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Sir Robert S. Ball's space of linear screws

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With two double tables I and II.

A purely geometric overview of linear manifolds of screws, their axis positions, and parameter distributions shall be offered here, as was first developed by *Sir Robert S. Ball* (*Theory of Screws, A Study in the Dynamics of a Rigid Body,"* Dublin 1876, German version by *Harry Gravelius*, as *Theoretische Mechanik starrer Systeme,"* Berlin 1889; finally, in *A treatise on the theory of screws*, Cambridge, 1900). This will be predominantly cloaked in mechanical garb. One finds that any rigid body that moves with an arbitrary number of degrees of freedom n (≤ 6) belongs to the screws of a *linear* manifold R_n of rank n, and will not be influenced by dynames, which belong to another screw manifold P_ν of rank $\nu = (6 - n)$ that is "reciprocal" to the first one. We will assume some familiarity with the simplest operations of *Hermann Grassmann's* theory of extensions (A_1, 1844, A_2, 1862). (Cf., *E. W. Hyde* in the Annals of Mathematics, vol. IV, no. 5, 1888, and by the same author: *The directional calculus, based upon the methods of Hermann Grassmann,* Boston, 1890). Our three-dimensional space is determined by four mass points e_i (i = 1, 2, 3, 4) that do not lie in a plane. Each point x in it can be represented by means of suitable numerical quantities x_i as the multiple sum $x = \sum x_i e_i$ of these four points. This summation is known as the center of mass determination in mechanics, where the x_i are the general linear point coordinates, and in particular, when e_1, e_2, e_3 are infinitely-distant points – i.e., segments of constant direction and length – they will be homogeneous *Hessian* parallel coordinates.

We will understand the *exterior* product of three mass points – e.g., e_1, e_2, e_3 – to mean the planar *plate* or plane segment ($e_1\, e_2\, e_3$), and not merely its surface area, and its position will be regarded as – e.g. – a force-couple, as is used in mechanics (rotating twin, the exterior product of two segments), but also its membership in a certain plane. Two plates will then be added like two rotating twins, except that one must establish their membership in the line of intersection of the planes that go through the given plates as part of the sum of plates. In that sense, every plate ξ in space can be represented as the multiple sum:

$$\xi = \xi_1 \cdot e_2\, e_3\, e_4 + \xi_2 \cdot e_3\, e_4\, e_1 + \xi_3 \cdot e_4\, e_1\, e_2 + \xi_4 \cdot e_1\, e_2\, e_3$$

of four basic plates of the reference tetrahedron. The numerical quantities ξ_i are thus the general linear plane coordinates; in particular, when e_1, e_2, and e_3 are segments, they will be the homogeneous *Hessian* plane coordinates.

We will understand the exterior product of two mass points – e.g., e_1 and e_2 – to be the line segment, or *rod* (e_1 e_2), which is not merely regarded as having a length and direction, but also membership in a certain line, like a force or also a rotational velocity around an axis in the mechanics of rigid systems. (The rod e_1e_2 can also be regarded as the *regressive* product of two plates e_1 e_2 e_3 and e_1 e_2 e_4.) In particular, if e_1, as well as e_2, is a segment then the infinitely-distant rod (e_1e_2) will be called a *field*; it will then correspond completely to the geometric image of the concept of a rotating twin or also a parallel translation quantity – i.e., an angular velocity component of a rotation around an infinitely-distant axis. Rods will be added like forces or angular velocities for axial rotations.

Any rod l can be linearly derived from the six basis rods of the elementary tetrahedron e_1 e_2 e_3 e_4 :

$$e_1e_2,\ e_1e_3,\ e_1e_4,\ e_2e_3,\ e_3e_4,\ e_4e_2,$$

by suitable numerical quantities $p_{i,k}$ (i, k = 1, 2, 3, 4):

$$l = \sum p_{i,k}\, e_i\, e_k\ .$$

When there is a likewise-to-be-discussed quadratic relationship:

$$p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0$$

between the determining pieces $p_{i,k}$, they will be the so-called *Plückerian* line coordinates, which, as *F. Klein* remarked (*Nichteuklidische Geometrie*, Göttingen, 1893), should actually be named for *Grassmann*, who presented his comprehensive theory in 1844, and thus two or three years before *Plücker*. If the aforementioned relationship does not exist then:

$$\sum p_{i,k}\, e_i\, e_k = L$$

will not be representable as a rod, but only a sum of at least two skew rods l_1 and l_2, which will then be called *conjugate:* $L = l_1 + l_2$. (Cf., *Moebius*, *Ges. Werke*, Leipzig, 1886, Bd. III; *Reye*, *Geom. d. Lage*, Leipzig, 1892, whose used the expression "reciprocal polars in the null system L," instead of "conjugate rel. L," for the skew lines on which l_1 and l_2 lie.) If is known that for a given sum L, if one of the conjugate lines – e.g., l_1 – is arbitrary then the length and sense of l_1, as well as the conjugate rod l_2 will then be determined already.

We call L a *screw*; it can be represented in a unique, canonical way as the sum of a certain rod l and a field f that is perpendicular to it:

$$L = l + f.$$

If one gives f the form of a right angle whose one side has the length l of l then, for *Plücker*, the length $\mathfrak{p}$ of the second side will be called the *parameter* of the screw. It shall be positive or negative, and the corresponding screw will be called *right-wound* or *left-wound*, according to whether the field f does or does not appear to describe a clockwise rotation to an observer that looks in the direction l, resp. The opposite convention would also be permissible. If we multiply L by an arbitrary numerical factor then the axes and the parameter of the resulting screw would remain the same. All screws that are generated in that way with the same axes and the same parameter will then define a *screwing motion*. A screwing motion then relates to the screw as the line relates to its rod. It is invertible and uniquely linked with *Moebius's* concept of a *null system*, since screwing motions and null systems are determined in an identical way by the association of conjugate lines. We remain at the standpoint of the Euclidian metric. l and f then define the single conjugate pair of L that is polar with respect to the absolute sphere circle. (If one had established another *Cayleyian* metric surface in a non-Euclidian geometry then there would also be, in general, a polar conjugate pair relative to it that would be suitable to a canonical representation.)

If one adds a parallel field to l, such that l is displaced parallel to l', then by a simultaneous subtraction of this field from f one will obtain:

$$L = l' + f';$$

i.e., as the sum of a rod l' that is equipollent to l and a field f' (that is no longer perpendicular to it). If the perpendicular separation of l and l' has the length ρ then f' will be rotated with respect to f through an angle α, and indeed around the direction of the plane ll', which is perpendicular to l, and one will satisfy the condition:

$$\rho = \mathfrak{p} \tan \alpha.$$

If w is an arbitrary point then the plane of the plate:

$$(w\,L) = (e\,f')$$

will be called its *null plane* relative to L [if l' goes through w then one will have $(w\,l') = 0$], and $\rho = \mathfrak{p} \tan \alpha$ will express the metric relationship between the length of the perpendicular ww_0 from w to the *axis* l of the screw L (i.e., the *axis* of the null system L), and the angle α around which the null plane $(w_0\,f')$ of the point w_0 is rotated when one goes from that point to w (through $w_0\,w$). The parameter is thus equal to the separation of those points of the screw axis whose null planes subtend an angle of 45° with that axis.

With the concept of a *linear* manifold, one can let six fixed screws L_i ($i = 1, \ldots, 6$) enter in place of the six basic rods $e_i\,e_k$, and derive any screw L from them, as linearly-independent *basic screws*, using suitable numerical quantities λ_i:

$$L = \sum \lambda_i\,L_i\,.$$

The λ_i are then the general *Ball* screw coordinates ([1]).

([1]) *Ball* always used numbers that were proportional to the λ_i above as coordinates, which made the rod lengths of L equal to unity. As a result of the latter requirement, we would not like to use that convention.

We understand the *product* $l_1\ l_2$ of two rods l_1 and l_2 to be mean the volume of the parallelepiped that they define. It shall be positive or negative according to whether a float (*Schwimmer*) that lies on one rod facing the other one seems to be directed towards the left or the right, respectively. One has $l_1\ l_2 = l_2\ l_1$, since the rods are also point products of rank two. The term "product" will then be justified by the fact that the distributive law is valid for the addition of rods, under which, $l_1\ l_2$ will remain unchanged when one these rods (or even both of them) decomposes into summands and one defines the algebraic sum of the resulting sub-volumes. One obtains the meaning of a *product of two screws* ($L\ \Lambda$) from the distributive law in the precisely the same way, and it will be a sum (4) of sub-volume numbers that is independent of the random choice of conjugate rods, and which *Ball* said was twice the *virtual coefficient* of the two screws. If it is zero then we, with *Ball*, will call the screws *reciprocal*. In that case, *Reye* said that L and Λ "supported" or "carried" the null system, while *F. Klein* (Mathematische Annalen II) spoke of an *involutory position.*

The number:

$$\tfrac{1}{2}L^2 = \tfrac{1}{2}\ L \cdot L = \tfrac{1}{2}(l_1 + l_2)\ (l_1 + l_2) = l_1\ l_2 = \tfrac{1}{2}(l + f)\ (l + f) = lf = \bar{l}^{\,2}\mathfrak{p}$$

is characteristic of any screw $L = l + f$, and shall be called the *volume* of the screw (*H. Grassmann, Jr.*, used the word *characteristic* for it in "Schraubenrechnung und Nullsystem," Halle, 1899). The relationship:

$$l_1\ l_2 = lf = \bar{l}^{\,2}\mathfrak{p} = \text{const.}$$

that it yields is an expression of *Chasles's theorem* on the invariance of the volume $l_1 l_2$ of the parallelepiped when l_1 and l_2 are arbitrary conjugate rods of L. (Our sign convention implies that the screw volume and the parameter always have the same sign.) As a consequence, a rod or a field can be regarded as a screw with zero volume; however, the parameter will become 0 in former case and ∞ in the latter, if the parameter is the number by which the length $\bar{l}$ of the rod axis l must be multiplied in order to obtain the surface area of f.

Conversely, if the volume of the screw is $\tfrac{1}{2}L^2 = l_1 l_2 = 0$ then no two conjugate rods can be skew; i.e., L will itself be a rod or a field, in particular.

We would also like to establish this condition equation for a screw to degenerate into a rod (or especially a field) when L is not real, but of the form:

$$L = L_1 + L_2\sqrt{-1}\,,$$

in which L_1 and L_2 mean real screws.

$$L^2 = (L_1^2 - L_2^2) + 2L_1L_2\sqrt{-1} \qquad \text{or} \qquad L_1^2 = L_2^2 \qquad \text{and} \qquad L_1L_2 = 0$$

then says that the screws L_1 and L_2 must be reciprocal and have the same volume. In this case, L is called a *complex rod*, and its screw is called an *imaginary Staudt line of the second kind.*

It is easy to convert the *line equation* $L^2 = 0$ into the relation $p_{12} \cdot p_{34} + \ldots = 0$ between the *Plückerian* line coordinates. One needs only to consider the squares in:

$$L = \sum p_{i,k}\, e_i\, e_k\,,$$

such as:

$e_1\, e_2 \cdot e_1\, e_2 = 0$	etc.,	(the volume of identical rods is zero)	
$e_1\, e_2 \cdot e_1\, e_3 = 0$	"	"	intersecting " "
$e_1\, e_3 \cdot e_4\, e_2 = e_1 e_2 e_3 e_4\,,$	"	"	the rods $e_1\, e_3$ and $e_4\, e_2$ is equal to that of $e_1\, e_2$ and $e_3\, e_4$.)

If we would like to write the line equation $L^2 = 0$ in, for example, simply-chosen *Ball* screw coordinates, instead of *Plückerian* line coordinates, then we could choose the six basic screws L_i ($i = 1, 2, 3, -1, -2, -3$) to be:

$$\begin{aligned} L_1 &= (e_1\, e_3 + e_3\, e_4), & L_{-1} &= (e_1\, e_2 + e_4\, e_3), \\ L_2 &= (e_1\, e_3 + e_4\, e_2), & L_{-2} &= (e_1\, e_3 + e_2\, e_4), \\ L_3 &= (e_1\, e_4 + e_2\, e_3), & L_{-3} &= (e_1\, e_4 + e_3\, e_2), \end{aligned}$$

which represents a system of co-reciprocal basic screws when each L_i is reciprocal to all of the five remaining ones, but not to itself, as exterior multiplication would yield. If one chooses, in particular, e_1, e_2, and e_3 to be mutually perpendicular unit segments and e_4 to be a point then the L_i will be in canonical form by themselves. Now, any screw L is determined by six λ_i according to:

$$L = \sum \lambda_i\, L_i \qquad (i = \pm 1, \pm 2, \pm 3).$$

If one recalls the fact that:

$$L_1^2 = L_2^2 = L_3^2 = -L_{-1}^2 = -L_{-2}^2 = -L_{-3}^2$$

then it will follow from $L^2 = 0$ that:

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_{-1}^2 + \lambda_{-2}^2 + \lambda_{-3}^2) = 0.$$

If we desire a co-reciprocal system with equal-volume basic screws then we need only to introduce – e.g. – the imaginary screws:

$$L_{-1}\sqrt{-1} = L_4, \quad L_{-2}\sqrt{-1} = L_5, \quad L_{-3}\sqrt{-1} = L_6\,,$$

in place of L_{-1}, L_{-2}, L_{-3} . If the deriving numbers that belong to the latter are λ_4, λ_5, λ_6 then $L^2 = 0$ will mean the same thing as:

$$\sum \lambda_i^2 = 0 \qquad (i = 1, \ldots, 6).$$

From the geometric meaning of *Ball's* λ_i, when referred to a system of co-reciprocal basic screws of equal volume 1, $L = \sum \lambda_i L_i$ will yield, upon multiplication by L_i :

$$(L_i\, L_k = 0,\ \tfrac{1}{2}L_i^2 = \tfrac{1}{2}L_k^2 = e_1\, e_2\, e_3\, e_4 = 1, \quad \text{where } k \neq i,\ i = 1, \ldots, 6)$$

$$L\, L_i = \lambda_i L_i^2 = 2\lambda_i\,,$$

i.e.,

$$\lambda_i = \tfrac{1}{2}(L\, L_i).$$

The λ_i *mean one-half the product (Ball's virtual coefficients) of L with the corresponding basic screws* L_i*,*

so one will have:

$$L = \sum (\tfrac{1}{2}L\, L_i)\, L_i$$

identically.

The meaning of *Plückerian* line coordinate $p_{i,k}$ follows from multiplying the equation $L = \sum p_{i,k}\, e_i\, e_k$ with $(e_m\, e_n)$, when $e_m\, e_n$ means the opposite edge to $(e_i\, e_k)$ of the basic tetrahedron, and thus assume that $e_i\, e_k\, e_m\, e_n = e_1\, e_2\, e_3\, e_4 = 1$. Under this assumption, one will get: $p_{i,k} = L \cdot (e_m\, e_n)$.

The $p_{i,k}$ *are the products (or moments, Ball's virtual coefficients) of a screw L (especially a rod L = l when* $L^2 = 0$*) with the opposite edges* $(e_m\, e_n)$ *to the* $e_i\, e_k$ *of the basic tetrahedron to which the deriving numbers* $p_{i,k}$*.*

One will then have $L = \sum [L \cdot (e_m\, e_n)]\, e_i\, e_k$, identically. In particular, if $L = l$, since $L^2 = 0$, then a rod $\sum p_{i,k}\, e_i\, e_k$ will be representable as the connecting (intersecting, resp.) rod of two points (plates, resp.) x and y, so one will get:

$$l = xy = \sum \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \cdot e_i\, e_k\,,$$

by exterior multiplication, then *in this case* the $p_{i,k}$ will be the six determinants of the matrices of coordinates of these points (plates, resp.). In the following section, we will prefer the λ_i over the p_{ik} .

Introduction of linear manifolds of screws and their reciprocal domains.

If the λ_i $(i = 1, \ldots, 6)$ are arbitrary then $L = \sum \lambda_i L_i$ can mean any screw in space. One can then define a linear space of screws R_{VI} of rank VI or dimension five that then contains ∞^5 screwing motions. (The dimension number is one less that *Grassmann's* rank number.) We assume that there exists a homogeneous, linear equation:

$$\text{(I)} \qquad \sum \alpha_{1i}\, \lambda_i = 0$$

between the λ_i . When one sets $A_1 = \sum \alpha_{1i}\, L_i$, this will be identical with $(A_1\, L) = 0$, which says that *any* screw L of the linear screw domain R_{V} – viz., the *web of screws* of L – that is cut from the domain of all screws in space by equations (I) is reciprocal to a certain screw A_1, and thus also reciprocal to any screw of the screwing motion $P_{\mathrm{I}} = x_1\, A_1$ (x_1, and arbitrary number) that is defined by A_1 . Conversely, a web R_{V} of reciprocal screws belongs to any screwing motion P_{I} that is fulfilled by any screw $\sum \lambda_i\, L_i$ whose λ_i satisfy the linear, homogeneous equation (I) when the α_{1i} are assumed to be the coordinates of a screw A_1 of the screwing motion P_{I} .

If two mutually-independent, homogeneous, linear equations:

$$\text{(II)} \qquad \left\{ \begin{array}{l} \sum \alpha_{1i}\lambda_i = 0, \\ \sum \alpha_{2i}\lambda_i = 0, \end{array} \right\}, \qquad \text{which are identical with} \left\{ \begin{array}{l} A_1 L = 0, \\ A_2 L = 0, \end{array} \right\},$$

exist between the λ_i of the screw $\sum \lambda_i\, L_i$, when A_1 and A_2 mean the screws:

$$\left\{ \begin{array}{l} A_1 = \sum \alpha_{1i} L_i , \\ A_2 = \sum \alpha_{2i} L_i , \end{array} \right\}$$

then that will say the same thing as saying that any arbitrary screw L of the linear space of screws of rank IV – viz., the *bush of rays*, R_{IV} – that is cut out from the screw domain in space by equations (II) is reciprocal to two linearly-independent screws A_1 and A_2, and thus also reciprocal to any screw of the rank-II linear screw space – viz., the *pencil of screws:*

$$P_{\mathrm{II}} = x_1\, A_1 + x_2\, A_2$$

(x_1 and x_2 are arbitrary numbers) that is determined by the latter two. Conversely, a bush R_{IV} of reciprocal screws $L = \sum \lambda_i\, L_i$ is established by two arbitrary linearly-independent screws:

$$\left\{ \begin{array}{l} A_1 = \sum \alpha_{1i} L_i , \\ A_2 = \sum \alpha_{2i} L_i \end{array} \right\}$$

of a pencil of screws P_{II} that they determine, when one demands the existence of linear, homogeneous equations:

$$\left\{ \begin{array}{l} \sum \alpha_{1i}\lambda_i = 0, \\ \sum \alpha_{2i}\lambda_i = 0 \end{array} \right\}$$

between the λ_i .

Analogously, if three linearly-independent, linear, homogeneous equations:

(III) $$\left\{\begin{array}{l} \sum \alpha_{1i}\lambda_i = 0, \\ \sum \alpha_{2i}\lambda_i = 0, \\ \sum \alpha_{3i}\lambda_i = 0, \end{array}\right\}, \text{ which are identical to } \left\{\begin{array}{l} A_1 L = 0, \\ A_2 L = 0, \\ A_3 L = 0, \end{array}\right\}$$

exist between the λ_i of the screw $L = \sum \lambda_i L_i$, where A_1, A_2, A_3 are the screws:

$$\left\{\begin{array}{l} A_1 = \sum \alpha_{1i} L_i, \\ A_2 = \sum \alpha_{2i} L_i, \\ A_3 = \sum \alpha_{3i} L_i, \end{array}\right\}$$

then the ∞^3 screws (∞^2 screws) L that are compatible with these three equations will fill up a linear, rank-III, screw domain – viz., a *sheaf of screws*, R_{III} – whose screws will all be, as we say briefly, reciprocal to themselves with respect to the screws of the sheaf:

$$P_{\text{III}} = x_1 A_1 + x_2 A_2 + x_3 A_3$$

(x_1, x_2 x_3 are arbitrary numbers). Since the relationship between the reciprocal sheaves R_{III} and P_{III} is completely mutual, each P_{III} will, conversely, correspond to a sheaf of reciprocal screws, namely, a *reciprocal* sheaf R_{III}.

A pencil of screws (that are all screws) will be cut out by four linearly-independent, linear, homogeneous equations between the λ_i of a screw $L = \sum \lambda_i L_i$, which will be identical to the four equations:

(IV) $$A_1 L = A_2 L = A_3 L = A_4 L = 0$$

($A_1, \ldots, A_4$ are linearly-independent screws), which is reciprocal (to itself) relative to the bush of screws:

$$P_{\text{IV}} = x_1 A_1 + \ldots + x_4 A_4$$

($x_1, \ldots, x_4$ are arbitrary numbers). Conversely, a bush P_{IV} that is established by arbitrary linearly-independent screws will determine the pencil of screws R_{II} that is *reciprocal* to it. Analogously, five linearly-independent equations between the λ_i :

(V) $$A_1 L = A_1 L = \ldots = A_1 L = 0$$

will determine, on the one hand, a screwing motion R_{I} of screws L when they establish the five ratios of the λ_i, and on the other hand, the web of screws:

$$P_{\text{V}} = x_1 A_1 + \ldots + x_5 A_5$$

(x_1, …, x_5 are arbitrary numbers). Conversely, if a P_{V} is given by five linearly-independent screws then the reciprocal screw will be determined by it.

