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Asymmetric elasticity

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To Antonio Signorini on his 70th birthday.

Translated by D. H. Delphenich

Summary. – *A general theory of elastic bodies with asymmetric stress characteristic is developed that is valid for finite deformations. The structure of the isothermal elastic potential will be determined in an explicit way in the case of slightly deformable isotropic bodies.*

To my knowledge, there does not exist a complete theory regarding the deformations and the state of stress of elastic bodies with symmetric stress characteristics, even in the case of infinitesimal deformations. Most works that treat the mathematical theory of general elasticity do not even mention this question, with some rare exceptions. Going back to 1910, a note ([1]) of C. SOMIGLIANA contains the general relations for the case of small deformations.

The problem was then subsequently considered by BODASZEWSKI ([2]), in which he also made an application to hydrodynamics. Nevertheless, one must observe that the results contained in these works are founded on intuitive postulates for the expression for the work done by the internal contact force (in the first place) and the linear relations between the strains (in the second place) that do not seem acceptable to me, as a result of the considerations that follow.

An asymmetry in the stress characteristic can present itself in the presence of body moments, namely, when the body force that acts on any element of volume is reducible to a force and a couple, which might occur, for example, in the presence of magnetic force, and it is this case that is regarded as the most interesting one.

Nevertheless, even when one excludes the presence of body moments one can give examples ([3]) in which one requires asymmetry in the characteristic of the stress for the solution, at least in certain parts of the body. In these cases, the usual theory of symmetric stress characteristic leads fatally to solutions with singularities (e.g., polydromic or infinite ones), and one should not exclude that exceeding the plastic limit is often due to just such singularities. These reasons, by themselves, might succeed in

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([1]) C. SOMIGLIANA, *Sopra un'estensione della teoria dell'elasticità*, Rend. Acc. dei Lincei, 1910, Vol. XIX, 1st issue.

([2]) S. BODASZEWSKI, *On the asymmetric state of stress and its applications to the mechanics of continuous mediums*, Archiwam Mechaniki Stosowanej, **5** (1953), pp. 351.

([3]) E. REISSNER, *Note on the theorem of the symmetry of the stress tensor*, Journal of Mathematics and Physics, Vol. XXIII, 1944, page 192.

attracting interest to a theory in which the stress characteristic might be asymmetric even in the absence of body moments. However, in this case, one cannot exclude *absolutely* the presence of surface moments. That is, one must necessarily assume that the totality of the internal contact forces that act across an arbitrary internal surface element of a body are generally reducible to a force that is applied at a point of the element and a couple whose moment is called the *surface moment*. In this way, one may treat, within the context of regularity ([4]) (and which explains its interest), problems that, within the ambit of the usual theory, admit only solutions with singularities whose existence is not entirely plausible from the physical point of view.

Some authors ([5]) found it difficult to admit the presence of surface moments for the reason that they could not conceive of a way of realizing them, but it seems to me that, in effect, it must be impossible (or difficult) for us to realize external surface moments, while, at the same time, excluding the possibility that they are present inside the body, if one supposes that its elements are small, but not vanishing. After all, I do not think that it is even easy to indicate the way in which one realizes the distribution of surface forces that are admitted by the usual theory of elastic bodies in full generality.

It is therefore my intention to outline the study of a theory with asymmetric stress characteristic that admits not only the presence of body moments – which are often inessential – as well as surface moments. This will be done under the hypothesis of finite deformations, which I do not think is a useless complication, even if one has the objective of establishing a linear theory that is valid for slightly deformable bodies. In fact, one can confirm that a treatment of the theory in that case that runs parallel to that of the symmetric case allows indeterminacies in the structure of the elastic potential (and therefore in the stress-deformation relations) that are difficult to remove without making the linear theory descend from that of finite deformations as a theory of the first approximation, and without considering a special condition that is imposed on the potential itself in the ambit of finite deformations.

If one thinks of the stress tensor as being decomposed into two tensors – a symmetric one and an anti-symmetric one (the latter one being null in the usual theory) – then one finds that the work done by the internal contact force for any infinitesimal displacement when one starts from the current state does not depend upon the anti-symmetric part and depends upon the local rotation only by way of its derivatives. If one supposes that the internal surface moments are null then the expression for that work becomes identical to that of the symmetric case even in the case where the stress characteristic is asymmetric due to the presence of body moments, which ultimately determine the asymmetry in the stress (in this case).

The knowledge of the expression for the work done by the internal contact force is essentially coupled with well-known thermodynamic considerations that establish the stress-deformation relationship, but, with the difference that the ones that occur in the symmetric case present the new circumstance that the stresses depend upon not only the knowledge of a thermodynamic function – the *free energy* – but also on the fact that a certain parameter also does not appear in the expression for work. Indeed, it presents a

([4]) See the example at the conclusion of this article.

([5]) VOIGT, *Theoretiche Studien über die Elasticitätsverhältnisse der Krystalle*, Abhand. K. Ges. Göttingen, 1887.

certain formal analogy with the way things happen in the theory of incompressible elastic body ([6]), but although in this case the condition of incompressibility, which represents an internal constraint, permits us to determine the parameter that presents itself, in the asymmetric elastic theory it is not obvious how one determines it, and, at least in the present state of the considerations that I have carried out, it seems, in general, to constitute a preventative element that one introduces as one does for the structure of the thermodynamic potential.

I have developed the arguments that follow by assuming static conditions, but it is obvious that the general relations are also valid under dynamic conditions, except for the addition of the force of inertia in the right-hand side of the indefinite equation. Things almost always take place for me under isothermal conditions, but the results are clearly applicable to all of the adiabatic cases, by means of only the use of the isentropic condition.

In the case of small deformations, one can admit the hypothesis – which seems quite tenable, to me – that the state of stress that is created by the effect of an arbitrary, infinitesimal, irrotational displacement on a natural state has internal surface moments that are all zero, so it is possible to prove – at least in the isotropic case – that the parameter that intervenes is set equal to zero and the resulting thermodynamic potential is the sum of a function of only the characteristics of deformation and one that is a function of only the derivatives of the local rotation. Of these functions (in the isotropic case), the former has the structure of an elastic potential in the classical theory of small deformations, while the latter one depends upon the knowledge of just one coefficient and is such that the indefinite equations, when written in the components of the displacement are all of fourth order.

It does not seem superfluous to me to give, in conclusion, an example in which the results, like the theory that was developed, plainly point to a regular solution in a case in which the solution that is obtained under the hypothesis of symmetry in the stress characteristic certainly has singularities.

I. – Statement of a theory of finite deformations.

1. Fundamental equations in Eulerian form.

Let $\mathcal{C}$ be the present configuration, $d\mathcal{C}$, its volume element, Σ, the boundary of $\mathcal{C}$, and $d\Sigma$, the boundary element.

Let the body force that acts on the element $d\mathcal{C}$ be reducible to the resultant $\mathbf{F}\ d\mathcal{C}$, which is applied to an interior point of $d\mathcal{C}$ and the couple moment $\mathbf{M}\ d\mathcal{C}$, while the surface force that acts on $d\Sigma$ is reducible to the surface force $\mathbf{f}\ d\Sigma$ that is applied to an interior point of $d\Sigma$ and a couple moment $\mathbf{m}\ d\Sigma$.

([6]) A. SIGNORINI, *Questioni di elasticità non linearizzata e semilinearizzata*, Rend. di Matematica, Roma, 1959, vol. 18, pp. 95.

One supposes that the internal contact force is representable by means of two vectors Φ_ν, Ψ_ν , the former of which is the usual specific stress, while the latter is defined in a manner that is analogous to Φ_ν, but in regard to the moments. In other words, one retains the fact that the applied vector (P, $\mathbf{v}$) is unitary in $\mathbf{v}$, so the totality of the internal contact forces that act across an infinitesimal element $d\sigma$ surrounding P that is contained in the plane π that is orthogonal to $\mathbf{v}$ at P is reducible to the vector (P, $\Phi_\tau\, d\sigma$) and the couple moment $\Psi_\nu\, d\sigma$. (By convention, in analogy to what is valid for Φ_ν, Ψ_ν, this refers to precisely all forces on the elements of the portion of the body that does not contain (P, $\mathbf{v}$) that act on the other portions.)

In reference to the static case, for a portion c that is completely interior to $\mathcal{C}$ and its boundary σ, give $\mathbf{n}$ the direction of the interior normal to σ, to the main equations are written:

(1) $$\int_\sigma \Phi_n d\sigma + \int_c \mathbf{F}\, d\mathcal{C} = 0,$$

(2) $$\int_\sigma OP \wedge \Phi_n d\sigma + \int_c OP \wedge \mathbf{F}\, d\mathcal{C} + \int_\sigma \Psi_n d\sigma + \int_c \mathbf{M}\, d\mathcal{C} = 0.$$

From this, and:

(3) $$\lim_{c\to P} \frac{1}{c} \int_\sigma \Phi_n d\sigma = -\mathbf{F},$$

one deduces that:

(4) $$\lim_{c\to P} \frac{1}{c} \int_\sigma \Psi_n d\sigma = \sum_s \mathbf{c}_s \wedge \Phi_s - \mathbf{M},$$

where ([7]) Φ_s has the usual the significance. In order to obtain (4), one must assume the validity of the relation:

(5) $$\Phi_\nu = \sum_s \Phi_s \nu_s ,$$

and the indefinite equations of CAUCHY, which naturally persist, according to (1), (3).

The ν_s are the direction cosines of $\mathbf{v}$ with respect to a presupposed triad of reference $\mathcal{T}$ that is tri-rectangular and right-handed.

By proceeding in a fashion that is parallel to the one by which one deduced (5) from (3) (CAUCHY tetrahedron), it follows from (4) that:

(6) $$\Psi_\nu = \sum_s \Psi_s \nu_s ,$$

with the obvious meaning for Ψ_s. (6) (like (5)) obviously has a general character, and also persists in the dynamical case, as one easily recognizes.

