"Contributo per una formulazione di tipo integrale della meccanica dei continui di Cosserat," Ann. Mat. pura ed Appl. **111** (1976), 175-183.

Contribution to a formulation of integral type of the mechanics of Cosserat continua (*).

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Abstract. – *With the goal of making a contribution to a formulation of integral type of the mechanics of Cosserat continua with finite deformations, we establish:* 1) *A variational property of stress,* 2) *A condition of integral type on strain.*

I recently established a variational property of stress for finite deformations of a classical continuum of hyperelastic type [**1**], [**2**]. I treated a property of stationarity – i.e., of minimum action – of a potential energy that not only characterized the real stress but also permitted one to give an existence theorem and to conceive of a pure integration procedure in the presence of unilateral surface constraints.

In the case of more complex continua – viz., ones with *micro-structure* – it is only in the linearized case that it is possible to establish a variational property that is analogous to this and of the same magnitude [**3**]. On the other hand, this is not possible in the case of finite deformation, and the fundamental reason consists in the fact that it does not seem possible to express the field equation without making the deformation intervene in it, unlike what happens in the classical case, in which one appeals to Kirchhoff's asymmetric representation of stress, as one can do. A final difficulty is then connected with the extreme complexity of the constitutive equations.

With the goal of expanding upon the question, while considering the case of a hyperelastic Cosserat continuum that is subject to finite deformations, we show how, in reality, the stress is not characterized by the stationarity of a potential energy B, and also establish what the first variation of B will equal that corresponds to real stress. More precisely, we established that the first variation will prove to be equal to a quantity that is annulled when one linearizes the problem – as with small deformations [**3**] – in which it has higher order than the first variation of B.

The integral property that is established for real stress cannot have the operational significance of the analogue that was established for the classical case, but certainly can be considered to be a first contribution to the formulation of the mechanics of Cosserat continua in integral form. With the goal of extending that contribution, and also because the question is directly linked to the integral property on which the variational property that we established is founded (which is, however, not invertible), I would like to point

(*) Submitted for editing on 4 August 1975.

out the possibility of giving a global form to the integrability conditions for the strain variables.

1. Introduction. Let C and C' be the reference configuration and the current one, resp., for a Cosserat continuum, let P and P' be corresponding points, and let y_i, x_i be their coordinates with respect to the same oriented, tri-rectangular reference. From the purely geometric and kinematic viewpoint, the transformation from C to C' is defined by the knowledge of the relations:

(1) $$x_i = x_i(y_1, y_2\, y_3;\, t), \qquad R_{rs} = R_{rs}(y_1, y_2\, y_3;\, t),$$

where R_{rs} is the matrix that expresses the rotation that is associated with P. One supposes that (1) satisfies all of the required analytical conditions for the one-to-one character of the correspondence between C and C'; e.g., continuity, etc.

The state of tension (i.e., stress) in C' is defined by the knowledge of two asymmetric matrices X_{rs}, ψ_{rs}, the former of which corresponds to the usual stress in Cauchy form, while the latter one expresses the density of contact couples.

If a comma denotes the derivative with respect to y_i and one supposes, as is legitimate, that $D = \text{Det}\,\|\, x_{r,s}\,\| > 0$ then one sets:

(2) $$X_{rs} = \frac{K_{rl}\, x_{s,l}}{D}, \qquad \psi_{rs} = \frac{\lambda_{rl}\, x_{s,l}}{D}.$$

The stress variables K_{rs}, λ_{rs}, whose significance is obvious, satisfy the field equations:

(3) $$K_{rs,\,l} = F_r, \qquad \lambda_{rs,\,l} + \varepsilon_{rms}\, K_{sl}\, x_{m,\,l} = M_r \qquad (\text{in } C),$$

(4) $$K_{rl}\, n_l = f_r, \qquad \lambda_{rl}\, n_l = m_r \qquad (\text{on } \sigma),$$

where F_r, M_r denote the volume force and couple densities, resp., when referred to C, which consist of the inertial force in the dynamical case, while f_r, m_r consist of the corresponding surface force and couple, resp., that are defined on the boundary σ of C. In (3), e_{rms} denotes the Ricci indicator of three-dimensional Euclidian space, while in (4), n_r is the interior normal to σ.

