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# General solution to the static one-body problem in Einstein’s theory of gravitation

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With the application of polar coordinates:

$$\begin{aligned} x_1 &= r \sin \vartheta \cos \varphi, \\ x_2 &= r \sin \vartheta \sin \varphi, \\ x_3 &= r \cos \vartheta, \end{aligned} \tag{1}$$

the general differential equation of EINSTEIN’s theory of gravitation, with spherical symmetry, will take on the form:

$$ds^2 = L\,du^2 + 2M\,du\,dr + N\,dr^2 + O\,r^2\,(d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2), \tag{2}$$

in which:

$$u = ct,$$

and $ds$ must be real for a time-like vector. In the static case, $L$, $M$, $N$, $O$ depend upon only $r$. The static one-body problem represents a special static, spherically-symmetric case that is characterized by the fact that for an increasing value of $r$, equation (2) should go to the one in the special theory of relativity, so one can set:

$$\begin{aligned} L &= 1 + \sum \frac{l_i}{r^i}, & M &= \sum \frac{m_i}{r^i}, \\ N &= -\left(1 + \sum \frac{n_i}{r^i}\right), & O &= -\left(1 + \sum \frac{o_i}{r^i}\right). \end{aligned} \tag{3}$$

The solution that SCHWARZSCHILD and HILBERT found is based upon the assumption that $M = O$, and is known to imply that:

$$L = 1 - \frac{\alpha}{r}, \qquad N = -\frac{1}{L}, \qquad O = -1.$$

It shall be shown here that when one applies rectangular Cartesian coordinates, the latter will represent only one special solution to the static one-body problem. In agreement with EINSTEIN's gravitational equations, we shall then set:

$$g = -1.$$

If one denotes the derivatives of $u$, $r$, $\vartheta$, $\varphi$ with respect to $s$ by $u'$, $u''$, … then the geodetic world-lines will be obtained from the condition:

$$\delta \int_1^2 F\, ds = 0,$$

in which:

$$F = L\, u'^2 + 2M\, r'u' + N\, r'^2 + O\, r^2\, (\vartheta'^2 + \sin^2\vartheta\, \varphi'^2),$$

and variation will result in the four equations:

$$\frac{\partial F}{\partial r} - \frac{d}{ds}\frac{\partial F}{\partial r'} = 0, \qquad \frac{\partial F}{\partial \vartheta} - \frac{d}{ds}\frac{\partial F}{\partial \vartheta'} = 0, \qquad \frac{d}{ds}\frac{\partial F}{\partial \varphi'} = 0, \qquad \frac{d}{ds}\frac{\partial F}{\partial u'} = 0.$$

The derivatives of $L$, $M$, $N$, $O$ with respect to $r$ shall now be denoted by $L'$, $L''$, …, with which, one will have:

$$\frac{dL}{ds} = L'\, r',$$

etc. Along with two equations that contain only second-order differential quotients $\varphi'$ ($\vartheta'$, resp.), differentiation will also yield two equations:

$$N\, r'' + M\, u'' = \tfrac{1}{2}(L'u'^2 - N'r'^2) + \left(Or + \frac{O'r^2}{2}\right)(\vartheta'^2 + \sin^2\vartheta\, \varphi'^2),$$

$$M\, r'' + L\, u'' = -M'r'^2 - L'r',$$

from which, one eliminates $r''$, and then $u''$, and one will get the coefficients $LN - M^2$ from these differential quotients. It shall be eliminated by means of the equation of the determinant:

$$O^2\, (LN - M^2) + 1 = 0.$$

After the term:

$$-O^2 r'^2 \left(MM' - \frac{LN'}{2}\right)$$

that appears in the equation for $r''$, with an application of the equation:

$$O^2\, (L\, N' + N\, L' - 2\, M\, M') - \frac{2O'}{O} = 0, \tag{4}$$

has been put into the form:

$$-\ r'^2\left(\frac{NO^2L'}{2}-\frac{O'}{O}\right),$$

the equations for the geodetic world-lines will have the form:

$$r''=-\frac{LO^2L'}{2}u'^2-MO^2L'r'u'-\left(\frac{NO^2L'}{2}-\frac{O'}{O}\right)r'^2-LO^2\left(Or+\frac{O'r^2}{2}\right)(\vartheta'^2+\sin^2\vartheta\varphi'^2)\,,$$