([7]) In the sequel, it will be intended that any indices in the summation range from one to three, unless stated to the contrary.

From (1), (2), taking into account (5), (6), one deduces, along with the CAUCHY equations, the indefinite equations, and on the boundary, the equations for the vectors $\boldsymbol{\Psi}_s$ that are valid in the static case.

Collectively, they are, in Eulerian form:

$$\sum_s \frac{\partial \Phi_s}{\partial x_s} = \mathbf{F}, \qquad (\text{in } \mathcal{C}), \tag{7}$$

$$\sum_s \Phi_s N_s = \mathbf{f}, \qquad (\text{on } \Sigma), \tag{8}$$

$$\sum_s \frac{\partial \Psi_s}{\partial x_s} = -\sum_s \mathbf{c}_s \wedge \Phi_s + \mathbf{M}, \qquad (\text{in } \mathcal{C}), \tag{9}$$

$$\sum_s \Psi_s N_s = \mathbf{m}, \qquad (\text{on } \Sigma), \tag{10}$$

where the x_s are the coordinates of P with respect to $\mathcal{T}$.

2. Eulerian expression for the work done by internal forces.

For what follows, set:

$$X_{rs} = \mathbf{c}_r \times \Phi_s, \qquad \Psi_{rs} = \mathbf{c}_r \times \Psi_s, \tag{11}$$

in which $\mathbf{c}_r$ denotes the unit vector of the axis with index r.

One supposes that the body is subjected to an infinitesimal displacement while starting from the current state, for which each and every point corresponds to the displacement $\delta P \equiv (\delta u_r)$ and the rotation of the neighboring element with components ([8]):

$$\delta \boldsymbol{\omega}_r = \frac{1}{2}\left[\frac{\partial \delta u_{r+2}}{\partial x_{r+1}} - \frac{\partial \delta u_{r+1}}{\partial x_{r+2}}\right]. \tag{12}$$

The corresponding work $\delta \mathcal{L}^{(i)}$ done by the internal contact force that acts in $\mathcal{C}$ is:

$$\delta \mathcal{L}^{(i)} = -\int_{\mathcal{C}} \sum_s \frac{\partial \Phi_s}{\partial x_s} \times \delta P \, d\mathcal{C} - \int_\Sigma \mathbf{f} \times \delta P \, d\Sigma \tag{13}$$

$$- \int_{\mathcal{C}} \sum_s \frac{\partial \Phi_s}{\partial x_s} \times \delta' \boldsymbol{\omega} \, d\mathcal{C} - \int_\Sigma \sum_s \mathbf{c}_s \times \Phi_s \times \delta' \boldsymbol{\omega} \, d\mathcal{C} - \int_\Sigma \sum_s \mathbf{m} \times \delta' \boldsymbol{\omega} \, d\Sigma .$$

([8]) Naturally, if the index $r + i$ exceeds 3 then it must be diminished by 3. This will always follow.

One arrives at (13) by taking (7), (9) into account.

If one also takes into account (8), (10) then one deduces from (13):

$$\delta\mathcal{L}^{(i)} = -\int_{\mathcal{C}}\left\{\sum_{r,s}\left[X_{rs}\frac{\partial\delta u_r}{\partial x_s}+\Psi_{rs}\frac{\partial\delta'\omega_r}{\partial x_s}\right]+\sum_r \delta'\omega_r(X_{r+1,r+2}-X_{r+2,r+1})\right\}d\mathcal{C}. \tag{14}$$

For the element $d\mathcal{C}$, the work done by the internal force is obtained by multiplying $d\mathcal{C}$ by the expression:

$$\delta l^{(i)} = \sum_{r,s}\left[X_{rs}\frac{\partial\delta u_r}{\partial x_s}+\Psi_{rs}\frac{\partial\delta'\omega_r}{\partial x_s}\right]+\sum_r (X_{r+1,r+2}-X_{r+2,r+1})\delta'\omega_r\ . \tag{15}$$

One immediately conforms that $\delta l^{(i)}$ *only apparently* depends upon $\delta'\omega_r$, but not their derivatives. In fact, taking into account (12), (15) may be written:

$$\delta l^{(i)} = \sum_{r,s}\left[X_{rr}\frac{\partial\delta u_r}{\partial x_r}+X_{r+1,r+2}\frac{\partial\delta u_{r+1}}{\partial x_{r+2}}\right. \tag{16}$$

$$\left. +\ X_{r+1,r+1}\frac{\partial\delta u_{r+2}}{\partial x_{r+1}}+\frac{1}{2}(X_{r+1,r+2}-X_{r+2,r+1})\left(\frac{\partial\delta u_{r+2}}{\partial x_{r+1}}-\frac{\partial\delta u_{r+1}}{\partial x_{r+2}}\right)\right]+\sum_{r,s}\Psi_{rs}\frac{\partial\delta'\omega_r}{\partial x_s},$$

and therefore:

$$\delta l^{(i)} = \sum_{r,s}\left[X_{rr}\frac{\partial\delta u_r}{\partial x_r}+\frac{1}{2}(X_{r+1,r+2}-X_{r+2,r+1})\left(\frac{\partial\delta u_{r+1}}{\partial x_{r+2}}-\frac{\partial\delta u_{r+2}}{\partial x_{r+1}}\right)\right]+\sum_{r,s}\Psi_{rs}\frac{\partial\delta'\omega_r}{\partial x_s}. \tag{17}$$

Therefore, if one sets:

$$\xi_{rs} = \frac{X_{rs}+X_{sr}}{2} = \xi_{sr} \tag{18}$$

then one has:

$$\delta l^{(i)} = \sum_{r,s}\left[\xi_{rs}\delta' e_{rs}+\Psi_{rs}\frac{\partial\delta'\omega_r}{\partial x_s}\right], \tag{19}$$

with:

$$\delta e_{rs} = \frac{1}{2}\left(\frac{\partial\delta u_r}{\partial x_s}+\frac{\partial\delta u_s}{\partial x_r}\right). \tag{20}$$

One finds as a result that $\delta l^{(i)}$ does not depend on the $\delta'\omega_r$, but on its derivatives, and as far as the X_{rs} are concerned, they depend upon only the *dilatation* that is contained in the Eulerian homography of stress, when one thinks of it as decomposed into the sum of a dilatation and an axial homography ([9]).

On the contrary, if one thinks of the **m** and the Ψ_{rs} as being zero, in the sense that the X_{rs} are assumed to be symmetric when **M** is present, then $\delta l^{(i)}$ *does not depend upon* $\delta\omega_r$ *in any way whatsoever.*

3. Fundamental equations in Lagrangian form.

In order to obtain the Lagrangian form of the fundamental equations in the case of finite deformations, recall (1), (2). In scalar form, they may be written:

$$\int_\sigma \sum_s X_{rs} n_s \, d\sigma + \int_c F_r \, d\mathcal{C} = 0, \tag{1′}$$

$$\int_\sigma \sum_s (x_{r+1} X_{r+2,s} - x_{r+2} X_{r+1,s}) \, n_s \, d\sigma + \int_c (x_{r+1} F_{r+2} - x_{r+2} F_{r+1}) \, d\mathcal{C} + \int_\sigma \sum_s \Psi_{rs} n_s \, d\sigma = 0. \tag{2′}$$

Let $\mathcal{C}^*$ denote a presupposed reference configuration, let P^* be the point in it that corresponds to P ([10]) and let y_r be the coordinate of P^* with respect to $\mathcal{T}$.

Set:

$$D = \left\| \frac{\partial x_r}{\partial y_s} \right\| > 0, \tag{21}$$

so one introduces the Lagrangian stress characteristic Y_{rs} of the stress by means of the formula ([11]):

$$X_{rs} = \frac{1}{D} \sum_{l,m} Y_{lm} x_{r,l} x_{s,m} \, . \tag{22}$$

Set:

$$\mathbf{F}^* \, d\mathcal{C}^* = \mathbf{F} \, d\mathcal{C}, \qquad \mathbf{f}^* \, d\Sigma^* = \mathbf{f} \, d\Sigma, \tag{23}$$

$$\mathbf{M}^* \, d\mathcal{C}^* = \mathbf{M} \, d\mathcal{C}, \qquad \mathbf{m}^* \, d\Sigma^* = \mathbf{m} \, d\Sigma. \tag{24}$$

The well-known Lagrangian equations ([12]) follow from (1′):

([9]) As a result of the expression that was postulated in *loc. cit.* in footnote ([1]) in the case of small deformations, $\delta l^{(i)}$ depends upon the $X_{rs} - X_{sr}$ and the $\delta'\omega_r$ directly. As for the other case, $\delta l^{(i)}$ is not zero for a generic rigid displacement.

([10]) Putting the asterisk on the symbol of an object C is always intended to mean that it refers to the corresponding one in C^*.

([11]) In order to indicate the derivative of a generic function f of the y_r with respect to one of them, put the index of its y coordinate after a comma: $\dfrac{\partial f}{\partial y_s} = f_{,s}$.

([12]) See A. SIGNORINI, *Transformazioni termoelastiche finite*, Memoria 1[a]; Annali di Matematica pura ed applicata, ser. IV, Tomo XXII (1943), pp. 33-143 [page 106].