Due to the reciprocity in the relationship between reciprocal domains R_n (n = I, …, V) and P_ν (ν = V, …, I), the cases (I) and (V), (II) and (IV) are not geometrically distinct, and for that reason we will have to seek:

(I) The screwing motion R_{I} and the reciprocal web of screws P_{V} that is connected with it,
(II) The pencil of screws R_{II} and the bush of screws P_{IV} that is connected with it,
(III) The sheaf of screws R_{III} and the reciprocal sheaf of screws P_{III} that it connected with it,

in order to provide an insight into all possible axis positions and parameter distributions of the linear space of screws. For many theorem that are otherwise suitable to what we are doing here, we refer to *E. Müller's* "Die Liniengeometrie nach den Prinzipien der Grassmannschen Ausdehnungslehre" in the Wiener Monatsheften, 1891, 2.

The line framework of the linear space of screws.

A curved manifold of screws that degenerate to lines will be cut out of any linear space of screws R_n (P_ν, resp.) of rank n (ν, resp.) $(n, \nu = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI})$ by the *line equation* $L^2 = 0$, namely, the *line framework* (*Liniengerippe*) r_n in R_n (ρ_ν in P_ν, resp.) in. Conversely, the entire associated linear screw space will be determined by such a framework r_n (ρ_ν, resp.) when the n screws that establish an R_n can be chosen from the ones that are cut out by the nonlinear equation $L^2 = 0$.

The framework r_{VI} of the rank-VI screw domain R_{VI} in space is the linear manifold itself.

The framework r_{V} of the rank-V screw domain R_{V} is *a linear complex*.

The framework r_{IV} of the rank-IV screw domain R_{IV} is *a linear congruence*.

The framework r_{III} of the rank-III screw domain R_{III} is *a second-degree ruled surface*.

The framework r_{II} of the rank-II screw domain R_{II} is *a skew pair of lines*.

One cannot speak of the framework r_{I} of the rank-I screw domain R_{I}, in general, since the screw itself would be line $R_{\mathrm{I}} = r_{\mathrm{I}}$, as a very special case.

In order to establish the validity of these assertions, one needs only to apply the *law of intersection of linear domains* (which is also understandable, with no further assumptions, in the consideration of systems of linear equations) to linear screws (ray-manifolds, resp.). It reads:

If a linear domain of rank α and one of rank β lies in one of rank γ, and no lower ranks ($\alpha < \beta$, $\beta < \gamma$, $\alpha + \beta > \gamma$) then the first two domains will have a linear domain of rank ($\alpha + \beta - \gamma$) in common.

(*Grassmann's* A_1, 1844, pp. 183, § 126; A_2, 1862, pp. 13, 14, no. 25, 26.)

In fact, for a framework r_{V} of a space of screws R_{V}, a planar pencil of screws will go through every point in space, and one likewise lies in any plane. The rays of a sheaf or a plane will define a special screw domain, namely, a ray domain of rank III, since all rods in it, and only them, can be derived from III linearly-independent ones of them. However, any R_{V} will have a rank V + III – VI = II linear domain, and thus, a plane pencil of rays, in common with a rank-III sheaf of rays (planar line field, resp.) that lies with it in a screw domain in space of rank VI and no lower. It will follow from this, as one can also infer directly in a completely analogous way (viz., V + II – VI = I), that any planar pencil of rays in the space I are associated with rays in r_{V} (the framework of R_{V}), or more briefly, that r_{V} is a *linear complex.*

For the framework r_{IV}, an R_{IV} will go through every point, and likewise a ray will lie in every plane, so every screw domain R_{IV}, and thus also its framework r_{IV}, will have a IV + III – VI = I linear domain – therefore, a line – in common with a rank-III sheaf of rays or a planar field of lines in the same space R_{IV} (and no lower-rank screw domain): i.e., r_{IV} is then a *linear congruence.*

The framework r_{II} of R_{II} that is cut out by $L^2 = 0$ is a skew pair of lines: Let R_{II} be determined by the screws L_1 and L_2 :

$$L = \lambda_1 L_1 + \lambda_2 L_2$$

and thus an arbitrary screw of R_{II} :

$$L^2 = \lambda_1^2 L_1^2 + 2\lambda_1\lambda_2 (L_1 L_2) + \lambda_2^2 L_2^2 = 0$$

will determine two ratios $\lambda_1 : \lambda_2$, and thus two lines, which must be skew, since otherwise it would not be true that conversely one could *not* derive screws that would *degenerate* to lines from L_1 or L_2 . This skew pair of lines will be real and separate, intersecting, or a *v. Staudt* imaginary line-pair of the second kind, according to whether the discriminant of L^2:

$$\begin{vmatrix} L_1^2 & (L_1 L_2) \\ (L_2 L_1) & L_2^2 \end{vmatrix} > 0, = 0, \text{ or } < 0,$$

resp.

The framework r_{III} of an R_{III} is a second-degree ruled family, so the surface that is filled up by ∞^1 rays of that domain will be met twice by an arbitrary line ρ_{I} : The rays that meet ρ_{I} will then belong to an R_{V} as the line framework r_{V} . However, R_{III} and R_{V} have a rank III + V – VI = II linear screw domain in common whose line framework consists of just those two lines of r_{III} that meet ρ_{I} .

This ruled family r_{III} can degenerate into a pair of pencils with common rays when the *Hessian* determinant:

$$\begin{vmatrix} L_1^2 & (L_1L_2) & (L_1L_3) \\ (L_2L_1) & L_2^2 & (L_2L_3) \\ (L_3L_1) & (L_3L_2) & L_3^2 \end{vmatrix}$$

of the line equation:

$$L^2 = (\lambda_1\, L_1 + \lambda_2\, L_2 + \lambda_3\, L_3)^2 = 0$$

(λ_1, λ_2, λ_3 are arbitrary numbers; L_1, L_2, L_3 are linearly-independent screws) vanishes. Cf., the aforementioned treatise of *E. Müller*.

If R_n and P_ν (n and ν positive whole numbers, $n + \nu = \mathrm{VI}$) are two linearly-reciprocal associated screw domains then their frameworks r_n and ρ_ν will also be reciprocal – i.e., any line of r_n will cut each of the ρ_ν, and conversely.

(I) If one can speak of a framework r_{I} of R_{I} in the case of reciprocal screw domains R_{I} and P_{V}, then $R_{\mathrm{I}} = r_{\mathrm{I}}$ must represent a line, in particular. ρ_ν will then be the linear complex that degenerates to the system of transversals of r_{I} .

(II) For reciprocal domains R_{II} and P_{IV}, each of the two lines of the skew pair R_{II} will cut each ray that belongs to the reciprocal congruence r_{IV}, so the two lines r_{II} will be the guiding lines of the congruence ρ_{IV} . In particular, as was already remarked, r_{II} can consist of two infinitely close skew lines, which can be thought of as being determined by one of them G and an infinitely-close line of the family that lies in a second-degree surface that contains G. This family can be replaced with an equilateral paraboloid T with the vertex line G. ρ_{IV} will then consist of the transversals of G that contact the surface T at its point of intersection with G. In particular:

(II′) r_{II} can be a pencil of rays (there are then no proper screws in the associated R_{II}), and ρ_{IV} can be a pencil of rays with the center of the pencil of ρ_{II} as the carrier *and* the line field of the plane of the pencil of r_{II} .

(III) The line frameworks r_{III} and ρ_{III} to two reciprocal sheaves of rays R_{III} and P_{III} are, in general, the two families of a second-degree surface. In particular:

(III′) If r_{III} is a pair of pencils of rays with a common ray then ρ_{III} will consist of a pair of just such pencils, each of which possesses the center of one of the two pencils r_{III} and the plane of the other. Even more especially:

(III″) r_{III} and ρ_{III} can be one and the same pencil of rays *or* planar line field; in this case, there will be no proper screws in the domain whose volume is non-zero, and one will have:

$$r_{\mathrm{III}} = R_{\mathrm{III}} = \rho_{\mathrm{III}} = P_{\mathrm{III}}\ .$$

Metric relation between reciprocal screws

I. The screwing motion R_{I} and the reciprocal web of screws P_{V} .

If two screws L and Λ are represented in canonical form, so they are sums of a rod l (λ, resp.) and a field f (φ , resp.) that is perpendicular to it whose volume is $\overline{l}\,\mathfrak{p}$ ($\lambda\pi$, resp.) if l (λ, resp.) means the (always positive) length of the rod and $\mathfrak{p}$ (π, resp.) means the parameter of the screw L (Λ, resp.), then the equation $L\,\Lambda = 0$ (which will make L and Λ *reciprocal* when it is true) will demand that one have:

$$L\,\Lambda = (l + f)\,(\lambda + \varphi) = l\varphi + \lambda f + l\lambda = 0 \qquad \text{(since } f\varphi = 0)$$

or

$$\overline{l\lambda}\,\pi\cos\lfloor\underline{l\lambda} + \overline{\lambda l}\,\mathfrak{p}\cos\lfloor\underline{l\lambda} - e\,\overline{l\lambda}\sin\lfloor\underline{l\lambda} = 0,$$

if we denote the angle that the axes l and λ of both screws make by $\lfloor\underline{l\lambda}$ and denote the shortest distance between them by e. By cancelling $\overline{l\lambda}$, this will give:

$$(\mathfrak{p} + \pi)\cos\lfloor\underline{l\lambda} - e\sin\lfloor\underline{l\lambda} = 0$$

or

$$(\mathfrak{p} + \pi) = e\tan\lfloor\underline{l\lambda}\,.$$

In particular, it follows that:

Two screws whose axes intersect are reciprocal when their parameters are equal and opposite or when the axes intersect perpendicularly.

The point of intersection of the axes can thus lie at a finite point ($e = 0$) or at infinity ($\lfloor\underline{l\lambda} = 0$); when the axes coincide, $\mathfrak{p} + \pi$ will serve as the reciprocity condition. Screws whose axes intersect perpendicularly are always reciprocal for *completely arbitrary* parameters $\mathfrak{p}$ and π. If the axes of reciprocal screws are perpendicular to each other then they must also intersect. If this were not the case then one of the two parameters would have to be infinitely large; i.e., no proper screw would be present, but only a field that would be perpendicular to the axis in question.

We now fix one of the two reciprocal screws – say, L – whose axis (which contains l) might be G, and examine the possible positions of the axis, as well as the parameter, of any screw Λ that is reciprocal to L, and thus, to any screw of the screwing motion R_{I} of L; i.e., we study the most general *web of screws* P_{V} (if we remark that each of them will also conversely determine *one* reciprocal screwing motion)!

Above all, we show that *any line* Γ in space can be the axis of a Λ in P_{V} whose parameter π is deduced from the reciprocity condition. On the other hand, if π is also capable of taking on an arbitrary value then an entire *linear complex* of axes Γ of the screws Λ in P_{V} will belong to every π = const.: Just as $e\tan\lfloor\underline{l\lambda} = \mathfrak{p}$ can be regarded as the metric defining equation of the linear complex ($\pi = 0$) of the line $\Gamma_{(\pi = 0)}$, relative to whose rod the product (cf., *Moebius's* definition of a linear complex) with L is zero:

$$e \tan \underline{\lfloor l\lambda} = \mathfrak{p} + \pi = \text{const.}$$

(π const.) can be regarded as the metric defining equation of a linear complex with the same axis G whose *dividing constant* (viz., the shortest separation between complex rays of G that subtend an angle of 45° with that axis) is greater than π. In addition to the complex $\pi = 0$ and the line framework ρ_{V} of the web P_{V}, the degenerate complexes:

$$\mathfrak{p} + \pi = 0, \qquad \text{i.e.,} \qquad \pi = -\varphi,$$

of lines Γ that cut G are worthy of note, as well as the other ones:

$$\varphi + \pi = \infty, \qquad \text{i.e.,} \qquad \pi = \infty,$$

for which G is perpendicular to Γ. The axes Γ that simultaneously belong to both of the latter complexes – i.e., they cut G perpendicularly – belong to *arbitrary* parameters in P_{V}.

Since only the sum $(p + \pi)$ of the parameters enters into the reciprocity condition, it will follow that:

The system of axes of two reciprocal domains R_n and P_ν ($n + \nu = \mathrm{VI}$) *remains unchanged when all parameters of one of those domains are increased by some amount and simultaneously the parameter of the other one is decreased by the same amount.*

In order to get a picture of the distribution of the parameters π on all lines Γ in space, which are regarded as axes in P_ν – we consider:

α) The parameter π of the line Γ that goes through an arbitrary spatial point w and

β) The π of the line Γ that lies in an arbitrary plane W.

α) Let e be the length of the altitude wx from w to G, so we first consider the parameter of the rays Γ_e that are perpendicular wx. Among them, one finds the ray wz that is parallel to G and one wy that is perpendicular to (wG). If we measure off the length $\rho = p + \pi$ (on both sides) on each Γ_e from w in the plane of the rectangular coordinate system w (y, z) then the endpoints with the coordinates y, z will trace out a curve $\mathfrak{C}$ that can be constructed according to $\rho = e \tan \vartheta$ (where $\vartheta = \underline{\lfloor l\lambda} = \underline{\lfloor G\Gamma_e}$), as is clear in Fig. 1 (†), where the coordinates:

$$y = \rho \sin \vartheta = e \frac{\sin^2 \vartheta}{\cos \vartheta}, \qquad z = \rho \cos \vartheta = e \sin \vartheta,$$

in which:

(†) Translator's note: The figures did not seem to be available in the version of this article that was used.

$$\sin\vartheta = \frac{z}{e},\ \cos\vartheta = \frac{z^2}{ey},$$

will satisfy the equation:

$$\frac{z^2}{e^2} + \frac{z^4}{e^2 y^2} - 1 = 0 \qquad \text{or} \qquad z^2 (y^2 + z^2) - e^2 y^2 = 0.$$

If we take *wx* to be the third rectangular coordinate axis and arrange all rays of the sheaf *w* into pencils that connect a Γ_e with *wx* then all rays Γ of each of these pencils must possess the same parameter as the ray Γ_e itself, since that parameter (like any arbitrary one) will also belong to (*wx*), which is regarded as the axis Γ_0 of P_ν, and any screw with the same parameters in the pencil thus-determined will be linearly derivable from two screws with the same parameters whose axes intersect. The endpoints (with the coordinates *x*, *y*, *z*) of the segments of length $\rho = \mathfrak{p} + p$ that were measured off on all sides on the axes Γ of the sheaf *w* will then trace out a surface *F* that one can think of as being constructed from a variable circle whose center is *w*, whose plane goes through *wx*, and which cuts the curve $\mathfrak{C}$. The equation of *F* in terms of *x*, *y*, *z* is obtainable by eliminating ϑ and η (viz., the angle between Γ and *wx*) from:

$$x = \rho\cos\eta = e\tan\vartheta\cos\eta, \qquad \frac{y}{z} = \tan\vartheta, \qquad \frac{y^2+z^2}{x^2} = \tan^2\eta,$$

so since:

$$\cos\eta = \frac{xz}{ey},\ \frac{y^2+z^2}{x^2} = \frac{\sin^2\eta}{\cos^2\eta}, \qquad \sin^2\eta = \frac{y^2+z^2}{x^2}\cdot\frac{x^2z^2}{e^2y^2} = \frac{z^2(y^2+z^2)}{e^2x^2},$$

one will have:

$$x^2 y^2 + z^2 (y^2 + z^2) = e^2 y^2 \text{ or } z^2 (x^2 + y^2 + z^2) - e^2 y^2 = 0.$$

Ball called *F* a "pectenoid" and gave an intuitive picture of that surface in his treatise (1900) on pp. 255.

If *w* is at infinity – so $\underline{|\, l\lambda}$ = const. – then $\rho = (\mathfrak{p} + \pi) = e \tan \underline{|\, l\lambda}$ will be proportional to *e*.

β) Let *W* be an arbitrary plane, so above all, the parameter of a line Γ_1 in it that is parallel to the orthogonal projection of the axis *G* onto *W* will be easy to imagine when one measures off the segment $\rho = \mathfrak{p} + \pi = e \tan \underline{|\, gW}$ on each of them – say, starting from their point of intersection *w* with the lines Γ_0 in *W* that cut *G* perpendicularly – such that the endpoints of the segments thus measured off will trace out a line. Any line Γ of *W* will then have a parameter that is equal to the Γ_1 that goes through its point of intersection with Γ_0 . Any arbitrary parameter will belong to Γ_0 itself.

If *W* is parallel to *G* then π = const. for all parallel rays in it, such that all rays of a well-defined direction of *W* can be represented by one of them that goes through a fixed point w in the plane W; we only need to choose it arbitrarily on the orthogonal projection of G onto W in order to be able to repeat the considerations that were discussed in α), in which we employed the curve $\mathfrak{C}$ for the purpose of illustration.

R_{I} and P_{V} are determined in a canonical way when one is given the axis G and parameter $\mathfrak{p}$ of R_{I}. The latter given can be replaced with the "characteristic" of the screwing motion – i.e., the length $\overline{l}$ of the rod l of any screw L of the screwing motion R_{I} whose volume is 1: $\frac{1}{2}L^2 = \overline{l}^{\,2}\mathfrak{p} = 1$. The characteristic and the parameter are thus coupled by the equation:

$$\overline{l} = \pm\sqrt{\frac{1}{\mathfrak{p}}} \qquad \text{or} \qquad \mathfrak{p} = \frac{1}{\overline{l}^{\,2}},$$

so the characteristic of a screw with a negative parameter will be imaginary; a finite line (i.e., a screw of volume 0) will have:

a characteristic of ∞, corresponding to $\mathfrak{p} = 0$,

while a field will have:

a characteristic of 0, corresponding to $\mathfrak{p} = \infty$.

II. The pencil of screws R_{II} and the reciprocal bush of screws P_{IV}.

The most general screw L of an R_{II} is derivable from *linearly-independent* basic screws L_1 and L_2 and arbitrary numerical quantities λ_1 and λ_2 by way of $L = \lambda_1 L_1 + \lambda_2 L_2$. The latter can also *not* have the same axis and the same parameter. Special cases:

1*) If the axes G_1 and G_2 of the screws L_1 and L_2, resp., coincide in a line G then R_{II} will consist of all screws with that axis and an arbitrary parameter, and in particular, the line framework r_{II} will consist of G and the field that is perpendicular to it. The axes Γ of the reciprocal domain P_{IV} that also belong to arbitrary parameters will exhaust the congruence of transversals that are perpendicular to G; this congruence will also define the line framework ρ_{IV} of P_{IV}. All of the remaining skew lines that are perpendicular to G will belong to $\pi = \infty$ in P_{IV}; i.e., they will represent fields of the pencil of fields that are parallel to G.

2*) If the axes G_1 and G_2 of L_1 and L_2, resp., are parallel then these basic screws can be replaced with a certain rod l_0 of a line G_0 of the pencil of parallels $G_1\,G_2$ and a field f_0 that contains the direction A in the plane $G_1\,G_2$ that is perpendicular to G. In fact, λ_1 and λ_2 can be chosen such that the field in $L = \lambda_1 L_1 + \lambda_2 L_2$ that is perpendicular to G vanishes in one case, and in the other case, such that rods of $\lambda_1 L_1$ and $\lambda_2 L_2$ contain equal and opposite segments such that a field f_0 with the property above will result for L. The line framework r_{II} of R_{II} here consists of the lines G_0 of l_0 and the infinitely-distant ones of f_0 ; the ρ_{IV} of the reciprocal bush P_{IV} consist of the transversals of l_0 that are parallel to f_0 .

If one adds a multiple λf_0 of f_0 to l_0 then the latter can be represented as the sum of two fields $\lambda f_0'$ and λf, the former of which lies in the plane G_1G_2 and can be constructed as a right angle with one side parallel to l_0 and of length $\overline{l_0}$ and the other side of length x in the direction A that is perpendicular to G, while λf_0 is perpendicular to l_0 and has a volume of $\overline{l_0}\,\mathfrak{p}$, where $\mathfrak{p}$ is the parameter of the variable screw $L = l_0 + \lambda f_0 = (l_0 + \lambda f_0') + \lambda f = l + \lambda f$, which appears to be the sum of the rod l and the field λf that is perpendicular to it in canonical form. l lies in the plane G_1G_2 that is parallel to l_0 with a perpendicular separation x from the latter rod and has the same length as l_0 . If φ_0 is the constant angle of f_0 with respect to l_0 – i.e., also with respect to the plane $l_0\, l = G_1G_2$ – then tan φ_0 will be equal to the ratio of the volumes $l_0\,\mathfrak{p}$ of the field λf and $\overline{l_0}\,x$ of the field , so $\tan \varphi_0 = \mathfrak{p} / x$.

The axis G of any screw L in R_{II} *thus lies in the plane* G_1G_2 *that is parallel to those lines and its parameter* $\mathfrak{p}$ *is proportional to the perpendicular separation x between G and* G_0 : $\mathfrak{p} = x \tan \varphi_0$.

If one then measures off the parameters $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *of these screws as segments on* G_1 *and* G_2, resp., *and lets A be the connecting line of the starting points of these lines (which is assumed to be perpendicular to the direction of G), and B is the endpoint of these segments then on any line G of the pencil of parallels* G_1G_2 *that appears in* R_{II} *as an axis the segment from the point of intersection with A to the point of intersection with B will specify the associated parameter* $\mathfrak{p}$. *In particular, the line* G_0 *in the pencil of screws that is associated with* $\mathfrak{p} = 0$ *goes through the point of intersection of A and B;* f_0 *, the field of the pencil, is parallel to A and the axis G will subtend an angle* φ_0 *with the plane* G_1G_2 *that is equal to the one between A and B.*

The axes Γ *of the reciprocal bush* P_{IV} *that belong to finite parameters fill up the totality of all lines that are parallel to* f_0, *and the parameter p that is associated with them in this screw domain is equal and opposite to the parameter* $\mathfrak{p}$ *of the axis G of the pencil* G_1G_2 *of* R_{II} *that meets* Γ. *The lines* Γ *in the plane* G_1G_2 *that are perpendicular to G thus belong to arbitrary parameters. The pencil of the fields that are parallel to l – i.e., to G – also belong to* P_{IV}, *which is why any line that is perpendicular to G can be regarded as the axis* Γ *in* P_{IV} *that is associated with* $\pi = \infty$.

