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# Questions of compatibility for Cosserat continua (*)

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In their publications during 1907-1909, E and F. Cosserat [**1**], [**2**] defined a continuum to be a set of points and tri-rectangular trihedra with their origins at $P'$. More precisely, consider such a three-dimensional continuum in a given configuration $C$, and associate each of its points $P$ with a tri-rectangular triad $T$. The deformation that takes the configuration $C$ to another one $C'$ is characterized by the displacement $PP'$ and the rotation that changes $T$ into its corresponding $T'$ in $C'$, with its origin at the point $P'$ that corresponds $P$.

The displacement $u = PP'$ and the rotation $R$ that changes the orientation of $T$ into that of $T'$ are independent *a priori*.

Suppose that there is a potential energy from which one derives the state of internal stress. In the Cosserat theory, it will depend substantially upon 18 variables: Six characteristics of the deformation, three parameters that characterize the orientation of $T'$ with respect to $T$, and their nine derivatives with respect to the coordinates of $P$. The stress variables *are defined* by means of the 18 partial derivatives of potential energy with respect to those variables, nine of which correspond to the classical stress tensor (which is asymmetric now), and the others to the contact couples.

In the Cosserat theory, one must take into account what currently goes by the name of the *principle of material indifference*, which imparts a certain type of dependency of the potential energy upon its dependent variables.

In the case of the three-dimensional Cosserat continuum, the theory had been developed very little until a few years ago. In the case of small deformations, the renascence was due to E. L. Aero and E. Kuvshinski [**3**]. However, in the case of finite deformations, it was only in 1960 that the study of continua with contact couples that were characterized by asymmetric stresses was taken up again. That reprise was initiated by the work of G. Grioli [**4-8**], Toupin [**9**], [**11**], Mindlin and Tiersten [**10**], which was followed by the work of other authors.

Nonetheless, one must observe that the work that was cited above was concerned with one particular type of Cosserat continuum, in which one supposes that the rotation that takes the triad $T$ to $T'$ is not independent of the field of displacements $u$, but will properly coincide with the local rotation that is subordinate to that field. By now, however, many studies regarding Cosserat continua, especially in the linearized case, and without the restrictions above, suppose that the rotation that takes $T$ to $T'$ is, in fact, free.

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Recently, the theory of *microstructures* has developed in the linear case of small deformations by the work of Mindlin [**12**].

In that theory, any point of the continuum is essentially associated with a deformable molecule. In effect, one is treating a generalization of the Cosserat case, which one will return to when one supposes that the molecule is rigid. In other words, a Cosserat continuum is a microstructure with rigid molecules.

One should observe that in both the case of a microstructure and that of a Cosserat continuum, the stress (in the broad sense of the word) is defined by the gradients of the potential energy with respect to the variables upon which it depends, which are variables that depend upon the kinematical-geometrical model that the continuum defines. Moreover, well-known variational principles will allow one to establish the field equations that the stress must satisfy.

Nevertheless, one can observe that once one has established what the variables are that the potential energy depends upon, the theory will ignore, in effect, the kinematical model that it has moved away from: Known variational principles will determine both the field equations and the stress variables. The inconvenience can certainly present itself that not all of the stress variables do work. That is, it can happen that one can postulate a certain type of potential energy for which the dynamical equations are not the only ones that are compatible with it, and which can include stress terms that do no work.

Certainly, inconveniences of this type will not present themselves if, while establishing the geometrico-kinematic behavior of the continuum and the stress variables to which it is subject, one places the theory upon the basis of the fundamental equations of mechanics or (if one prefers) the differential equations of the field, except for the verification of the compatibility of the conclusions to which one arrives. In that way, one will always be dealing with physically-meaningful variables.

This is the path that one follows in the only case that is considered for Cosserat continua: The fundamental variables that one considers are naturally the ones that will be established shortly and coincide with the ones that were considered by the Cosserats themselves and other authors, namely, the displacements and rotations by which one regards the kinematical aspect, and the stress characteristics and contact couples by which one regards the internal stresses.

By definition, the kinematical behavior of the continuum is characterized by two fields: viz., those of the displacements $u = PP'$ and those of the rotations $R$, which are *a priori* independent of $u$, while the mechanical behavior is characterized by two asymmetric matrices, namely, the matrix of stress and that of contact couples.

The choice of variables upon which the thermodynamic potential depends cannot be postulated, but must be derived from the general equations of mechanics.