(25) $$\sum_{m,l}(Y_{lm}x_{r,l})_{,m} = F_r^* .$$

Take into account the relations ([13]):

(26) $$n_s\, d\sigma = d\sigma^* \sum_{r,t} C_{rt} n_t^* \mathbf{c}_r \times \mathbf{c}_s = d\sigma^* \sum_t C_{st} n_t^*$$

and set:

(27) $$\Psi_{rs} = \frac{1}{D}\sum_{lm}\varphi_{lm}x_{r,l}x_{s,m} ,$$

and, as a consequence of (22), (26), (27), (2′) assumes the Lagrangian aspect:

(28) $$\int_{\sigma^*}\frac{1}{D}\sum_{mls}x_{s,m}(x_{r+1}x_{r+2,l}-x_{r+2}x_{r+1,l})Y_{lm}\sum_t C_{st}n_t^* d\sigma^*$$

$$+\int_{C^*}(x_{r+1}F_{r+2}^* - x_{r+2}F_{r+1}^* + M_r^*)\,d\mathcal{C}^* + \int_{\sigma^*}\frac{1}{D}\sum_{lmst}\varphi_{lm}x_{r,m}C_{st}n_t^* d\sigma^* = 0.$$

It follows that:

(29) $$\int_{\sigma^*}\sum_{ml}Y_{lm}(x_{r+1}x_{r+2,l}-x_{r+2}x_{r+1,l})n_m^* d\sigma^* + \int_{C^*}(x_{r+1}F_{r+2}^* - x_{r+2}F_{r+1}^* + M_r^*)\,d\mathcal{C}^*$$

$$+\int_{\sigma^*}\sum_{lm}\varphi_{lm}x_{r.l}n_m^* d\sigma^* = 0,$$

from which, taking (25) into account, one deduces that:

(30) $$\int_{C^*}\sum_{lm}Y_{lm}(x_{r+2}x_{r+1,l}-x_{r+1}x_{r+2,l})d\mathcal{C}^* + \int_{C^*}\sum_{lm}(\varphi_{lm}x_{r,l})_{,m}d\mathcal{C}^* - \int_{C^*}M_r^*\, d\mathcal{C}^* = 0.$$

Given the arbitrariness in C^* and the independence of the integrand functions of the domain of integration, it follows from (30) that *almost everywhere:*

(31) $$\sum_{lm}(\varphi_{lm}x_{r,l})_{,m} = \sum_{lm}(x_{r+2}x_{r+1,l}-x_{r+1}x_{r+2,l})Y_{lm} + M_r^* \ .$$

The indefinite equations of equilibrium are therefore the ones that follow from (25) and (31) when one sets:

(32) $$\lambda_{rm} = \sum_{lm}\varphi_{lm}x_{r,l} ,$$

([13]) C_{rt} denotes the algebraic complement of $x_{r,t}$ in the determinant (21).

which may then be written in the form:

(33)
$$\sum_m \lambda_{rm,m} = \sum_{lm} (x_{r+1} x_{r+2,l} - x_{r+1} x_{r+2,l}) Y_{lm} + M_r^* \ .$$

Along with (25), (33), one may associate the boundary conditions:

(34)
$$\begin{cases} \sum_{lm} Y_{lm} x_{r,l} N_m^* = f_r^*, \\ \sum_m \lambda_{rm} N_m^* = m_r^*. \end{cases} \qquad \text{on } \Sigma^*.$$

4. Lagrangian expression for the work done by the internal contact force.

The determination of a Lagrangian expression for the work done by the internal forces may be accomplished by starting with the general Lagrangian equations that we previously established, but one may arrive at them more simply by conveniently transforming the Eulerian expression (19).

To that end, begin with the observation that:

(35)
$$\begin{cases} \dfrac{\partial \delta u_r}{\partial x_s} = \dfrac{1}{D} \sum_l C_{sl} (\delta u_r)_{,l}, \\ \dfrac{\partial \delta' \omega_r}{\partial x_s} = \dfrac{1}{D} \sum_l C_{sl} (\delta' \omega_r)_{,l}, \end{cases}$$

(36)
$$d\mathcal{C} = D\, d\mathcal{C}^*,$$

and that (18), (22) imply that:

(37)
$$\xi_{rs} = \frac{1}{D} \sum_{lm} T_{lm} x_{r,l} x_{s,m} \ ,$$

with:

(38)
$$T_{lm} = T_{ml} = \frac{Y_{lm} + Y_{ml}}{2} \ .$$

(27), (35), (36), (37) permit us to alter (19) into the expression ([14]):

(39)
$$\delta^* l^{(i)} = \frac{1}{D} \sum_{rslmq} x_{r,l} x_{s,m} \left\{ \frac{1}{2} T_{lm} [C_{sq} (\delta u_r)_{,q} + C_{rq} (\delta u_s)_{,q}] + \varphi_{lm} C_{sq} (\delta' \omega_r)_{,q} \right\},$$

([14]) The product $d\mathcal{C}^* \, \delta^* l^{(i)}$ obviously expresses the work that is done by the internal contact force relative to the element dC^*.

which is equivalent to:

(40) $$\delta^* l^{(i)} = \sum_{lm}\left[T_{lm}\sum_r(\delta u_r)_{,m}x_{r,l} + \varphi_{lm}\sum_r x_{r,l}(\delta'\omega_r)_{,m}\right],$$

which, setting:

(41) $$b_{rs} = \sum_i x_{i,r}x_{i,s}\ ,$$

may also be written:

(42) $$\delta^* l^{(i)} = \sum_{lm}\left[T_{lm}\frac{\delta b_{lm}}{2} + \lambda_{lm}(\delta'\omega_r)_{,m}\right].$$

5. A convenient transformation of the expression for the internal force.

In the case of finite deformations, it is not clear what sort of functions of y_r may be regarded as the variations $\delta'\omega_r$ in the passage from the configuration that is characterized by the current values of the x_r to the one that corresponds to the varied values $x_r + \delta x_r$.

Such an inconvenience renders it impossible to make the elegant application of thermodynamics ([15]) that, in the case of symmetric elasticity, permits one to equate – $\delta^* l^{(i)}$ to the variation of a thermodynamic function: *the Helmholtz free energy*. With it, one may conveniently transform the expression (42) for $\delta l^{(i)}$ in the manner that will be established.

Begin with the observation that if one calls $u_i = x_i - y_i$ the components of the displacement $\mathcal{C}^* \to \mathcal{C}$ and interprets the δu_i as the variations of the u_i under the passage from $\mathcal{C}$ to a neighboring configuration then it obviously results that:

(43) $$\delta(u_{i,s}) = \delta(u_i)_{,s} \qquad \delta(u_{i,sm}) = (\delta u_i)_{,sm}\ ,$$

while $F(x)$ is a function that depends on the y_r only by way of the $x_{i,h}$, then one has, at the same time:

(44) $$\begin{cases} \delta(F_{,s}) = \delta\left[\sum_{ih}\frac{\partial F}{\partial x_{i,h}}u_{i,hs}\right] = \sum_{ih}\left[\frac{\partial F}{\partial x_{i,h}}(\delta u_{i,h})_{,s} + \sum_{pq}\frac{\partial^2 F}{\partial x_{i,h}\,\partial x_{i,h}}u_{i,hs}(\delta u_{p,q})\right], \\ \delta(F)_{,s} = \left[\sum_{ih}\frac{\partial F}{\partial x_{i,h}}\delta(u_{i,h})\right] = \sum_{ih}\left[\frac{\partial F}{\partial x_{i,h}}(\delta u_{i,h})_{,s} + \sum_{pq}\frac{\partial^2 F}{\partial x_{i,h}\,\partial x_{p,q}}u_{p,qs}(\delta u_{i,h})\right]. \end{cases}$$

It then results that:

(45) $$(\delta F)_{,s} = \delta(F_{,s}).$$

([15]) E. and F. COSSERAT, *Sur la théorie de l'élasticité*, Premier mémoire; page 59, *et seq.*

On the basis of (35.1), one recognizes that the expression (12) for the $\delta'\omega_r$ may be put into the Lagrangian form:

$$\delta'\omega_r = \frac{1}{2D}\sum_m [(\delta u_{r+2})_{,m} C_{r+1,m} - (\delta u_{r+1})_{,m} C_{r+2,m}]\,. \tag{46}$$

From (43), (44), (45), (46), it follows that:

$$\begin{aligned}(\delta'\omega_r)_{,s} = \frac{1}{2}\sum_{pqm}\Bigg\{&\left[\delta(u_{r+2,m})\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{r+1,m}}{D}\right) - \delta(u_{r+1,m})\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{r+2,m}}{D}\right)\right]u_{p,sq} \\ &- \left[u_{r+2,sm}\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{r+1,m}}{D}\right) - u_{r+1,sm}\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{r+2,m}}{D}\right)\right]\delta(u_{p,q})\Bigg\} \\ &+ \delta\left[\frac{1}{2D}\sum_m (u_{r+2,sm}C_{r+1,m} - u_{r+1,sm}C_{r+2,m})\right].\end{aligned} \tag{47}$$

In (42), set:

$$\mu_{rs} = \frac{1}{2D}\sum_m [u_{r+2,sm}C_{r+1,m} - u_{r+1,sm}C_{r+2,m}]\,, \tag{48}$$

$$\begin{cases} M_{rs} = \dfrac{1}{2}\displaystyle\sum_{pqm}\left[u_{p+1,qm}\frac{\partial}{\partial x_{r,s}}\left(\frac{C_{p+2,m}}{D}\right) - u_{p+2,qm}\frac{\partial}{\partial x_{r,s}}\left(\frac{C_{p+1,m}}{D}\right)\right]\lambda_{pq}, \\ N_{rs} = \dfrac{1}{2}\displaystyle\sum_{pqm}\left[\lambda_{p+1,qm}\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{p+2,s}}{D}\right) - \lambda_{r+2,m}\frac{\partial}{\partial x_{p,q}}\left(\frac{C_{p+1,s}}{D}\right)\right]u_{p,qm}, \end{cases} \tag{49}$$

and due to (47), one gets:

$$\delta^* l^{(i)} = \sum_{rs}\left[T_{rs}\frac{\delta b_{rs}}{2} + (M_{rs} + M_{rs})\delta(u_{r,s}) + \lambda_{rs}\delta\mu_{rs}\right], \tag{50}$$

which may also be written [see (41)]:

$$\delta^* l^{(i)} = \sum_{rs}\left[\sum_l T_{rs}x_{r,l} + M_{rs} + M_{rs}]\delta(u_{r,s})\right] + \lambda_{rs}\delta\mu_{rs}\,. \tag{51}$$

(51) constitutes the previously-announced expression for the work done by the internal contact force.