The screwing motions $\Gamma(\pi)$ thus obtained, and only them, are in fact reciprocal under f_0, as well as under any screwing motion $G(\mathfrak{p})$ whose axis G is met by Γ.

General case R_{II} (P_{IV}, resp.)

Principal screws and parameter distribution in the domain R_{II} .

If the axes G_1 and G_2 of the screws L_1 and L_2, resp., that determine a pencil of screws R_{II} subtend a non-zero angle then two screws L_{I} and L_{II} will always be contained in R_{II} that:

1) Are reciprocal to each other and
2) Possess mutually-perpendicular axes that intersect according to the reciprocity condition ($\underline{|\,l\lambda}$ = 90°, $\mathfrak{p} + \pi$ is finite, so $e = 0$).

In order to arrive at these *principal screws* L_{I} and L_{II}, we can first replace L_1 and L_2 with L_1 and $(L_1 + \kappa L_2)$, and make the latter screw reciprocal to L_1 by a suitable choice of the number $\kappa = -\dfrac{L_1^2}{(L_1 L_2)}$. In order to not complicate the notations, we will assume that L_1 and L_2 are *reciprocal*, and that they are *of equal volume* ([1]):

$$L_1 L_2 = 0, \qquad L_1^2 = L_2^2,$$

since that can be achieved by multiplying one of the two screws by a number.

Let $L_1 = l_1 + f_1$ and $L_2 = l_2 + f_2$ be represented in canonical form, so for any arbitrary φ the linearly-independent screws of R_{II} , namely:

$$\begin{aligned} L_{\mathrm{I}} &= L_1 \cos\varphi - L_2 \sin\varphi = l_{\mathrm{I}} + f_{\mathrm{I}}, \\ L_{\mathrm{II}} &= L_1 \sin\varphi + L_2 \cos\varphi = l_{\mathrm{II}} + f_{\mathrm{II}}, \end{aligned}$$

(canonical form) will also be reciprocal, since $L_{\mathrm{I}}\, L_{\mathrm{II}} = (L_1^2 - L_2^2)\sin\varphi\cos\varphi = 0$. The rods $\left.\begin{aligned} l_{\mathrm{I}} &= l_1 \cos\varphi - l_2 \sin\varphi, \\ l_{\mathrm{II}} &= l_1 \sin\varphi + l_2 \cos\varphi, \end{aligned}\right\}$ can be made perpendicular to each other by a suitable choice of φ. If an underlined symbol denotes the segment of a rod and an overbar means the rod length then in order for the product $\underline{l}_{\mathrm{I}} \,|\, \underline{l}_{\mathrm{II}} = (\underline{l}_1 \cos\varphi - \underline{l}_2 \sin\varphi) \,|\, (\underline{l}_1 \sin\varphi + \underline{l}_2 \cos\varphi)$ to be zero, one needs only to take $(L_1^2 - L_2^2)\sin\varphi\cos\varphi + \underline{l}_1\, \underline{l}_2 (\cos^2\varphi - \sin^2\varphi) = 0$; i.e., $\tan 2\varphi = \dfrac{2\,\underline{l}_1 \underline{l}_2}{-\bar{l}_1^2 + \bar{l}_2^2}$.

If we set the volume of the principal screws (which prove to be equal $L_{\mathrm{I}}^2 = L_{\mathrm{II}}^2 = L_1^2 = L_2^2$) to 1 then the lengths of the rods of these screws will immediately become their characteristics. Each screw in R_{II} of volume 1:

([1]) If the parameters and volumes of L_1 and L_2 have opposite signs then the following development can be replaced with one that operates merely with *real* screws, in which one assumes that $L_1^2 + L_2^2 = 0$ and the corresponding hyperbolic functions are used in place of $\cos\varphi$, $\sin\varphi$. Cf., the remark on page 22, moreover.

$$L = L_{\mathrm{I}} \cos \omega + L_{\mathrm{II}} \sin \omega \qquad (\omega \text{ is an arbitrary number})$$

possesses a rod segment $\underline{l} = \underline{l}_{\mathrm{I}} \cos \omega + \underline{l}_{\mathrm{II}} \sin \omega$, which is the radius of the conic section $\mathfrak{K}$ that is determined by the semi-axes l_{I} and l_{II} and is "characteristic" of R_{II} . $\mathfrak{K}$ has the equation:

$$\frac{x^2}{\bar{l}_{\mathrm{I}}^2} + \frac{y^2}{\bar{l}_{\mathrm{II}}^2} = 1, \qquad \text{or} \qquad (\text{since } \tfrac{1}{2} L_a^2 = \bar{l}_a^2\, \mathfrak{p}_a = 1,\ a = \mathrm{I, II})$$

$$(\mathfrak{K}) \qquad \mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2 = 1,$$

when referred to G_{I} and G_{II} as the x and y axes, resp.

The radii l of the characteristic conic section $\mathfrak{K}$ are the characteristics of the screws in R_{II} that are parallel to them.

We obtain the parameter $\mathfrak{p}$ of any screw whose axis G subtends the angle ϑ with G_{I} from this. We set $x = \bar{l} \cos \vartheta$, $y = \bar{l} \sin \vartheta$ in $\mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2 = 1$, and find:

$$\bar{l}^{\,2} (\mathfrak{p}_{\mathrm{I}} \cos^2 \vartheta + \mathfrak{p}_{\mathrm{II}} \sin^2 \vartheta) = 1 \ ;$$

i.e., (since $\bar{l}^{\,2} \mathfrak{p} = 1$)

$$\mathfrak{p} = \mathfrak{p}_{\mathrm{I}} \cos^2 \vartheta + \mathfrak{p}_{\mathrm{II}} \sin^2 \vartheta .$$

If one measures off not the characteristic, but the parameter $\mathfrak{p}$, of the screw in R_{II} whose axis has that direction directly from the point $p = G_{\mathrm{I}}\, G_{\mathrm{II}}$ in the plane of these principal axes then the endpoints of the segments thus-obtained will trace out the *parameter curve* $\mathfrak{P}$ in R_{II} . Since $\mathfrak{p} = \sqrt{x^2 + y^2} = \mathfrak{p}_{\mathrm{I}} \cos^2 \vartheta + \mathfrak{p}_{\mathrm{II}} \sin^2 \vartheta$, $\cos^2 \vartheta = \dfrac{x^2}{x^2 + y^2}$, $\sin^2 \vartheta = \dfrac{y^2}{x^2 + y^2}$, the equation for it relative to $G_{\mathrm{I}}\, G_{\mathrm{II}}$ will be:

$$(\mathfrak{P}) \qquad (x^2 + y^2)^3 - (\mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2)^2 = 0.$$

The forms of $\mathfrak{K}$ are indicated in Fig. 2, 3, 4 when L_{II} (and thus l_{II}) is real, so $\mathfrak{p}_{\mathrm{II}}$ can be assumed to be positive, and

Fig. 2: L_{I} (and thus l_{I}) is real, so $\mathfrak{p}_{\mathrm{I}} > 0$,

Fig.3: L_{I} has volume 0, $L_{\mathrm{I}} = G_{\mathrm{I}}$, $\bar{l}_{\mathrm{I}} = \infty$, $\mathfrak{p}_{\mathrm{I}} = 0$ (limiting case) ([1]),

Fig. 4: L_{I} (and thus l_{I}) is imaginary, so $\mathfrak{p}_{\mathrm{I}} < 0$ ([1]).

([1]) In the limiting case, $\mathfrak{K}$ will be a pair of parallel lines, and not a parabola: *Ball-Gravelius*, 1889, pp. 272, *Ball*, 1900, pp. 111.

The associated parameter curves $\mathfrak{P}$ are exhibited in Fig. 2′, 3′, 4′, and in fact the curve segments that belong to endpoints with negative parameters are recorded with primes. The construction that is apparent in the figures according to the equation:

$$\mathfrak{p}(\cos^2\vartheta + \sin^2\vartheta) = \mathfrak{p}_{\mathrm{I}}\cos^2\vartheta + \mathfrak{p}_{\mathrm{II}}\sin^2\vartheta \qquad [(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}})\cos^2\vartheta = (\mathfrak{p}_{\mathrm{II}}-\mathfrak{p})\sin^2\vartheta, \text{ resp.}]$$

or

$$-\ \mathfrak{p}-\mathfrak{p}_{\mathrm{I}} = \tan^2\vartheta, \qquad (\mathfrak{p}_{\mathrm{II}}-\mathfrak{p} = \tan^2\vartheta, \text{ resp.})$$

is the following: Two circles $\mathfrak{K}_{\mathrm{I}}$ and $\mathfrak{K}_{\mathrm{II}}$ with radii $\mathfrak{p}_{\mathrm{I}}$ and $\mathfrak{p}_{\mathrm{II}}$, resp., are drawn around $p = G_{\mathrm{I}}\,G_{\mathrm{II}}$ as their centers, and made to intersect an arbitrary radius at $\mathfrak{P}_{\mathrm{I}}$ and $\mathfrak{P}_{\mathrm{II}}$, resp. (in case $\mathfrak{p}_{\mathrm{I}}$ and $\mathfrak{p}_{\mathrm{II}}$ are the same; otherwise, one takes the extension of the radius over p in one case). One drops a perpendicular to the radius that is employed through the point of intersection M of the perpendiculars that are drawn through $\mathfrak{P}_{\mathrm{I}}$ to G_{I} and through $\mathfrak{P}_{\mathrm{II}}$ to G_{II} . Its foot will describe $\mathfrak{P}$ when the radius changes with ϑ.

In particular, for $\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}} = 0$, $\mathfrak{K}$ will be an equilateral hyperbola and $\mathfrak{P}$ will be star-like: $\mathfrak{P} = \mathfrak{P}^*$. (Figs. 5 and 6.)

As was remarked above (pp. 13), any linear screw domain will have the property of still being one when one changes all parameters $\mathfrak{p}$ by an equal amount for equal axis positions. This is connected with the fact that new $\mathfrak{P}$ will emerge from $\mathfrak{P}$ by enlarging all radius vectors $\mathfrak{p}$ by the same amount:

$$\mathfrak{p} = \mathfrak{p}_{\mathrm{I}}\,\frac{1+\cos 2\vartheta}{2} + \mathfrak{p}_{\mathrm{II}}\,\frac{1-\cos 2\vartheta}{2} = \frac{\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}}}{2} - \frac{\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{I}}}{2}\cos 2\vartheta$$

which will change independently of ϑ by a constant amount when $\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}}$ varies, although $\frac{\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{I}}}{2} = \mathfrak{h}$ will remain constant. A family of curves $\mathfrak{P}$ is represented in Fig. 7 that belong to constant $\frac{\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{I}}}{2} = \mathfrak{h}$, and each of which can be generated from one of them – e.g., from the star-like $\mathfrak{P}^*$ ($\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}} = 0$) by uniformly changing the radius vectors $\mathfrak{p}$ of p. From the remark that was made on pp. 13, *this family of curves belongs to a well-defined axis surface* and *specifies all possible parameter distributions* of the domain R_{II} that belongs to this surface. This family $\mathfrak{P}$ corresponds to the pencil $(\mathfrak{p}_{\mathrm{I}} + \kappa)\,x^2 + (\mathfrak{p}_{\mathrm{II}} + \kappa)\,y^2 = 1$ of characteristic conic sections (Fig. 8).

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(1) Since $\frac{1}{2}L^2 = \bar{l}^{\,2}\,\mathfrak{p} = 1$, the real screws L of volume 1 will be pure imaginary and imaginary real when we switch the convention that we made on pp. 2 regarding the parameter sign with the opposite one, which is likewise permissible.

The same parameter $\mathfrak{p}$ belongs to $+\vartheta$ and $-\vartheta$. Screws in R_{II} whose axes possess symmetric directions relative to the principal axes G_I and G_{II} will have the same parameter.

Screws in R_{II} whose axes are perpendicular to each other – which corresponds to ϑ and $(90^o + \vartheta)$ – or have symmetric directions with respect to the angle bisectors of G_I and G_{II} – which corresponds to ϑ and $(90^o - \vartheta)$ – will possess parameters whose sum is constant and equal to $\mathfrak{p}_I + \mathfrak{p}_{II}$.

$\mathfrak{P}$ moves entirely between the limits $\mathfrak{p}_I$ and $\mathfrak{p}_{II}$; if they are equal then $\mathfrak{p}$ will always have the same value. For arbitrary ω, $\left.\begin{aligned} L_1 &= L_I \cos\omega + L_{II} \sin\omega, \\ L_2 &= -L_I \sin\omega + L_{II} \cos\omega, \end{aligned}\right\}$ will give all pairs of reciprocal screws in R_{II} with equal volume, and their characteristics $\left.\begin{aligned} \underline{l}_1 &= \underline{l}_I \cos\omega + \underline{l}_{II} \sin\omega, \\ \underline{l}_2 &= -\underline{l}_I \sin\omega + \underline{l}_{II} \cos\omega, \end{aligned}\right\}$ will be conjugate radii of $\mathfrak{K}$:

The axes of reciprocal screws in R_{II} are parallel to conjugate diameters of $\mathfrak{K}$.

$\underline{l}_1$ and $\underline{l}_2$ satisfy the equation:

$$\overline{l}_1^2 + \overline{l}_2^2 = \overline{l}_I^2 + \overline{l}_{II}^2 = \text{const.},$$

from which (if one recalls that $\overline{l}^2 \mathfrak{p} = 1$), it will follow that:

$$\frac{1}{\mathfrak{p}_1} + \frac{1}{\mathfrak{p}_2} = \frac{1}{\mathfrak{p}_I} + \frac{1}{\mathfrak{p}_{II}} = \text{const.}$$

The sum of the reciprocal values of the parameters of reciprocal screws of a pencil R_{II} is constant.

Two screwing motions in R_{II} that correspond to $\mathfrak{p} = 0$ will degenerate to lines and define the line framework r_{II} of this domain; they will subtend the angle ϑ_0 with G_I which is such that:

$$\tan\vartheta_0 = \sqrt{-\frac{\mathfrak{p}_I}{\mathfrak{p}_{II}}},$$

and will be parallel to the asymptotes of $\mathfrak{K}$ and to the tangents to $\mathfrak{P}$ at the origin p.

Positions of the axes in the domain R_{II}. Plücker's cylindroid.

The principal screws L_{I} and L_{II} of volume 1 are sums of semi-axis rods l_{I} (l_{II}, resp.) of $\mathfrak{K}$ and a field that is perpendicular to it that is representable in right-angle form with the other semi-axis segments $\underline{l}_{\mathrm{II}}$ ($\underline{l}_{\mathrm{I}}$, resp.) of $\mathfrak{K}$ as one side and the segment c that is perpendicular to the principal plane $G_{\mathrm{I}}\,G_{\mathrm{II}}$ whose length is:

$$\overline{c} = \frac{1}{\underline{l}_{\mathrm{I}}\,\underline{l}_{\mathrm{II}}} = \sqrt{\mathfrak{p}_{\mathrm{I}}\,\mathfrak{p}_{\mathrm{II}}}\,,$$

namely:

$$\left.\begin{aligned} L_{\mathrm{I}} &= l_{\mathrm{I}} + (\underline{l}_{\mathrm{I}} \cdot c), \\ L_{\mathrm{II}} &= l_{\mathrm{II}} + (\underline{l}_{\mathrm{I}} \cdot c). \end{aligned}\right\}.$$

The arbitrary screw in R_{II} of volume 1:

$$\begin{aligned} L &= L_{\mathrm{I}} \cos\omega + L_{\mathrm{II}} \sin\omega, \\ &= (l_{\mathrm{I}} \cos\omega + l_{\mathrm{II}} \sin\omega) + [-\underline{l}_{\mathrm{I}} \sin\omega + \underline{l}_{\mathrm{II}} \cos\omega]\, c \end{aligned}$$

is then given – but still not, like perhaps the principal screws, in canonical form – as a sum of an arbitrary radius rod:

$$l = l_{\mathrm{I}} \cos\omega + l_{\mathrm{II}} \sin\omega$$

of $\mathfrak{K}$ and a field:

$$f = [-\underline{l}_{\mathrm{I}} \sin\omega + \underline{l}_{\mathrm{II}} \cos\omega] \cdot c$$

over the radius l' ($\underline{l}' = -\underline{l}_{\mathrm{I}} \sin\omega + \underline{l}_{\mathrm{II}} \cos\omega$) and the segment c. The position of the axis G of L that might subtend the angle ϑ with G_{I} will be determined by the rod that l goes to when the field f is projected orthogonally onto the plane (lc). This summand, which is a field of volume:

$$\overline{c}\,\overline{l}' \cos\underline{\lfloor ll'},$$

will displace the rod l along the Z-axis, which we draw through the principal point $p = G_{\mathrm{I}}\,G_{\mathrm{II}}$ and c, through the segment:

$$z = \frac{\overline{c}\,\overline{l}' \cos\underline{\lfloor ll'}}{\overline{l}} = \frac{\overline{c}\,\overline{l}\,\overline{l}' \cos\underline{\lfloor ll'}}{\overline{l}^{\,2}} = \mathfrak{p}\,\cos\underline{\lfloor ll'}$$

$$(\text{since } \overline{c}\,\overline{l}\,\overline{l}' = 1 \text{ and } \overline{l}^{\,2}\mathfrak{p} = 1);$$

l has the direction cosines $\cos\vartheta$, $\sin\vartheta$, when measured with respect to G_{I} and G_{II}, resp.,

l' " " " , $\dfrac{-\mathfrak{p}_{\mathrm{I}} \sin\vartheta}{\mathfrak{p}}$, $\dfrac{-\mathfrak{p}_{\mathrm{II}} \cos\vartheta}{\mathfrak{p}}$, such that:

$$\cos \underline{|ll'} = \frac{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}}{\mathfrak{p}} \sin \vartheta \cos \vartheta,$$

so

$$z = \frac{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}}{2} \sin 2\vartheta = \mathfrak{h} \sin 2\vartheta$$

will become:

$$\mathfrak{h} = \frac{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}}{2}.$$

One will obtain the axis surface of a pencil of screws R_{II} when one displaces any ray G of the pencil of rays that is determined by the principal axes G_{I} and G_{II} in the direction that is perpendicular to the "principal plane" through a distance $z = \mathfrak{h} \sin 2\vartheta$. ($\vartheta = \underline{|GG_{\mathrm{I}}}$)

The invariance of z – and thus, the entire axis surface – is immediately obvious in the event that all parameters – and thus also $\mathfrak{p}_{\mathrm{I}}$ and $\mathfrak{p}_{\mathrm{II}}$ – are changed by equal amounts ([1]). $z = \mathfrak{h} \sin 2\vartheta$ tells us the axis surface by the following construction:

One draws two sine waves of arbitrary wave lengths and amplitude $\mathfrak{h}$ in a plane, and rolls them onto a cylinder of rotation in such a way that the starting point A of the first wave (Fig. 9) coincides with the endpoint of the second one, by which, the double wave will form a fourth-order curve (one of the simplest Lissajous figures of oscillation, next to the ellipse). Now, if one always connects two points of that curve that lie symmetrically with respect to the cylinder's Z-axis with a line G (which cuts A perpendicular) then the locus of these G will be the desired axis surface.

This axis surface is known by the name of *cylindroid*, and was investigated by *Plücker, Ball, et al.* (*Ball* applied the name of "cylindroid" to the pencil of screws R_{II} itself.)

The cylindroid is a conoid whose generator G cuts the nodal line Z (viz., the dividing axis, which is a double line of this surface) at right angles, so it will also *meet the line U at infinity that is perpendicular to Z.* (So it is a double line in the dual sense when the cylindroid is regarded as the locus of its tangential planes, but not its points.) Two generators of the surface will intersect at each point of Z that lies symmetrically with respect to the angle-bisecting planes of:

$$G_{\mathrm{I}}\, Z \quad \text{and} \quad G_{\mathrm{II}}\, Z.$$

([1]) Therefore, the case that was touched upon on pp. 17, rem. 1, as well as the limiting case of pp. 19, rem. 1, does not need to be treated again for the determination of the axis surface, since one can think of the axis surface in these cases as being first determined when one has made all screw volumes that enter into consideration positive by increasing all parameters by a constant amount.

At the limiting points $z = \pm\,\mathfrak{h}$ of Z – viz., the *pinch points* of the surface – these two generators will coincide in a line, namely, the *external* edge or *pinch edge* in the angle-bisecting plane considered, and for $|\,z\,| > \mathfrak{h}$ they will no longer be real. G_{I} G_{II} and Z, which intersect at the *principal point p* of the cylindroid, will be symmetry axes for the surface; if we make them into coordinates axes (x, y, z) then since:

$$\tan\vartheta = \frac{y}{x}, \qquad \sin 2\vartheta = \frac{2\tan\vartheta}{1+\tan^2\vartheta} = \frac{2xy}{x^2+y^2},$$

the equation of the cylindroid surface, namely:

$$z = \frac{2\mathfrak{h}xy}{x^2+y^2},$$

will become:

$$\begin{vmatrix} z & y \\ 2\mathfrak{h}\,x & x^2+y^2 \end{vmatrix} = 0$$

in determinant form.