$$u''=\frac{LO^2L'}{2}u'^2+NO^2L'r'u'+O^2\left(NM'-\frac{MN'}{2}\right)r'^2+MO^2\left(Or+\frac{O'r^2}{2}\right)(\vartheta'^2+\sin^2\vartheta\varphi'^2)\,,$$

$$\varphi''=-\,2\cot\vartheta\,\varphi'\,\vartheta'-\left(\frac{2}{r}+\frac{O'}{O}\right)r'\varphi',$$

$$\vartheta''=\sin\vartheta\cos\vartheta\,\varphi'^2-\left(\frac{2}{r}+\frac{O'}{O}\right)r'\varphi'.$$

The components $B_{\mu\nu} = R_{\mu\nu}$ of the contracted CHRISTOFFEL tensor shall now be ascertained for the coordinate system $\varphi = 0$, $\vartheta = \pi / 2$. In order to use them with the simplification that comes about for $g = -1$, we should emphasize that only transformations with a substitution determinant 1 are admissible, so from now on, we shall introduce coordinates $x_i$ such that $x_i'$, $x_i''$ shall denote their derivatives with respect to $s$. Since the $\left\{\begin{matrix}\mu\nu\\ \sigma\end{matrix}\right\}$ must be differentiated with respect to $x_0$, the first powers of $x_2$ ($x_3$, resp.) must be taken in the equations for $x_2''$ ($x_3''$, resp.) Differentiating equations (1) twice will yield:

$$x_1''=r''-r\,(\vartheta'^2+\varphi'^2),$$

$$x_2''=\frac{x_2\,r''}{r}+r\,\varphi''+2r'\varphi'-x_2\,(\vartheta'^2+\varphi'^2),$$

$$x_3''=\frac{x_3\,r''}{r}-r\,\vartheta''-2r'\vartheta'-x_3\vartheta'^2,$$

and after eliminating $\varphi$ and $\vartheta$, while considering the fact that one has:

$$r'=x_1'+\frac{x_2\,x_2'+x_3\,x_3'}{r}\,,\qquad \varphi'=\frac{x_2'}{r}-\frac{x_2\,x_1'}{r^2}\,,\qquad \vartheta'=-\frac{x_2'}{r}+\frac{x_3\,x_1'}{r^2}\,,$$

with the prescribed approximation, one will get the equations:

$$x_1'' = r'' - \frac{x_2'^2 + x_3'^2}{r}, \qquad x_2'' = \frac{O' x_1' x_2'}{O} + \frac{x_2}{r}\left( x_1'' + \frac{O'(x_1'^2 - x_1'^2)}{O} \right),$$

$$x_4'' = u'', \qquad x_3'' = -\frac{O' x_1' x_3'}{O} + \frac{x_3}{r}\left( x_1'' + \frac{O'(x_1'^2 - x_1'^2)}{O} \right).$$

Now since one has:

$$x_\sigma'' = -\sum_{\mu,\nu} \begin{Bmatrix} \mu\,\nu \\ \sigma \end{Bmatrix} x_\mu' \, x_\nu',$$

all of the $\begin{Bmatrix} \mu\,\nu \\ \sigma \end{Bmatrix}$ will be known. Furthermore, for $g = -1$, one will have:

$$B_{\mu\nu} = R_{\mu\nu} = -\sum_\sigma \frac{\partial}{\partial x_\sigma} \begin{Bmatrix} \mu\,\nu \\ \sigma \end{Bmatrix} + \sum_{\alpha,\beta} \begin{Bmatrix} \mu\,\alpha \\ \beta \end{Bmatrix} \begin{Bmatrix} \nu\,\beta \\ \alpha \end{Bmatrix}.$$

In order to ascertain $R_{44}$, one finds that the sum of the products contains only the terms:

$$\begin{Bmatrix} 1\,4 \\ 1 \end{Bmatrix}^2 + 2\begin{Bmatrix} 1\,4 \\ 4 \end{Bmatrix}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 4\,4 \\ 4 \end{Bmatrix}^2.$$