If we let a denote the matrix of components $x_{r,s}$ then the strain is characterized by the four matrices:

(5) $$\varepsilon = \tfrac{1}{2}(a^{(\mathrm{T})}a - 1), \qquad v = a^{(\mathrm{T})}R, \qquad v^{(s)} = a^{(\mathrm{T})}R_{,\,s}\,.$$

One observes that if one sets:

(6) $$Z^{(l)} = \tfrac{1}{2}R^{(\mathrm{T})}R_{,\,l}$$

then it not only results that:

(7) $$Z_{rs}^{(l)} = \tfrac{1}{2} R_{pr}\, R_{ps,l} = -\tfrac{1}{2}\, R_{pr,\,l}\, R_{ps}\,, \qquad R_{ps,l} = 2\, Z_{rs}^{(l)} R_{pr}\,,$$

but also that:

(8) $$\varepsilon = \tfrac{1}{2}(v v^{(\mathrm{T})} - 1), \qquad v^{(s)} = 2v\, Z^{(s)}.$$

One concludes that ultimately the strain is characterized by the knowledge of the matrices v, $Z^{(s)}$.

2. A particular form for the constitutive equations. – The new quantity R_{rs} can be expressed by means of three parameters Q_i . One sets:

(9) $$B_{il}(Q_s) = \frac{1}{2} e_{isp} \frac{\partial R_{pt}}{\partial Q_l} R_{st}\,.$$

Under the hypothesis of hyperelasticity, there exists a potential energy density W from which the stress is derived. It will depend upon the fundamental variables $x_{r,s}$, R_{rs} , $R_{rs,t}$, which characterize the geometrical behavior of a Cosserat continuum only through the agency (1) of the matrices v_{rs}, $Z_{rs}^{(l)}$, and one has (2):

(10) $$\begin{cases} K_{rs} = -\dfrac{\partial W}{\partial v_{sp}} R_{rp} = -\dfrac{\partial W}{\partial x_{rs}}, \\ \lambda_{rs} B_{rl} = -\dfrac{\partial W}{\partial Q_{l.s}}. \end{cases}$$

In the sequel, it will be convenient to give a more appropriate form to the constitutive equations (10). To that end, one observes that on the basis of (7), (9), it results that:

(11) $$Z_{rs}^{(l)} = -Z_{sr}^{(l)} = \tfrac{1}{2} e_{rqs}\, R_{pq}\, B_{pl}\, Q_{l,t}\,.$$

Since W can depend upon $Q_{r,\,s}$ only by means of the $Z_{rs}^{(l)}$, from (10, 2) it follows that:

(12) $$\lambda_{rs}\, B_{rl} = -\frac{\partial W}{\partial Z_{rt}^{(s)}} \frac{\partial Z_{rt}^{(p)}}{\partial Q_{l,s}} = -\frac{1}{2} \frac{\partial W}{\partial Z_{rt}^{(s)}}\, e_{rqt}\, R_{pq}\, B_{pl}\,.$$

One knows that Det | B_{rl} | > 0. One therefore deduces from (12) that:

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(1) One can treat this as a consequence of the principle of material indifference. Naturally, W will depend on other variables, in general, e.g., temperature, etc. Nevertheless, such circumstances will not be considered here.
(2) See (40) in [**5**], if one assumes that the coupling $K_{rs} = x_{r,m}\, Y_{ml}$ is valid in it.

(13) $$\lambda_{rs} = -\frac{1}{2} e_{pmq} R_{rm} \frac{\partial W}{\partial Z_{pq}^{(s)}} .$$

Set:

(14) $$\tau_{pq}^{(s)} = e_{pmq}\, R_{rm}\, \lambda_{rs} , \qquad \lambda_{rs} = \tfrac{1}{2}\, e_{pmq}\, R_{rm}\, \tau_{pq}^{(s)} ,$$

so from (13), one ultimately has that:

(15) $$\tau_{pq}^{(s)} = -\frac{\partial W}{\partial Z_{pq}^{(s)}} .$$

A convenient transformation of (10, 1) can be obtained from:

(16) $$N_{sp} = K_{rs}\, R_{rp} , \qquad K_{rs} = R_{rp}\, N_{sp} .$$

It follows immediately that:

(17) $$N_{sp} = -\frac{\partial W}{\partial v_{sp}} .$$

Equations (15), (17) constitute a particular form that is adapted to the context that applies to the constitutive equations of a Cosserat continuum with free rotations.