One can make the imaginary lines (*v. Staudt's* first kind), which meet in an arbitrary point $|\,z\,| > \mathfrak{h}$ of Z, more intuitive by means of two real cylindroid edges that are at a height of $\mathfrak{h}^2 / z$ over the principal plane G_{I} G_{II} . If one draws these edges through the point z of a parallel to Z then they will define a harmonic quadruple of rays with their angle bisectors, which will represent the imaginary line-pair in *Staudt's* way of thinking: The cylindroid and the auxiliary surface:

$$\begin{vmatrix} \mathfrak{h}/z & y \\ 2x & x^2+y^2 \end{vmatrix} = 0,$$

which will be traced out by the parallels above, will have the equations:

$$\begin{vmatrix} z & \xi^2-\eta^2 \\ \mathfrak{h} & \xi^2+\eta^2 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \mathfrak{h} & \xi^2-\eta^2 \\ z & \xi^2+\eta^2 \end{vmatrix} = 0,$$

resp., with respect to the system ξ, η, z that is the previous one after it has been rotated around Z by 45°, from which, it will emerge that both surfaces are affine with respect to each other, with ξz-plane as the affine plane, when the η-ordinate of one surface goes to that of the other one by multiplying by $\sqrt{-1}$. It follows from this that under these affine transformations of the one surface into the other, the real line-pairs in the plane z = const. will become imaginary, and conversely, so the stated *Staudt* representation is applicable.

We can obtain the aforementioned auxiliary surface simultaneously with the cylindroid when (Fig. 9), analogously to the sine-double wave that was employed in the construction of the cylindroid, roll the associated cosecant line of equal amplitude onto the cylinder and the connect its diametrically-opposite points with lines.

In particular:

The tangents t_1, t_2 to the cylindroid go from the infinitely-distant point of Z to the absolute sphere-circle.

The intersection curve of the cylindroid with an arbitrary plane *W* is a line $S = S_0^{3,4}$ of order three and class four, and genus 0,

that is collinearly generatable by any line of the types:

$$y^2 = x^2 (x \pm 1),$$

(Fig. 10 and 11)

The projection cone of the cylindroid onto an arbitrary plane *w* projects that surface onto an arbitrary plane *m* in a line $s = s_0^{4,3}$ of order four and class three, and genus 0,

and its type is, e.g., that of the cardioid of Fig. 12 or the *Steiner* hypocycloid (1) in Fig. 13,

and is *real* collinear to the first or second one, according to whether

the double point (*WZ*) possesses real tangents, and correspondingly (2) only one of the three inflection points (which lie on a line) will be real, *or* the (real) double point (*WZ*) will have imaginary tangents, and therefore all three inflection points of the curve will be real.

the double tangents (i.e., the trace of *wU*) contact the curve at real points, and correspondingly (2) only one of the three vertices (whose tangents converge to a point) is real, *or* the (real) double tangent is isolated (i.e., it contacts the curve at imaginary points), and therefore all three vertices will be real.

Case 1 or 2 will apply to the cylindroid according to whether

W cuts the *Z*-axis at point (*WZ*) that lies between or outside of the pinch points $z = \pm\mathfrak{h}$, as the behavior of the double-point tangents would suggest.

If *W* cuts *Z* at a pinch point then the double-point tangents will coincide, and the double point will become a cusp at this pinch point, so *S* must be real collinear to a curve of class 3:

w is found in space between or outside of the *pinch planes* $z = \pm\,\mathfrak{h}$, as would emerge from the behavior of the contact points of the double tangents, namely, the trace of *wU* (projected onto the parallels to the cylindroid edges that are parallel to *w* and found at an equal height).

If *w* lies in a pinch plane then the contact points with the double tangents will

(1) Cf., *Cremona*, "Sur l'hypocycloïde à trois rebroussements," in Crelles Journal **64** (1865).

(2) Cf., *Cayley* in the Encylop. Brit., 9th ed., "Curve," and *Salmon*, "Higher plane curves," pp. 141 (Dublin, 1852).

$$S = S_0^{3,3}$$

coalesce into one, which will then represent an inflection point, so s must be real collinear to a curve of order 3:

$$s = s_0^{3,3}$$

with a cusp and an inflection point; e.g., the *Neil* parabola $y^2 = x^3$ (Fig. 14)

Remark. If w is infinitely-distant then the double tangents of $s_0^{4,3}$ will be projected onto the plane at infinity, and their contact points will be projected to rays that lead to the circle points of the principal plane:

The parallel projection of the cylindroid onto any plane that is parallel to the principal plane $G_{\mathrm{I}}\, G_{\mathrm{II}}$ is a Steiner hypocycloid.

The conic section S^2 of the cylindroid and the second-degree cone (ws^2) that is circumscribed by that surface.

The intersection S^2 of the cylindroid with a tangential plane W to that surface – i.e., with plane that contains an edge G of the cylindroid – has an infinitely-distant point in W, through which, the aforementioned infinitely-distant cylindroid edges Z_1, Z_2 will emanate, so it will be an ellipse in the cylinder of rotation through the nodal line Z. (G, and thus W, as well, will meet a cylindroid edge K at a point of Z.)

The second-degree cone that projects from a point w of the cylindroid has a *parabola* s^2 for its trace in any plane that is parallel to the principal plane (which contacts the two cylindroid edges G_1 and G_2 that fall in this plane), has the projection w' of the point w onto that image plane for its focal point, and whose axis is parallel to the cylindroid edge K that will be cut by the edge G that goes through w. (The cone $(ws^2) = \mathfrak{K}_d$ ([1]), when projected from the point (KU), has the tangential plane (wU) along the edge in question and projects the line t_1t_2 that was introduced on pp. 23.)

The cylindroid can be constructed as the locus

of altitudes from an arbitrary point of one of the ∞^2 ellipses S^2 on the nodal line Z.

the perpendicular transversals to Z in arbitrary tangential planes to one of the ∞^2

([1]) The cone (ws^2) belongs to *Reye's* cones of *category d:*

$$(ws^2) = \mathfrak{K}_d ,$$

since the focal axis ww', which is the line of intersection of wt_1 and wt_1, is perpendicular to the tangential plane (wU). Cf., *Reye, Geometrie der Lage I*, page 220 (Leipzig, 1886). Any such cone will be enveloped by planes that cut the tangential planes to it that are perpendicular to the focal axis in a normal pair.

cones:

$$ws^2 = \mathfrak{K}_d\,.$$

Thus, a cylindroid of given *span* $2\mathfrak{h}$ can be generated in the following way:

(Fig. 15)

One chooses an ellipse S^2 on an arbitrary cylinder of rotation whose edge directions are perpendicular to the two contact planes to S^2, E_1 and E_2, which are separated by $2\mathfrak{h}$.

E_1 and E_2 might contact S^2 at points B_1 and B_2, resp. Let Z be *an arbitrary edge of the cylinder* that meets E_1 at A_1 and E_2 at A_2, resp., so the locus of the altitudes from the points of S^2 to Z will be a cylindroid with the pinch edges A_1B_1 and A_2B_2 and a span of $2\mathfrak{h}$.

(Fig. 16)

One gives oneself a *cone* $\mathfrak{K}_d$ when one projects a parabola s^2 from a point w of the altitude that is erected from its focal point F onto its plane, and a *line Z that is parallel to the focal axis* (Fw), whose points of intersection A_1 and A_2 are separated from $\mathfrak{K}_d$ at a distance of $2\mathfrak{h}$ from each other. The perpendicular transversals of Z, which contact $\mathfrak{K}_d$, trace out a cylindroid with a span of $A_1\,A_2 = 2\mathfrak{h}$, whose pinch edges are the tangents to $\mathfrak{K}_d$ at A_1 and A_2 that are perpendicular to Z.

The *construction of the cylindroid* as the locus of *shortest transversals* to the rays of a plane *pencil* and a fixed *line Z* then follows from this, since these pencils play a role in the discussion above of

the plane W that converges to any point of S^2 that is diametrically opposite to the point of intersection of S^2 with Z.

the rays through the point w that link the latter with the points of the vertex tangent to the parabola s^2.

The *parameter curve* $\mathfrak{P}^*$:

$$\mathfrak{p} = -\,\mathfrak{h}\cos 2\vartheta$$

that belongs to $\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}} = 0$, from which one can obtain every parameter curve $\mathfrak{P}$ of the family:

$$\frac{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}}{2} = \mathfrak{h} = \text{const.}$$

that belongs to a certain cylindroid as its axis surface by uniform variation of the radius vectors, is the *projection of the line of intersection* P^* of the cylindroid with the sphere K from the principal point p on it and the radius $\mathfrak{h}$ onto the principal plane, as the equation:

$$z^2 + \mathfrak{p}^2 = \mathfrak{h}^2$$

teaches us, when we recall the fact that $z = \mathfrak{h} \sin 2\vartheta$. As *Plücker* remarked, it follows from this that any $\mathfrak{P}$ of that family is the projection of an intersection *P* of the cylindroid with a torus that is made orthogonally to the principal plane, and the latter will be enveloped by all spheres of radius $\mathfrak{h}$ whose centers lie on the circle in the principal plane around the principal point *p* with a radius of:

$$\frac{\mathfrak{p}_{\rm I} + \mathfrak{p}_{\rm II}}{2}.$$

(Cf., the *Lewis* construction of the cylindroid that was mentioned by *Ball.*)

The distance $2\mathfrak{h} \cos 2\vartheta$ *between the intersection points of any cylinder edge G with K – i.e., and also with* P^* *– is equal to the dividing parameter l of the equilateral tangential paraboloid to the cylindroid along the edge G.*

Namely, if *e* is the distance from an arbitrary point *w* on the vertex tangent *G* to the paraboloid to the central point *GZ*, as measured on *G*, and $\mathfrak{w}$ is the angle between the tangential plane to the cylindroid at *w* with the central plane *GZ* then:

$$k = e \cot \mathfrak{w} = e\,\frac{dz}{d(e\vartheta)} = \frac{dz}{d\vartheta} = 2\mathfrak{h} \cos 2\vartheta.$$

P^*, as the intersection of the cylindroid with *K* of order six, will have the simple equations in *polar coordinates* $\mathfrak{r}$, ϑ, ψ:

$$\mathfrak{r} = \mathfrak{h}; \qquad \psi = 2\vartheta,$$

when we let ψ denote the angle between the radius vector $\mathfrak{r}$ (= $\mathfrak{h}$) that leads from *p* to one of its points and the principal plane, and therefore also with *G*, if one recalls that $z = \mathfrak{h} \sin \psi$.

Two intersecting (on *Z*) cylindroid edges:

$$G_{(\vartheta)} \quad \text{and} \quad G_{(\vartheta' = 90^\circ = \vartheta)}$$

belong to the parameters:

$$\left\{\begin{aligned} \mathfrak{p} &= \frac{\mathfrak{p}_{\rm I} + \mathfrak{p}_{\rm II}}{2} - \mathfrak{h} \cos 2\vartheta, \\ \mathfrak{p}' &= \frac{\mathfrak{p}_{\rm I} + \mathfrak{p}_{\rm II}}{2} + \mathfrak{h} \cos 2\vartheta, \end{aligned}\right\}$$

whose sum is:

(1) $$\mathfrak{p} + \mathfrak{p}' = \mathfrak{p}_{\rm I} + \mathfrak{p}_{\rm II} = \text{const.},$$

while their product is:

$$(2) \qquad \mathfrak{p}\,\mathfrak{p}' = \left(\frac{\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}}}{2}\right)^2 - \mathfrak{h}^2\cos^2 2\vartheta = \left(\frac{\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}}}{2}\right)^2 - \left(\frac{\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}}}{2}\right)^2 + z^2$$
$$= \mathfrak{p}_{\mathrm{I}}\,\mathfrak{p}_{\mathrm{II}} + z^2,$$

which differs from the product of the principal parameters by the square of the distance from the axis to the principal point p. One of the two parameters will be zero for $z^2 = -\,\mathfrak{p}_{\mathrm{I}}\,\mathfrak{p}_{\mathrm{II}}$, and as was already remarked on pp. 20, this will be true for $\vartheta = \vartheta_0$ with:

$$\tan\vartheta_0 = \sqrt{-\frac{\mathfrak{p}_{\mathrm{I}}}{\mathfrak{p}_{\mathrm{II}}}}\,.$$

We can now also determine the principal axes G_{I} and G_{II} of an R_{II} even better than before, as well as the principal parameters $\mathfrak{p}_{\mathrm{I}}$ and $\mathfrak{p}_{\mathrm{II}}$, when this pencil of screws is given by two screws L_1 and L_2 with the axes G_1 and G_2, resp., and the parameters $\mathfrak{p}_1$ and $\mathfrak{p}_2$, resp. (Fig. 17).

Let:

$\mathfrak{e}$	be the shortest distance between G_1 and G_2
$2\alpha\ (\neq 0)$	be the angle " " "
φ	" " between the angle-bisectors G_1 and G_2 and the desired direction of G_{I},
z_1	be the shortest distance between G_1 and G_{I},
z_2	" " " " " G_2 " ",

so the relations ($\vartheta = \varphi \mp \alpha$):

$$\left\{\begin{aligned} \mathfrak{p}_1 &= \frac{\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}}}{2} - \frac{\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}}}{2}\cos 2(\varphi-\alpha), \\ \mathfrak{p}_2 &= \frac{\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}}}{2} - \frac{\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}}}{2}\cos 2(\varphi+\alpha), \end{aligned}\right\}$$

that were derived already on pp. 19 and:

$$\left\{\begin{aligned} z_1 &= \frac{\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}}}{2}\sin 2(\varphi-\alpha), \\ z_2 &= \frac{\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}}}{2}\sin 2(\varphi+\alpha), \end{aligned}\right\}$$

which were derived on pp. 21, along with $z_2 - z_1 = \mathfrak{e}$, allow one to express the five data $\mathfrak{p}_{\mathrm{I}}$, $\mathfrak{p}_{\mathrm{II}}$, $\tan 2\varphi$, z_1, z_2 :

$$\left.\begin{aligned} \mathfrak{e} = z_2 - z_1 &= (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}) \cos 2\varphi \sin 2\alpha, \\ \mathfrak{p}_2 - \mathfrak{p}_1 &= (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}) \sin 2\varphi \cos 2\alpha, \end{aligned}\right\}$$

(1) $$\tan 2\varphi = \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\mathfrak{e}},$$

$$\left.\begin{aligned} \mathfrak{p}_2 + \mathfrak{p}_1 &= (\mathfrak{p}_{\mathrm{II}} + \mathfrak{p}_{\mathrm{I}}) - (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}) \cos 2\varphi \sin 2\alpha, \\ \mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}} &= \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\sin 2\varphi \cos 2\alpha}, \end{aligned}\right\}$$

$$\left\{\begin{aligned} \mathfrak{p}_{\mathrm{II}} + \mathfrak{p}_{\mathrm{I}} &= (\mathfrak{p}_2 + \mathfrak{p}_1) + \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\tan 2\varphi \tan 2\alpha}, \\ &= (\mathfrak{p}_2 + \mathfrak{p}_1) + \mathfrak{e} \cot 2\alpha, \\ \mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}} &= \sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha, \end{aligned}\right\}$$

$$\left(\text{since} \quad \sin 2\varphi = \frac{\tan 2\varphi}{\sqrt{1 + \tan^2 2\varphi}} = \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2}}\right)$$

(2′) $$\mathfrak{p}_{\mathrm{I}} = \tfrac{1}{2}[(\mathfrak{p}_2 + \mathfrak{p}_1) + \mathfrak{e} \cot 2\alpha - \sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha],$$

(2″) $$\mathfrak{p}_{\mathrm{II}} = \tfrac{1}{2}[(\mathfrak{p}_2 + \mathfrak{p}_1) + \mathfrak{e} \cot 2\alpha + \sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha],$$

$$\left.\begin{aligned} z_1 &= \tfrac{1}{2}\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha \sin 2(\varphi - \alpha), \\ z_2 &= \tfrac{1}{2}\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha \sin 2(\varphi + \alpha), \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin 2(\varphi - \alpha) &= \sin 2\varphi \cos 2\alpha - \cos 2\varphi \sin 2\alpha, \\ \sin 2(\varphi + \alpha) &= \sin 2\varphi \cos 2\alpha + \cos 2\varphi \sin 2\alpha, \end{aligned}\right\} \qquad \sin 2\varphi = \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2}},$$

$$\cos 2\varphi = \frac{\mathfrak{e}}{\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2}},$$

$$\left.\begin{aligned} z_1 &= \tfrac{1}{2}\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha \frac{(\mathfrak{p}_2 - \mathfrak{p}_1)\cos 2\alpha - \mathfrak{e} \sin 2\alpha}{\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2}}, \\ z_2 &= \tfrac{1}{2}\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2} \csc 2\alpha \frac{(\mathfrak{p}_2 - \mathfrak{p}_1)\cos 2\alpha + \mathfrak{e} \sin 2\alpha}{\sqrt{\mathfrak{e}^2 + (\mathfrak{p}_2 - \mathfrak{p}_1)^2}}, \end{aligned}\right\}$$

(3′) $z_1 = \frac{1}{2}[(\mathfrak{p}_2 - \mathfrak{p}_1)\cot 2\alpha - \mathfrak{e}],$

(3″) $z_2 = \frac{1}{2}[(\mathfrak{p}_2 - \mathfrak{p}_1)\cot 2\alpha + \mathfrak{e}]\,.$

G_{I} and G_{II}, which are themselves parallel to the plane of the directions G_1, G_2, intersect perpendicularly at the point p of the shortest transversals of $G_1$and G_2 that has a distance of z_1 (z_2, resp.) from G_1 (G_2, resp.). One will get the directions of the principal axes G_{I} and G_{II} from those of the angle-bisectors of G_1 and G_2 by rotating them through φ, where:

$$\tan 2\varphi = \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\mathfrak{e}}.$$

For example, for $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_0$, G_{I} and G_{II} will be the symmetry axes of G_1 and G_2, and:

$$\mathfrak{p}_{\mathrm{I}} = \mathfrak{p}_0 + \frac{\mathfrak{e}}{2}\cot\alpha,$$

$$\mathfrak{p}_{\mathrm{II}} = \mathfrak{p}_0 - \frac{\mathfrak{e}}{2}\tan\alpha.$$

If G_1 and G_2 are perpendicular to each other then G_{I} and G_{II} will go through the midpoint of their shortest distance:

$$\left(z_1 = -\frac{\mathfrak{e}}{2},\quad z_2 = +\frac{\mathfrak{e}}{2}\right).$$

If G_1 and G_2 intersect then the span $2\mathfrak{h}$ of the axis cylindroid will be equal to $\frac{\mathfrak{p}_2 - \mathfrak{p}_1}{\sin 2\alpha}$. It will be zero – i.e., one will have the pencil G_1G_2, instead of the cylindroid – in the event that one has $\mathfrak{p}_1 = \mathfrak{p}_2$, in addition; R_{II} will be called "circular" in this case.

If Z is the perpendicular to the pencil G_1G_2 then the tangential planes through Z to the absolute sphere-circle will also belong to the degenerate form of the cylindroid that they represent when they cut any plane that is parallel to G_1G_2 in a line g that can be regarded as an axis of that one of two screws of the circular R_{II} that belongs to an *arbitrary* parameter $\mathfrak{p} = \frac{1}{2}\frac{L^2}{l^2}$ (pp. 15), and whose volume is $\frac{1}{2}L^2$ and the square of the length of its axis rod $\overline{l}^{\,2}$ is zero, and which represents the imaginary, circular, line-pair r_{II} (v. Staudt's second kind) that belongs to R_{II} . The concept of "axis" of such a circular screw is no longer unique, since every g can be regarded as an axis. (Cf., pp. 3 and *F. Klein* in volume 47 of this Zeitschrift, pp. 253.)

System of axes and the parameter distribution of the reciprocal domain P_{IV} .

The most general bush of screws P_{IV}, as the reciprocal screw domain to a pencil of screws R_{II}, belongs to a cylindroid as its axis surface. In order for a screw A in P_{IV} with axis Γ and parameter π to be reciprocal to three screws in R_{II} whose axes G that lie in the cylindroid surface are met by Γ, Γ must intersect one of these three axes G, and the parameter π must be equal and opposite to that of the other two. The latter two axes G must therefore be symmetric on the cylindroid of R_{II} relative to its principal axes G_{I} and G_{II}, since they belong to one and the same parameter $\mathfrak{p} = -\pi$. The axes Γ in P_{IV} that go through an arbitrary point w or lie in an arbitrary plane W can be generated as the ∞^1 transversals over any pair of cylindroid edges that are symmetric with respect to the principal axes. All Γ in P_{IV} that belong to a certain parameter π = const. will fill up the linear congruence that has the cylindroid edges G as its guiding lines and whose parameter is $\mathfrak{p} = -\pi$. For $\pi = 0$, one will get the line framework ρ_{IV} of P_{IV} . Two particular examples of these congruences are the systems of cylindroid tangents along G_{I} or G_{II} ; instead of the cylindroid, the equilateral tangential paraboloid to it can therefore be used, which has G_{I} and G_{II} as its vertex lines and $k = 2\mathfrak{h}$ for its dividing parameter. One of these congruences can also be ρ_{IV} in the case where the line-pair r_{II} of R_{II} converges to a pair of infinitely-close skew lines of the cylindroid along G_{I} or G_{II} (the limiting case on pp. 19).