6. Introduction of the thermodynamic potential.

Suppose the system has reversible transformations, and introduce the function of *free thermodynamic energy:*

(52) $$\mathcal{T} = \mathcal{U} - eTE$$

($\mathcal{U}$ is internal energy, T is absolute temperature, E is the entropy, e is the mechanical equivalent to the heat), so well-known thermodynamic considerations ([16]) imply the equality:

(53) $$\delta^* l^{(i)} + eE\ \delta T = -\ \delta\mathcal{T}$$

for any infinitesimal transformation of the system that starts in the current state.

(51), (53) plainly show that $\mathcal{T}$ must be thought of as depending on the current state only by means of the $x_{r,s}$, μ_{rs}, and T.

In the case of symmetry in the stress characteristic, the relations that express it in terms of the derivatives of $\mathcal{T}$ with respect to the deformation characteristic follow immediately from (53).

In the asymmetric case, however, (53) does not lead to analogous formulas. In fact, one must then take into account the fact that the μ_{rs}, and consequently the $\delta(u_{p,q})$ and the $\delta\mu_{rs}$, are not only independent of them, but, as one easily confirms on the basis of (48), the fact that it results identically that:

(54) $$\sum_{rs} C_{rs}\mu_{rs} = 0,$$

as well as:

(55) $$\sum_{rs}\left[\sum_{rs}\frac{\partial C_{rs}}{\partial x_{p,q}}\delta(u_{p,q})\mu_{rs} + C_{rs}\delta\mu_{rs}\right] = 0.$$

It then follows that (53) must remain valid for not just arbitrary variations $\delta(u_{p,q})$, $\delta\mu_{rs}$, but for all of them that verify (55), and only those.

In the absence of internal constraints, one then deduces from (53), (55) that:

(56) $$\sum_l T_{ls}x_{r,l} + M_{rs} + N_{rs} + \tau\sum_{pq}\frac{\partial C_{pq}}{\partial x_{r,s}}\mu_{pq} = -\frac{\partial\mathcal{T}}{\partial x_{r,s}},$$

(57) $$\lambda_{rs} + \tau C_{rs} = -\frac{\partial\mathcal{T}}{\partial\mu_{rs}},$$

(58) $$e\,E = -\frac{\partial\mathcal{T}}{\partial T},$$

where τ is a parameter, for which one may not exclude, *a priori*, its dependency on the current state; i.e., on the $x_{r,s}$ and μ_{rs} .

([16]) *Loc. cit.*, in note ([15]).

It is almost superfluous to warn that in (56), (57), (58), the derivation of $\mathcal{T}$ with respect to the $x_{r,s}$, μ_{rs} , T must be thought of as independent of those variables. From (56), it follows immediately that:

$$T_{rs} = -\frac{1}{D}\sum_l C_{lr}\left[\frac{\partial \mathcal{T}}{\partial x_{l,s}} + M_{ls} + N_{ls} + \tau \sum_{pq} \frac{\partial C_{pq}}{\partial x_{ls}} \mu_{pq}\right]. \tag{59}$$

The knowledge of $\mathcal{T}$ and the parameter τ – i.e., generally, the two functions of the x_{rs}, μ_{rs}, T – will determine, on the basis of (56), (57), (58), the expressions for the T_{rs}, λ_{rs}, and E. In the isothermal case, it is sufficient to equate the number of unknowns in the system of indefinite equations (25), (33) to the number of equations, taking into account the relations (38). Analogously to the adiabatic case, as one can demonstrate with considerations that are analogous to the ones that are made in the symmetric case, it is then obvious that the expressions that are obtained maintain their validity in the dynamical case.

7. Equations of condition for the free energy.

One may finally affirm that the thermodynamic function $\mathcal{T}$ and the parameter τ must satisfy equations that express the symmetry of T_{rs} . On the basis of (59), they are:

$$\begin{aligned}\sum_l \Bigg\{ C_{lr}\left[\frac{\partial \mathcal{T}}{\partial x_{l,s}} + M_{ls} + N_{ls} + \tau \sum_{pq} \frac{\partial C_{pq}}{\partial x_{l,s}} \mu_{pq}\right] \\ - C_{ls}\left[\frac{\partial \mathcal{T}}{\partial x_{l,r}} + M_{lr} + N_{lr} + \tau \sum_{pq} \frac{\partial C_{pq}}{\partial x_{l,s}} \mu_{pq}\right]\Bigg\} = 0.\end{aligned} \tag{60}$$

One might believe that (60) serve to determine the parameter τ, but, in reality, things are not so, since (60) *only apparently* contains the parameter τ.

In fact, set:

$$\left\{\begin{aligned} \alpha_{rs} &= \sum_l (C_{lr} M_{ls} - C_{ls} M_{lr}), \\ \beta_{rs} &= \sum_l (C_{lr} N_{ls} - C_{ls} N_{lr}), \\ \gamma_{rs} &= \sum_l \left(C_{lr} \frac{\partial C_{pq}}{\partial x_{l,s}} - C_{ls} \frac{\partial C_{pq}}{\partial x_{l,s}} \right), \end{aligned}\right. \tag{61}$$

and with a little patience, one finds that:

(62)
$$\begin{cases} \alpha_{s+1,s} = \dfrac{1}{2D} \displaystyle\sum_{lmpqtv} u_{p,qm} \left[C_{l,s+1} \dfrac{\partial C_{tm}}{\partial x_{l,s}} - C_{ls} \dfrac{\partial C_{tm}}{\partial x_{l,s+1}} \right] \lambda_{vq} \mathbf{c}_v \times \mathbf{c}_p \times \mathbf{c}_t , \\ \beta_{s+1,s} = \dfrac{1}{2D} \displaystyle\sum_{lq} \left[D u_{l,q,s+2} - x_{l,s+2} \sum_{mt} C_{ts} u_{t,qm} \right] \lambda_{vq} , \\ \gamma_{s+1,s} = -\dfrac{1}{2D} \displaystyle\sum_{lmpqtv} u_{p,qm} \left[C_{l,s+1} \dfrac{\partial C_{tm}}{\partial x_{l,s}} - C_{ls} \dfrac{\partial C_{tm}}{\partial x_{l,s+1}} \right] \lambda_{vq} \mathbf{c}_p \times \mathbf{c}_v \times \mathbf{c}_t , \end{cases}$$

and it is easy to see that if one identifies λ_{rs} with $-C_{rs}$ then one has:

(63) $$\beta_{s+1,\,s} = 0, \qquad \text{(for } \lambda_{rs} = -C_{rs}\text{)}$$

(64) $$\alpha_{s+1,\,s} + \gamma_{s+1,\,s} = 0, \qquad \text{(for } \lambda_{rs} = -C_{rs}\text{)},$$

which exhibits the fact that the functions in (60) are completely exhausted when one considers $s = 1, 2, 3$, and $r = s + 1$.

One concludes that (60) are satisfied identically if – taking (49) into account – one identifies the λ_{rs} with $-\tau\, C_{rs}$ in them, and furthermore, that *the coefficients of* τ *in* (60), taking into account (49), *are identically zero.*

From the above, one sees that equations (60) are equivalent to the system:

(65) $$\alpha_{s+1,s} + \beta_{s+1,s} + \sum_l \left[C_{l,s+1} \frac{\partial \mathcal{T}}{\partial x_{l,s}} - C_{ls} \frac{\partial \mathcal{T}}{\partial x_{l,s+1}} \right] = 0,$$

from the consideration of (62.1, 2), if one identifies the λ_{rs} with $-\partial\mathcal{T}/\partial\mu_{rs}$.

The system (65), taking (62) into account, presents itself (at this point) with coefficients that depend upon the second derivatives of the components of the displacement, so this shows that $\mathcal{T}$ depends upon them only by means of μ_{rs}. We would like to say that (65) must be (in appearance) valid identically with respect to the second derivatives of the u_r. It is not, however, superfluous to prove, as one does immediately, that in the system (65) the coefficients depend upon those second derivatives only by means of the μ_{rs}, and that (65) therefore constitutes a system of three partial differential equations that influence the structure of $\mathcal{T}$.