It is important to observe that if one would like to make a deeper study of Cosserat continua then it would be convenient to keep in mind that the field of displacements $u$ is subordinate to a field of local rotations that one is advised to exhibit explicitly. One can assume, as is certainly legitimate, that the global rotation $R$ is the product of a certain number of local rotations that are due to the displacement $u$ with a second rotation that is completely free.

If one develops the theory of Cosserat continua in that way then one will be surprised to find that there will generally be incompatibilities in the case of free rotations. By that, we mean to say that if the system is assumed to be free of internal constraints and has reversible transformations then free energy must satisfy a system of differential conditions – some which will translate into the principle of material indifference – that are generally incompatible.

That signifies, at least, that a Cosserat continuum with free rotations cannot generally constitute a system with reversible transformations.

The incompatibility will cease to exist in the linearized case, since it originates in the nonlinear terms. Nevertheless, if one interprets the linear theory as a first approximation to the complete theory then one will remain perplexed about its significance.

A very special case of compatibility exists in the case of free rotation. In it, the contact couples prove to be zero, while the stress characteristics are still asymmetric.

However, no incompatibility will be present in a theory in which the local rotations are not free, but coincide with the ones that are subordinate to the field of displacements, which is what happened in the work that was developed in 1960.

## 1. Some observations on rotational displacements.

Some facts of a geometric character will prove useful. In particular, it will be useful to exhibit a certain property of composite rotational displacements that will prove to have fundamental importance for the study of Cosserat continua.

Let $T$ and $T'$ be two left-handed tri-rectangular triads that have the same origin and let $R$ be the rotation that takes $T$ to $T'$. Let $e_s$ , $e'_s$ be the respective unit vectors, so one will have $e'_s = Re_s$ , and it will follow immediately that the operator $R$ will be represented by the matrix:

$$R \equiv | R_{rs} |, \tag{1}$$

if $R_{rs}$ denotes the cosine of the angle between $e_r$ and $e'_s$ .

It is well-known that the new quantity $R_{rs}$ can be expressed by means of the parameters $Q_1$, $Q_2$, $Q_3$ (which are called *rotational parameters*), and they might be the three Euler angles of $T'$ with respect to $T$, the three components of the rotation vector, or the Rodriguez parameters. One therefore says that the operator $R$ is a function of the three parameters $Q_i$ .

The rotation $R$ can always be imagined to be the product of two successive rotations that are characterized by the values $q_i$ , $\eta_i$ of the rotation parameters. Therefore, it is like saying that one can always write:

$$R(Q) = R(q)\, R(\eta), \tag{2}$$

or, given the general invertibility of the rotation operator, one can fix any two of the rotations that appear in (2) and determine the third one uniquely.

One now considers a third triad $T''$ that is left-handed and tri-rectangular, has the same origin as $T$, $T'$, and is *close to* $T'$. By the locution *close to*, we mean to say that the rotation ([1]) $R_{\delta Q}$ that takes $T'$ to $T''$ is evaluated by taking into account only the linear

([1]) One must be careful to note that the rotation $R_{\delta Q}$ is not obtained by simply substituting $\delta Q_i$ for $Q_i$ in $R(Q)$.

terms in the variations $\delta Q_i$ of the parameters $Q_i$ , or also that the displacement of $T'$ to $T''$ is of the infinitesimal rotational displacement type.

The rotation that takes $T$ to $T''$ is obviously given by:

(3) $$R\,(Q + \delta Q) = R_{\delta Q}\, R(Q),$$

from which, it follows that

(4) $$R_{\delta Q} = R\,(Q + \delta Q)\, R^{-1}\,(Q).$$

Suppose that (2) is valid, the displacement from $T$ to $T''$ is subordinate to (not independent) variations $\delta Q_i$ , $\delta q_i$ , $\delta\eta_i$ of the of the parameters $Q_i$ , $q_i$ , $\eta_i$ , and along with (3), one has the equalities:

(5) $$R\,(q + \delta q) = R_{\delta q}\, R\,(q), \qquad R\,(\eta + \delta\eta) = R_{\delta\eta}\, R\,(\eta),$$

as well as:

(6) $$R\,(Q + \delta Q) = R\,(q + \delta q)\, R\,(\eta + \delta\eta),$$

which gives one a link between the variations of the rotational parameters.