Since any Γ must intersect a cylindroid edge G perpendicularly, the complex of Γ will be defined to be the locus of the perpendicular transversals of the edges of the cylindroid that belongs to R_{II} .

The complex cone through an arbitrary point w will be defined by the perpendiculars from w to the cylindroid edges.

The complex curves in an arbitrary plane W be enveloped by the perpendicular transversals of the cylindroid edges that lie in W.

In particular, for $\mathfrak{p}_{\mathrm{I}} = \mathfrak{p}_{\mathrm{II}} = \mathfrak{p}$, the pencil of principal planes with p as its center will enter in place of the cylindroid surface in the case of a "circular" domain, and

the complex cones w will be orthogonal cones over the circles in the principal plane with pw' as the diameter, if w' is the orthogonal projection of the point w onto the principal plane.

the complex curves in W will be parabolas whose vertex tangent is the trace of the principal plane in W and whose focal point is the orthogonal projection of the principal point p onto W.

Otherwise, in the general case $\mathfrak{p}_{\mathrm{I}} \neq \mathfrak{p}_{\mathrm{II}}$

the feet of the perpendiculars that are dropped from w on the cylindroid edges onto any plane that is parallel to the

the planes $G\Gamma$ that connect any cylindroid edge G with its perpendicular transversals that lie in W will envelop *a cone $\mathfrak{K}_d$ that* principal plane will project to points of a circle that has the orthogonal projection w' of w onto that plane and the piercing point of the latter with Z for its diametrically-opposite points. The foot-curve itself is then the intersection of the cylindroid with the cylinder of rotation that has Z and the line through w that is parallel to it as its diametrically-opposite edges. This line of intersection can thus be only one of the ellipses S^2 on the cylindroid ([1]), since otherwise the cylindroid, which is a third-order surface, and the cylinder of rotation would have in common:

1) The doubly-counted Z itself, and,
2) The infinitely-distant edges t_1 and t_2.

A cone of the complex of the axes Γ *of* P_{IV} will go through every point w, whose basis the ellipse S^2 of the cylindroid that projects onto the principal plane as the circle with diameter pw', if p refers to the principal point of the cylindroid, and w' is the orthogonal projection of the point w onto the principal plane.

The plane of S^2 connects the point of intersection of the parallels to z that go through w and the cylindroid with the cylindroid edge that is symmetric with respect to G_{I} and G_{II} to the cylindroid edge that meets these parallels.

The same ellipse S^2 belongs to *all* points w that lie on a parallel to Z.

circumscribes the cylindroid ([1]), if the direction cone of the developable that is enveloped by these planes represents a conic section in the plane at infinity, whose common developables with the cylindroids are already:

1) The doubly-counted pencil U, and
2) The planar pencil through the infinitely-distant cylindroid edges t_1, t_2.

Thus, the only developable that remains whose trace in W will be the desired complex curve will be only the cone $\mathfrak{K}_d$ that circumscribes the cylindroid, which has focal axes that are normal to W and U, and whose vertex w can be constructed as the point of intersection of the tangential plane to the cylindroid that is parallel to W with thw cylindroid edge that is symmetric with respect to G_{I} and G_{II} in that tangential plane.

In any plane W, a parabola s will be enveloped by the axes Γ *of* P_{IV}, whose axis will be parallel to the orthogonal projection of Z onto W.

s will be projected from w through *a cone* $\mathfrak{K}_d$ that will be cut, not just W, but *any plane that is parallel to it*, in a parabola that will be enveloped by axes Γ of P_{IV} .

The complex of axes Γ of a pencil of screws P_{IV} is quadratic.

If w is at infinity then the complex cone through that point will decompose into the pencil of fields in that direction, which will not be considered, and the pencil of normals from the direction w to the cylindroid edge that is perpendicular to w.

If W is parallel to Z then the parabola of the complex will decompose into the pencil of parallels to Z, which shall not be considered, since it belongs to $\pi = \infty$, and a pencil whose center is the point of intersection of the cylindroid edge that is perpendicular to W with W.

([1]) Cf., pp. 25.

In particular, if

w is on the cylindroid, and indeed on the edge $G(\mathfrak{p})$, then the complex cone will decompose into:

1) The pencil of normals to G through w and

2) The pencil of transversals.

W is the tangential plane to the cylindroid, and indeed the one through the edge $G(\mathfrak{p})$, then the parabola will decompose into:

1) The pencil of normals to G and

2) The pencil Γ ($\pi = -\mathfrak{p}$) whose center is the trace point in W of the cylindroid edge that lies symmetric to $G(\mathfrak{p})$ relative to to G_{I} and G_{II} .

In particular, if

w is a point of the nodal line Z then we will draw a plane W that is perpendicular to Z through it:

W is a plane through U – i.e., it is perpendicular to Z – then we will determine its point of intersection w with Z:

Now, there are two axes Γ' and Γ'' in P_{IV} through w in W, namely, that ones that lie in W and go through w and are perpendicular to the cylindroid edges G' and G'' of R_{II} and possess parameters π' (π'', resp.) that are equal and opposite to the other ones.

The complex cone w of axes Γ now degenerates into the two pencils of rays $Z\Gamma'$ and $Z\Gamma''$, which belong to the parameters π' (π'', resp.).

If w is one of the two pinch points

The complex curve in W that is enveloped by the axes Γ degenerates here into the two pencils of parallels Γ' and Γ'', which belong to the parameters π' (π'', resp.).

If W is one of the two pinch-planes

of the cylindroid of R_{II} then both pencils will coalesce into one whose planes (infinitely-distant center, resp.) are perpendicular to the pinch-edges in question.

One can obtain all of the axes Γ of P_{IV} that cut Z (are perpendicular to Z, resp.) when one reflects the cylindroid in R_{II} relative to its principal plane $G_{\mathrm{I}}\,G_{\mathrm{II}}$, and

constructs the pencil that connects the edges of the mirror-cylindroid thus-obtained with Z.

subjects every edge of the mirror-cylindroid thus-obtained to a parallel displacement in a direction that is perpendicular to Z.

Any Γ thus-obtained in P_{IV} will have the parameter $\pi = -\mathfrak{p}$, which is equal and opposite to the $\mathfrak{p}$ of the cylindroid edge G in R_{II} that Γ is obtained from by reflection.

P_{IV} is representable in a canonical way in terms of the distribution axis Z that belongs to an arbitrary parameter and the principal screws, which have the axes $\Gamma_{\mathrm{I}} = G_{\mathrm{I}}$, $\Gamma_{\mathrm{II}} = G_{\mathrm{II}}$, and the parameters $\pi_{\mathrm{I}} = -\mathfrak{p}_{\mathrm{I}}$, $\pi_{\mathrm{II}} = -\mathfrak{p}_{\mathrm{II}}$. The principal axes Γ_{I} and Γ_{II} are the only axes in P_{IV} that cut Z perpendicularly at *one* point and are perpendicular to each other. It is only in the case of equal principal parameters $\mathfrak{p}_{\mathrm{I}} = \mathfrak{p}_{\mathrm{II}} = \mathfrak{p}_0$ (and thus, a "circular" bush of screws P_{IV}) that a pencil that is perpendicular to Z and goes through the point p of Z will enter in place of the principal screws ([1]). Any line of the plane of the latter, as well as any line in the pencil p will belong to P_{IV} as an axis Γ that is endowed with the parameter $\pi_0 = -\mathfrak{p}_0$. Due to the reciprocity condition $\pi - \pi_0 = e \cot g \underline{|\Gamma Z}$, the remaining rays of the axis complex Γ can be assigned to "circular" linear congruences with π = const. of Γ that relate to the G of P_{II}, which is parallel to the plane ΓZ; i.e., onto ones whose rays can all be obtained from rays of that Z by *rotation around* Z as the *family* of equilateral hyperbolic *paraboloids* with its vertex at p that contains the vertex line, and whose distribution parameter is $k = \pi - \pi_0$. The guiding rays of any such linear congruence will be appropriately called "circular" lines ("imaginary" in v. Staudt's second kind)

Transition to the canonical representation

The transition to the canonical representation of a P_{IV} that is given by four screws Λ with the axes Γ_i (i = 1, 2, 3, 4) and well-defined parameters. Corresponding to 1*) and 2*) on pp. 15, we first deal with two special cases.

1*) If the four Γ_i cut a certain line G perpendicularly then P_{IV} will consist of screws with arbitrary parameters whose axes Γ cut the G perpendicularly. R_{II} will belong to G and an arbitrary parameter. The following cases will also belong here: Γ_1, Γ_2, Γ_3 cut some G perpendicularly, Γ_4 is replaced with a field that is parallel to G, and Γ_3, Γ_4 are replaced with fields that are parallel to the shortest transversals of Γ_1 and Γ_2, resp.

2*) If the Γ_i are parallel to a plane E then its field f_0 will belong to R_{II}. A pencil of fields will belong to P_{IV} whose direction l can be determined by two fields, each of which will be found by adding three screws of the four given ones, when one demands that their rod sum should be zero. The screws of finite parameters in P_{IV} whose *axes are perpendicular to* l (and parallel to E) and whose parameters are arbitrary will fill up the *principal plane* in R_{II} (which is parallel to l); i.e., the plane in which *the perpendicular transversals* to the latter (viz., the parallels to l) are screw axes G of R_{II}. One of the G is in R_{II} and endowed with the parameter 0, so it will be representable in terms of its rod l_0. l_0 and f_0 will represent R_{II}, and therefore P_{IV}, in a canonical way.

In the general case, one first provides the field U that is present in P_{IV} when one forms a multiple sum of the four given screws in such a way that the sum of its rod segments is zero. All analogously derivable screws whose rods are perpendicular to U will have the same axes, namely, the dividing axis Z. The shortest transversal over Z and the Γ will determine the cylindroid of R_{II}; its principal axes will also be those of P_{IV}.

([1]) Cf., pp. 30.

We can now also represent any *web of screws* P_{V} that is given by *five screws* A_i (i = 1, …, 5) in *canonical form*, namely, by being given the reciprocal screwing motion R_{I} . Two quadruples from the five screws will determine a field that belongs to the P_{IV} in question. The axis of R_{I} will have same direction as these two fields and will be identical with the cylindroid edge with that direction in that R_{II} that is reciprocal to one of the P_{IV} above. The parameter $\mathfrak{p}$ of R_{I} can be determined from any Λ by means of the reciprocal relation.

The parameter π of any axis Γ of P_{IV} that must cut a cylindroid edge $G(\vartheta)$ of R_{II} in a point w that is at a distance of e from Z perpendicularly is equal and opposite to that of the two screws in R_{II} whose axes encounter Γ, in addition. However, since the latter must not be real, that would suggest the determination of π from the reciprocal relation of the $\Gamma_{(\pi)}$ with respect to any screw $G_{(90^{\circ} + \vartheta)}$ of the R_{II} that has the parameter:

$$\mathfrak{p} = \mathfrak{p}_{\mathrm{I}} \sin^2 \vartheta + \mathfrak{p}_{\mathrm{II}} \cos^2 \vartheta = \frac{\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}}}{2} + \frac{\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}}}{2} \cos 2\vartheta,$$

which is parallel to the projection of Γ onto the principal plane of the cylindroid: $\pi + \mathfrak{p} = e \cot \angle \Gamma Z$.

The sheaf of screws R_{III} and the reciprocal sheaf P_{III} .

An R_{III} is established by three screws L_1, L_2, L_3 with axes G_1, G_2, G_3 and parameters $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ that do not belong to a pencil. We first consider the special case:

1*) G_1, G_2, G_3 are parallel *and* lie in a plane E; the axes G of R_{III} are then (cf., pp. 15, 2*) all rays of this pencil of parallels, and indeed every G is endowed with an arbitrary parameter. In addition, the pencil of fields that are parallel to the perpendicular transversals Γ of G in E belongs to R_{III}. All of these Γ belong to P_{III} as axes that are endowed with an arbitrary parameter, as well as the pencil of fields that are parallel to G.

2*) G_1, G_2, G_3 are parallel, but *without* lying in a plane. If one measures off the parameter segments $\mathfrak{p}_i$ on the G_i (i = 1, 2, 3) from their intersection points with an arbitrary plane A, and lets B be the connecting plane of the endpoints of the segments thus-defined then any screw will belong to R_{III} whose axis is parallel to G_i and will have a parameter $\mathfrak{p}$ that, when measured off on G, will reach from the point of intersection with A to the one with B (pp. 16). In particular, all of the *transversals* to the intersecting edge AB that are parallel to G, whose plane might be called $\mathfrak{E}$, will define the line framework r_{III} of R_{III} when united with the following *pencil of fields* l: If we take A to be perpendicular to the G_i then if x means the normal distance to an arbitrary axis G of Γ then G will be associated with the parameter $\mathfrak{p} = x \tan \angle AB$. If we give the rod l_0 of the R_{II} of screws $L = l_0 + f_0$ with that axis that falls on G the length 1 then we can represent the field f_0 that is perpendicular to l_0 in the form of a right angle whose volume is $\mathfrak{p}$ and whose sides might then have the lengths x and $\mathfrak{p} / x$. Let the former side be perpendicular to $\mathfrak{E}$, and let the latter be parallel to the edge AB. The rectangular field f that connects the former right-angle side with l_0 will produce a *field* $f = f + f_0$ when it is added to an f_0 that *belongs to* R_{III}, since it is equal $L - l_0' = (l_0 - l_0') + f_0$, if l_0' means the rod in the plane $\mathfrak{E}$ that belongs to r_{III} (R_{III}, resp.) that one obtains by projecting l_0 onto $\mathfrak{E}$. The field f in R_{III} that is perpendicular to $\mathfrak{E}$ will subtend an angle with G that is equal to $\underline{|AB}$, since its tangent is equal to the ratio of the lengths of the right-angle sides of f_0 and f that are not common; i.e., it will satisfy $\mathfrak{p} / x = \tan \underline{|AB}$.

The field of $\mathfrak{E}$ also belongs to r_{III} (R_{III}, resp.), *so each of them do, as well,* and is derivable from f – i.e., *every field of the pencil whose axis* l lies in f and subtends the constant angle $\underline{|AB} = \text{arc tan } \mathfrak{p} / x$ with G (from the associated side).

The *reciprocal sheaf* P_{III} is filled up with all screws whose axes are parallel to l and possess a parameter that is equal and opposite to that of the screws of R_{III}, whose axes they intersect, such that the associated parameters π for the screws of R_{III} will be proportional to the separation between the P_{III}-axis Γ and $\mathfrak{E}$ (which is parallel to l), and indeed to the equal and opposite proportionality factor of $\tan \underline{|AB}$ that appears in R_{III}. Here, the line framework ρ_{III} of P_{III} will consist of the rods in the plane $\mathfrak{E}$ that are parallel to l and the pencil of fields that are parallel to G.

3*) The G_i ($i = 1, 2, 3$) have a common shortest transversal Γ_0. There then exists a field f in the sheaf of screws R_{III} that is parallel to Γ_0, since one can determine a multiple sum of the three screws L_i such that the sums of the rod segments are zero. Furthermore, there is a screw in R_{III} whose axis G_0 cuts Γ_0 perpendicularly at a point p and whose parameter is *completely arbitrary;* namely, if a screw L is linearly derivable from the L_i then its axis will always cut Γ perpendicularly, since Γ_0 is the common division axis of all the cylindroids that belong to R_{III}; if one chooses the deriving numbers of the screw L such that *its* rod segment is perpendicular to f then an arbitrary multiple of f can be added to L in the screws of this sheaf R_{III} – i.e., the parameter that belongs to the axis G_0 of the screw L will be *arbitrary*.

Now, R_{III} is determined just as well as before by the L_i, the screws with the axis G_0 and arbitrary parameter, and the screws with the axis G_1 and the parameter $\mathfrak{p}_1$ (Fig. 18). The associated reciprocal sheaf P_{III} is determined in a completely analogous way by the shortest Γ_0 of G_0 and G_1 that are assigned an arbitrary parameter and an arbitrary transversal Γ_1 of G_0 and G_1 that is perpendicular to G_0, which must be assigned the parameter $\pi_1 = -\mathfrak{p}_1$, since it is a screw axis in P_{III}. In precisely the same way that the single field f that is perpendicular to Γ_0 belongs to R_{III}, one finds only the field φ that goes through G_0 perpendicular to Γ_0 in P_{III}.

Since any screw that is reciprocal to P_{III} will belong to R_{III}, and conversely, every transversal of Γ_0 and Γ_1 (G_0 and G_1, resp.) that is endowed with the parameter $\mathfrak{p}_1$ (π_1, resp.) ($\mathfrak{p}_1 + \pi_1 = 0$) and is perpendicular to Γ_0 (G_0, resp.) will define an axis in R_{III} (P_{III}, resp.). Let e_1 ($\mathfrak{e}_1$, resp.) be the length of the shortest transversals of G_0 and G_1 (Γ_0 and Γ_1, resp.), so there will be an equilateral *hyperbolic paraboloid* $\mathfrak{P}_1$ with the vertex lines G_0 and Γ_0 (which intersect perpendicularly at its vertex p), which includes G_1 and Γ_1, and has the equation $xy = \kappa_1\, z$ in the coordinate system that has Γ_0 as the x-axis, G_0 as the –axis, and the line through p that is perpendicular to $\Gamma_0\, G_0$ as the z-axis.

$$\kappa_1 = e_1 \cot \underline{|G_0 G_1} = \mathfrak{e}_1 \cot \underline{|\Gamma_0 \Gamma_1}$$

as the dividing parameter of $\mathfrak{P}_1$. The family of lines G (Γ, resp.) of the paraboloid $\mathfrak{P}_1$ is filled up with the axes that belong to R_{III} (P_{III}, resp.) whose parameter has the constant value $\mathfrak{p}_1$ (π_1, resp.) ([1]). Since no confusion is possible, from now on, let G_1 (Γ_1, resp.) denote any line of the family G (Γ, resp.) on $\mathfrak{P}_1$ that subtends a 45° angle with the central plane, and thus has the shortest distance of $e_1 = \mathfrak{e}_1 = \kappa$ from G_0 (Γ_0 resp.).

Any perpendicular transversal of Γ_0 (G_0, resp.) *is an axis* G (Γ, resp.) *in* R_{III} (P_{III}, resp.) *that is endowed with a well-defined parameter* $\mathfrak{p}$ (π, resp.) – *except for the perpendicular transversals that fall in the principal plane* $G_0\Gamma_0$, *which are associated with the parameter* ∞, *in addition to the* G_0 (Γ_0, resp.) *that are thought of as linked with arbitrary parameters, and for that reason they merely represent the field* f (φ, resp.).

Any equilateral hyperbolic paraboloid $\mathfrak{P}$ ($\kappa y = xz$) *(κ arbitrary) with* G_0 *and* Γ_0 *as vertex lines represents the family of lines that belongs to* G_0 *(Γ_0, resp.), namely, the system of axes in R_{III} (P_{III}, resp.) that belong to a constant parameter* $\mathfrak{p}$ *(π, resp.)* ($\mathfrak{p} + \pi = 0$). $\mathfrak{p}$ *(π, resp.) differs in absolute value from* $\mathfrak{p}_1$ *(π_1, resp.) by exactly as much as the dividing parameter* κ *of the paraboloid* $\mathfrak{P}$ *differs from* κ_1, *which is that of the paraboloid* $\mathfrak{P}_1$.

The Z-axis of our coordinate system will belong to R_{III} (P_{III}, resp.) as a screw axis when we endow it with the parameter:

$$\mathfrak{p}_0 = \pm\, \kappa_1 + \mathfrak{p}_1 \qquad (\pi_0 = \mp\, \kappa_1 + \pi_1, \text{ resp.}).$$

This follows from the reciprocal relation that relates to $\Gamma_1(\pi_1)$ [$G_1(\mathfrak{p}_1)$, resp.]:

$$\pm\,(\mathfrak{p}_0 + \pi_1) = \kappa_1 \tan 45^\circ \qquad [\mp\,(\pi_0 + \varphi_1) = \kappa_1 \tan 45^\circ, \text{ resp.}]$$
$$(\mathfrak{p}_0 + \pi_0 = 0) \qquad (\mathfrak{p}_1 + \pi_1 = 0, \text{ resp.}),$$

([1]) Only G_0 and Γ_0 are capable of being assigned arbitrary parameters.

in which the upper or lower sign is true – and both simultaneously – according to whether the sign of the parameter is chosen as on pp. 2 or the opposite one. The axis family in R_{III} (P_{III}, resp.) with the parameter $\mathfrak{p}_0$ (π_0, resp.) will then be:

1) The pencil p of planes $G_0\varphi = yz$ ($\Gamma_0 f = xz$, resp.), because it is derivable from the Z and G_0 (Z and Γ_0, resp.) that belong to $\mathfrak{p}_0$ (π_0, resp.).

2) The pencil of rays in the plane $\Gamma_0 f = xz$ ($G_0\varphi = yz$, resp.) that are parallel to Z, because they are derivable from f and $Z(\mathfrak{p}_0)$ [φ and $Z(\pi_0)$, resp.].