The required quantities are then:

$$\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix} = \frac{LO^2 L'}{2}, \qquad \begin{Bmatrix} 4\,4 \\ 2 \end{Bmatrix} = \frac{x_2}{r}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix}, \qquad \begin{Bmatrix} 4\,4 \\ 3 \end{Bmatrix} = \frac{x_3}{r}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix},$$

$$\begin{Bmatrix} 1\,4 \\ 1 \end{Bmatrix} = \frac{MO^2 L'}{2}, \qquad \begin{Bmatrix} 1\,4 \\ 4 \end{Bmatrix} = -\frac{NO^2 L'}{2}, \qquad \begin{Bmatrix} 4\,4 \\ 4 \end{Bmatrix} = -\begin{Bmatrix} 1\,4 \\ 1 \end{Bmatrix},$$

and that will imply that:

$$R_{44} = -\frac{d}{dr}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix} - \frac{2}{r}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix} + 2\begin{Bmatrix} 1\,4 \\ 1 \end{Bmatrix}^2 + 2\begin{Bmatrix} 1\,4 \\ 4 \end{Bmatrix}\begin{Bmatrix} 4\,4 \\ 1 \end{Bmatrix}$$

$$= -\frac{1}{2}\left( LO^2 L'' + O^2 L'^2 + 2LOL'O' + \frac{2LO^2 L'}{r} - M^2 O^4 L'^2 + LNO^4 L'^2 \right).$$

The three terms that contain $L'^2$ are collectively:

$$\frac{O^2L'^2}{2}\{1 + O^2(LN - M^2)\} = 0,$$

such that:

$$R_{44} = -\frac{L\,O^2L'}{2}\left(\frac{L''}{L'} + \frac{2O'}{O} + \frac{2}{r}\right),$$

and after integrating this for $R_{44} = 0$, it will result that:

$$r^2\,O^2\,L' = \text{const.} \tag{5}$$

In order to ascertain $R_{14}$, one still needs the values:

$$\begin{Bmatrix}11\\1\end{Bmatrix} = \frac{NO^2L'}{2} - \frac{O'}{O}, \qquad \begin{Bmatrix}11\\4\end{Bmatrix} = -O^2\left(NM' - \frac{M\,N'}{2}\right),$$

$$\begin{Bmatrix}14\\2\end{Bmatrix} = \frac{x_2}{r}\begin{Bmatrix}14\\1\end{Bmatrix}, \qquad \begin{Bmatrix}14\\3\end{Bmatrix} = \frac{x_3}{r}\begin{Bmatrix}14\\1\end{Bmatrix},$$

and one will then find that:

$$R_{14} = -\frac{d}{dr}\begin{Bmatrix}14\\1\end{Bmatrix} - \frac{2}{r}\begin{Bmatrix}14\\1\end{Bmatrix} + \begin{Bmatrix}11\\1\end{Bmatrix}\begin{Bmatrix}14\\1\end{Bmatrix} + \begin{Bmatrix}14\\1\end{Bmatrix}\begin{Bmatrix}14\\4\end{Bmatrix} + \begin{Bmatrix}11\\4\end{Bmatrix}\begin{Bmatrix}44\\1\end{Bmatrix} + \begin{Bmatrix}14\\4\end{Bmatrix}\begin{Bmatrix}44\\4\end{Bmatrix}$$

$$= -\frac{1}{2}\left[MO^2L'' + O^2L'M' + 2MOL'O' + \frac{2MO^2L'}{r} - MO^2L'\left(\frac{NO^2L'}{2} - \frac{O'}{O}\right) + \frac{LO^4L'}{2}(2NM' - MN')\right].$$

Since one has $R_{44} = 0$:

$$M\,O^2\,L'' + 2\,M\,O\,L'O' + \frac{2MO^2L'}{r} = 0,$$

and therefore:

$$R_{14} = \frac{MO^2L'}{4}\left(NO^2L' - \frac{2O'}{O} + LO^2N'\right) - \frac{O^2L'M'}{2}(1 + LNO^2)\,.$$

Since the last term can be written in the form:

$$-\frac{MO^2L'}{4}\cdot 2O^2\,M\,M',$$

equation (4) will imply that $R_{14}$ will vanish identically when $R_{44} = 0$.