One easily convinces oneself that relations (15), (17) are invertible. One can set ([3]):

(18) $$v_{rs} = \alpha_{rs}(N;\, \tau^{(l)}), \qquad Z_{rs}^{(l)} = \beta_{rs}^{(l)}(N;\, \tau^{(l)}),$$

where the α_{rs} , $\beta_{rs}^{(l)}$ are functions of the variables N_{pq}, , $\tau_{pq}^{(l)}$ that are deduced from the inversion of (15), (17).

Set:

(19) $$W' = -\, W[\alpha(N;\, \tau^{(l)});\, \beta(N;\, \tau^{(l)})] - \alpha_{pq}\, N_{pq} - \beta_{pq}^{(l)} Z_{pq}^{(l)} ,$$

so W' defines a second form for the potential energy, and one easily deduces that:

(20) $$v_{rs} = -\frac{\partial W'}{\partial N_{rs}} , \qquad Z_{rs}^{(l)} = -\frac{\partial W'}{\partial \tau_{pq}^{(l)}} .$$

3. A variational property of real stress. – Suppose that a surface force and couple are given on some part σ_1 of σ, while some translations and rotations are given on the remaining part σ_2 . On σ_2 , one has $x_r \equiv \bar{x}_r$, $R_{rs} \equiv \bar{R}_{rs}$, where $\bar{x}_r$, $\bar{R}_{rs}$ denote functions

([3]) This does not begin to address the complex and interesting question of the possible *a priori* breakdown of uniqueness in the α_{rs} , $\beta_{rs}^{(l)}$ that are deduced from the inversion (18). Furthermore, the same question presents itself in the case of classical continua.

that are defined on σ_2 . Let V be the class of possible reactions and reaction couples, while ϕ_r and μ_r are allowable constraints. Let K^*_{rs}, λ^*_{rs} be the matrices that express the real stress that corresponds to the current configuration C', which are characterized by the values of x_r, R_{rs} at the instant t, values that will be indicated by x^*_r, R^*_{rs}.

Any other stress that satisfies (3), (4) on σ_1 and corresponds to the current configuration C' and to the volume, inertial, and surface (on σ_1) forces at the instant t is obtained by adding increments ΔK_{rs}, $\Delta\lambda_{rs}$ to K^*_{rs}, λ^*_{rs} that satisfy the equations:

$$\Delta K_{rs,l} = 0, \qquad \Delta\lambda_{rs,l} + e_{rms}\,\Delta K_{sl}\,x^*_{m,l} = 0, \tag{21}$$

$$\begin{cases} \Delta K_{rl}n_l \; \left\langle \begin{array}{ll} = 0 & (\text{on } \sigma_1), \\ = \Delta\phi_r & (\text{on } \sigma_2), \end{array} \right. \\ \Delta\lambda_{rl}n_l \; \left\langle \begin{array}{ll} = 0 & (\text{on } \sigma_1), \\ = \Delta\mu_r & (\text{on } \sigma_2), \end{array} \right. \end{cases} \tag{22}$$

where $\Delta\phi_r$, $\Delta\mu_r$ denote increments in the ϕ_r, μ_r that are allowed by the constraints.

From (21), it follows immediately that:

$$\int_C \left\{ \Delta K_{rl,l}\,x^*_r - \tfrac{1}{2}[\Delta\lambda_{rl,l} + e_{rms}\Delta K_{sl}\,x^*_{m,l}]\,e_{rqp}R^*_{qp} \right\} dC = 0. \tag{23}$$

Taking (22) into account, one deduces from (23) that:

$$\int_C \left\{ \Delta K_{rl,l}\,x^*_r - \tfrac{1}{2}e_{rqp}[R^*_{qp,l}\Delta\lambda_{rl} + R_{qp}e_{rms}\Delta K_{sl}\,x^*_{m,l}] \right\} dC + \int_{\sigma_2} [\Delta\phi_r\bar{x}_r - \tfrac{1}{2}e_{rqp}\Delta\mu_r\bar{R}_{qp}]\,d\sigma_2 = 0. \tag{24}$$