One ray $G(\mathfrak{p}_0)$ [$\Gamma(\pi_0)$, resp.] from the pencil 1) and *one line* from the family $G(\mathfrak{p}_1)$ [$\Gamma(\pi_1)$, resp.] of the paraboloid $\mathfrak{P}_1$ lies in *any* plane E through Γ_0 ($\mathfrak{E}$ through G_0, resp.). The latter line, in particular, can be G_1 (Γ_1, resp.) itself when E ($\mathfrak{E}$, resp.) is inclined with respect to the principal plane by 45°. *Any* line in E ($\mathfrak{E}$, resp.) that is parallel to *both* of them is then an axis G in R_{III} (Γ in P_{III}, resp.), since it is linearly derivable from the two representative screwing motions. In particular, we choose E through G_1 ($\mathfrak{E}$ through Γ_1, resp.), so it will be 45° from the principal plane. Any line G (Γ, resp.) in that plane that is perpendicular to Γ_0 (G_0, resp.) and might have the shortest distance of κ from G_0 (Γ_0, resp.) – such that κ is the dividing parameter of the paraboloid $\mathfrak{P}$ [through G (Γ, resp.) with the vertex lines G_0 and Γ_0] – is then representable by a screwing motion in R_{III} (P_{III}, resp.) that is endowed with a parameter $\mathfrak{p}$ (π, resp.) ($\mathfrak{p} + \pi = 0$) that can be determined from the reciprocal relation for $G(\mathfrak{p})$ [$\Gamma(\pi)$, resp.] with respect to $Z(\pi_0)$ [$Z(\mathfrak{p}_0)$, resp.]:

$$\mathfrak{p}_0 = \pm\,\kappa + \mathfrak{p} \qquad (\pi_0 = \mp\,\kappa + \pi, \text{ resp.})$$

[while previously:

$$\mathfrak{p}_0 = \pm\,\kappa_1 + \mathfrak{p}_1 \qquad (\pi_0 = \mp\,\kappa_1 + \pi_1, \text{ resp.})]$$

and will then actually behave according to the stated equation:

$$\mp(\mathfrak{p} - \mathfrak{p}_1) = \kappa - \kappa_1 \qquad [\pm(\pi - \pi_1) = \kappa - \kappa_1, \text{ resp.}].$$

The line framework r_{III} of R_{III} (ρ_{III} of P_{III}, resp.) consists of the family of paraboloids $\mathfrak{P}$ ($xy = \kappa z$) that belong to G_0 (Γ_0, resp.) and whose κ, $\mathfrak{p} = 0$ are correspondingly $\kappa = \mp\,\pi_0 = \pm\,\mathfrak{p}_0$; for $\mathfrak{p}_0 = \pi_0 = 0$, in particular, r_{III} (ρ_{III}, resp.) will be the pair of pencils that was mentioned in (1), (2).

A pencil of axes in R_{II} (P_{III}, resp.) that are perpendicular to Γ_0 (G_0, resp.) goes through any point $\mu(e, 0, 0)$ of Γ_0 [$m(0, \mathfrak{e}, 0)$ of G_0, resp.]. If one measures off along that axis, not the associated parameter $\mathfrak{p}$ (π, resp.) itself, but $\rho = -\,\mathfrak{p}_0 + \mathfrak{p}$ ($-\,\pi_0 + \mathfrak{p}$, resp.), and indeed starting from μ (m, resp.), then for fixed μ (m, resp.) the endpoints of the segments thus-defined – viz., $\rho = e \tan \lfloor GZ$ ($\mathfrak{e} \tan \lfloor \Gamma Z$, resp.) ([1]) – will trace out the curve $\mathfrak{C}$ in Fig. 1 [in which $\vartheta = \lfloor GZ$ ($\lfloor \Gamma Z$, resp.)] whose equation reads $z^2 (y^2 + z^2) - e^2 y^2 = 0$ [$z^2 (x^2 + z^2) - \mathfrak{e}^2 x^2 = 0$, resp.], so when the point μ varies on Γ_0 (m varies on G_0, resp.), one will get the fourth-order cone $z^4 + y^2 (z^2 - z^2) = 0$ [$z^4 + x^2 (z^2 - y^2) = 0$, resp.], which the $\mathfrak{C}$ above will project from p.

4*) Let the axes G_1, G_2, G_3 of the three screws that determines a domain R_{III} be parallel to a planar field φ, *without however* possessing the same shortest transversal ([2]). On the two linear pencils of screws that, e.g., connect the first of these screws with any of the other ones, we can determined a certain real screw – in place of the one that belongs to G_2 (G_3 resp.) – that belongs to the same parameter $\mathfrak{p}_1$ as G_1, such that we can likewise assume (in order to minimize the notations) that the axes G_1, G_2, G_3 that are parallel to φ *belong to the same parameter* $\mathfrak{p}_1$. (If two of these three axes – say, G_1 and G_2, so G_3 will be parallel to the plane G_1G_2 – intersect then in addition to the pencil G_1G_2, the pencil of parallels that likewise belongs to $\mathfrak{p}_1$ and connects G_3 with the ray in the pencil that is parallel to it will also belong to R_{III}. This case will then present no peculiarities, insofar as also in the most general case that belongs to 4*) will also yield a pair of pencils of that sort with constant parameters, which was given here only from the outset.)

A peculiarity will arise when the G_1, G_2, G_3 that determine an R_{III} and belong to $\mathfrak{p}_1$ lie in a plane. It entire line field will then be the locus of the axes in R_{III} that are endowed with $\mathfrak{p}_1$ and, at the same time, the locus of the axes in the reciprocal P_{III} that are provided with the parameter $\pi_1 = -\mathfrak{p}_1$. In addition to the field $f = \varphi$ of the axis plane, the totality of all circular imaginary lines of *v. Staudt's* second kind will belong to the line framework r_{III} of R_{III} (ρ_{III} of P_{III}, resp.), and the latter lines will be representable in the form $L = L_1 + \sqrt{-1}\, L_2$, where L_1 and L_2 mean screws with the same volume whose equal-length axis rods that belong to $\mathfrak{p}_1$ (π_1, resp.) lie in the axis plane and intersect perpendicularly. The name "line" for L itself is justified by the validity of the equation $L^2 = 0$ ($L_1^2 = L_2^2$, $L_1L_2 = 0$). Thus, L can be ascribed to any arbitrary parameter, and one can regard any *Staudtian* imaginary circular line (of the first kind) of the field $\mathfrak{E} = \varphi$ as the axis of such an L. (Cf., pp. 30).

In the general case 4*), the paraboloid G_1, G_2, G_3 is not equilateral, since otherwise one would be dealing with case 3*). The *entire family* G_1, G_2, G_3 of them represents the family of axes in R_{III} that belong to $\mathfrak{p}_1$. The other family of paraboloids, which is endowed with the parameter $\pi_1 = -\mathfrak{p}_1$, belongs to the reciprocal domain P_{III}. The

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([1]) The reciprocal relation of $G(\mathfrak{p})$ with respect to $Z(\pi_0)$ [$\Gamma(\pi)$ with respect to $Z(\mathfrak{p}_0)$, resp.] says, in fact:

$$\pi_0 + \mathfrak{p} = e \tan \lfloor GZ \qquad (\mathfrak{p}_0 + \pi = \mathfrak{e} \tan \lfloor \Gamma Z \text{, resp.}).$$

([2]) *N. Zanichevski* treated this case 4*) analytically in his treatise: "Die Schraubenlehre und ihre Anwendung auf die Mechanik," pp. 63-67. (In Russian, Odessa, 1889).

direction field f of the latter family belongs to R_{III}, just as φ belongs to P_{III}. The same argument teaches us that the axes of all ∞^1 screws in R_{III} (P_{III}, resp.) that possess an arbitrary constant parameter $\mathfrak{p}$ (π, resp.) will trace out families of non-equililateral paraboloids whose direction plane is φ (f, resp.), and indeed both of them will be on the same paraboloid in case one has $\mathfrak{p} + \pi = 0$. We would like to examine the positions of these paraboloids.

Let $G_0(\mathfrak{p}_1)$ be the vertex line, and let $\mathfrak{e}_1$ be the dividing parameter of the family G_1, G_2, G_3, which equals the shortest separation of the vertex lines of this paraboloid from the lines of the same family that subtend an angle of 45° with the former, which is the separation S_1E_1 (Fig. 19), measured along the vertex lines $\Gamma_0(\pi_1)$ of the other family, from the vertex $S_1 = G_0\,\Gamma_0$ to the line $G(\mathfrak{p}_1)$ whose connecting line with $\Gamma_0(\pi_1)$ in inclined by 45° with respect to the central plane ($G_0\,\Gamma_0$). The family of paraboloids $\mathfrak{p} = -\pi =$ const. with their axes G in R_{III} (Γ in P_{III}, resp.) has the direction field φ (f, resp.), so its vertex lines $G_0(\mathfrak{p})$ [$\Gamma_0(\pi)$, resp.] will be parallel to those of the family $\mathfrak{p}_1 = -\pi_1 =$ const., and thus to $G_0(\mathfrak{p}_1)$ [$\Gamma_0(\pi_1)$, resp.]. Therefore, the screw that belongs to the axis $G_0(\mathfrak{p})$ [$\Gamma_0(\pi)$, resp.] must be linearly derivable from the screw $G_0(\mathfrak{p}_1)$ and the field f [$\Gamma_0(\pi_1)$ and φ, resp.], and one will obtain $G_0(\mathfrak{p})$ from $G_0(\mathfrak{p}_1)$ [$\Gamma_0(\pi)$ from $\Gamma_0(\pi_1)$, resp.] by parallel displacement along the Z-axis of the paraboloid ($\mathfrak{p}_1$) through an amount z that satisfies the equation $\mathfrak{p} - \mathfrak{p}_1 = z \tan \alpha$ $\big(\alpha = \underline{|\,G_0\Gamma_0} = \underline{|\,G_0 f}$, the angle between the direction plane φ and $f\big)$. (pp. 15, 2*).

In order to understand the now-known vertex lines $G_0(\mathfrak{p})$ [$\Gamma_0(\pi)$, resp.] of the family $\mathfrak{p} = -\pi =$ const., as well as its dividing parameter $\mathfrak{e}$, and thus, the paraboloid in question, we consider the fact that any line $G(\mathfrak{p})$ in that family that is parallel to $G_0(\mathfrak{p})$ and has a distance $\mathfrak{e}$ from the vertex line $G_0(\mathfrak{p})$, as measured along $\Gamma_0(\pi)$, will belong to R_{III} as the axis of a screw in it that must be linearly derivable from the screws $G(\mathfrak{p}_1)$ and f, and must therefore be linked to $G(\mathfrak{p}_1)$ by a plane that has a trace in the field f that is perpendicular to $G(\mathfrak{p})$, such that this trace H of the plane $G(\mathfrak{p}_1) \cdot G(\mathfrak{p})$ in the plane $\Gamma_0 Z = Zf$ must subtend an angle of α with Z. [$G(\mathfrak{p}_1)$ lies in the angle-bisecting plane of $\Gamma_0\, G_0$ and $\Gamma_0\, Z$ and is perpendicular to the altitude E_1L to G_0 in the former plane and to the line H in the latter plane, so one has:

$$\underline{|\,H\Gamma_0} = \underline{|\,(LE_1)\cdot\Gamma_0} = 90^{\circ} - \alpha,$$

from which, it will follow that $\underline{|\,HZ} = \alpha$.]

$G(\mathfrak{p})$ can then – and for arbitrary $\mathfrak{p}$ – be any transversal of H that is parallel to $G(\mathfrak{p}_1)$. The altitude that is dropped from the point of intersection E of $G(\mathfrak{p})$ and H onto Z is a vertex line $\Gamma_0(\mathfrak{p})$ of the paraboloid that belongs to $\mathfrak{p}$ ($\pi = -\mathfrak{p}$, resp.), and its length ES is the dividing parameter $\mathfrak{e}$ of that paraboloid. It is (in absolute value):

$$\mathfrak{e} - \mathfrak{e}_1 = z \tan \alpha = \mathfrak{p} - \mathfrak{p}_1 \,;$$

i.e., the dividing parameter and screw parameter vary from one paraboloid to another by the same amount.

The family of paraboloids $\mathfrak{p} = -\pi =$ const. *of axes in* R_{III} (P_{III}, *resp.*) *all have parallel vertex lines and the same paraboloid axis Z. A fixed line H* (H', *resp.*) *that subtends an angle of* α *with Z, as do the direction planes f and* φ *of the paraboloid, connects those points of all vertex lines* Γ_0 (G_0, *resp.*) *whose tangential planes to the paraboloid in question are inclined by* 45° *with respect to Z.*

One of these families of paraboloids degenerates to a pair of pencils, namely, the one for which $\mathfrak{e} = 0$ and one correspondingly has:

$$\mathfrak{p} = \mathfrak{p}_1 - \mathfrak{e}_1 = \mathfrak{p}_0 \,, \qquad \pi = -\mathfrak{p}_0 = \pi_0 \,;$$

they will degenerate to the pencil with its center at $M = HZ = H'Z$ in the plane $ZG_0 = Z\varphi$ ($Z\Gamma_0 = Zf$, resp.) and the pencil of rays in the plane $Z\Gamma_0 = Zf$ ($ZG_0 = Z\varphi$, resp.) that are parallel to Z.

One of the families of paraboloids – viz., the one that corresponds to $\mathfrak{p} = -\pi = 0$ – defines the line framework r_{III} in R_{III} (ρ_{III} in P_{III}, resp.). In particular, if the pair of pencils that we just mentioned is the line framework then we will be dealing with a special case of III′), pp. 11.

The ∞^1 *paraboloids* $\mathfrak{p} = -\pi =$ const., *or – what amounts to the same thing – all of the* ∞^2 *axes G in* R_{III} (Γ *in* P_{III}, *resp.*), *envelop a second-degree cone* $\mathfrak{K}_d$ *that has M for its vertex, the planes* $Z\varphi$ *and Zf for its tangential planes, and focal axes d and* δ *that are perpendicular to these planes.*

(Fig. 20) In any arbitrary plane $\mathfrak{E}$ that is parallel to φ, there will be ∞^1 rays G that are axes of R_{III} . Of them, G will go through every point P of the line of intersection t of $\mathfrak{E}$ with (Zf), namely, the one that belongs to the paraboloid $\mathfrak{P}$ ($\mathfrak{p} = -\pi =$ const.) whose vertex line $\Gamma_0(\pi)$ is the altitude that is dropped from P to Z. (The other vertex line $G_0(\mathfrak{p})$ of that paraboloid goes through the vertex $S = \Gamma_0 Z$ and will lie in the plane $Z\varphi$ that is perpendicular to Z. – moreover, t itself will represent an axis in R_{III}, since it is a ray of the pencil of parallels that belong to $\mathfrak{p}_0$.) If we were to cut E with the line H in $\Gamma_0(\pi)$ that runs through M in the plane Zf at an angle of α with respect to Z then the piece $\mathfrak{e} = SE$ will be the dividing parameter of the paraboloid $\mathfrak{P}$. The tangential plane $\Gamma_0(\pi)\, G$ to $\mathfrak{P}$ at P will thus subtend an angle with the central plane of that paraboloid whose goniometric tangent will be equal to $x / \mathfrak{e}$, if the piece SP is denoted by x. The line G that goes through P will thus subtend an angle of ω with its orthogonal projection G' onto the central plane whose tangent is:

$$\tan\,\omega = \frac{x}{\mathfrak{e}}\sin\,\alpha,$$

so α will also be the angle between G' with $\Gamma_0(\pi)$.

The line d that goes through M perpendicular to $(Z\varphi)$, which we have referred to as the focal axis of $\mathfrak{K}_d$, might meet $\mathfrak{E}$ at the point F. The angle ω' that PF subtends with its orthogonal projection PI onto the line t has the tangent:

$$\tan\,\omega' = \frac{IF}{PI};$$

now, if $IF = x\cos\,\alpha$, as the rectangular triangle MFI for F would imply, since $IM = PS = x$ and $\angle FIM = \alpha$, then:

$$PI = SM = \mathfrak{e}\cot\,\alpha,$$

as would follow from the rectangular triangle SME for S, for which $SE = \mathfrak{e}$ and $\angle SME = \alpha$. As a result, we will have:

$$\tan\,\omega' = \frac{x\cos\alpha}{\mathfrak{e}\cot\alpha} = \frac{x}{\mathfrak{e}}\sin\,\alpha = \tan\,\omega, \quad \omega' = \omega,$$

so G will be *perpendicular to FP*. Since F and t lie fixed in the plane $\mathfrak{E}$ of the pencil of parallels φ, we can recognize that the rays G (which are axes in R_{III}) that lie in $\mathfrak{E}$ will envelop a *parabola* with the focal point F and the vertex tangent t. However, since $\mathfrak{E}$ can also be parallel displaced, such that F moves along d and t moves parallel to Zf, all G will envelop the cone $\mathfrak{K}_d$, which will project the parabola above onto M and have d for its focal axis.

All tangents to $\mathfrak{K}_d$ that are parallel to φ represent axes G in R_{III}, only the lines that lie in $Z\varphi$ and go through M [which contact $\mathfrak{K}_d$ at a point of the edge $G_0(\mathfrak{p}_0)$] will exhaust the axes G in R_{III}, *not merely* the pencil of points M of axes G in R_{III} that belongs to $\mathfrak{p}_0$.

The system of axes G of the tangents to $\mathfrak{K}_d$ that are parallel to φ is thus of order two and class one, so the two tangents G to the parabola of intersection of $\mathfrak{K}_d$ with $\mathfrak{E} = w\varphi$ will go through every point w in space, and the tangents to the conic section $(W \cdot \mathfrak{K}_d)$ that are parallel to φ and do not lie in $(Z\varphi)$ will lie in any plane.

This happens in precisely the same way that any axis Γ in P_{III} (that is parallel to f) will contact the same cone $\mathfrak{K}_d$ that also possesses the focal axis δ that is perpendicular to f, and that, conversely, all tangents to that cone that are parallel to f will be axes in P_{III}, *except for* the lines in the plane Zf that do not go through M.

If an arbitrary axis G in R_{III} (Γ in P_{III}, resp.) meets the plane Zf ($Z\varphi$, resp.) at a point P (Π, resp.) that has a distance of z from the plane that is drawn through M perpendicular to Z then one will determine its parameter $\mathfrak{p}$ (π, resp.) by recalling that one must have (in terms of absolute values):

$$(\text{cf., } \mathfrak{p} - \mathfrak{p}_1 = \mathfrak{e} - \mathfrak{e}_1), \qquad \mathfrak{p} - \mathfrak{p}_0 = \mathfrak{e}\,(-\,0) = z \tan \alpha.$$

The general case R_{III} (P_{III}, resp.).

The general case R_{III} (P_{III}, resp.) of a sheaf of screws whose axes possess all possible directions in space will occur when the screws $G_1(\mathfrak{p}_1)$, $G_2(\mathfrak{p}_2)$, $G_3(\mathfrak{p}_3)$ that determine R_{III} have axes that are *not* parallel to a plane. As in the special case 4*), we can once more assume, with no loss of generality, that $\mathfrak{p}_2 = \mathfrak{p}_3 = \mathfrak{p}_1$, such that all lines in the family of hyperboloids G_1, G_2, G_3 that belong to the parameter $\mathfrak{p}_1$ will represent screw axes in R_{III}, while the other family of these hyperboloids F_i will be filled up by the axes Γ in the reciprocal sheaf P_{III} that are endowed with the parameter $\pi_1 = -\,\mathfrak{p}_1$.

The axes of all ∞^1 screws in R_{III} (P_{III}, resp.) that possess an arbitrary parameter $\mathfrak{p}$ (π, resp.) will belong to one family $G(\mathfrak{p})$ ($\Gamma(p)$, resp.) of a hyperboloid $F(\mathfrak{p})$, and indeed the same hyperboloid, when one has $\mathfrak{p} + \pi = 0$. Thus, for a certain parameter – and this will actually happen, as we will show – the one family $G(\mathfrak{p})$ will degenerate into a pair of pencils (center M, plane μ and center N, plane ν) with a common ray $MN = \mu\nu$, where the family $\Gamma(\pi)$ ($\mathfrak{p} + \pi = 0$) will be the pair of pencils with the same planes μ, ν, but with the centers N, M switched.

The family of lines in one of these hyperboloids $\mathfrak{p} = -\,\pi = 0$ will serve as the line framework r_{III} of R_{III}, while the other will serve as the line framework ρ_{III} of the reciprocal P_{III}. [The case III′ (pp. 11) is the one that yields a pair of pencils $M\mu$, $N\nu$ ($M\mu$, $M\nu$, resp.) instead of a hyperboloid for $\mathfrak{p} = -\,\pi = 0$, precisely.]

How do the ∞^1 hyperboloids $F(\mathfrak{p})$ ($\mathfrak{p} = -\,\pi$ arbitrary, constant) lie, and what kind of ray system exhaust the axes G (Γ, resp.) in the sheaf R_{III} (P_{III}, resp.)? *This ray system is identical with the congruence $K(G)$ [$K(\Gamma)$, resp.] (of order three and class two) that E. Waelsch (inter alia) examined* ("Über eine Strahlencongruenz beim Hyperboloide," Wiener Ber., Bd. 95, pp. 781-802, "Über das Normalsystem u. die Centralfl. alg. Fl.," Halle, 1888), *which is defined by the shortest transversals to two generators of the same family $\Gamma(\pi_1)$ [$G(\mathfrak{p}_1)$, resp.] of a hyperboloid F_1*, into which any hyperboloid $F(\mathfrak{p})$ with its family of lines $\Gamma(\pi)$ [$G(\mathfrak{p})$, resp.] can enter in place of F_1, with its family.

