, and conversely, the line of the rod Y cuts the null line of Z, one defines the two bilinear forms:

$$[Z \cdot Y\,\mathfrak{S}] \qquad \text{and} \qquad [Y \cdot Z\,\mathfrak{S}].$$

Due to (72), one will have:

$$Y\mathfrak{S} = [Y\,S]\,S - \mathfrak{c}\,Y, \qquad \text{and} \qquad Z\mathfrak{S} = [Z\,S]\,S - \mathfrak{c}\,Z,$$

so

$$[Z \cdot Y\,\mathfrak{S}] = [Y\,S]\,[Z\,S] - \mathfrak{c}\,[Z\,Y] \qquad \text{and} \qquad [Y \cdot Z\,\mathfrak{S}] = [Z\,S]\,[Y\,S] - \mathfrak{c}\,[Y\,Z].$$

However, since two rod factors Y and Z commute with each other with no change of sign, the right-hand sides of these two equations will be equal to each other. Therefore, the relation:

$$[Z \cdot Y\,\mathfrak{S}] = [Y \cdot Z\,\mathfrak{S}] \tag{98}$$

will exist between the bilinear forms in question, from which, it will, in fact, follow that the two equations (96) and (97) are invariantly coupled with each other.

That then yields the theorem:

If the line of the rod Z cuts the null line of the rod Y then conversely the line of the rod Y will cut the null line of the rod Z.

It can now be suspected that the relationship $\mathfrak{S}$ is also *involutory*, so the two-fold application of the transformation $\mathfrak{S}$ to an arbitrary rod X will again *take it to that rod*, except for possibly a numerical factor.

In order to prove this property, one again employs the first representation of the relationship $\mathfrak{S}$:

(72) $$\mathfrak{S} = [L\ S]\ S - \mathfrak{c}\ .$$

A *single* multiplication of an arbitrary rod X by the expression $\mathfrak{S}$ will convert that rod into the rod:

(99) $$X\ \mathrm{S} = [X\ S]\ S - \mathfrak{c}\ X.$$

However, if one *once more* subjects the rod $X\ \mathfrak{S}$ thus-obtained to the transformation (72) then one will get the expression:

$$X\mathfrak{S}\mathfrak{S} = [X\mathfrak{S} \cdot S]\ S - \mathfrak{c}\ X\mathfrak{S}$$

for the null rod $X\mathfrak{S}\mathfrak{S}$ of the null rod $X\mathfrak{S}$ of X; that is, due to (99):

$$\begin{aligned} X\mathfrak{S}\mathfrak{S} &= [\{[X\ S]\ S - \mathfrak{c}\ X\}\ S]\ S - \mathfrak{c}\ \{[X\ S]\ S - \mathfrak{c}\ X\} \\ &= ([X\ S]\ [S\ S]\ S - \mathfrak{c}\ [X\ S]\ S - \mathfrak{c}\ [X\ S]\ S + \mathfrak{c}^2\ X. \end{aligned}$$

However, since the product $[S\ S] = 2\mathfrak{c}$, from (33), this expression will simplify to:

(100) $$X\mathfrak{S}\mathfrak{S} = \mathfrak{c}^2 X,$$

and one will have the theorem:

The rod relationship $\mathfrak{S}$ is involutory; that is, when it is applied twice to any rod X on an arbitrary line, it will convert that rod into a rod on the same *line.*

One can give this theorem yet another statement. Namely, if one sets, as above:

(101) $$X\ \mathfrak{S} = Y$$

then equation (100) can also be written in the form:

(102) $$Y\ \mathfrak{S} = \mathfrak{c}^2 X.$$

However, from the simultaneous validity of equations (101) and (102), one can give the theorem above the new form:

If the line of the rod Y is the null line of the rod X then, conversely, the line of the rod X will be the null line of the rod Y.

Due to the reciprocality in the relation between the two lines of the rods X and Y, one calls two lines in space, one of which is the null line of the other relative to a null system, *conjugate to each other relative to the null system* and also refers to two such lines as *two conjugates* of the null system.

If one can describe the rod X of a pencil of rays by:

$$X = [p\,(\mathfrak{g}_1\, x_1 + \mathfrak{g}_2\, x_2)] = [\rho\,(\mathfrak{n}_1\, \varphi_1 + \mathfrak{n}_2\, \varphi_2)] \tag{103}$$

then one will obtain the following expressions for its null rod $X\mathfrak{S}$, with consideration given to (87) and (88):

$$X\mathfrak{S} = [p\,(\mathfrak{g}_1\, x_1 + \mathfrak{g}_2\, x_2)]\,\mathfrak{S} = [p\,S\,(\mathfrak{g}_1\,[x_1\,S] + \mathfrak{g}_2\,[x_2\,S]) = [\pi\,(\mathfrak{g}_1\,\xi_1 + \mathfrak{g}_2\,\xi_2)] \tag{104}$$

and

$$Y\mathfrak{S} = [\rho\,(\mathfrak{n}_1\,\varphi_1 + \mathfrak{n}_2\,\varphi_2)]\,\mathfrak{S} = [\rho\,S\,(\mathfrak{n}_1\,[\varphi_1\,S] + \mathfrak{n}_2\,[\varphi_2\,S]) = [r\,(\mathfrak{n}_1\,f_1 + \mathfrak{n}_2\,f_2)]. \tag{105}$$