Letting $R_{[qp]}$ denote the anti-symmetric part of R_{qp} and keeping (7), (14) in mind, it follows from (24), after some calculations, that:

$$\int_C \left\{ R^*_{ps}Z^{*(l)}_{tp}\Delta\tau^{(l)}_{ts} - R^*_{[sm]}x^*_{m,l}\Delta K_{sl} + \Delta K_{rl}\,x^*_{r,l} \right\} dC + \int_{\sigma_2} [\Delta\phi_r\bar{x}_r - \tfrac{1}{2}e_{rqp}\Delta\mu_r\bar{R}_{qp}]\,d\sigma_2 = 0, \tag{25}$$

where v^*_{rs} and $Z^{*(l)}_{rs}$ indicate the expressions for v_{rs} and $Z^{(l)}_{rs}$ that are provided by (5), (7) when one identifies x_r and R_{rs} in them with x^*_r and R^*_{rs}, respectively.

After some final calculations, (25) becomes:

$$\int_C \left\{ R^*_{ps}Z^{*(l)}_{tp}\Delta\tau^{(l)}_{ts} + [\delta_{pl} - R^*_{[pl]}]v^*_{lt}\Delta N_{lp} \right\} dC + \int_{\sigma_2} [\Delta\phi_r\bar{x}_r - \tfrac{1}{2}e_{rqp}\Delta\mu_r\bar{R}_{qp}]\,d\sigma_2 = 0. \tag{26}$$

Taking into account the fact that the real stress satisfies the constitutive equations (20), it ultimately follows from (26) that:

(27) $$\int_C \left\{ \frac{\partial W'}{\partial \tau_{tp}^{(l)}} \Delta \tau_{ts}^{(l)} R_{ps}^* + \frac{\partial W'}{\partial N_{lt}} \Delta N_{lp} [\delta_{pl} - R_{[ps]}^*] \right\} dC \; - \int_{\sigma_2} \left[\Delta \phi_r \bar{x}_r - \tfrac{1}{2} e_{rqp} \Delta \mu_r \bar{R}_{qp} \right] d\sigma_2 = 0.$$

Set $R'_{rs} = R_{rs} - \delta_{rs}$, and additionally:

(28) $$B = \int_C W' dC - \int_{\sigma_2} [\phi_r \bar{x}_r - \tfrac{1}{2} e_{rsp} \mu_r \bar{R}_{sp}] \, d\sigma_2 \; .$$

One immediately recognizes that (27) can be presented in the form:

(29) $$\Delta B = -\int_C \left\{ R'_{ps} \frac{\partial W'}{\partial \tau_{pt}^{(l)}} \Delta \tau_{ts}^{(l)} - R'_{[sm]} \frac{\partial W'}{\partial N_{lm}} \Delta N_{ls} \right\} dC \, ,$$

about which, one asserts that – as opposed to what happens for classical continua – the real stress does not render the potential energy *B* stationary for the class of stresses that are in equilibrium with the given volume, inertial, and surface forces, which can give rise to constraint reactions that are allowed by the constraints. Nevertheless, one can observe that in the linearized case *B* will result in stationarity, properly speaking, that corresponds to the real stress, as was observed in [**3**]. Indeed, one easily recognizes that the right-hand side of (29) is set equal to zero in the case of small deformations that are consistent with the linearization of the problem, in which, regarding – as is necessary – the quantity $R_{rs} - \delta_{rs}$ to be of first order, it follows that it is of higher order with respect to the left-hand side.