In order to get $R_{11}$, in addition to the values that were given already, one still needs:

$$\begin{Bmatrix}1\,2\\2\end{Bmatrix} = \begin{Bmatrix}1\,3\\3\end{Bmatrix} = \frac{O'}{2O}, \qquad \begin{Bmatrix}1\,1\\2\end{Bmatrix} = \frac{x_2}{r}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix} - \frac{O'}{O}\right), \qquad \begin{Bmatrix}1\,1\\3\end{Bmatrix} = \frac{x_3}{r}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix} - \frac{O'}{O}\right).$$

That will yield:

$$R_{11} = -\frac{d}{dr}\begin{Bmatrix}1\,1\\1\end{Bmatrix} - \frac{2}{r}\begin{Bmatrix}1\,1\\1\end{Bmatrix} + \frac{2O'}{Or} + \begin{Bmatrix}1\,1\\1\end{Bmatrix}^2 + \begin{Bmatrix}1\,4\\4\end{Bmatrix}^2 + 2\begin{Bmatrix}1\,4\\1\end{Bmatrix}\begin{Bmatrix}1\,1\\4\end{Bmatrix} + 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}^2$$

$$= -\frac{d}{dr}\begin{Bmatrix}1\,1\\1\end{Bmatrix} - \frac{NO^2L'}{r} + \frac{4O'}{Or} + \left(\frac{O'}{O} - \frac{NO^2L'}{2}\right)^2 + \frac{N^2O^4L'^2}{4} - MO^4L'\left(NM' - \frac{MN'}{2}\right) + \frac{O'^2}{2O^2}.$$

Now, $MM'$ and $M^2$ can be eliminated from the last two terms by means of equation (4) (the determinant equation, resp.), with which they will assume the form:

$$-N\,O^2\,L'\left[\frac{O^2}{2}(LN' + NL') - \frac{O'}{O}\right] + \frac{O^2L'N'}{2}(1 + L\,N\,O^2),$$

and one will then get:

$$R_{11} = -\frac{d}{dr}\begin{Bmatrix}1\,1\\1\end{Bmatrix} - \frac{NO^2L'}{r} + \frac{O^2L'N'}{2} + \frac{4O'}{Or} + \frac{3O'^2}{2O^2}.$$

Since one further has:

$$-\frac{d}{dr}\begin{Bmatrix}1\,1\\1\end{Bmatrix} = -\frac{NO^2L'}{r} - NOL'O' - \frac{O^2L'N'}{2} + \frac{d}{dr}\frac{O'}{O},$$

with an application of the equation $B_{44} = 0$, one will then get:

$$R_{11} = \frac{d}{dr}\frac{O'}{O} + \frac{4O'}{Or} + \frac{3O'^2}{2O^2} = \frac{O'}{O}\left(\frac{O''}{O'} + \frac{O'}{2O} + \frac{4}{r}\right).$$

In order for $R_{11}$ to vanish, one must have either $O$ = const. or:

$$r^4O'\sqrt{-O} = \text{const.}$$

With consideration given to the equation (3) in question, integration will yield:

$$O = -\left(1 + \frac{\beta}{r^3}\right)^{2/3}, \tag{6}$$

and the former alternative is characterized in this equation by saying that the integration constant $\beta$ assumes the value zero.

In order to ascertain the value of $R_{22}$, one still needs:

$$\left\{\begin{matrix}2\,2\\1\end{matrix}\right\} = LO^2\left(\frac{O}{r}+\frac{O'}{2}\right)+\frac{1}{r}, \quad \left\{\begin{matrix}2\,2\\2\end{matrix}\right\} = \frac{x_2}{r}\left(\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}+\frac{O'}{O}\right), \quad \left\{\begin{matrix}2\,2\\3\end{matrix}\right\} = \frac{x_3}{r}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\},$$

and one has:

$$\begin{aligned} R_{22} &= -\frac{d}{dr}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\} - \frac{2}{r}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\} - \frac{O'}{Or} + 2\left\{\begin{matrix}1\,2\\2\end{matrix}\right\}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\} \\ &= -\frac{1}{r^2}(1+LO^3) - \frac{1}{r}(O^3L' + 3LO^2O') - \frac{1}{2}(LO^2O'' + LOO'^2 + O^2L'O'). \end{aligned}$$