Here, however, the product:

$$X\,\mathfrak{S} = [\pi\,(\mathfrak{g}_1\,\xi_1 + \mathfrak{g}_2\,\xi_2)]$$

in equation (104) represents the pencil of rays that is cut out (from pp. 13) of the perspective pencil of planes $\mathfrak{g}_1\,\xi_1 + \mathfrak{g}_2\,\xi_2$ that are null planes to the point sequence $\mathfrak{g}_1\,x_1 + \mathfrak{g}_2\,x_2$ by the null plane π to the point p. This pencil of rays is then itself perspective to the point sequence $\mathfrak{g}_1\,x_1 + \mathfrak{g}_2\,x_2$, and thus, from (103), also projective to the pencil of rays $[p\,(\mathfrak{g}_1\,x_1 + \mathfrak{g}_2\,x_2)]$ that is described by the rod X. Moreover, the plane p of the pencil of rays $X\,\mathfrak{S}$, as the null plane of the point p, goes through the vertex p of the original pencil of lines X.

On the other hand, the product:

$$X\,\mathfrak{S} = [r\,(\mathfrak{n}_1\,f_1 + \mathfrak{n}_2\,f_2)]$$

in equation (105) represents the pencil of rays that is cut out by the null plane π of the point r from the pencil of planes $\mathfrak{n}_1\,\varphi_1 + \mathfrak{n}_2\,\varphi_2$, which is perspective to the point sequence $\mathfrak{n}_1\,f_1 + \mathfrak{n}_2\,f_2$ that consists of the null points of its planes. This pencil of rays is itself perspective to the pencil of planes $\mathfrak{n}_1\,\varphi_1 + \mathfrak{n}_2\,\varphi_2$, and therefore, from (103), also projective to the pencil of rays $[\rho\,(\mathfrak{n}_1\,\varphi_1 + \mathfrak{n}_2\,\varphi_2)]$ that is described by the rod X. Moreover, the vertex r of the pencil of rays $X\,\mathfrak{S}$, as the null point of the plane ρ, lies in the plane ρ of the original pencil of rays X.

One then obtains the theorem:

Any pencil of rays will be taken to a projective pencil of rays by the rod relationship $\mathfrak{S}$ *whose plane goes through the vertex of the latter pencil of rays and whose vertex lies in its plane.*

The connecting line of the vertices p and r of these two pencils of rays thus coincides with the edge of intersection of their planes, and since the planes of each of the two pencils of rays is the null plane of the vertex of the other pencil of rays, the connecting line of the two vertices will likewise be the edge of intersection of their two null planes, and thus *its proper null line*, or as one says, a *double line* or *guiding line* of the null system. In this, lies the theorem:

Two conjugate pencils of rays of a null system are projectively related to each other in such a way that the connecting line of their vertices is a self-corresponding ray.

Furthermore, if X is an arbitrary ray of the pencil of rays, so:

$$X = [p\,(\mathfrak{g}_1\, x_1 + \mathfrak{g}_2\, x_2 + \mathfrak{g}_3\, x_3)], \tag{106}$$

and Y is an arbitrary ray of a planar system of rays, that is:

$$Y = [\rho\,(\mathfrak{n}_1\, \varphi_1 + \mathfrak{n}_2\, \varphi_2 + \mathfrak{n}_3\, \varphi_3)], \tag{107}$$

then one will have:

$$\begin{aligned} X\mathfrak{S} &= [p\,(\mathfrak{g}_1\, x_1 + \mathfrak{g}_2\, x_2 + \mathfrak{g}_3\, x_3)]\,\mathfrak{S} = [p\,S\,(\mathfrak{g}_1\,[x_1\,S] + \mathfrak{g}_2\,[x_2\,S] + \mathfrak{g}_3\,[x_3\,S])], \\ &= [\pi\,(\mathfrak{g}_1\,\xi_1 + \mathfrak{g}_2\,\xi_2 + \mathfrak{g}_3\,\xi_3)] \end{aligned} \tag{108}$$

and

$$\begin{aligned} Y\,\mathfrak{S} &= [\rho\,(\mathfrak{n}_1\, \varphi_1 + \mathfrak{n}_2\, \varphi_2 + \mathfrak{n}_3\, \varphi_3)]\,\mathfrak{S} = [\rho\,S\,(\mathfrak{n}_1\,[\varphi_1\,S] + \mathfrak{n}_2\,[\varphi_2\,S] + \mathfrak{n}_3\,[\varphi_3\,S])] \\ &= [r\,(\mathfrak{n}_1\,f_1 + \mathfrak{n}_2\,f_2 + \mathfrak{n}_3\,f_3)]\,, \end{aligned} \tag{109}$$

in which lies the theorem:

A pencil of rays will go to a projective planar system of rays whose plane goes through the vertex of the pencil of rays by the rod relationship $\mathfrak{S}$ *of a null system.*

A planar system of rays will go to a projective pencil of rays whose vertex lies in the plane of the system of rays by the rod relationship $\mathfrak{S}$ *of a null system.*

One might further seek the condition for the line of a rod X to have a point in common with the line of its null rod $X\,\mathfrak{S}$.