To that end, in the first place, observe that (62.1) may be written:

(66) $$\alpha_{s+1,\,s} = \frac{1}{2D} \sum_{lmq} \left[u_{l+1,\,qm} \left(C_{l+1,\,m}\, x_{l,\,s+2} - C_{l\,m}\, x_{l+1,\,s+2} \right) + u_{l+1,\,qm} \left(C_{l+2,\,m}\, x_{l,\,s+2} - C_{l\,m}\, x_{l+2,\,s+2} \right) \right] \lambda_{pq}$$

and that (65), taking (62.2), (66) into account, becomes:

(67) $$\sum_l \left\{ C_{l,s+1} \frac{\partial \mathcal{T}}{\partial x_{l,s}} - C_{ls} \frac{\partial \mathcal{T}}{\partial x_{l,s+1}} - \frac{1}{2D} \sum_q [D u_{l,q,s+2} - \sum_{mt} C_{lm} x_{t,s+2} u_{t,qm}] \frac{\partial \mathcal{T}}{\partial \mu_{lq}} \right\} = 0.$$

Taking into account the equalities:

(68) $$\begin{cases} C_{\nu+1,s+1} C_{\nu,s+2} - C_{\nu+1,s+2} C_{\nu,s+1} = -x_{\nu+2,s} D, \\ C_{\nu+2,s+1} C_{\nu,s+2} - C_{\nu+2,s+2} C_{\nu,s+1} = x_{\nu+1,s} D, \end{cases}$$

it is easy to confirm the result:

(69) $$\sum_l C_{l,s+2} \left[C_{l,s+1} \frac{\partial \mathcal{T}}{\partial x_{l,s}} - C_{ls} \frac{\partial \mathcal{T}}{\partial x_{l,s+1}} \right] = D \sum_s \left[\frac{\partial \mathcal{T}}{\partial x_{\nu+2,s}} x_{\nu+2,s} - \frac{\partial \mathcal{T}}{\partial x_{\nu+1,s}} x_{\nu+1,s} \right]$$

and in addition, taking (48) into account, the fact that one has:

(70) $$\sum_{lqs} C_{\nu,s+2} \left[D u_{l,q,s+2} - \sum_{mt} C_{tm} x_{t,s+2} u_{t,qm} \right] \frac{\partial \mathcal{T}}{\partial \mu_{lq}} =$$

$$= D \sum_{lqm} (C_{\nu m} u_{l,qm} - C_{lm} u_{\nu,qm}) \frac{\partial \mathcal{T}}{\partial \mu_{lq}} = 2D^2 \sum_q \left[\frac{\partial \mathcal{T}}{\partial \mu_{\nu+2,q}} \mu_{\nu+1,q} - \frac{\partial \mathcal{T}}{\partial \mu_{\nu+1,q}} \mu_{\nu+2,q} \right].$$

Since the determinant of C_{rs} is certainly non-zero, (67) is equivalent to the system that can be deduced by multiplying them generically by $C_{\nu,s+2}$, summing over s, and making the ν vary from 1 to 3. Having done this, the system (67), taking (67), (70) into account, finally changes into the system:

(71) $$\sum_q \left\{ \frac{\partial \mathcal{T}}{\partial x_{\nu+2,q}} x_{\nu+2,q} - \frac{\partial \mathcal{T}}{\partial x_{\nu+2,q}} x_{\nu+2,q} + \frac{\partial \mathcal{T}}{\partial \mu_{\nu+2,q}} x_{\nu+2,q} - \frac{\partial \mathcal{T}}{\partial \mu_{\nu+1,q}} \mu_{\nu+2,q} \right\} = 0$$

whose coefficients depend upon only the $x_{r,s}$ and $\mu_{r,s}$, and that certainly constitutes the most convenient form for the system (60).

Naturally, if the system in question is an elastic body then the work done by the internal force for an arbitrary isothermal non-rigid displacement when one starts with the configuration $\mathcal{C}^*$ of spontaneous equilibrium is negative. Supposing that $\mathcal{T}$ is zero in $\mathcal{C}^*$, it follows from (53) in that case that:

(72) $$\mathcal{T} > 0$$

in any configuration that is distinct from $\mathcal{C}^*$ and not obtainable from it by means of a rigid displacement.

One sees that (71) are certainly satisfied when $\mathcal{T}$ depends upon the $x_{r,s}$ only by means of the b_{rs} [see (41)] and the μ_{rs} only by means of the ν_{rs}, where:

$$\nu_{rs} = \sum_i \mu_{ir}\mu_{is} \,. \tag{73}$$

II. – Linearized theory of isothermal asymmetric elasticity.

1. Solutions that depend upon one parameter – linearized theory.

One replaces the vectors F^*, etc., with the vectors $h\mathbf{F}^*$, $h\mathbf{M}^*$, $h\mathbf{f}^*$, $h\mathbf{m}^*$, where h is a parameter that is independent of the coordinates. The x_r, Y_{lm}, etc., are thought of as functions of h.

Suppose that they are *differentiable* with respect to *h at least once* in the neighborhood of zero, and for the generic function η of h, set:

$$\lim_{h\to 0}\left(\frac{d^n\eta}{dh^n}\right) = h^{(n)}, \qquad (n = 0, 1, \ldots, \bar{n}), \qquad \bar{n} \geq 1. \tag{74}$$

If one supposes that $\mathcal{C}^*$ is the configuration of the system that corresponds to $h = 0$ then one must set:

$$u_r^{(0)} \equiv 0, \tag{75}$$

while one can have:

$$Y_{rs}^{(0)} \neq 0, \qquad \lambda_{rs}^{(0)} \neq 0. \tag{76}$$

In addition, let:

$$\lim_{h\to 0}\frac{d^n}{dh^n}\left(\frac{\partial\eta}{\partial y_l}\right) = \frac{\partial\eta^{(n)}}{\partial y_l} \tag{77}$$

for any function of h and y that appears in the sequel.

In particular, one has:

$$(x_{r,l})^{(0)} = \delta_{rl}\,, \qquad (x_{r,l})^{(1)} = u_{r,l}^{(1)}, \tag{78}$$

where δ_{rl} is the KRONECKER symbol, while ε_{rs}, the deformation characteristic ([17]), will be defined by:

$$\varepsilon_{rs}^{(0)} = 0, \qquad \varepsilon_{rs}^{(1)} = \frac{1}{2}\left(u_{r,s}^{(1)} + u_{s,r}^{(1)}\right), \tag{79}$$

([17]) For simplicity, let this denote the expression for ε_{rs} that is derived from $\varepsilon_{rs}^{(1)}$, despite the fact that for $r \neq s$, this notation applies to all of the quantities $2\varepsilon_{rs}$.

(80) $$\mu_{rs}^{(0)} = 0, \qquad \mu_{rs}^{(1)} = \omega_{r,s}^{(1)},$$

where:

(81) $$\omega_r^{(1)} = \frac{1}{2}\left(u_{r+2,r+1}^{(1)} - u_{r+1,r+2}^{(1)}\right)$$

represents the components of the local rotation inherent to the passage by means of the displacement with components $u_r^{(1)}$ of the configuration $\mathcal{C}^*$ to a neighboring configuration.

Taking (74), (57), (77), (78) into account, substitute hF_r^*, etc., in (25), (33), (34), in place of F_r, etc. (this will always be assumed in what follows), which gives, for $h \to 0$:

(82) $$\begin{cases} \sum_m Y_{rm,m}^{(0)} = 0, \\ Y_{r+1,r+2}^{(0)} - Y_{r+2,r+1}^{(0)} = \lambda_{rm,m}^{(0)}, \end{cases}$$

(83) $$\begin{cases} \sum_m Y_{rm}^{(0)} N_m^* = 0, \\ \sum_m \lambda_{rm}^{(0)} N_m^* = 0, \end{cases}$$

which possibly constrain the pre-existing state of stress in the presupposed configuration of spontaneous equilibrium.

One conforms immediately that in a theory in which one necessarily considers the λ_{mn} to be zero, the possible pre-existing stress state *cannot be symmetric*.

One now evaluates the limits as $h \to 0$ of the first derivative with respect to h of the two sides of equations (25), (33), (34). Taking into (78) account, one finds that:

(84) $$\begin{cases} \sum_m Y_{rm,m}^{(1)} = F_r^* - \sum_{lm} [Y_{lm}^{(0)} u_{r,l}^{(1)}]_{,m} & (\text{in } \mathcal{C}^*) \\ \sum_m \lambda_{rm,m}^{(1)} = Y_{r+1,r+2}^{(1)} - Y_{r+2,r+1}^{(1)} + \sum_l [u_{r+1,l}^{(1)} (Y_{l,r+2}^{(0)} - Y_{r+2,l}^{(0)}) & \\ \qquad + u_{r+2,l}^{(0)} (Y_{r+1,l}^{(0)} - Y_{l,r+1}^{(0)})] + M_r^* & (\text{in } \mathcal{C}^*), \end{cases}$$

(85) $$\begin{cases} \sum_m Y_{rm}^{(1)} N_m^* = f_r^* - \sum_{lm} Y_{lm}^{(0)} u_{r,l}^{(1)} N_m^* \\ \sum_m \lambda_{rm}^{(1)} N_m^* = m_r^*. \end{cases} \qquad (\text{on } \Sigma^*).$$

On the basis of (38), one must set:

(86) $$T_{rs}^{(i)} = \frac{1}{2}(Y_{rs}^{(i)} + Y_{sr}^{(i)}).$$

As a consequence, (82.1), (83.1) become:

$$
(87) \qquad \begin{cases} \sum_m T^{(0)}_{rm,m} + \frac{1}{2}\sum_s [\lambda^{(0)}_{r+2,s,r+1} - \lambda^{(0)}_{r+2,s,r+1}]_{,s} = 0, & (\text{in } \mathcal{C}^*) \\ \sum_m T^{(0)}_{rm,m} N^*_m + \frac{1}{2}\sum_s [\lambda^{(0)}_{r+2,s,r+1} - \lambda^{(0)}_{r+2,s,r+1} N^*_{r+2}] = 0, & (\text{on } \Sigma^*) \end{cases}
$$

while (84.1), (85.1) become:

$$
(88) \qquad \begin{cases} \sum_m T^{(1)}_{rm,m} + \frac{1}{2}\sum_s [\lambda^{(1)}_{r+2,s,r+1} - \lambda^{(1)}_{r+1,s,r+2}]_{,s} \\ \qquad = -\sum_{lm} [u^{(1)}_{r,l} T^{(0)}_{lm} - \frac{1}{2} u^{(1)}_{m,l} (Y^{(0)}_{rl} - Y^{(0)}_{lr})]_{,m} + F^*_r + \frac{1}{2}(M^*_{r+2,r+1} - M^*_{r+1,r+2}), \\ \\ \sum_m T^{(1)}_{rm} N^*_m + \frac{1}{2}\sum_s [\lambda^{(1)}_{r+2,s,r+1} N^*_{r+1} - \lambda^{(1)}_{r+1,s,r} N^*_{r+2}]_{,s} \\ \qquad = -\sum_{lm} [u^{(1)}_{r,l} T^{(0)}_{lm} - \frac{1}{2} u^{(1)}_{m,l} (Y^{(0)}_{rl} - Y^{(0)}_{lr})] N_m + f^*_r + \frac{1}{2}(M^*_{r+2} N^*_{r+1} - M^*_{r+1} N^*_{r+2}). \end{cases}
$$