If one keeps in mind that the inverse of the product of two rotations is equal to the product of the inverse rotations, in the opposite order, then it will follow from (4), (5), (6) that:

(7) $$\begin{aligned} R_{\delta Q} &= R\,(q + \delta q)\, R\,(\eta + \delta\eta)\, R^{-1}\,(\eta)\, R^{-1}\,(q) \\ &= R_{\delta q}\, R\,(q)\, R_{\delta\eta}\, R(\eta)\, R^{-1}\,(\eta)\, R^{-1}\,(q) = R_{\delta q}\, R\,(q)\, R_{\delta\eta} R^{-1}\,(q). \end{aligned}$$

Since $R_{\delta q}$ , $R_{\delta\eta}$ represent infinitesimal rotations, there will exist two vectors $\delta\omega$, $\delta\omega''$ for which it will result that:

(8) $$R_{\delta q} = 1 + \delta\omega'\times, \qquad R_{\delta\eta} = 1 + \delta\omega''\times,$$

in which the vectors $\delta\omega'$, $\delta\omega''$ depend upon $\delta q_i$ , $\delta\eta_i$ , respectively.

If one takes only the first-order terms into account then it will follow from (7), (8) that:

(9) $$\begin{aligned} R_{\delta Q} &= (1 + \delta\omega'\times)\, R\,(q)\, (1 + \delta\omega''\times)\, R^{-1}\,(q) \\ &= 1 + \delta\omega'\times + R\,(q)\, \delta\omega''\times R^{-1}\,(q). \end{aligned}$$

Furthermore, if one keeps in mind the known property of the rotation operators then one will observe that even though it is not essential, it will follow that (9) simplifies to:

(10) $$R_{\delta q} = 1 + \delta\omega'\times + (R\,(q)\, \delta\omega'')\times .$$

Obviously, the vectors $\delta\omega'$, $\delta\omega''$ depend upon $\delta q_i$ , $\delta\eta_i$ linearly. By definition, one can then assert that if one sets:

(11) $$R_{\delta q} = 1 + \delta\omega\times,$$

as is certainly legitimate, then there will exist an operator $B\,(q)$ *that depends upon only* the parameters $q_i$ , and another one $C\,(q, \eta)$ for which one has, however, that when one is given $q_i$ and $\eta_i$ , it will result that:

(12) $$\delta\omega_r = B_{rl}\,(q)\;\delta q_l + C_{rl}\,(q,\,\eta)\;\delta\eta_l\,.$$

It is worth observing that, together with (11), the linear dependency of $\delta\omega_r$ on $\delta q_l$, $\delta\eta_l$ will be predictable, but *it will certainly not be predictable* that the matrix $|\,B_{rl}\,|$ is independent of the $\eta_i$ , which is a fact that will prove important in what follows.

The lack of formal symmetry in (12) with respect to the parameters $q_l$ , $\eta_l$ should not be surprising. Obviously, it is due to the non-commutability of the factors in the product of matrices.

## 2. On the local rotations of Cosserat continua.

What was said in the preceding section is applicable to the case of deformations of a Cosserat continuum that passes from a reference configuration $C$ to the present one $C'$ in regard to the rotations that change the orientation of the triad $T$ that is associated with the generic element $P$ of $C$ to that of the corresponding triad $T'$ that is associated with the element $P'$ that corresponds to $P$ in $C'$.

Naturally, one lets $R$ denote that global (and free) rotation and $Q_i$ denotes the parameters upon which it depends.

The vectorial field $u = PP'$ is subordinate to a local rotation, that is assumed to be characterized by the parameters $\eta_i$ : That rotation will therefore be $R\,(\eta)$.

The rotations $R\,(Q)$, $R\,(\eta)$ will define a third rotation $R\,(q)$ uniquely, that is characterized by the parameters $q_i$ and the three aforementioned rotations that were given already. One can, in fact, state that any chosen global rotation $R\,(Q)$ can be realized by associating it with an opportune choice of parameters $q_i$ , in such a way that one will realize a rotation that will give $R\,(Q)$ when it is multiplied by the one that is subordinate to the displacement $u$.

By definition, one can believe that the geometrico-kinematic behavior of a Cosserat continuum is characterized completely by knowing the vectorial field $u$ and the *free* parameters $q_i$ , rather than the parameters $Q_i$ .