This immediately yields the equation:

$$[X \cdot X\mathfrak{S}] = 0. \tag{110}$$

Now, from the first formula for the relationship factor $\mathfrak{S}$, namely, the formula:

(72) $$\mathfrak{S} = [L\ S]\ S - \mathfrak{c},$$

the product will become:

(111) $$X\mathfrak{S} = [X\ S]\ S - \mathfrak{c}\ X.$$

Equation (110) will then be converted into:

(112) $$[X\ \{[X\ S]\ S - \mathfrak{c}\ X\}] = 0,$$

or, since the product:

(113) $$[X\ X] = 0$$

for any rod X, into:

(114) $$[X\ S]\ [X\ S] = 0,$$

or finally into the equation:

(115) $$[X\ S] = 0.$$

Thus, if a rod X satisfies equation (110) – that is, if it lines in a plane with its null rod $X\mathfrak{S}$ – then it will also satisfy equation (115).

Conversely, however, when X is a rod (and not a screw), so equation (113) is fulfilled, equation (112), as well as equation (110), will also follow from equation (115).

However, if the rod X satisfies equation (115) then equation (111) will simplify to:

(116) $$X\mathfrak{S} = -\ \mathfrak{c}\ X\ ;$$

that is, the relationship $\mathfrak{S}$ will take the line of the rod X to itself.

The names of *double line* or *guiding line of the null system* were already introduced above for such a line X that is mapped to itself by the rod relationship of a null system.

The results that were obtained can then be summarized in the theorem:

If a line has a point in common with its null line then it will coincide with itself, so it will be a double line of the null system.

Equation (115) is especially suitable for giving a presentation of the spatial distribution of the double lines of a null system.

Namely, if one asks what the double lines of a null system would be that go through a given point p or lie in a given plane then one will set:

(117) $$X = [x\ p]$$

in the former case, or:

(118) $$X = [\tau\ \rho]$$

in the latter. Equation (115) will then assume the two forms:

$$[x\ p\ S] = 0 \quad \text{and} \quad [\tau\ \rho\ S] = 0,$$

or also

(119) $$[x \cdot pS] = 0 \quad \text{and} \quad (120) \quad [\tau \cdot \rho\ S] = 0.$$

However, equation (119) is satisfied by all points x that lie in the null plane $[p\ S]$ of p, and equation (120), by all plates τ whose planes go through the null point $[\rho\ S]$ of ρ. It follows from this that:

All double lines $X = [x\ p]$ *of the null system that go through a fixed point p will lie in a plane, namely, the null plane* $[p\ S]$ *of the point p, and conversely, all rays of a pencil of rays that has an arbitrary point p in space for its vertex and whose plane is its null plane will belong to the double line of the null system.*

Furthermore:

All double lines $X = [\tau\ \rho]$ *of the null system that lie in a fixed plane* ρ *go through one and the same point, namely, through the null point* $[\rho\ S]$ *of the plane* ρ*, and conversely, all lines of a pencil of rays that belongs to an arbitrary plane* ρ *and whose vertex is its null point will belong to the double lines of the null system.*

The double lines of the null system thus define a *linear complex.*

Finally, in order to ascertain *the connection between the double lines and the conjugates of a null system*, we consider two conjugates Y and $Y\mathfrak{S}$, and ask what it would mean for a rod X to have a line that cuts both conjugates. For that, one might only assume that the two conjugates do not coincide in one line, so Y is not, perhaps, a double line of the null system. One will then have that the product satisfies:

(121) $$[Y\ S] \neq 0,$$

and since the line of the rod X should cut the two conjugates Y and $Y\mathfrak{S}$, one will have the equations:

(122) $$[X\ Y] = 0$$

and

(123) $$[X \cdot Y\mathfrak{S}] = 0.$$

However, if one recalls the value of $\mathfrak{S}$ from (72) then equation (123) can also be written in the form:

$$[X\ \{[Y\ S]\ S - \mathfrak{c}\ Y] = 0,$$

and due to (122), it will go to:

(124) $$[X\ S]\ [Y\ S] = 0,$$

or, due to (121), to the equation:

(125) $$[X\ S] = 0,$$

which, from the above, says that the line of the rod X is *a double line of the null system*.

One has then proved the theorem:

Any line that cuts two conjugates of a null system is a double line of the null system.

Obviously, the converse is also true:

If a double line of a null system cuts an arbitrary line Y then it will also cut its null line $Y\mathfrak{S}$.