In fact, one such shortest transversal G (Γ, resp.) can possess an arbitrary parameter $\mathfrak{p}$ (π, resp.) as the axis of a screw L (Λ, resp.) in R_{III} (P_{III}, resp.), if it should merely be reciprocal to the two screws of P_{III} (R_{III}, resp.) that have axes $\Gamma(\pi_1)$ [$G(\mathfrak{p}_1)$, *resp.*] that intersect them perpendicularly. Now, if one is provided with such a parameter $\mathfrak{p}$ (π, resp.) that makes it reciprocal to a third screw in R_{III} (P_{III}, resp.) that is linearly-independent to the previous two then it will represent a screw that is reciprocal to the entire sheaf of screws P_{III} (R_{III}, resp.), and will thus belong to R_{III} (P_{III}, resp.). The same congruence $K(G)$ [$K(\Gamma)$, resp.] will belong to all of the ∞^1 hyperboloids $F(\mathfrak{p})$, in the same way that is true for F_1 . The symmetry axes $G_{\mathrm{I}} = \Gamma_{\mathrm{I}}$, $G_{\mathrm{II}} = \Gamma_{\mathrm{II}}$, $G_{\mathrm{III}} = \Gamma_{\mathrm{III}}$ of the hyperboloid F_1 are symmetry axes of the congruence $K(G)$ [$K(\Gamma)$, resp.], and thus, one of the ∞^1 hyperboloids $F(\mathfrak{p})$. $K(G)$ and $K(\Gamma)$ – viz., the "right" and "left" congruences, resp. – will go into each other, just like the families of any one of the hyperboloids $F(\mathfrak{p})$, when one performs a reflection in one of the symmetry planes $G_{\mathrm{II}}\, G_{\mathrm{III}}$, $G_{\mathrm{III}}\, G_{\mathrm{I}}$, $G_{\mathrm{I}}\, G_{\mathrm{II}}$. The $G_i = \Gamma_i$ (i = I, II, III) will present themselves to us when we provide them with suitable parameters $\mathfrak{p}_i$ (π_i, resp.) (i = I, II, III) – viz., the principal parameters of the three "principal screws" L_i in R_{III} (Λ_i in P_{III}, resp.) that give the canonical representation of that sheaf of screws. The $\mathfrak{p}_i$ (π_i, resp.) are derivable from the semi-axes a_i of the hyperboloid F_1, but also just as well from any of the other ∞^1 hyperboloids $F(\mathfrak{p})$, which is why we shall drop the index 1 from $\mathfrak{p}_1$ from now on. $\frac{x^2}{a_{\mathrm{I}}^2}+\frac{y^2}{a_{\mathrm{II}}^2}+\frac{z^2}{a_{\mathrm{III}}^2} = 1$ (which is a square of semi-axes – e.g., a_{III}^2 is negative, a_{III} is imaginary, and $a_{\mathrm{III}}\sqrt{-1}$ is real) is the equation of the hyperboloid $F(\mathfrak{p})$, when referred to the axis system G_i . In order to determine $\mathfrak{p}_i$, which are the parameters that belong to the same symmetry axes, we employ the reciprocity condition $\mathfrak{p} + \pi = \mathfrak{e} \tan \underline{|G\Gamma}$ of the $G_i(\mathfrak{p}_i)$ relative to any screw in P_{III} whose axis Γ cuts one of the other two symmetry axes of the hyperboloid, such that $\tan \underline{|G_i\Gamma}$ is deduced from the semi-axis ratios of two of the a_i, while $\mathfrak{e}$ is equal to the third a_i :

$$\left.\begin{aligned}
\mathfrak{p}_{\mathrm{I}} &= \mathfrak{p} - \frac{a_{\mathrm{III}}a_{\mathrm{II}}\sqrt{-1}}{a_{\mathrm{I}}},\\
\mathfrak{p}_{\mathrm{II}} &= \mathfrak{p} - \frac{a_{\mathrm{III}}a_{\mathrm{I}}\sqrt{-1}}{a_{\mathrm{II}}},\\
\mathfrak{p}_{\mathrm{III}} &= \mathfrak{p} - \frac{a_{\mathrm{I}}a_{\mathrm{II}}\sqrt{-1}}{a_{\mathrm{III}}},
\end{aligned}\right\}
\quad
\begin{aligned}
&\text{from which, it will result that:}\\
&a_{\mathrm{I}}a_{\mathrm{II}}a_{\mathrm{III}}\sqrt{-1}\\
&= -(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}}),\\
&\text{which is why it can follow that:}
\end{aligned}
\quad
\left\{\begin{aligned}
a_{\mathrm{I}}^2 &= -(\mathfrak{p}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}}),\\
a_{\mathrm{II}}^2 &= -(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}}),\\
a_{\mathrm{III}}^2 &= -(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}}).
\end{aligned}\right.$$

The hyperboloid $F(\mathfrak{p})$ *thus has the equation:*

$$\frac{x^2}{(\mathfrak{p}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}})}+\frac{y^2}{(\mathfrak{p}-\mathfrak{p}_{\mathrm{III}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}})}+\frac{z^2}{(\mathfrak{p}-\mathfrak{p}_{\mathrm{I}})(\mathfrak{p}-\mathfrak{p}_{\mathrm{II}})} + 1 = 0,$$

or

$$F(\mathfrak{p}) = (\mathfrak{p} - \mathfrak{p}_{\mathrm{I}})\, x^2 + (\mathfrak{p} - \mathfrak{p}_{\mathrm{II}})\, y^2 + (\mathfrak{p} - \mathfrak{p}_{\mathrm{III}})\, z^2 + (\mathfrak{p} - \mathfrak{p}_{\mathrm{I}})\,(\mathfrak{p} - \mathfrak{p}_{\mathrm{II}})\,(\mathfrak{p} - \mathfrak{p}_{\mathrm{III}}) = 0.$$

For values of $\mathfrak{p}$ that are found between the extremes of the principal parameters $\mathfrak{p}_i$, this equation will represent a one-sheeted hyperboloid, and only when $\mathfrak{p}$ is chosen to be outside of those limits will it be an imaginary midpoint surface of order two that is traced out by imaginary lines of *von Staudt's second kind.* The hyperboloid that the line framework r_{III} of R_{III} (ρ_{III} of P_{III}, resp.) carries, viz.:

$$-F(0) = \mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2 + \mathfrak{p}_{\mathrm{III}}\, z^2 + \mathfrak{p}_{\mathrm{I}}\, \mathfrak{p}_{\mathrm{II}}\, \mathfrak{p}_{\mathrm{III}} = 0,$$

is real when one of the $\mathfrak{p}_i$ has a different sign from the other two.

One of the hyperboloids $F(\mathfrak{p})$ – and two conjugate-imaginary ones, in addition to this real one – will degenerate to the pair of planes μ, ν: Let $\mathfrak{p}_{\mathrm{II}}$ be the value that lies algebraically between $\mathfrak{p}_{\mathrm{I}}$ and $\mathfrak{p}_{\mathrm{III}}$, so μ, ν will be represented by $F(\mathfrak{p}_{\mathrm{II}}) = 0$ or $\frac{z}{x} = \pm\sqrt{-\frac{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}}}{\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}}}}$. The points M, N ($x = 0$, $z = 0$, $y = \pm\sqrt{-(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{I}})}$) are the centers of those pencils $M\mu$, $N\nu$ ($N\mu$, $M\nu$, resp.) that represent the one real degenerate family of hyperboloids, and should be called *basic pencils.* Their planes – viz., the *basic planes* μ and ν *– are the common cyclic planes of all hyperboloids* $F(\mathfrak{p})$. The basic pencil determines the congruence $K(G)$ [$K(\Gamma)$, resp.] as the system of shortest transversals over any ray of either of them that:

can be constructed from the rays G_1, G_2, G_3 (Γ_1, Γ_2, Γ_3, resp.) of the congruence $K(G)$ [$K(\Gamma)$, resp.] that go through an arbitrary point w as the common edges of the orthogonal cone that has the orthogonal edges:

1) wM and altitude from w to v (μ, resp.).

2) wN and the altitude from w to μ (ν, resp.).

The edge of both cones that intersects the principal axis $MN = \mu\nu = G_{\mathrm{II}}$ perpendicularly will be omitted.

can be constructed from the two rays G_1, G_2 (Γ_1, Γ_2, resp.) of the congruence $K(G)$ [$K(\Gamma)$, resp.] that lie in an arbitrary plane W as the common tangents to the two parabolas that have:

1) The trace of ν in W for their vertex tangents and the orthogonal projections of M (N, resp.) onto W for their focal points.

2) The trace of μ in W for their vertex tangents and the orthogonal projections of N (M, resp.) onto W for their focal points.

The common tangents to the parabola that cut G_{II} perpendicularly, as well as the infinitely-distant one, will be omitted.

If we arrange that:

$$F(\mathfrak{p}) = \mathfrak{p}^3 - A_1\, \mathfrak{p}^2 + A_2\, \mathfrak{p} - A_3 = 0,$$

in which:

$$\left.\begin{aligned} A_1 &= \mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}} + \mathfrak{p}_{\mathrm{III}}, \\ A_2 &= \mathfrak{p}_{\mathrm{II}}\mathfrak{p}_{\mathrm{III}} + \mathfrak{p}_{\mathrm{III}}\mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{I}}\mathfrak{p}_{\mathrm{II}} + x^2 + y^2 + z^2, \\ A_3 &= \mathfrak{p}_{\mathrm{I}}\mathfrak{p}_{\mathrm{II}}\mathfrak{p}_{\mathrm{III}} + \mathfrak{p}_{\mathrm{I}}x^2 + \mathfrak{p}_{\mathrm{II}}y^2 + \mathfrak{p}_{\mathrm{III}}z^2, \end{aligned}\right\}$$

then it will follow for the parameters $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ of the three screws in R_{III} whose axes G_1, G_2, G_3, resp., go through the arbitrary point $w(x, y, z)$, which will then, as we would like to say, belong to w, that:

1) $\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 = \mathfrak{p}_{\text{I}} + \mathfrak{p}_{\text{II}} + \mathfrak{p}_{\text{III}}$

is constant for all points w in space, and:

2) $\mathfrak{p}_2\,\mathfrak{p}_3 + \mathfrak{p}_3\,\mathfrak{p}_1 + \mathfrak{p}_1\,\mathfrak{p}_2 = \mathfrak{p}_{\text{II}}\,\mathfrak{p}_{\text{III}} + \mathfrak{p}_{\text{III}}\,\mathfrak{p}_{\text{I}} + \mathfrak{p}_{\text{I}}\,\mathfrak{p}_{\text{II}} + (x^2 + y^2 + z^2)$.

Thus, the sum of the products of any two of the parameters that belong to w differs from the one that belongs to the analogous product sum of the principal parameters at the principal point p by the square of the amount pw, so it will be constant for all spheres around p.

3) $\mathfrak{p}_1\,\mathfrak{p}_2\,\mathfrak{p}_3 = \mathfrak{p}_{\text{I}}\,\mathfrak{p}_{\text{II}}\,\mathfrak{p}_{\text{III}} + \mathfrak{p}_{\text{I}}\,x^2 + \mathfrak{p}_{\text{II}}\,y^2 + \mathfrak{p}_{\text{III}}\,z^2$.

All points w in space can this be arranged into ∞^2 curves $\mathfrak{C}$ of order four, along which, the three parameters $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ individually remain constant. *The* $\mathfrak{C}$ *are the intersections of the spheres* around the principal point p with the *cones* that go through p *whose cyclic planes are* μ, ν.

The three values $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ of $\mathfrak{p}$ that belong to w correspond to three hyperboloids $F(\mathfrak{p}_1)$, $F(\mathfrak{p}_2)$, $F(\mathfrak{p}_3)$, which amount to the $\mathfrak{C}$ that go *through* w. The lines through w from the other family of each of these three hyperboloids are the axes Γ_1, Γ_2, Γ_3 in P_{III} that belong to w. The tangential planes $G_1\Gamma_1$, etc., to the three hyperboloids that belong to w intersect in the tangents C to the $\mathfrak{C}$ at w.

The three *axes* G_1, G_2, G_3 that belong to an arbitrary *point* w in R_{III} and the three axes Γ_1, Γ_2, Γ_3 in P_{III} that go through w *define polar vertices*, for which the edges of both vertices that are not perpendicular to each other belong to equal and opposite

Two of the hyperboloids $F(\mathfrak{p})$ *contact an arbitrary plane W*, since two edges G_1,G_2 (Γ_1, Γ_2, resp.) of the R_{II} (P_{II}, resp.) that are contained in R_{III} (P_{III}, resp.) whose cylindroid edges are parallel to w will then lie in W ([1]). Each of the two G must be

([1]) One also recognizes this fact from the equation for $F(\mathfrak{p})$ in plane coordinates u, v, w, 1, which is quadratic in $\mathfrak{p}$:

$$(\mathfrak{p} - \mathfrak{p}_{\text{II}})\,(\mathfrak{p} - \mathfrak{p}_{\text{III}})\,u^2 + (\mathfrak{p} - \mathfrak{p}_{\text{III}})\,(\mathfrak{p} - \mathfrak{p}_{\text{I}})\,v^2 + (\mathfrak{p} - \mathfrak{p}_{\text{I}})\,(\mathfrak{p} - \mathfrak{p}_{\text{II}})\,w^2 + 1 = 0$$

or

$$\mathfrak{p}^2\,(u^2 + v^2 + w^2) - \mathfrak{p}\,[(\mathfrak{p}_{\text{II}} + \mathfrak{p}_{\text{III}})\,u^2 + (\mathfrak{p}_{\text{III}} + \mathfrak{p}_{\text{I}})\,v^2 + (\mathfrak{p}_{\text{I}} + \mathfrak{p}_{\text{II}})\,w^2]$$
$$+ [\mathfrak{p}_{\text{II}}\,\mathfrak{p}_{\text{III}}\,u^2 + \mathfrak{p}_{\text{III}}\,\mathfrak{p}_{\text{I}}\,v^2 + \mathfrak{p}_{\text{I}}\,\mathfrak{p}_{\text{II}}\,w^2 + 1] = 0,$$

which implies the two $F(\mathfrak{p})$ that contact any given plane.

parameters; otherwise, not all of the three screws in R_{III} that belong to w could be reciprocal to three similar ones in P_{III}. These six axes lie in a cone of order two and cut the common altitude planes of both polar vertices in the line C at the point w. normal to one of the two Γ, and possess the equal and opposite parameter in R_{III} to the other Γ in P_{III}; otherwise, the two screws that are represented by G_1, G_2 could not be reciprocal to the ones that belong to Γ_1, Γ_2. Thus, since G_1 G_2 and Γ_1 Γ_2 are normal angles, they will intersect one or the other of the lines in W that belong to the surface $F(\mathfrak{p})$, *which contacts W, at equal angles.*

All $F(\mathfrak{p})$ have the four altitudes of the basic pencil, in addition to the four circle points of the two basic planes μ, ν, as common focal axes; i.e., all $F(\mathfrak{p})$ contact the four pairs of conjugate-complex tangential planes of the absolute sphere-circle, which can be drawn through the perpendiculars to μ and ν that are erected in M and N, resp.

One obtains the coordinates u, v, w, 1 of these tangential planes, in fact, when one sets the coefficients of $\mathfrak{p}^2$, $\mathfrak{p}$, 1 in the equation (cf., pp. 46, rem. 1) of the system $F(\mathfrak{p})$, when written in plane coordinates, equal to zero.

$$\left.\begin{array}{rrrl} u^2 + & v^2 + & w^2 & = 0\ , \\ (\mathfrak{p}_{\mathrm{II}}+\mathfrak{p}_{\mathrm{III}})u^2 + & (\mathfrak{p}_{\mathrm{III}}+\mathfrak{p}_{\mathrm{I}})v^2 + & (\mathfrak{p}_{\mathrm{I}}+\mathfrak{p}_{\mathrm{II}})w^2 & = 0\ , \\ \mathfrak{p}_{\mathrm{II}}\mathfrak{p}_{\mathrm{III}}u^2 + & \mathfrak{p}_{\mathrm{III}}\mathfrak{p}_{\mathrm{I}}v^2 + & \mathfrak{p}_{\mathrm{I}}\mathfrak{p}_{\mathrm{II}}w^2 & = -1, \end{array}\right\} \quad \text{i.e.,} \quad \begin{aligned} u^2 &= \frac{1}{(\mathfrak{p}_{\mathrm{III}}-\mathfrak{p}_{\mathrm{I}})(\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}})}, \\ v^2 &= \frac{1}{(\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{III}})}, \\ w^2 &= \frac{1}{(\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{III}})(\mathfrak{p}_{\mathrm{III}}-\mathfrak{p}_{\mathrm{I}})}. \end{aligned}$$

These equations teach us that the common tangential planes to all $F(\mathfrak{p})$ can be obtained from one of them by reflecting through the symmetry planes; i.e., from one that cuts out the pieces:

$$+\sqrt{(\mathfrak{p}_{\mathrm{III}}-\mathfrak{p}_{\mathrm{I}})(\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}})}\ ,\ +\sqrt{(\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{III}})}\ ,\ +\sqrt{(\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{III}})(\mathfrak{p}_{\mathrm{III}}-\mathfrak{p}_{\mathrm{I}})}\ ,$$

from the axes G_{I}, G_{II}, G_{III}, and thus contains the real line (viz., the *focal axis*) that is drawn through the center $M(0, +\sqrt{(\mathfrak{p}_{\mathrm{I}}-\mathfrak{p}_{\mathrm{II}})(\mathfrak{p}_{\mathrm{II}}-\mathfrak{p}_{\mathrm{III}})}\ , 0)$ and it perpendicular to a basic plane.

The theorem above can also be expressed as:

All hyperboloids $F(\pi)$ project orthogonally onto one of the basic planes μ, ν in a (doubly-counted) confocal system of conic sections with the centers M, N of the basic pencil as common focal points.

This also follows without the aid of plane coordinate from a construction of the surfaces $F(\mathfrak{p})$ that are concyclic with respect to μ, ν, and when one recalls that any $F(\mathfrak{p})$ is symmetric to the symmetry plane of the two pairs of basic pencils and is traced out by the axes of the sheaf of screws, which can be constructed as the shortest transversals over a ray of one of the pencils of a basic pair, this will imply:

Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be two circles around the principal point p (i.e., the midpoint of MN) that have equal, but *arbitrary*, radius and lie in the planes μ and ν, resp. Furthermore, let P_1 be a variable point – e.g., on $\mathfrak{K}_1$ – so the two transversals of $\mathfrak{K}_2$ that go through P_1 perpendicular to P_1M (let one of them be P_1P_2, with the point P_2 on $\mathfrak{K}_2$, while the other is $P_1 P_2'$) will trace out a family of the same kind as one of the two hyperboloids $F(\mathfrak{p})$ that go through $\mathfrak{K}_1$ and $\mathfrak{K}_2$. In fact, the perpendicular to P_1P_2 that is drawn through P_2 in ν, for example, goes through N; therefore, P_1P_2 is a shortest transversal over some rays P_1M in the pencil $M\mu$ and P_1N in the pencil $N\nu$, so it will be a general one of the two hyperboloids $F(\mathfrak{p})$ that go through $\mathfrak{K}_1$ and $\mathfrak{K}_2$.

P_1P_2 then projects orthogonally onto μ in the altitude that goes through P_1 to P_1M in μ, so it will always contact the conic section in μ that will have M and N for its focal points when P_1 varies on $\mathfrak{K}_1$. This conic section is thus the orthogonal projection of the hyperboloid $F(\mathfrak{p})$ that is traced out by P_1P_2 , which cuts μ and ν in $\mathfrak{K}_1$ and $\mathfrak{K}_2$, resp. (Yet a second surface of the system that goes through the same circles $\mathfrak{K}_1$ and $\mathfrak{K}_2$ will project onto it, since one can also take P_2' on $\mathfrak{K}_2$, instead of the point P_2 .)

In addition to the axes G in R_{III} (Γ in P_{III}, resp.) that are assigned to hyperboloids $F(\mathfrak{p})$ that belong to a certain parameter, all lines g (γ, resp.) (*Staudt's first kind*, cf., pp. 30, rem.) that belong to an indeterminate parameter should also be regarded as axes in the sheaf of screws that emerge from the circular rays that are present in each of two basic pencils $M\mu$, $N\nu$ in R_{III} ($M\nu$, $N\mu$ in P_{III}, resp.) by parallel displacement along the altitude to the pencil.

The $F(\mathfrak{p})$ can intersect in only one of the concyclic, spherical, conic sections (pp. 47), so the *focal surface of the congruence* $K(G)$ [$K(\Gamma)$, *resp.*] *that is enveloped by the* $F(\mathfrak{p})$, namely, the sixth-order surface (that comes from $F(\mathfrak{p}) = 0$, $\partial F / \partial\mathfrak{p} = 0$) ([1]):

$$4A_1^3A_3 - A_1^2A_2^2 - 18A_1A_2A_3 + 27A_3^2 + 4A_2^3 = 0$$

(with the absolute sphere-circle as its cuspidal curve) will also be contacted by *each* $F(\mathfrak{p})$ along a $\mathfrak{C}$. Thus contact curve $\mathfrak{C}$ is simultaneously the complement of the intersection (*Schnittrest*) of the focal surface with another hyperboloid $F(\mathfrak{p}')$. *At each point w* of an arbitrary $\mathfrak{C}$ of that kind, *the focal surface will degenerate* into the aforementioned *polar vertices* when two of the three axes G (Γ, resp.) that belong to w coalesce into two

([1]) Order six and class *four*, as one infers from setting the discriminant of the equation (pp. 46, rem. 1) that is quadratic in $\mathfrak{p}$ equal to zero.