The passage from a current configuration $C'$ to a *neighboring* of $C' + \delta C'$ is consequently characterized by a field of displacements $\delta u$ and a triad of variations $\delta q_i$ of the parameters $q_i$ .

Let $y_s$, $x_s$ denote the coordinates of $P$, $P'$, respectively, with respect to the same left-handed, tri-rectangular, reference triad. It is known that the local rotation that is subordinate to the vector $\delta u$ is characterized by the vector whose components are:

(13) $$\delta\omega_s'' = \tfrac{1}{2}\,e_{spt}\,(\delta u_t)_{/\,p}$$

in which $e_{spt}$ is the Ricci indicator, and the stroke denotes the derivative with respect to $x_p$.

Since the second term on the right-hand side of (12) represents the global rotation vector in the case of $\delta q_i = 0$, one will deduce that it must coincide with the right-hand side of (13). Therefore, one has, by definition, the expression:

(14) $$\delta\omega_r = B_{rl}\,(q)\;\delta q_l + \tfrac{1}{2}\,e_{spt}\,(\delta u_t)_{/p}$$

for the vector that characterizes the global rotation for the passage from $C'$ to $C' + \delta C'$.

Do not be surprised that one sees no trace of the parameters $\eta_i$ in (13) and in the last term in (14), while they do enter into last term of (12). In reality, the identification of the right-hand side of (13) with the last term in (12) serves to distinguish the $\delta\eta_i$ as functions of the $q_i$, $\eta_i$, $\delta u_r$, and those $\delta\eta_i$ are not arbitrary when one has assigned the vectorial field $\delta u$.

## 3. Lagrangian form of the work done by internal forces.

Let $| X_{rs} |$ denote the Eulerian matrix of (Cauchy) forces, and let $| \psi_{rs} |$ denote that of the contact couples, always with reference to the tri-rectangular triad to which one refers the continuum. Along with those matrices, consider the Piola-Kirchhoff matrix $| Y_{rs} |$ and the matrix $| \lambda_{rs} |$. The link between those matrices is expressed by the equalities:

(15) $$X_{rs} = \frac{1}{D}\,Y_{lm}\,x_{r,\,l}\,x_{s,\,m}\,, \qquad\qquad \psi_{rs} = \frac{1}{D}\,\lambda_{rl}\,x_{s,\,l}\,,$$

in which $D$ is the Jacobian determinant:

(16) $$D = \left\| x'_{r,s} \right\| > 0.$$

In (15), (16), and in what follows, the comma will denote the derivation with respect to $y_s$.

The fundamental equations of the statics of Cosserat continua, in Eulerian form, are presented in the form of (2):

(17) $$\begin{cases} X_{rs/s} = F_r\,, \\ \psi_{rs/s} = e_{rpt}\,X_{pt} + M_r\,, \end{cases} \qquad \text{(in } C'\text{)},$$

(18) $$\begin{cases} X_{rs}\,n_s = f_r\,, \\ \psi_{rs}\,n_s = m_r\,, \end{cases} \qquad \text{(in } \sigma'\text{)},$$

in which $F_r$, $M_r$ denote the body forces and moments per unit volume of the current state, $\sigma'$ is the contour of $C$, $f_r$, $m_r$ are the surface forces and moments per unit area of the current state, and $n$ is the interior normal to $\sigma'$.

The Eulerian expression for the work done by internal contact forces that correspond to an arbitrary displacement from $C'$ to $C' + \delta C'$ follows from (17), (18). It is:

(2) See, e.g., [**4**]. It is not at all essential in what follows to consider the equations for dynamics, instead of those of statics.

$$\delta l^{(i)} = X_{rs}\,(\delta u_r)_{/\,s} + \psi_{rs}\,(\delta\omega_r)_{/\,s} + e_{rlm}\,X_{lm}\,\delta\omega_r \tag{19}$$

per unit volume in the current state.

In (19), $\delta\omega_r$ represents the vector that characterizes the global rotation and can be considered to be expressed by (14).

If one separates the symmetric part $X_{(rs)}$ of $X_{rs}$ from the antisymmetric part $X_{[rs]}$ and keeps (14) in mind then the expression (19) for $\delta l^{(i)}$ can be put into the form:

$$\delta l^{(i)} = X_{(rs)}\,(\delta u_r)_{/\,s} + \psi_{rs}\{[B_{rl}\,(q)\;\delta q_l]_{/s} + \tfrac{1}{2}\,e_{rtm}\,(\delta u_m)_{/\,ts}\} + e_{rlm}\,X_{[lm]}\,B_{rt}\,(q)\;\delta q_t\,, \tag{20}$$

in which one confirms the independence of $\delta l^{(i)}$ from antisymmetric part of stress when the rotations of the elements are not free ($\delta q_t \equiv 0$).