4. A possible integral formulation of the compatibility conditions for the strain matrices. – Suppose that the strain matrices $\varphi_{rs}, \gamma_{lp}^{(l)}$ are given, consider a rotation matrix ρ_{rs}, and set:

(30) $$\varphi_{rs} = \xi_{mr} \, \rho_{ms} \, , \qquad \gamma_{tp}^{(l)} = \tfrac{1}{2} \rho_{mt} \eta_{mp}^{(l)} ,$$

where the matrices ξ_{mr} , $\eta_{mp}^{(l)}$ are uniquely determined and the $\gamma_{tp}^{(l)}$ are assumed to be skew-symmetric (*emi-simmetrica*) in the lower indices. In addition, one sets:

(31) $$N'_{sp} = K'_{rs} \rho_{rp} \, , \qquad \tau'^{(s)}_{pq} = e_{pmq} \, \rho_{rm} \, \lambda'_{rs} \, ,$$

where K'_{rs} , λ'_{rs} represent an arbitrary solution of the differential system:

(32) $$K'_{rl,l} = 0, \qquad \lambda'_{rl,l} + e_{rms} K'_{sl} \xi_{ml} = 0 \qquad \text{(in } C\text{)},$$

(33) $$K'_{rl} n_l = 0, \qquad \lambda'_{rl} n_l = 0 \qquad \text{(on } \sigma\text{)}.$$

One considers a system of integral equations:

(34)
$$\begin{cases} \int_C N'_{sp}\varphi_{sp}dC = 0, \\ \int_C [\rho_{ps}\gamma^{(l)}_{tp}\tau'^{(l)}_{ts} - \rho_{[pt]}\varphi_{lt}N'_{lp}]\,dC = 0. \end{cases}$$

One has the theorem:

A necessary and sufficient condition for the matrices $\varphi_{rs}, \gamma^{(l)}_{lp}$ *, the second of which is skew-symmetric, to represent a strain in a Cosserat continuum is that it satisfy* (34) *for any choice of* N'_{sp} , $\tau'^{(l)}_{st}$ *that are constructed on the basis for* (31) *with any possible solution of the system of equations* (32), (33). *In that case, the transformation from which the strain is derived descends precisely from the rotation* ρ_{rs} [*and a suitable displacement vector*].

The condition is necessary: Suppose that:

(35)
$$\xi_{mr} = \xi_{m,r}\ , \qquad \eta^{(l)}_{mp} = \rho_{mp,l}\ .$$

It follows from (32), (33) that:

(36)
$$\begin{cases} \int_C K'_{rl,l}\xi_r dC = 0, \\ \int_C [\lambda'_{rl,l} + e_{rms}K'_{sl}\xi_{m,l}]\,dC = 0. \end{cases}$$

One deduces (34, 1) from (36, 1) immediately.

When one takes (30), (31) into account, (36, 2) becomes:

(37)
$$\int_C [e_{rmp}\lambda'_{rl}\rho_{mp,l} + 2\rho_{[pt]}N'_{lp}\varphi_{lt}]\,dC = 0,$$

which, again, on the basis of (30), (31), transforms into (34, 2).

The condition is sufficient ([4]): Assuming (30), (31), one suppose that (34) is satisfied by the φ_{rs}, $\gamma^{(l)}_{rs}$ for any choice of $N'_{lp}, \tau^{(l)}_{st}$ that are constructed from solutions of (32). Introduce an arbitrary double system of functions χ_{pq} that are differentiable and zero on the boundary of C and set:

(38)
$$K'_{rs} = e_{sti}\,\chi_{rt,i}\ , \qquad N'_{sp} = e_{sti}\,\rho_{rp}\,\chi_{rt,i}\ .$$

One easily recognizes that the K'_{rs} that are defined in (38) satisfy (32). (34, 1) then becomes:

(39)
$$\int_C e_{sti}\xi_{rs}\chi_{rt,i}dC = 0,$$

([4]) Observe that the hypothesis that (33) is satisfied never enters into the proof of sufficiency. It is enough to suppose that the χ_{pq} are zero on the boundary of C, but arbitrary everywhere else.

and from the arbitrariness of χ_{pq} this implies that:

(40) $$e_{sti}\ \xi_{rs,\,i} = 0.$$

(40) shows the existence of three functions ξ_r that satisfy (35, 1). As a consequence, it then results that:

(41) $$\varphi_{rs} = \xi_{m,\,r}\,\rho_{ms}\ .$$

Taking (40) into account, one recognizes that the functions that were defined by (38, 1), as well as:

(42) $$\lambda'_{rs} = e_{rpm}\, e_{\sigma qs}\ \xi_{m,\sigma}\chi_{pq}\,,$$

satisfy (32, 2) for arbitrary χ_{pq} . It then follows that:

(43) $$\tau_{pt}^{(l)} = e_{pst}\,\rho_{rs}\ e_{rvm}\ e_{\sigma ql}\ \xi_{m,\sigma}\chi_{vq}\,,$$

and on the basis of (30), (38), (41), and (43), (34, 2) becomes:

(44) $$\int_C \left\{ \tfrac{1}{2} e_{s\varphi t} e_{rpm} e_{\sigma ql} \rho_{r\varphi} \rho_{\varepsilon s} \rho_{\tau t} \eta_{\tau\varepsilon}^{(l)} \xi_{m\sigma} \chi_{pq} + \rho_{[pm]} e_{lqi} \xi_{m,l} \chi_{pq,i} \right\} dC = 0.$$

(44) simplifies to:

(45) $$\int_C \left\{ e_{\sigma ql} \xi_{m,\sigma} [\tfrac{1}{2} e_{\varepsilon r\tau} e_{rpm} \eta_{\tau\varepsilon}^{(l)} - \rho_{[pm],i}] \chi_{pq} \right\} dC = 0,$$

which, upon taking into account the equivalence:

(46) $$\rho_{[pm]} = \tfrac{1}{2} e_{rpm}\ e_{r\tau\varepsilon}\,\rho_{\tau\varepsilon}\,,$$

and the arbitrariness of the χ_{pq}, gives:

(47) $$e_{rpm}\ e_{\sigma qi}\ e_{r\tau\varepsilon}\ \xi_{m,\sigma}\,(\eta_{\tau\varepsilon}^{(i)} - \rho_{\tau\varepsilon,\,i}) = 0.$$

Set:

(48) $$c_{rpqi} = e_{rpm}\ e_{\sigma qi}\ \xi_{m,\sigma}\,, \qquad g_{ri} = e_{r\tau\varepsilon}(\eta_{\tau\varepsilon}^{(i)} - \rho_{\tau\varepsilon,\,i}),$$

so (47) assumes the form:

(49) $$c_{rpqi}\ g_{ri} = 0.$$

This constitutes a new homogeneous, linear system of equations in the new unknowns g_{ri}.

One can prove that the determinant of the coefficients is non-zero, in general: i.e., Det $|\,c_{rpqi}\,| \neq 0$. It then follows that:

(50) $$\eta_{\tau\varepsilon}^{(i)} = \rho_{\tau\varepsilon,\,i} + L_{\tau\varepsilon}^{(i)}\,,$$

where the $L_{\tau\varepsilon}^{(i)}$ constitute an arbitrary system for any i that is symmetric with respect to the lower indices. From (30, 2), one obtains:

$$\gamma_{tp}^{(l)} = \tfrac{1}{2}\rho_{mt}(\rho_{mp,i} + L_{mp}^{(i)}), \tag{51}$$

and the condition of skew-symmetry for the $\gamma_{tp}^{(l)}$ implies that:

$$\rho_{mt}L_{mp}^{(i)} + \rho_{mp}L_{mt}^{(i)} = 0. \tag{52}$$

The system of six equations (52) – for any value of i – in the six unknowns $L_{rs}^{(i)}$ admits the zero solution as its unique solution if one is given that the determinant of the coefficients is non-zero ([5]). Upon taking (41) and (51) into account and setting $L_{rs}^{(i)} \equiv 0$, one thus concludes that the matrices φ_{rs}, $\gamma_{tp}^{(l)}$ define an effective strain. It is provided by the deformation that is characterized by the displacement ξ_r and the rotation ρ_{rs}. Q.E.D.

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([5]) Under the correspondence of the element P of C, one assumes that the reference triad has its third axis parallel to the rotational axis that is defined by the matrix ρ_{rs} : Let θ be the angle of rotation, so one has:

$$\rho_{11} = \rho_{22} = \cos\theta, \qquad \rho_{33} = 1, \qquad \rho_{21} = -\rho_{12} = \sin\theta, \qquad \rho_{i3} = \rho_{3i} = 0 \qquad (i = 1, 2).$$

One easily recognizes the validity of what we asserted in such a situation.