E. W. Hyde has investigated this surface in the Annals of Mathematics, ser. II, vol. 2, no. 4 (Mass., 1901), and produced some intuitive Figures of it. (Esp., Fig. 2 in the treatise: "On a surface of sixth order which is touched by the axes of all screws reciprocal to three given screws.")

infinitely-close lines G_0 (Γ_0, resp.) of a certain plane [viz., the focal plane of G_0 (Γ_0, resp.)]. In addition to these lines G_0 (Γ_0, resp.) of the hyperboloid $F(\mathfrak{p})$ through w, yet another line G' (Γ', resp.) will go through that point that is an axis in R_{III} (P_{III}, resp.) with the parameter $\mathfrak{p}'$ (π', resp.), for which one must have $\mathfrak{p}' + \pi' = 0$, since G' and Γ' are attached to the hyperboloid $F(\mathfrak{p}')$ that belongs to w, along with the two infinitely-close $F(\mathfrak{p})$. The G_0 (Γ_0, resp.) of $F(\mathfrak{p})$ that belong to the parameter $\mathfrak{p}$ (π, resp.) (with $\mathfrak{p} + \pi = 0$) must be perpendicular to the plane $\Gamma_0\,\Gamma'$ ($G_0\,G'$, resp.) and possess focal planes that are perpendicular to G' (Γ', resp.), since otherwise both of the two screws $G_0(\mathfrak{p})$ [$\Gamma_0(\pi)$, resp.], as well as $G'(\mathfrak{p}')$ [$\Gamma'(\pi')$, resp.], that are infinitely close to w in the focal plane could not be reciprocal to the analogous ones $\Gamma_0(\pi)$ [$G_0(\mathfrak{p})$, resp.] and the $\Gamma'(\pi')$ [$G'(\mathfrak{p}')$, resp.].

Therefore, each of the concyclic, spherical, conic sections $\mathfrak{C}$ of the focal surface an "orthogonal point curve" of the hyperboloid $F(\mathfrak{p})$ that contacts the focal surface along $\mathfrak{C}$ and likewise a "limit point curve" of the other hyperboloid $F(\mathfrak{p}')$, which (contacts the focal surface along another $\mathfrak{C}'$) and has $\mathfrak{C}$ for the complement of its intersection with the focal surface. We understand *Kummer's* term "limit point" of a ray G (Γ, resp.) of a congruence to mean, as would be useful, one of the two points *between* which the shortest transversals over G' (Γ', resp.) and the infinitely-close rays of the congruence $K(G)$ [$K(\Gamma)$, resp.] [i.e., the perpendicular transversals Γ of G' (G of Γ', resp.) that trace out the axis cylindroid of the P_{II} in P_{III} (R_{II} in R_{III}, resp.) whose principal plane is perpendicular to G' (Γ', resp.)] are *real*, in which the aforementioned shortest transversals [at the pinch edges; e.g., Γ_0 (G_0, resp.) in the aforementioned cylindroid] themselves draw together, and *outside* of which these shortest (cylindroid edges) will be imaginary. [On the relationship between the radii of the "orthogonal point curve" of a *Monge* sphere that implies an arbitrary hyperboloid $F(\mathfrak{p})$ and the other sphere that cuts out the "limit point curve" in $F(\mathfrak{p})$, etc., cf., the cited treatise of *E. Waelsch*.]

The "focal surface" and the "limit surface" of $K(G)$ [$K(\Gamma)$, resp.] are identical. Any ray G' (Γ', resp.) of this congruence whose limit points are *met* by Γ_0 (G_0, resp.) and the other pinch edge of the cylindroid above *will contact* the focal surface in addition at *two points* – viz., its focal points – at which it is met by the *reciprocal domain that belongs to the equal and opposite parameter* and the axes that *intersect it perpendicularly* (viz., edges of the aforementioned cylindroid). Each of the latter, together with G' (Γ', resp.), will then play the same role that was played above by the pinch edges Γ_0 and G_0 (G_0 and Γ_0, resp.) that went through the point w, namely, they, along with G' (Γ', resp.), will determine the tangential plane to the focal surface at the focal point in question. The focal planes of every focal point on G' (Γ', resp.) are perpendicular to the second cylindroid edge, which goes through the focal point, in addition to the above. The former two limit points, through which the "limit point sphere" that belongs to the rays in question of the hyperboloid $F(\mathfrak{p})$ meet, and the "focal points" (or "orthogonal points" relative to $F(\mathfrak{p})$) on the concentric *Monge* sphere are symmetric with respect to the point m – viz., the "midpoint" (principal point of the cylindroid above) – at which the principal point p of R_{III} (P_{III}, resp.) projects orthogonally onto the ray of the congruence. The "midpoint surface" that is traced out by these points m of the rays of the congruence shall, as soon as it is given, shall be the first point at which we can recognize the *relationship between the direction of a screw axis* in R_{III} (P_{III}, resp.) and *its parameter* $\mathfrak{p}$ (π, resp.). The axes Γ in P_{III} belong to screws in that domain whose parameter is equal and opposite to that ($\mathfrak{p}$) of the parallel axis in R_{III}, such that examination that relates to R_{III} will suffice:

From the intersection laws for linear domains, for every screw L in R_{III} there is a reciprocal to the latter in R_{II}. We give the principal screws L_i (i = I, II, III) in R_{III} the volume 1, such that their rod lengths $\overline{l_i}$ will be the "characteristics" of the principal screws. From the development on pp. 18, etc., on the characteristic conic section $\mathfrak{K}$ of an R_{III}, it follows that the midpoint surface of order two with the semi-axes $l_i = \pm\sqrt{\frac{1}{\mathfrak{p}_i}}$ ($\mathfrak{p}_i$ principal parameter) that fall on the axes G_i of the L_i will be a *characteristic surface* of order two ($\mathfrak{K}$) for R_{III}:

$$\mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2 + \mathfrak{p}_{\mathrm{III}}\, z^2 = 1,$$

in which *every radius l of* ($\mathfrak{K}$) (which might have the direction cosines c_1, c_2, c_3) will be the characteristic of the screw L in R_{III} whose axis is parallel to it, such that $\mathfrak{p} = 1 / \overline{l}^{\,2}$ will prove to be the parameter of the latter. The pencil of screws R_{II} that is reciprocal to L and contained in R_{III} has the plane $\mathfrak{p}_{\mathrm{I}}\, c_1\, x + \mathfrak{p}_{\mathrm{II}}\, c_2\, y + \mathfrak{p}_{\mathrm{III}}\, c_3\, z = 0$, which is conjugate to $\overline{l}$ relative to ($\mathfrak{K}$), for its direction plane. Incidentally, it follows from this or the theorem that:

The sum of the squares of three conjugate radii of a surface ($\mathfrak{K}$) *is constant*

that

The sum of the reciprocal values of the parameters of three co-reciprocal screws in R_{III} *is constant:*

$$\frac{1}{\mathfrak{p}_1}+\frac{1}{\mathfrak{p}_2}+\frac{1}{\mathfrak{p}_3} = \frac{1}{\mathfrak{p}_{\mathrm{I}}}+\frac{1}{\mathfrak{p}_{\mathrm{II}}}+\frac{1}{\mathfrak{p}_{\mathrm{III}}},$$

since the associated screws have the parallel conjugate three radii of ($\mathfrak{K}$) for their characteristics. This is connected with *H. Everett's* proof of the theorem:

The sum of the reciprocal values of the parameters ($\mathfrak{p}_1$, …, $\mathfrak{p}_6$) *of six co-reciprocal screws in space* (R_{VI}) *is zero.*

Three of the six screws determine an R_{III}, while the remaining ones will determine its reciprocal domain P_{III} . Thus, in addition to the equation above, one will have

$\frac{1}{\mathfrak{p}_4}+\frac{1}{\mathfrak{p}_5}+\frac{1}{\mathfrak{p}_6} = \frac{1}{\pi_{\mathrm{I}}}+\frac{1}{\pi_{\mathrm{II}}}+\frac{1}{\pi_{\mathrm{III}}}$, in which the $\pi_i = -\mathfrak{p}_i$ are the principal parameters of the P_{III}. The equations that were written down will imply the stated theorem:

$$\frac{1}{\mathfrak{p}_4}+\cdots+\frac{1}{\mathfrak{p}_6} = 0.$$

From this theorem, and the one that is derived for co-reciprocal screws in an R_{II} (P_{II}, resp.) (pp. 20), namely, $\frac{1}{\mathfrak{p}_1}+\frac{1}{\mathfrak{p}_2}$ = const., it can be proved that:

For four co-reciprocal screws in an R_{IV} (P_{IV}, *resp.*), *the sum of the reciprocal values of their parameters is constant.*

To that end, P_{II} (R_{II}, resp.) needs only to be chosen to be the reciprocal pencil to R_{IV} (P_{IV}, *resp.*). Analogous statements will be true for the parameters of five co-reciprocal screws in a web of screws. – The sum of the parameters that belong to all triples of normal axes in R_{III} is constant; i.e., one has:

$$\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 = \mathfrak{p}_{\mathrm{I}} + \mathfrak{p}_{\mathrm{II}} + \mathfrak{p}_{\mathrm{III}}\,,$$

when $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ stand for three screws in R_{III} *whose axes are mutually perpendicular*, as would follow from the fact that the sum of the parameters of two screws is constant in each R_{II} when its axes are normal to each other. –

Instead of representing the parameters of the screws in all axis direction by ($\mathfrak{K}$), one can measure from a point p parallel to the axes of every screw in R_{III} whose parameter is $\mathfrak{p} = 1 / \overline{l}^2$ and employ the "parameter surface" ($\mathfrak{P}$) that is arrived at as the locus of endpoints thus-obtained in order to visualize the parameter distribution. (Cf., pp. 19) On the ray through p that has the direction cosines c_1, c_2, c_3 (so $c_1^2+c_2^2+c_3^2 = 1$), the parameter:

$$\mathfrak{p} = \mathfrak{p}_{\mathrm{I}}c_1^2+\mathfrak{p}_{\mathrm{II}}c_2^2+\mathfrak{p}_{\mathrm{III}}c_3^2$$

$$\Big(x = \lambda\, c_1, y = \lambda\, c_2, z = \lambda\, c_3\,; \quad \lambda^2(\mathfrak{p}_{\mathrm{I}}c_1^2 + \ldots) = 1; \qquad x^2 = \frac{c_1^2}{\mathfrak{p}_{\mathrm{I}}c_1^2+\cdots}, \text{ etc.;}$$

$$\overline{l}^2 = x^2 + y^2 + z^2 = \frac{1}{\mathfrak{p}_{\mathrm{I}}c_1^2+\cdots} = \frac{1}{\mathfrak{p}}\Big),$$

such that *the equation of* ($\mathfrak{P}$), in which x, y, z should mean the coordinates of the points on that surface from now on, so one sets $\mathfrak{p} = \sqrt{x^2+y^2+z^2}$, $c_1^2 = \frac{x^2}{x^2+y^2+z^2}$, etc., will read:

$$(x^2 + y^2 + z^2)^3 - (\mathfrak{p}_{\mathrm{I}}\, x^2 + \mathfrak{p}_{\mathrm{II}}\, y^2 + \mathfrak{p}_{\mathrm{III}}\, z^2)^2 = 0.$$

Since $\mathfrak{p}$ can be regarded as the length of the projection of a segment ($\mathfrak{p}_{\mathrm{I}}\, c_1$, $\mathfrak{p}_{\mathrm{II}}\, c_2$, $\mathfrak{p}_{\mathrm{III}}\, c_3$) onto the segment ($c_1$, c_2, c_3), that will yield the following construction of the parameter surface ($\mathfrak{P}$):

Let $\mathfrak{K}_i$ (i = I, II, III) *be the spheres that are described around p with radii* $\mathfrak{p}_i$ *, and let an arbitrary radius of each* $\mathfrak{K}_i$ *meet it at the point* $\mathfrak{P}_i$ *(when the* $\mathfrak{p}_i$ *have the same signs; if one of these three principal parameters has a sign that is different from the other two then the extension of the radius beyond p should cut the* $\mathfrak{K}_i$ *in question); if the three planes in M that are drawn through* $\mathfrak{P}_i$ *perpendicular to the principal axes* G_i *in* R_{III} *intersect (OM has the projections* $\mathfrak{p}_{\mathrm{I}}\, c_1$, $\mathfrak{p}_{\mathrm{II}}\, c_2$, $\mathfrak{p}_{\mathrm{III}}\, c_3$ *onto the principal axes) then the orthogonal projection of M onto the radius employed will be a point of* ($\mathfrak{P}$) *and will trace out this surface when the radius varies through all p.*

All of the radius vectors that belong to a certain axis congruence $K(G)$ [$K(\Gamma)$, resp.] and are obtained by varying all radius vectors that are based at p by a constant amount will be obtained from a parameter surface ($\mathfrak{P}$) that is so constructed (cf., pp. 13). The family of ($\mathfrak{P}$) thus-obtained will represent the parameter distribution that is possible for a given axis congruence as the pencil:

$$(\mathfrak{p}_{\mathrm{I}} + \lambda)\, x^2 + (\mathfrak{p}_{\mathrm{II}} + \lambda)\, y^2 + (\mathfrak{p}_{\mathrm{III}} + \lambda)\, z^2 = 1 \qquad (\lambda \text{ arbitrary})$$

of the associated ($\mathfrak{K}$) that also contain the sphere that leads to the absolute sphere-circle.

The axis congruences $K(G)$ in R_{III} and $K(\Gamma)$ in P_{III} are symmetric relative to the principal point p and the principal planes $G_{\mathrm{I}}G_{\mathrm{II}}$, etc., so the same thing will also be true for *Kummer's* middle surface; we thus need only to consider R_{III} . Any screw L in R_{III} (with volume 1), since it is linearly derivable from the three principal screws, will be representable as the sum of a radius rod l (with direction cosines c_1, c_2, c_3) of the characteristic surface of order two ($\mathfrak{K}$) ($\mathfrak{p}_{\mathrm{I}}\, x^2 + \ldots = 1$) and a field of the plane $\mathfrak{E}$ ($\mathfrak{p}_{\mathrm{I}}\, c_1\, x + \ldots = 0$, which is conjugate to l relative to ($\mathfrak{K}$) (cf., pp. 2) In order to go from this form of representation to the canonical one, we need only to displace the rod l that goes through p in the direction in $\mathfrak{E}$ that is perpendicular to l through an amount pm. m will then trace out the desired middle surface when l assumes all possible radii in ($\mathfrak{K}$). The direction cosines of pm, which are proportional to the determinants of the matrix (the projections of the altitudes to $\mathfrak{E}$ and l)

$$\begin{vmatrix} \mathfrak{p}_{\mathrm{I}}\, c_1 & \mathfrak{p}_{\mathrm{II}}\, c_2 & \mathfrak{p}_{\mathrm{III}}\, c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

will be:

$$\frac{c_2 c_3 (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})}{\sqrt{c_2^2 c_3^2 (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \cdots}}, \ldots, \ldots$$

and the length ρ of the displacement amount must be $\rho = \mathfrak{p}\tan\alpha$, according to (pp. ?), when $\mathfrak{p} = \mathfrak{p}_{\mathrm{I}}\, c_1^2 + \mathfrak{p}_{\mathrm{II}}\, c_2^2 + \mathfrak{p}_{\mathrm{III}}\, c_3^2$ means the parameter of the screw L and α is the angle between l and the altitude to $\mathfrak{E}$. If one recalls that:

$$\cos\alpha = \frac{\mathfrak{p}_{\mathrm{I}}\, c_1^2 + \mathfrak{p}_{\mathrm{II}}\, c_2^2 + \mathfrak{p}_{\mathrm{III}}\, c_3^2}{\sqrt{\mathfrak{p}_{\mathrm{I}}^2\, c_1^2 + \mathfrak{p}_{\mathrm{II}}^2\, c_2^2 + \mathfrak{p}_{\mathrm{III}}^2\, c_3^2}} = \frac{\mathfrak{p}}{\sqrt{\mathfrak{p}_{\mathrm{I}}^2\, c_1^2 + \mathfrak{p}_{\mathrm{II}}^2\, c_2^2 + \mathfrak{p}_{\mathrm{III}}^2\, c_3^2}}$$

and

$$\sin\alpha = \frac{1}{\sqrt{\mathfrak{p}_{\mathrm{I}}^2 c_1^2 + \cdots}} \sqrt{c_2^2 c_3^2 (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \cdots}$$

then one will have:

$$\rho = \sqrt{c_2^2 c_3^2 (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \cdots},$$

and we will get:

$$x = c_2\, c_3\, (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}}), \qquad y = c_3\, c_1\, (\mathfrak{p}_{\mathrm{III}} - \mathfrak{p}_{\mathrm{I}}), \qquad z = c_1\, c_2\, (\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}}),$$

as the projections of the displacement amount $\rho = pm$, from which one eliminates c_1, c_2, c_3 with the help of $c_1^2 + c_2^2 + c_3^2 = 1$:

$$\left.\begin{aligned} c_2^2 c_3^2 (\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 &= x^2, \\ c_3^2 c_1^2 (\mathfrak{p}_{\mathrm{III}} - \mathfrak{p}_{\mathrm{I}})^2 &= y^2, \\ c_1^2 c_2^2 (\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}})^2 &= z^2, \end{aligned}\right\} \qquad \left.\begin{aligned} c_1^2 &= \frac{1}{x^2}(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2, \\ c_2^2 &= \frac{1}{y^2}(\mathfrak{p}_{\mathrm{III}} - \mathfrak{p}_{\mathrm{I}})^2, \\ c_3^2 &= \frac{1}{z^2}(\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}})^2, \end{aligned}\right\}$$

$$\lambda\left[\frac{(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})}{x^2} + \cdots\right] = 1, \qquad \lambda = \frac{x^2 y^2 z^2}{(\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{III}})^2 y^2 z^2 + \cdots},$$

$$c_1^2 c_2^2 c_3^2 [(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \cdots] = c_1^2 x^2 + c_2^2 y^2 + c_3^2 z^2,$$

$$\frac{\lambda^2}{x^2 y^2 z^2}(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 (\mathfrak{p}_{\mathrm{III}} - \mathfrak{p}_{\mathrm{I}})^2 (\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}})^2 [(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \ldots] = \lambda\, [(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})^2 + \ldots],$$

$$\lambda = \frac{\pm xyz}{(\mathfrak{p}_{\mathrm{II}} - \mathfrak{p}_{\mathrm{III}})(\mathfrak{p}_{\mathrm{III}} - \mathfrak{p}_{\mathrm{I}})(\mathfrak{p}_{\mathrm{I}} - \mathfrak{p}_{\mathrm{II}})}.$$

This yields the *Steiner* surface of order four:

$$xyz\,(\mathfrak{p}_{II} - \mathfrak{p}_{III})(\mathfrak{p}_{III} - \mathfrak{p}_{I})(\mathfrak{p}_{I} - \mathfrak{p}_{II})$$
$$\mp\ [(\mathfrak{p}_{II} - \mathfrak{p}_{III})^2\,y^2\,z^2 + (\mathfrak{p}_{III} - \mathfrak{p}_{I})^2\,z^2\,x^2 + (\mathfrak{p}_{I} - \mathfrak{p}_{II})^2\,x^2\,y^2] = 0.$$

as the middle surface of the congruences $K(G)$ and $K(\Gamma)$.

(Cf., *Kummer, Schröter, Cayley* in Crelles Journal, Bd. 64, 1864, etc.) ([1]) An affine picture of this surface is obtainable as a model from *L. Brill's* Darmstadt publication (series 9, no. 3).

In particular, if $\mathfrak{p}_I = \mathfrak{p}_{II}$ then the $F(\mathfrak{p})$ will be hyperboloids of rotation, and the cyclic plane μ, ν, as the well as the middle surface of the axis congruences $K(G)$, $K(\Gamma)$ in R_{III} and P_{III}, resp., will lie in the principal plane $G_I\,G_{II}$, and the centers M, N will lie at its principal point p. The focal surface will be a surface of rotation ([2]), as will the characteristic surface of order two ($\mathfrak{K}$) and the parameter surface ($\mathfrak{P}$). $K(G)$ will be traced out by rays that can be obtained from the edges of the cylindroid that is determined by the screws $G_I(\mathfrak{p}_I)$ and $G_{III}(\mathfrak{p}_{III})$ by rotation around G_{III}.

In particular, for $\mathfrak{p}_I = \mathfrak{p}_{II} = \mathfrak{p}_{III}$, all axes in R_{III} will go through the principal point p and have the same parameter $\mathfrak{p} = \mathfrak{p}_I$. There will be no real screws with other parameters. The line framework r_{III} is one of the systems of generators (*Staudt's* line of the second kind) of the sphere $x^2 + y^2 + z^2 + \mathfrak{p}_1^2 = 0$ with an imaginary radius whose square is $(-\mathfrak{p}_1^2)$; any circular line g of *v. Staudt's* first kind that belongs to an arbitrary parameter, and which lies in a tangential plane to the cone $x^2 + y^2 + z^2 = 0$ can (cf., pp. 30) be regarded as the "axis" of such a line. The other system of generators of that sphere defines the line framework ρ_{III} of the reciprocal P_{III}. In addition to the circular axes g of ρ_{III}, in turn, the rays of the sheaf p will belong to P_{III} as axes, but endowed with the parameter $\pi = -\mathfrak{p} = -\mathfrak{p}_I$.

([1]) Closely-related to *Ernesto Pascal Repertorio di Matematiche superiori: II. Geometria,* Milano, 1900, pp. 474, etc. "La superficie romana di Steiner."

([2]) C., Fig. 3 in the cited treatise of *E. W. Hyde* (pp. 48, rem. 1).