(88), together with (84.2) and (85.2), represent the fundamental differential system for the linearized theory. If $\mathcal{C}^*$ represents a configuration of *natural equilibrium* – so $Y^{(0)}_{rs}$ = $\lambda^{(0)}_{rs}$ = 0 – then (88), (84.2), (85.3) become:

$$
(89) \qquad \begin{cases} \sum_m T^{(1)}_{rm,m} + \frac{1}{2}\sum_s [\lambda^{(1)}_{r+2,s,r+1} - \lambda^{(1)}_{r+1,s,r+2}]_{,s} = F^*_r + \frac{1}{2}(M^*_{r+2,r+1} - M^*_{r+1,r+2}), \\ \sum_m T^{(1)}_{rm} N^*_m + \frac{1}{2}\sum_s [\lambda^{(1)}_{r+2,s,r+1} N^*_{r+1} - \lambda^{(1)}_{r+1,s,r} N^*_{r+2}]_{,s} = f^*_r + \frac{1}{2}(M^*_{r+2} N^*_{r+1} - M^*_{r+1} N^*_{r+2}), \end{cases} \quad (\text{in } \mathcal{C}^*)
$$

$$
(90) \qquad \begin{cases} \sum_m \lambda^{(1)}_{rm,m} = Y^{(1)}_{r+1,r+2} - Y^{(1)}_{r+2,r+1} + M^*_r, \\ \sum_m \lambda^{(1)}_{rm,m} N^*_m = m^*_r, \end{cases} \quad (\text{on } \Sigma^*)
$$

and one can associate:

$$
(91) \qquad Y^{(1)}_{rs} + Y^{(1)}_{sr} = 2T^{(1)}_{rs}.
$$

(89), (90) can obviously be obtained as an approximation directly from (7), (8), (9), (10) when one identifies the X_{rs} with the Y_{rs}, the ψ_{rs} with the λ_{rs}, and the x_r with the y_r .

2. Isothermal elastic potential for the small transformations of isotropic bodies.

In the sequel, we suppose that $\mathcal{C}^*$ is the configuration of natural equilibrium. Under the hypothesis of small deformations, there is no reason to not consider a law of HOOKE type to be valid, and to not regard the $T_{rs}^{(1)}$, $\lambda_{rs}^{(1)}$ as homogeneous polynomials of first degree in the $u_{r,s}^{(1)}$ and the $\mu_{rs}^{(1)}$. To that end, one adjoins, moreover, the first derivatives of the general relations (57), (59) with respect to the parameter h, when one lets h tend to zero and regards the τ and $\mathcal{T}$ as depending upon h by means of the $x_{r,s}$, μ_{rs}, and T. Indeed, the search for the relations that are valid in the case of small deformations by deduction from the case of finite deformations, rather than by the path pointed out, is certainly more useful, at least, in the problems that are considered, given that this will lead to a complete characterization of the relations between forces and deformations, and of the structure of the *free energy*, which is otherwise difficult to obtain.

On the basis of (57), one has, in the first place:

(92) $$\lambda_{rs}^{(0)} = -\,\tau^{(0)}\,\delta_{rs} - \left(\frac{\partial \mathcal{T}}{\partial \mu_{rs}}\right)^{(0)},$$

from which, one sees that – if $\mathcal{C}^*$ is, as assumed, the *natural state* – one must have:

(93) $$\tau^{(0)} + \left(\frac{\partial \mathcal{T}}{\partial \mu_{rs}}\right)^{(0)} = 0, \qquad \left(\frac{\partial \mathcal{T}}{\partial \mu_{rs}}\right)^{(0)} = 0, \text{ for } r \neq s.$$

From (59), it follows that:

(94) $$T_{rs}^{(0)} = -\left(\frac{\partial \mathcal{T}}{\partial x_{r,s}}\right)^{(0)},$$

which implies:

(95) $$\left(\frac{\partial \mathcal{T}}{\partial x_{r,s}}\right)^{(0)} = 0.$$

From (59), when it is derived with respect to h and one lets h tend to zero, it will result that:

(96) $$T_{rs}^{(1)} = -\,\tau^{(0)} \sum_{pq} \left(\frac{\partial C_{pq}}{\partial x_{r,s}}\right)^{(0)} \mu_{pq}^{(1)} - \sum_{pq} \left[\left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,s}\,\partial x_{p,q}}\right)^{(0)} u_{p,q}^{(1)} + \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,s}\,\partial \mu_{pq}}\right)^{(0)} \mu_{pq}^{(1)}\right],$$

(97) $$\lambda_{rs}^{(1)} = -\,\tau^{(0)} C_{rs}^{(1)} - \tau^{(1)} C_{rs}^{(0)} - \sum_{pq} \left[\left(\frac{\partial^2 \mathcal{T}}{\partial x_{p,q}\,\partial \mu_{rs}}\right)^{(0)} u_{p,q}^{(1)} + \left(\frac{\partial^2 \mathcal{T}}{\partial \mu_{rs}\,\partial \mu_{pq}}\right)^{(0)} \mu_{pq}^{(1)}\right],$$

and it is easy to confirm that:

(98)
$$\begin{cases} C_{rs}^{(0)} = \delta_{rs}, \\ C_{rs}^{(1)} = \delta_{rs}(u_{r+1,r+1}^{(1)} + u_{r+2,r+2}^{(2)}) - \delta_{r,s+1}u_{r+2,r}^{(1)} - \delta_{r,s+2}u_{r+1,r}^{(1)}, \\ \sum_{pq} \left(\dfrac{\partial C_{pq}}{\partial x_{r,s}} \right)^{(0)} \mu_{pq}^{(1)} = -\delta_{rr}\mu_{rr}^{(1)} - \delta_{r,s+1}\mu_{r+2,r}^{(1)} - \delta_{r,s+2}\mu_{r+1,r}^{(1)}. \end{cases}$$

For $r \neq s$, it follows from (96), (97), on the basis of (98) that:

(99) $T_{rs}^{(1)} =$
$$\tau^{(0)}(\delta_{r,s+1}\mu_{r+2,r}^{(1)} + \delta_{r,s+2}\mu_{r+1,r}^{(1)}) - \sum_{pq} \left[\left(\frac{\partial^2 x}{\partial x_{r,s}\,\partial x_{p,q}} \right)^{(0)} u_{p,q}^{(1)} + \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,s}\,\partial \mu_{pq}} \right)^{(0)} \mu_{pq}^{(1)} \right] \quad (r \neq s),$$

(100) $\lambda_{rs}^{(1)} =$
$$\tau^{(0)}(\delta_{r,s+1}u_{r+2,r}^{(1)} + \delta_{r,s+2}u_{r+1,r}^{(1)}) - \sum_{pq} \left[\left(\frac{\partial^2 \mathcal{T}}{\partial x_{p,q}\,\partial \mu_{rs}} \right)^{(0)} u_{p,q}^{(1)} + \left(\frac{\partial^2 \mathcal{T}}{\partial \mu_{rs}\,\partial \mu_{pq}} \right)^{(0)} \mu_{pq}^{(1)} \right] \quad (r \neq s).$$

From (71), when one differentiates it with respect to h and lets h tend to zero, and on the basis of (93), one deduces that:

(101)
$$\sum_{lm} \left\{ \left[\left(\frac{\partial^2 \mathcal{T}}{\partial x_{\nu+2,\nu+1}\,\partial x_{l,m}} \right)^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{\nu+1,\nu+2}\,\partial x_{l,m}} \right)^{(0)} \right] u_{l,m}^{(1)} \right.$$
$$\left. + \left[\left(\frac{\partial^2 \mathcal{T}}{\partial x_{\nu+2,\nu+1}\,\partial \mu_{lm}} \right)^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{\nu+1,\nu+2}\,\partial \mu_{lm}} \right)^{(0)} \right] \mu_{lm}^{(1)} \right\} + \tau^{(0)}(\mu_{\nu+2,\nu+1}^{(1)} - \mu_{\nu+1,\nu+2}^{(1)}) = 0.$$

Suppose – as is always plausible – that for any irrotational infinitesimal displacement that acts on $\mathcal{C}^*$ (for which, the $\mu_{rs}^{(1)}$ have zero resultant) the $\psi_{rs}^{(1)}$, and therefore the $\lambda_{rs}^{(1)}$, can all be equal to zero.

It follows from this that whenever one has:

(102)
$$u_{r,r+1}^{(1)} - u_{r+1,r}^{(1)} = 0,$$

and consequently the $\mu_{rs}^{(1)}$ are zero, the $\lambda_{rs}^{(1)}$ must all become equal to zero. From (100), for $s = r + 1$, one deduces that in order for this to have occurred, as for the other ones, one must have:

(103)
$$\tau^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r+1,r}\,\partial \mu_{r,r+1}}\right)^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,r+1}\,\partial \mu_{r,r+1}}\right)^{(0)} = 0,$$

while from (101), one deduces that:

(104)
$$\tau^{(0)} + \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,r+1}\,\partial \mu_{r,r+1}}\right)^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r+1,r}\,\partial \mu_{r,r+1}}\right)^{(0)} = 0.$$

From (103), (104), it follows that:

(105)
$$\left(\frac{\partial^2 \mathcal{T}}{\partial x_{r,r+1}\,\partial \mu_{r,r+1}}\right)^{(0)} = 0, \qquad \tau^{(0)} - \left(\frac{\partial^2 \mathcal{T}}{\partial x_{r+1,r}\,\partial \mu_{r,r+1}}\right)^{(0)} = 0.$$

Other consequences may be deduced from (101) and the condition that was posed for the $\lambda_{rs}^{(1)}$. However, one ceases to consider (105) as being satisfied in order to deduce that in the isotropic case – which is all that shall be considered in the sequel – one may assume that the parameter τ is null and the $\mathcal{T}$ is the sum of two functions, one of which, $\mathcal{T}_1$, is a function of only the $x_{r,s}$ and the other of which, $\mathcal{T}_2$, is a function of only the μ_{rs}. At the moment, (105.1) permits us to assert that *the expression for* $T_{r,r+1}$ that is deduced from (99) for $s = r + 1$ *is missing the term* $\mu_{r,r+1}$.