It will be useful to observe that if one lets $A_{rs}$ denote the algebraic complement of $x_{r,\,s}$ in the determinant (16) then one will have:

$$f_{/\,i} = f_{,\,l}\,\frac{A_{il}}{D} \tag{21}$$

for any function $f$ of $x_i$ .

(21) permits one to express the derivatives with respect to $x_s$ that appear in (20) in terms of derivatives with respect to $y_t$ . In particular, with some development, one will find that:

$$(\delta\omega''_r)_{/\,s} = \tfrac{1}{2}\,e_{rtm}\,(\delta u_m)_{/\,ts} = \frac{A_{sl}}{D}\,[\delta\mu_{rl} + B_{rlmp}\;\delta u_{m,\,p}], \tag{22}$$

in which one means that:

$$\mu_{rl} = \frac{1}{2D}\;e_{rlm}\,A_{tp}\,u_{m,\,pl}\,, \tag{23}$$

$$B_{rlmp} = \frac{u_{\nu,\sigma l}}{2D^2}\,[e_{rl\nu}\,A_{tp}\,A_{t\sigma} - e_{rtm}\,A_{t\sigma}\,A_{\nu p}]. \tag{24}$$

The Lagrangian expression $\delta^* l^{(i)}$ for the work done by internal forces per unit volume of the reference state is coupled to $\delta l^{(i)}$ by the relation:

$$\delta^* l^{(i)} = \delta l^{(i)}\;\frac{dC}{dC^*} = D\,\delta l^{(i)}. \tag{25}$$

If one takes (15), (21), (22), (25) into account, along with the permutability of the operator $\delta$ with derivation with respect to $y_s$ , then one can deduce from (20) that:

$$\begin{aligned}\delta^* l^{(i)} = {}& [Y_{(pl)}\,x_{m,\,l} + B_{rlmp}\,\lambda_{rl}]\;\delta\,(u_{m,\,p}) + \lambda_{rl}\,[\delta\mu_{rl} + B_{ri}\;\delta\,(q_{i,\,l})] \\ & + [\lambda_{rl}\,B_{rit}\,q_{t,\,l} + e_{rpt}\,Y_{[lm]}\,x_{p,\,l}\,x_{t,\,m}\,B_{ri}]\;\delta q_i\,,\end{aligned} \tag{26}$$

in which $B_{rit}$ denotes the derivative of $B_{ri}\,(q)$ with respect to $q_t$ .

It is necessary to observe that $\mu_{rl}$ verifies the equality:

(27) $$A_{rl}\,\mu_{rl} = 0,$$

from which, it follows that:

(28) $$\delta A_{rl}\,\mu_{rl} + A_{rl}\,\delta\mu_{rl} = 0,$$

which can be written:

(29) $$A_{rl}\left[\delta\mu_{rl} - \frac{A_{mp}}{D}\,\mu_{rp}\,\delta\mu_{m,l}\right] = 0.$$

## 4. Constitutive equations.

Let $U$ denote the internal energy, $\theta$, the absolute temperature, $E$, the entropy, and let:

(30) $$\mathcal{F} = U - E\theta$$

be the free energy of the continuum, all referred to the unit volume in the reference state.

From now on, suppose that the continuum is free of internal constraints and has reversible transformations. It will then follow that for any infinitesimal transformation from $C'$ to $C' + \delta C'$, the work that is done by internal forces, with the opposite sign, is equal to the corresponding variation of the free energy, augmented by $E\,d\theta$. If one takes equation (29) into account then that can be translated into the equation:

(31) $$\delta^*\,l^{(i)} = -\,\delta\mathcal{F} - E\,\delta T - \tau A_{rl}\left[\delta\mu_{rl} - \frac{A_{mp}}{D}\,\mu_{rp}\delta u_{m,l}\right],$$

in which $\tau$ is a parameter, and that equation is considered to be valid for any choice of the variations $\delta u_{m,p}$, $\delta\mu_{rl}$, $\delta q_i$, $\delta q_{i,l}$.