With the abbreviation:

$$\frac{\beta}{r^3} = \varepsilon,$$

equation (6) will imply that:

$$O^3 = -(1+\varepsilon)^2, \qquad O^2O' = \frac{2\varepsilon}{r}(1+\varepsilon),$$

$$OO'^2 = -\frac{4\varepsilon^2}{r^2}, \qquad O^2O'' = -\frac{2\varepsilon}{r^2}(4+3\varepsilon),$$

from which, it will result that:

$$R_{22} = \frac{1}{r^2}(L + r\,L' - 1) + \frac{\varepsilon\,L'}{r}.$$

$L'$ is eliminated from this equation by means of equation (5), in which its integration constant will be denoted by $\alpha$. It will then result that:

$$L = 1 - \frac{\alpha}{r}(1+\varepsilon)^{-2/3},$$

and differentiation of this equation will, in turn, imply equation (5).

It will then follow that the equations:

$$O = -\sqrt[3]{\left(1+\frac{\beta}{r^3}\right)^2}, \qquad L = 1 - \frac{\alpha}{r\sqrt{-O}}, \qquad O^2(LN - M^2) + 1 = 0 \tag{7}$$

represent *the general solution of the one-body problem.* $\beta$ can be chosen freely, while $\alpha$ is determined by NEWTON's theory. $M$ or $N$ can then be chosen freely, as long as one observes the restriction that is established by equations (3). In the latter case, the freedom of choice is restricted by the fact that $M$ cannot take on an imaginary value. *The one-body problem then has a two-fold infinitude of exact solutions.*

In order to also introduce a cosmic term into the general solution, let the initially-non-zero $B_{\mu\nu}$ be combined with the above:

$$B_{44} = -\frac{L}{2}\left(O^2L'' + 2OL'O' + \frac{2O^2L'}{r}\right) = -\frac{L}{2r^2}\frac{d}{dr}(r^2 O^2 L')$$

$$B_{14} = \frac{M}{L}\cdot B_{44},$$

$$B_{11} = \frac{N}{L}\cdot B_{44} + \frac{O'}{O}\left(\frac{O''}{O'} + \frac{O'}{2O} + \frac{4}{r}\right) = \frac{N}{L}\cdot B_{44} + \frac{1}{r^4 O\sqrt{-O}}\frac{d}{dr}(r^4 O'\sqrt{-O})$$

$$B_{22} = B_{33} = \frac{1}{r^2}(L - 1 - rL'O\sqrt{-O}) = \frac{1}{r^2}\left(L - 1 + \frac{\alpha}{r\sqrt{-O}}\right).$$

One then gets:

$$B_{44} - \lambda\, g_{44} = -\frac{L}{2r^2}\frac{d}{dr}\left(r^2O^2L' + \frac{2\lambda r^3}{3}\right),$$

and as a result:

$$r^2 O^2 L' + \frac{2\lambda r^3}{3} = \alpha. \tag{8}$$

Furthermore:

$$B_{22} - \lambda\, g_{22} = \frac{1}{r^2}(L - 1 - r\,L'O\sqrt{-O} - \lambda r^2 O),$$

and the elimination of $L'$ from these equations will yield:

$$L = 1 - (1+\varepsilon)^{-1/3}\left[\frac{\alpha}{r} + \frac{\lambda r^2}{3}(1+3\varepsilon)\right] = 1 - \frac{\alpha}{r\sqrt{-O}} + \lambda r^2\left(O + \frac{2}{3\sqrt{-O}}\right). \tag{7a}$$

Differentiating this equation will once more yield equation (8). Since one further has:

$$B_{14} - \lambda\, g_{14} = \frac{M}{L}(B_{44} - \lambda\, g_{44}),$$

$$B_{11} - \lambda\, g_{11} = \frac{N}{L}\,(B_{44} - \lambda\, g_{44}) + \frac{1}{r^4 O\sqrt{-O}} \cdot \frac{d}{dr}\left(r^4 O' \sqrt{-O}\right),$$

that illuminates the fact that *when one adds a cosmic term*, one will only need to replace equation (7) with equation (7a).