Nevertheless, a linear, homogeneous expression in $T_{rs}^{(1)}$:

(106)
$$T_{rs}^{(1)} = \sum_{pq} [m_{rspq} u_{p,q}^{(1)} + n_{rspq} \mu_{p,q}^{(1)}],$$

corresponds to the isotropic case when and only when the tensors m_{pqrs}, n_{pqrs} are isotropic; i.e., when, with reference to a Cartesian frame, it results that ([18]):

(107)
$$\begin{cases} m_{rspq} = \alpha\delta_{rs}\delta_{pq} + \beta\delta_{rp}\delta_{sp} + \gamma\delta_{rq}\delta_{sp}, \\ n_{rspq} = \alpha'\delta_{rs}\delta_{pq} + \beta'\delta_{rp}\delta_{sp} + \gamma'\delta_{rq}\delta_{sp}, \end{cases}$$

where *a*, *b*, etc., are arbitrary coefficients.

It follows that:

(108)
$$T_{rs}^{(1)} = \alpha\,\delta_{rs}\sum_p u_{p,p}^{(1)} + \beta\, u_{r,s}^{(1)} + \gamma\, u_{s,r}^{(1)} + \alpha'\delta_{rs}\sum_p \mu_{p,p}^{(1)} + \beta'\mu_{rs} + \gamma'\mu_{sr},$$

which, by the symmetry of $T_{rs}^{(1)}$, implies that:

([18]) See, e.g., B. FINZI, *Calcolo tensoriale e applicazioni*, Zanichelli, page 100.

(109) $\beta = \gamma, \qquad \beta' = \gamma'.$

One thus has:

(110) $$T_{rs}^{(1)} = \alpha\,\delta_{rs}\sum_p \varepsilon_{pp}^{(1)} + 2\beta\,\varepsilon_{rs}^{(1)} + \beta'(\mu_{rs} + \mu_{sr})\,,$$

and in order for the expression for $T_{r,r+1}^{(1)}$ *be missing* the term $\mu_{r,r+1}^{(1)}$, one must not have that:

(111) $\beta' = 0.$

The expression for $T_{rs}^{(1)}$ *is therefore that of the symmetric case.*

Analogous considerations indicate that a linear, homogeneous expression in the $\lambda_{rs}^{(1)}$ in the isotropic case is of the type:

(112) $$\lambda_{rs}^{(1)} = A\delta_{rs}\sum_p \varepsilon_{pp}^{(1)} + B'\,u_{r,s}^{(1)} + C'u_{s,r}^{(1)} - (B\mu_{rs} + C\mu_{sr})\,,$$

which, since it must be zero for any irrotational displacement, implies:

(113) $A' = 0, \qquad B' + C' = 0,$

while (100), when written for $s = r + 1$, shows that on the basis of (105.1) *the expression for* $\lambda_{r,r+1}^{(1)}$ *may not contain the term* $u_{r,r+1}^{(1)}$. In (112), therefore, one must have:

(114) $B' = C' = 0.$

It is obvious that on the basis of (110), (111), (112), (113), (114), the expressions for $T_{rs}^{(1)}$ depend upon only $\varepsilon_{rs}^{(1)}$, while those of $\lambda_{rs}^{(1)}$ depend upon only the $\mu_{rs}^{(1)}$. Nevertheless, if one takes the second derivative of $\mathcal{T}$ with respect to h and lets h tend to zero then, taking into account that $\mathcal{T}$ depends upon h only by means of the $x_{r,s}$ and $\mu_{r,s}$, one sees that the coefficients of the quadratic form $\mathcal{T}^{(2)}$ that one obtains coincide with the second derivatives of $\mathcal{T}$ with respect to $x_{r,s}$, $\mu_{r,s}$, and that in all of the preceding considerations and formulas it is legitimate, in the limits as h tends to zero, to replace the second derivatives of $\mathcal{T}$ with respect to $x_{r,s}$, $\mu_{r,s}$ with the analogous second derivatives of the quadratic form that expresses $\mathcal{T}^{(2)}$. In summation, one concludes from the preceding argument that – at least in the isotropic case of small deformations – one may regard $\tau = 0$ and $\mathcal{T} = \mathcal{T}^{(2)}$, with $\mathcal{T}^{(2)} = W_1 + W_2$ and W_1 being quadratic forms in just $\varepsilon_{rs}^{(1)}$, while W_2 is a quadratic form in just $\mu_{rs}^{(1)}$.

It follows from (97), (112), (113), (114), in particular, that:

(115) $$W_2 = \frac{1}{2}\sum_{pq}(B\mu_{pq} + C\mu_{qp})\,\mu_{pq}\,.$$

If one differentiates (71) twice with respect to h and then lets h tend to zero, taking (93) into account, one finds, among other things, that:

(116) $$\sum_q \left[\left(\frac{\partial\mathcal{T}}{\partial\mu_{\nu+1,q}}\right)^{(1)}\mu^{(1)}_{\nu+2,q} - \left(\frac{\partial\mathcal{T}}{\partial\mu_{\nu+2,q}}\right)^{(1)}\mu^{(1)}_{\nu+1,q}\right] = 0,$$

which translates into:

(117) $$\sum_{qlm}\left[\left(\frac{\partial^2\mathcal{T}^{(2)}}{\partial\mu_{\nu+1,q}\,\partial\mu_{lm}}\right)\mu^{(1)}_{\nu+2,q} - \frac{\partial^2\mathcal{T}^{(2)}}{\partial\mu_{\nu+2,q}\,\partial\mu_{lm}}\mu^{(1)}_{\nu+1,q}\right]\mu^{(1)}_{lm} = 0,$$

and on the basis of (115), that becomes:

(118) $$C\sum_q\left(\mu^{(1)}_{\nu+2,q}\mu^{(1)}_{q,\nu+1} - \mu^{(1)}_{\nu+1,q}\mu^{(1)}_{q,\nu+2}\right) = 0,$$

which implies that:

(119) $$C = 0.$$

In the case of elastic bodies, (72), (115), (119) imply that W_1 and W_2 are individually positive-definite.

In particular, one must have:

(120) $$B > 0.$$

3. Recapitulation of the general equations that are valid for small isothermal transformations. Some observations.

At this point, it does not seem pointless to me to recapitulate the fundamental equations that are valid in the statics of isothermal small deformations of a natural state for isotropic bodies with asymmetric stress characteristic. Suppressing the notation of superscript (1) and asterisk, for simplicity, which are superfluous by now, they are:

(121) $$Y_{rs} + Y_{sr} = 2T_{rs}\,,$$

(122) $$Y_{r+1,r+2} - Y_{r+2,r+1} = \sum_m \lambda_{rm,m} - M_r\,, \qquad \text{(in } \mathcal{C}\text{)}$$

(123) $$\sum_m \lambda_{rm}N_m = m_r\,, \qquad \text{(on } \Sigma\text{)}$$

$$(124)\quad \begin{cases} \sum_m T_{rm,m} + \frac{1}{2}\sum_s(\lambda_{r+2,s,r+1} - \lambda_{r+1,s,r+2}) = F_r + \frac{1}{2}\sum_s(M_{r+2,r+1} - M_{r+1,r+2}) & \text{(in } \mathcal{C}\text{)}, \\ \sum_m T_{rm,m}N_m + \frac{1}{2}\sum_s(\lambda_{r+2,s,r+1}N_{r+1} - \lambda_{r+1,s,r+2}N_{r+2}) = f_r + \frac{1}{2}(M_{r+2}N_{r+1} - M_{r+1}N_{r+2}) & \text{(on } \Sigma\text{)}. \end{cases}$$

After introducing the LAMÉ constants γ, ν, we may associate these equations with the relations:

$$(125)\qquad W(\varepsilon, \mu) = W_1(\varepsilon) + W_2(\mu),$$

$$(126)\qquad W_1(\varepsilon) = \frac{1}{2}[(\gamma + 2\nu)\left(\sum_r \varepsilon_{rr}\right)^2 - 4\nu\sum_r(\varepsilon_{rr}\varepsilon_{r+1,r+1} - \varepsilon^2_{r,r+1})],$$

$$(127)\qquad W_2(\mu) = \frac{B}{2}\sum_{rs}\mu^2_{rs}, \qquad B > 0,$$

$$(128)\qquad \varepsilon_{rs} = \frac{1}{2}(u_{r,s} + u_{r,s}), \qquad \mu_{rs} = \frac{1}{2}(u_{r+2,r+1} - u_{r+1,r+2})_{,s},$$

$$(129)\qquad T_{rs} = -\frac{\partial W}{(2-\delta_{rs})\partial\varepsilon_{rs}}, \qquad \lambda_{rs} = -\frac{\partial W}{\partial\mu_{rs}},$$

$$(130)\qquad \delta l^{(i)} = \sum_{rs}[T_{rs}\delta\varepsilon_{rs} + \lambda_{rs}\delta\mu_{rs}].$$

(122) show that in a theory in which one admits the possibility of non-zero surface moments λ_{rs} the stress characteristic can generally be asymmetric ([19]), even in the absence of body moments and external surfaces ($M_r = m_r = 0$). What can be removed from the solutions in the classical theory of singularities is not always possible from the physical point of view, as we will show by an example.