If one compares (26) with (31) then one will see that the free energy must depend upon the variables $u_{m,p}$, $\mu_{rl}$, $q_i$, $q_{i,l}$, $\theta$, and that the constitutive equation, in its initial form, will take on the appearance of:

(32) $$Y_{(pl)}\,x_{m,l} + B_{rlmp}\,\lambda_{rl} - \frac{\tau}{D}A_{ml}\,A_{rp}\,\mu_{rl} = -\,\frac{\partial\mathcal{F}}{\partial x_{m,p}},$$

(33) $$\lambda_{rl}\,B_{rit}\,q_{t,l} + e_{rpt}\,x_{p,l}\,x_{l,p}\,B_{ri}\,Y_{[lm]} = -\,\frac{\partial\mathcal{F}}{\partial q_i},$$

(34) $$\begin{cases} \lambda_{rl} + \tau A_{rl} = -\dfrac{\partial\mathcal{F}}{\partial\mu_{rl}}, \\ \lambda_{rt}B_{ri} = -\dfrac{\partial\mathcal{F}}{\partial q_{i,l}}. \end{cases}$$

On the basis of (34.1), one has the relation:

(35) $$B_{rlmp}\,\lambda_{rl} - \frac{\tau}{D} A_{ml}\,A_{rp}\,\mu_{rl} = -\,B_{rlmp}\,\frac{\partial \mathcal{F}}{\partial \mu_{rl}} - \tau\,A_{rl}\,B_{rlmp} - \frac{\tau}{D} A_{ml}\,A_{rp}\,\mu_{rl}\ .$$

If one keeps (23), (24) in mind then one will deduce that:

(36) $$\begin{aligned} A_{rl}\,B_{rlmp} + \frac{1}{D} A_{ml}\,A_{rp}\,\mu_{rl} \\ &= -\frac{A_{rl}}{2D^2}\,u_{v,\,\sigma l}\,[e_{rtm}\,A_{t\sigma}\,A_{vp} - e_{rtv}\,A_{tp}\,A_{m\sigma}] + \frac{A_{ml}\,A_{rp}}{2D^2}\,e_{rtv}\,A_{t\sigma}\,u_{r,\,\sigma l} \\ &= -\,\frac{e_{rtm}}{2D^2}\,u_{v,\,\sigma l}\,e_{rtv}\,A_{rl}\,A_{tp}\,A_{m\sigma} + \frac{1}{2D^2}\,u_{v,\,l\sigma}\,e_{rtv}\,A_{tp}\,A_{m\sigma}\,A_{rl} \\ &= 0, \end{aligned}$$

if one takes into account that the first term in the last expression is zero, from the antisymmetry of the Ricci indicator.

It follows from (32), (35), (36) that the dependency of $Y_{(pl)}$ on the parameter $\tau$ is only apparent. In reality, one has:

(37) $$Y_{(pt)} = \frac{A_{mt}}{D}\left[\frac{\partial \mathcal{F}}{\partial \mu_{rl}} B_{rlmp} - \frac{\partial \mathcal{F}}{\partial x_{m,p}}\right].$$

(37) coincides formally with what one finds in the case of Cosserat continua with rotations that are not free, but coincide with the ones that are subordinate to the field of displacements.

## 5. General incompatibility of Cosserat continua with free rotations.

If one formally sets:

(38) $$\mathcal{F} = \mathcal{F}' - \tau\,A_{rl}\,\mu_{rl}$$

then it will follow from (34) that:

(39) $$\lambda_{rl} = -\frac{\partial \mathcal{F}'}{\partial \mu_{rl}}, \qquad \lambda_{rl}\,B_{ri} = -\frac{\partial \mathcal{F}'}{\partial q_{i,l}}.$$

(39) implies the compatibility equations:

(40) $$\frac{\partial \mathcal{F}'}{\partial \mu_{rl}} B_{ri} - \frac{\partial \mathcal{F}'}{\partial q_{i,l}} = 0.$$

Set:

(41) $$\xi_{rs} = \mu_{rs}\,, \qquad z_{rs} = \mu_{rs} + B_{ri}\,q_{i,\,s}\,.$$

Given the general invertibility of the matrix $| B_{ri} |$, one can replace the variables $\mu_{rs}$, $q_{i,s}$ with the variables $\xi_{rs}$, $z_{rs}$ on the basis of (41) and think of the functions $\mathcal{F}$, $\mathcal{F}'$ as being functions of $\xi_{rs}$, $z_{rs}$, instead of $\mu_{rs}$, $q_{i,s}$, in addition to $x_{r,s}$, $q_i$, $\theta$. As a consequence of the system of equations (40), one will have:

$$\frac{\partial \mathcal{F}'}{\partial \xi_{rl}} = 0. \tag{42}$$

(42) shows clearly, when one takes (41) into account, that the functions $\mathcal{F}$, $\mathcal{F}'$ depend upon $\mu_{rs}$, $q_{i,l}$ only by way of the $z_{rs}$ :

$$\mathcal{F}' = \mathcal{F}(x_{r,s}, z_{rs}, q_i, \theta). \tag{43}$$

*

* *

The condition that is expressed by (43) is not the only condition that $\mathcal{F}'$ must satisfy. In fact, (37) imposes the symmetry condition upon its right-hand side that:

$$e_{spt}\frac{A_{mt}}{D}\left[\frac{\partial \mathcal{F}}{\partial \mu_{rl}}B_{rlmp} - \frac{\partial \mathcal{F}}{\partial x_{m,p}}\right] = 0. \tag{44}$$

If one replaces the system of equations (44) with the entirely-equivalent ones:

$$e_{spt}\frac{A_{vs}A_{mt}}{D}\left[\frac{\partial \mathcal{F}}{\partial \mu_{rl}}B_{rlmp} - \frac{\partial \mathcal{F}}{\partial x_{m,p}}\right] = 0 \tag{45}$$

then after some calculation, one will see that they can be presented in the form:

$$e_{spt}\left[x_{p,l}\frac{\partial \mathcal{F}}{\partial x_{t,l}} + \mu_{pl}\frac{\partial \mathcal{F}}{\partial \mu_{tl}}\right] = 0. \tag{46}$$

In reality, the symmetry conditions (44) and their equivalent ones (46) express (as one can prove) *the principle of material indifference*, or if one prefers, the condition that the free energy does not vary under any rigid, isothermal displacement, but is deduced as a simple consequence of (37).

With some calculation, one will find that the expression $\tau A_{rl}\, m_{rl}$ satisfies (46). That signifies that one can substitute $\mathcal{F}'$ for $\mathcal{F}$ in (46) and write it in the form:

(47) $$e_{spt}\left[x_{p,l}\frac{\partial\mathcal{F}'}{\partial x_{t,l}}+\mu_{pl}\frac{\partial\mathcal{F}'}{\partial\mu_{tl}}\right]=0.$$

(46), (47) are formally identical to the equations that are conditions on the free energy in the case of non-free rotations ([3]). The most general solution of (47) is an arbitrary function of the quantity ([4]):

(48) $$b_{rs}=x_{i,r}\,x_{i,s}\,,\qquad l_{rs}=x_{i,r}\,\mu_{is}\,,$$

along with $q_i$, $q_{i,r}$, $\theta$, it is understood.

If one takes (41.2), (43) into account then one will see that $\mathcal{F}'$ can only be a function of $b_{rs}$, $q_i$, $\theta$ and the quantity:

(49) $$L_{rs}=x_{i,r}\,z_{is}\,.$$

That gives rise to an incompatibility: In fact, the possible dependency of $\mathcal{F}'$ upon $L_{rs}$ implies that $\mathcal{F}'$ will depend upon $x_{i,r}$, as well as the quantities $x_{i,r}\,B'_{il}\,q_{ls}$, which, however, cannot be expressed as functions of the $b_{rs}$, as is required, given that the $B_{il}$ *do not depend upon the* $x_{\sigma,\nu}$, due to the fact that, on the basis of what was proved in no. 1, *they do not depend upon* the characteristic parameters of the local rotation that is subordinate to the field of displacements $u$.

Moreover, the incompatibility will obviously come about in a different way if one takes (41.2) into account and writes (47) in the form:

(50) $$e_{spt}\left[x_{p,l}\frac{\partial\mathcal{F}'}{\partial x_{t,l}}+(z_{pl}-B_{pi}\,q_{i,l})\frac{\partial\mathcal{F}'}{\partial z_{tl}}\right]=0,$$

which will reduce to the form:

(51) $$e_{spt}B_{pi}\,q_{i,l}\frac{\partial\mathcal{F}'}{\partial z_{tl}}=0,$$

since $\mathcal{F}'$ depends upon the $x_{i,r}$, $z_{ir}$ only by way of $b_{rs}$, $L_{rs}$, while the $B_{pl}$ do not depend upon the $x_{i,r}$.