In order to examine the precession of perihelion, one needs only the equations that were quoted in the introduction:

$$L\, u'^2 + 2M\, u' r' + N\, r'^2 + O\, r^2\, \varphi'^2 = F, \qquad \frac{d}{ds}\frac{\partial F}{\partial u'} = 0, \qquad \frac{d}{ds}\frac{\partial F}{\partial \varphi'} = 0$$

for $\vartheta \equiv \pi / 2$. The last two imply that:

$$O\, r^2 \varphi' = A, \qquad L\, u' + M\, r' = B,$$

in which $A$ and $B$ are integration constants. If the first equation were multiplied by $L$ then, after eliminating $u'$, one would get:

$$B^2 - F\, L + (L\, N - M^2)\, r'^2 + L\, O\, r^2\, \varphi'^2 = 0,$$

and an application of the determinant equation would yield:

$$\frac{r'^2}{O^2} = B^2 - F\, L + L\, O\, r^2\, \varphi'^2.$$

After introducing the notation:

$$\rho = \frac{1}{r},$$

one will obtain:

$$\frac{d\rho}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi} = -\frac{\rho^2 r'}{\varphi'},$$

and therefore:

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{O^2 r'^2}{A^2} = O^4\left(\frac{B^2 - F}{A^2} + \frac{\alpha\rho F}{A^2\sqrt{-O}} + \frac{\rho^2}{O} - \frac{\alpha\rho^3}{O\sqrt{-O}}\right)$$

$$= \frac{B^2 - F}{A^2}(1 + \varepsilon)^{8/3} + \frac{\alpha\rho F}{A^2}(1 + \varepsilon)^{7/3} - \rho^2\,(1 + \varepsilon)^2 + \alpha\,\rho^3\,(1 + \varepsilon)^{5/3}. \qquad (9)$$

In order to make this exact expression more manageable, it can be developed into a series of increasing powers of $\rho$:

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{B^2 - F}{A^2} + \frac{\alpha\rho F}{A^2} - \rho^2 + \rho^3\left[\alpha + \frac{8\beta(B^2 - F)}{3A^2}\right] + \dots \tag{10}$$

One sets $F$ = 1 for a mass-point and $F$ = 0 for the motion of light. In the latter case, $u'$, … mean the derivatives with respect to an arbitrary coordinate, which will also be applied after the variation. In EINSTEIN's theory, a planet is treated as a mass-point, and one tacitly assumes that the Sun is at rest in the reference system in which the planetary orbits represent ellipses in the first approximation.

The last term in equation (10), which is absent in NEWTON's theory, determines the precession of the perihelion when the equation contains no other terms, and due to the fact that $\beta$ can be chosen freely, the equation will give the impression that one can predict an arbitrary precession of the perihelion in EINSTEIN's theory.

When I proposed this equation to my dear friend OSEEN, that distinguished authority on the theory of relativity, he pointed out to me that with the transformation:

$$\rho' = \frac{\rho}{\sqrt{-O}} = \rho\,(1 + \beta\rho^3)^{-1/3},$$

from which, it will follow that:

$$\frac{d\rho'}{d\rho} = (1 + \beta\rho^3)^{-1/3} - \beta\rho^3\,(1 + \beta\rho^3)^{-4/3} = (1 + \beta\rho^3)^{-4/3} = \frac{1}{O^2},$$

one can give equation (9) the form:

$$\left(\frac{d\rho'}{d\varphi}\right)^2 = \frac{B^2 - 1}{A^2} + \frac{\alpha\rho'}{A^2} - \rho'^2 + \alpha\,\rho'^3$$

for $F$ = 1, with which, the precession of perihelion can be calculated in the usual way. In order to not be forced to repeat all of the calculations, I will refer to WEYL's presentation (*), in which the equation:

$$\varphi = \int \frac{d\theta}{\sqrt{\alpha\left(\rho'_0 - \frac{\rho'_1 + \rho'_2}{2} - \frac{\rho'_1 - \rho'_2}{2}\cos\theta\right)}}$$

is deduced. Now, since:

$$\rho'_1 = \rho_1(1 + \beta\,\rho_1^3)^{-1/3},$$

it follows that for $\rho = \rho'_1$, one will also have $\rho = \rho_1$, so the planet can also be found to be at perihelion in this calculation when $\theta = 2n\pi$. For that reason, WEYL obtained the angle between two successive perihelia as:

(*) HERMANN WEYL, *Raum, Zeit, Materie*, 3rd ed., Berlin 1920, pp. 221, *et seq.*

$$\varphi = 2\pi G^{-1/2},$$

$$G = 1 - \frac{3\alpha}{2}(\rho_1' + \rho_2') = 1 - \left[\rho_1 + \rho_2 - \frac{\beta}{3}(\rho_1^4 + \rho_2^4) + \cdots\right].$$

Since one further has:

$$\rho_1 + \rho_2 = \frac{2}{\alpha(1-e^2)}, \quad \rho_1^4 + \rho_2^4 = \frac{2(1+6e^2+e^4)}{a^4(1-e^2)^4} = \frac{6K}{\alpha(1-e^2)},$$

it will follow that:

$$G = 1 - \frac{3\alpha}{\alpha(1-e^2)}(1 - \beta K + \ldots)$$

and

$$\varphi = 2\pi\left[1 + \frac{3\alpha}{2a(1-e^2)}(1-\beta K) + \cdots\right].$$

Since $\beta$ can be chosen freely, this equation will imply that *if the precession of perihelion has been calculated in the same way as it always has been in* EINSTEIN's *theory then one can predict an arbitrary precession of perihelion with it.* Whether or not EINSTEIN's theory of gravitation agrees with the discovery of LEVERRIER will then remain unknown until the influence of the curvature of the universe on the astronomical observations in question is taken into account in the calculations of perturbation theory, and until it has therefore been shown that the precession of perihelion that is found in that way is independent of $\beta$. Namely, since the transformation affects only the radius vector, the true precession of perihelion must be independent of $\beta$ no matter how large it gets.

Since $\beta$ is contained in $L$, the expected redshift of spectral lines in the gravitational field of the Sun will also depend upon $\beta$.

By contrast, the bending of light in the gravitational field of the sun will not be affected by $\beta$. In order to see this, one needs only to apply the transformation that was given above to the equation that EDDINGTON ([1]) deduced:

$$x = R - \frac{\alpha}{2R}\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}},$$

in which $R$ represents the distance from the vertex of the light path to the center of the Sun, for infinitely-large values of $y$. One has:

$$x^2 + y^2 = (x_1^2 + x_2^2)\left(1 + \frac{\beta}{r^3}\right)^{2/3} = (x_1^2 + x_2^2)\left(1 + \frac{2\beta}{3r^3} + \cdots\right).$$

([1]) A. S. EDDINGTON, *Space, Time, Gravitation,* Cambridge 1921, pp. 207.

Since $r$ is also infinitely large for infinitely-large values of $y$, and $x_2$ and approaches the value of $x_3$ asymptotically, the directions of the asymptotes of the light path will remain unaffected by $\beta$.

If one sets $\beta = 0$ then $L$ will have the same value as before in EINSTEIN's theory, which is, as a result, also true for the redshift of spectral lines. In that way, the terms in equation (10) that were not written down will also vanish, such that not only the redshift of the spectral lines and the bending of light in the gravitational field of the Sun will take on the values that EINSTEIN gave for them, but also the precession of perihelion. Therefore, there are still infinitely-many exact solutions to the one-body problem, and among them, a two-fold infinitude that make space Euclidian, as well. They will be characterized by the equations:

$$N = O = -1, \quad M = \pm\sqrt{\frac{\alpha}{r}},$$

and if $d\sigma$ denotes the line element of space then one will get:

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) du^2 \pm \frac{2}{r}\sqrt{\frac{\alpha}{r}}\, du\, (x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3) - d\sigma^2 .$$

The calculations above teach us that the independence of the calculated precession of perihelion is based upon the choice of solution to the equation $g = -1$. *For* $\beta = 0$, *one can then describe space as Euclidian or non-Euclidian in infinitely-many ways, and the observed phenomena will be demanded exclusively by the non-Euclidian coupling of time with space.*

If one would like to go deeper into the question of whether EINSTEIN's theory agrees with the observed shift of the perihelion of Mercury then it would probably be advisable to initially describe space as Euclidian, because one would then have to deal with only the curvature of the light rays in the astronomical observations. However, since the precession of perihelion is treated as a residue that remains after the influence of perturbations has been calculated, one cannot overlook the demand that the perturbations should also be treated with EINSTEIN's theory, although the mathematical means to do that does not exist.