In a theory in which one necessarily supposes that the λ_{rs} are zero ($B = 0$), along with the m_r, the equations (122), (124) become:

$$(131)\qquad Y_{r+1,r+2} - Y_{r+2,r+1} = -M_r,$$

$$(132)\qquad \begin{cases} \sum_m T_{rm,m} = F_r + \frac{1}{2}(M_{r+2,r+1} - M_{r+1,r+2}), \\ \sum_m T_{rm,m}N_m = f_r + \frac{1}{2}(M_{r+2,r+1}N_{r+1} - M_{r+1,r+2}N_{r+2}), \end{cases}$$

([19]) In (122), one also notices the difference between the results that were already cited of BODASZEWSKI and SOMIGLIANA and the fact that the quantity $Y_{rs} - Y_{sr}$ depends linearly upon the local rotations.

while (123) does not have to be dragged into consideration. (130) becomes, moreover:

(133) $$\delta l^{(i)} = \sum_{rs} T_{rs}\,\delta\varepsilon_{rs} .$$

From the identity of $\delta l^{(i)}$ with $-\,\delta W$ for any isothermal infinitesimal transformation, it follows that:

(134) $$\delta W = -\sum_{rs} T_{rs}\,\delta\varepsilon_{rs} .$$

The expressions (133), (134) are formally identical to the ones that one has under the hypothesis of symmetry in the loads and *the structure of W does not change under the passage from the symmetric case to the asymmetric one if one does not assume the existence of surface contact moments* ([20]).

(132) show that the same property of the medium ([21]) that is valid for the Y_{rs} in the symmetric case is true for the T_{rs} , when one conveniently modifies the definitions of the astatic, hyperstatic, etc., coordinates by taking M_r, m_r into account.

4. An important example.

We consider a prism with a square section of side a that is stressed on the lateral surface, but devoid of body forces.

The reference frame $\mathcal{F}$ has its origin at the midpoint of an edge, the y_3 axis is parallel to that edge, and the y_1 and y_2 axes are parallel to the edges of the square section.

One supposes that the vector **f** that characterizes the external surface stress does not depend upon y_3 and is orthogonal to the y_3 axis and one looks for solutions for which one has:

(135) $$u_3 \equiv 0 \qquad u_1 \text{ and } u_2 \text{ are independent of } y_2 .$$

Let f_{is} denote the i^{th} component of **f** on the face $y_s = 0$, and let $f_{is}^{(a)}$ denote the components on the edge $y_s = a$ ($s = 1, 2$).

On the basis of (125), (126), (127), (128), equations (124.1) reduce to two and, more precisely, to the equations:

(136) $$\nu\,\Delta_2\, u_r + (\nu + \gamma) \sum_{s=1}^{2} u_{s,rs} + \frac{B}{4}\Delta_2\left[\sum_{s=1}^{2} u_{s,rs} - \Delta_2 u_r\right] = 0 \qquad (r = 1, 2),$$

while the boundary equations (123), (124.2) that are associated with them are:

([20]) (131), (132) are equivalent to (3), (3′) of SOMIGLIANA (*loc. cit.*, in note (1)). On the other hand, (134) does not differ noticeably, since δW also depends upon the $\delta\omega_r$.

([21]) G. GRIOLI, *Relazione quantitative per lo stato tensionale di un qualunque sistema continuo e per la deformazione di un corpo elastico in equilibrio*, Annali di Matematica pura ed applicata, Series IV, v. XXXIII, 1952.

(137) $(u_{2,1} - u_{1,2})_{,1} = 0,$ for $y_1 = 0, a,$

(138) $(u_{2,1} - u_{1,2})_{,2} = 0,$ for $y_2 = 0, a,$

(139) $$(\gamma + 2\nu)\, u_{2,1} + \gamma u_{2,2} = \begin{cases} -f_{11}, & \text{for } y_1 = 0, \\ f_{11}^{(a)}, & \text{for } y_1 = a, \end{cases}$$

(140) $$(\gamma + 2\nu)\, u_{2,2} + \gamma u_{1,1} = \begin{cases} -f_{22}, & \text{for } y_2 = 0, \\ f_{22}^{(a)}, & \text{for } y_2 = a, \end{cases}$$

(141) $$\nu(u_{2,1} + u_{1,2}) - \frac{B}{4}\Delta_2 (u_{2,1} - u_{1,2}) = \begin{cases} -f_{21}, & \text{for } y_1 = 0, \\ f_{21}^{(a)}, & \text{for } y_1 = a, \end{cases}$$

(142) $$\nu(u_{2,1} + u_{1,2}) + \frac{B}{4}\Delta_2 (u_{2,1} - u_{1,2}) = \begin{cases} -f_{12}, & \text{for } y_2 = 0, \\ f_{12}^{(a)}, & \text{for } y_2 = a. \end{cases}$$

One supposes that:

(143) $f_{11} = 0, \qquad f_{12} = 0.$

Therefore, one is dealing with a shear stress on the face $y_1 = 0$ that is normal to the face $y_2 = 0$.

By a simple verification, one sees that the pair:

(144) $$\begin{cases} u_1 = b\left\{\left[y_1^2 - \dfrac{\gamma+2\nu}{\nu} y_2^2\right] - B\dfrac{\gamma+2\nu}{2\nu^2}\dfrac{sh\,ky_2 + sh\,k(a-y_2)}{sh\,ka}\right\}, \\ u_2 = 0, \end{cases}$$

where b is a non-zero constant and:

(145) $$k = \sqrt{\frac{4\nu}{B}}$$

verifies the indefinite equations (136) and, taking (143), (145) into account, the boundary conditions (137), (138), (139.1), (142.1), as well.

In addition, on the face $y_1 = 0$, and on the basis of (141.1), (145), it results that:

(146) $$f_{21} = 2(\gamma + 2\nu)\left[y_2 + \sqrt{\frac{B}{\nu}}\frac{ch\,ky_2 - ch\,k(a-y_2)}{sh\,ka}\right],$$

and one has:

(147) $$\lim_{y_2=0} f_{21} = 2(\gamma + 2\nu)\sqrt{\frac{B}{\nu}}\frac{1-ch\,ka}{sh\,ka}.$$

In other words, on the face $y_1 = 0$, one has a shear stress that tends to a non-zero limit when $y_2 \to 0$, while on the edge $y_2 = 0$ one has a purely normal stress. Therefore:

$$\text{(148)} \qquad \begin{cases} \lim_{y_2 \to 0} Y_{21}(0, y_2) = 2(\gamma + 2\nu)\sqrt{\dfrac{B}{\nu}}\,\dfrac{1 - ch\,ka}{sh\,ka} \neq 0, \\ \lim_{y_1 \to 0} Y_{12}(y_1, 0) = 0. \end{cases}$$

Thus, it generally results that:

$$\text{(149)} \qquad Y_{21} - Y_{12} = -\sum_s \lambda_{3s,s} = \frac{B}{2} u_{1,222} = -2b(\gamma + 2\nu)\sqrt{\frac{B}{\nu}}\,\frac{ch\,ky_2 - ch\,k(a - y_2)}{sh\,ka}.$$

Therefore, in a problem for which the givens are such that the shear force that is applied to the face $y_1 = 0$, $y_2 = 0$ has different limits when it tends to the edge, the introduction of the λ_{rs} leads to a regular solution and, in particular, to a *monodromic* load, as one easily recognizes on the basis of (144).

On the other hand, the classical theory, which requires the λ_{rs} to be zero, implies singularities on the edge.

Mind you, such singularities are due, not to the existence of the angular point, but to the fact that the givens on the shear load present the peculiarity above.

In fact, it is easy to see that if one lets B tend to zero then the boundary problem tends to that of the well-known symmetric case, while (144) [see (145)] tends to the solution:

$$\text{(150)} \qquad u_1 = b\left(y_1^2 - \frac{\gamma + 2}{\nu}\right) y_2^2, \qquad u_2 = 0.$$

It is regular, along with all of its derivatives, but it corresponds to the applied shear load that tends to the same limit when one goes to the edge of the prism $y_1 = y_2 = 0$.

In fact, one has, as one easily sees, and taking (150) into account:

$$\text{(151)} \qquad \lim_{B \to 0} (Y_{21} - Y_{12}) = 0,$$

$$\text{(152)} \qquad \lim_{B \to 0} f_{21} = 2b(\gamma + 2\nu)\, y_2,$$

$$\text{(153)} \qquad \lim_{y_2 \to 0} \lim_{B \to 0} f_{21} = 0.$$

Meanwhile, in the symmetric case ($B = 0$), a applied shear load on the face $y_1 = 0$, $y_2 = 0$ that has different limits when one tends to the common edge generally implies polydromy in the loads and divergences in their derivatives.

As we did above ([22]), for example, suppose that we are given a shear load that is zero for $y_2 = 0$ and a non-zero constant for $y_1 = 0$.

From the considerations above, one deduces that it might be convenient to have a theory in which one supposes that the λ_{rs} are non-zero in order to avoid polydromy in the solution, which is certain implausible from the physical viewpoint. Moreover, one may also object that one does not know the way of realizing surface moments that will *rigorously* imply (as we have observed) that one must regard the external load on the varied surface elements as reducible to a force $f\, d\sigma$ (and generally, it is not easy to indicate the manner of realizing its surface distribution either), without a couple ($m = 0$), but *without excluding, a priori*, the possibility that λ_{rs} is non-zero, relative to the internal stress state.

([22]) *Loc. cit.*, in note ([3]).