(51) implies the independence of $\mathcal{F}'$ from the $z_{tl}$, etc., due to the arbitrariness of the quantity $B_{pi}\,q_{i,l}$.

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From that, one deduces that Cosserat systems with free rotations are generally incompatible systems. An exception is the case (which is very special indeed) in which

(3) See, e.g., [**4**].

(4) See [**14**].

$\mathcal{F}'$ does not depend upon the $z_{rs}$ . As a consequence of that, it does not depend upon the variables $\mu_{rs}$ , $q_{rs}$ , and one will have:

$$\mathcal{F} = \mathcal{F}'(b_{rs}, q_i, \theta) - \tau A_{rs}\, \mu_{rs} , \tag{52}$$

which will imply the constitutive equations:

$$\begin{cases} Y_{(pl)} = 2\dfrac{\partial \mathcal{F}'}{\partial b_{pt}}, \quad \lambda_{rl} = 0, \\ e_{rpt}\, x_{p,l}\, x_{t,m} B_{ri} Y_{[lm]} = -\dfrac{\partial \mathcal{F}'}{\partial b_{pt}}. \end{cases} \tag{53}$$

One sees immediately that in the particular case in question, there are no contact couples, but the stress can still be asymmetric.

More precisely, it is enough to consider equations (17) and (18) in order to see that the stress will be not symmetric in the static case when and only when body moments are present, just as in the case of non-free rotations. However, unlike what happens in this case, the global local rotations do not coincide with the ones that are subordinate to the displacements $u$, due to the presence of the parameters $q_i$ .

On the contrary, in the dynamical case, one can have asymmetry in the stress, even in the absence of body moments, as long as one takes into account the moment of the quantity of motion of the individual system elements ([5]).

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It is worth pointing out that if one has constructed a theory of Cosserat continua that supposes that the rotations are exclusively the ones that are subordinate to the field of displacements $u$ then the incompatibility will cease.

In fact, in that case, if one must consider the $\delta q_l$ in (14) to be zero then the expression (20) will lack corresponding terms, and similarly for (26). In addition, (33), (34.2) will cease to exist, while (34.1), (37) will remain valid.

The unique differential system for compatibility is the system (46), which expresses the principle of material indifference, and (38) will be valid with $\mathcal{F}'$ as a function of the variables $b_{rs}$ , $\mu_{rs}$ , $\theta$.

Obviously, that recalls the case that has been studied already since 1960.

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It is worth pointing out that if one linearizes the general theory of Cosserat continua with free rotations then (46) will lack terms that depend on the $\mu_{pl}$ explicitly.

([5]) See [**7**].

Incompatibility will cease to exist as a consequence. In fact, in that case, (46) will express the idea that $\mathcal{F}$ depends upon $x_{r,s}$ only by way of linearized $b_{rs}$ :

(54) $$b_{rs} = 1 + u_{r,s} + u_{s,r} ,$$

while $\mu_{rs}$ , $B_{rs}$ reduce to:

(55) $$\mu_{rs} = e_{rpt}\, u_{t,ps} , \qquad B_{rs} = \delta_{rs} ,$$

in which $\delta_{rs}$ is the Kronecker symbol.

On the basis of (40), (43), it will follow that $\mathcal{F}'$ depends upon the linearized $b_{rs}$ and $z_{rs} = \mu_{rs} + q_{r,s}$ without there being incompatibility any longer.

Correspondingly, one has:

(56) $$\begin{cases} Y_{(pl)} = -2\dfrac{\partial \mathcal{F}'}{\partial b_{pl}}, \quad Y_{[pl]} = -\dfrac{e_{ptl}}{2}\dfrac{\partial \mathcal{F}'}{\partial q_i}, \\ \lambda_{pl} = -\dfrac{\partial \mathcal{F}'}{\partial \mu_{pt}} = -\dfrac{\partial \mathcal{F}'}{\partial q_{p,t}} = -\dfrac{\partial \mathcal{F}'}{\partial z_{pt}}. \end{cases}$$

One then recovers the known formulas of the linear theory with free rotations ([6]).

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([6]) See, e.g., [**15**], [**17**].

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