"Zur Statik und Kinematik des Cosseratschen Kontinuums," Abh. d. Braunschw. Wiss. Ges. 10 (1958), 195-213.

On the Statics and Kinematics of Cosserat Continua

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With 4 figures

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Summary: The "Cosserat continuum" is a continuum of points, each of which are provided with a space direction. In this paper, the kinematics and statics of this continuum are investigated, and the author wishes to emphasize the analogies between the equations for kinematical and static quantities. Furthermore, it is shown that the continuum used in dislocation theory is an incompatible Cosserat continuum.

1. Introduction

Straight beams with length, shear, and bend deformations provide the simplest model of a *Cosserat* continuum: In the typical bending problem, a succession of rigid slices – the "cross-sections" are cut out, which depend on each other in a suitably elastic way (Fig. 1a). We allow the following variations of length:

1. The displacements $u(x)$ and $w(x)$ of the cross-sections in the x- and y-directions, without rotation of the slices, in which the beam axis is stretched by:

$$\varepsilon(x) = u'(x), \tag{1.1}$$

into the "bending line," which is given by $z = w(x)$, and has been rotated relative to the normal to the cross-section through the angle $w'(x)$ (Fig. 1b).

2. A rotation $\varphi(x)$ of the cross-section that is independent of the displacements (Fig. 1c), such that the end result is that the cross-section has been rotated through the angle:

$$y(x) = w'(x) + \varphi(x), \tag{1.2}$$

around the bending line normal. If we then introduce the quantity:

$$\kappa(x) = \varphi'(x), \tag{1.3}$$

then we have the following system of deformations: the dilatation $\varepsilon(x)$, the shear $\psi(x)$, and the rotation $\kappa(x)$. Conversely, if these deformations are given then $u(x)$, $w(x)$, and $\varphi(x)$ are obtained by integration, up to a rigid motion:

$$u(x) = u_0(x) + \int_{x_0}^{x} \varepsilon(x)\, d\xi,$$
$$\varphi(x) = \varphi_0(x) + \int_{x_0}^{x} \kappa(x)\, d\xi, \tag{1.4}$$
$$w(x) = w_0(x) - \varphi_0(x - x_0) + \int_{x_0}^{x} \kappa(x) - (x - \xi)\,\kappa(\xi)]\, d\xi,$$

with $u_0 = u(x_0)$, etc.

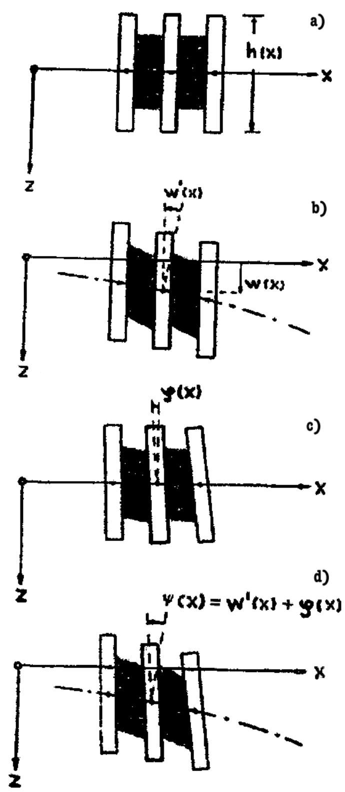

Fig. 1.

Which static quantity is associated with the three deformations is dictated by the principal of virtual displacement: The beam carries the external length, cross-sectional, and moment loads $l(x)$, $q(x)$, and $m(x)$, and is stressed, perhaps the free right end $x = a$, by the forces $L^{(a)}$, $Q^{(a)}$, and the moment $M^{(a)}$; the left end is unstressed. We assume rigid-body equilibrium and thus obtain:

$$-\int_0^a [l(x)\ \delta u(x) + q(x)\ \delta w(x) + m(x)\ \delta\varphi(x)]\ dx -$$
$$- [L^{(a)}\delta u(a) + Q^{(a)}\delta w(a) + M^{(a)}\delta\varphi(a)] = 0. \tag{1.5}$$

The rigidity conditions read:

$$\delta\varepsilon(x) = 0, \qquad \delta\psi(x) = 0, \qquad \delta\kappa(x) = 0, \tag{1.6}$$

with:

$$\delta\varepsilon(x) = \frac{d}{dx}[\delta u(x)], \qquad \text{etc.} \tag{1.7}$$

We successively multiply each of them by the "*Lagrange multipliers,*" $L(x)$, $Q(x)$, $M(x)$, and add this to the integral in (1.5):

$$-\int_0^a \{[L(x)\ \delta u(x) + Q(x)\ \delta w(x) + M(x)\ \delta\varphi(x)] -$$
$$- [l(x)\ \delta u(x) + q(x)\ \delta w(x) + m(x)\ \delta\varphi(x)]\}\ dx - \tag{1.8}$$
$$- [L^{(a)}\delta u(a) + Q^{(a)}\delta w(a) + M^{(a)}\delta\varphi(a)] = 0.$$

Now, the beam can be regarded as non-rigid, and then the *Lagrange* multipliers, which were reaction forces in the rigid body case, become imprinted force quantities. With (1.7), a partial integration and consideration of the stress-free conditions:

$$\delta u(x) = 0, \qquad \delta w(x) = 0, \qquad \delta\varphi(x) = 0, \tag{1.9}$$

gives:

$$-\int_0^a \{[L'(x) + l(x)]\ \delta u(x) + [Q'(x) + q(x)]\ \delta w(x) +$$
$$+ [M'(x) - Q(x) + m(x)]\ \delta\varphi(x)\}\ dx +$$
$$+ [L(a) - L^{(a)}]\delta u(a) + [Q(a) - Q^{(a)}]\delta w(a)$$
$$+ [M(a) - M^{(a)}]\delta\varphi(a) = 0.$$

This gives:

$$L'(x) + l(x) = 0,$$
$$Q'(x) + q(x) = 0, \tag{1.11}$$
$$M'(x) - Q(x) + m(x) = 0,$$

along the beam axis, and:

$$L(a) - L^{(a)} = 0,$$
$$Q(a) - Q^{(a)} = 0, \tag{1.12}$$
$$M(a) - M^{(a)} = 0,$$

at the free end. (1.12) gives the static interpretation for the *Lagrange* multipliers:

$L(a)$ = stretching force
$Q(a)$ = shear force,
$M(a)$ = bending moment;

(1.11) defines the equilibrium conditions for these static quantities. For the sake of what follows, it is now convenient to abstract from the particular nature of the model that we just discussed. For that, we replace each cross-section, which we collectively think of as constantly arrayed along the beam axis, with a local rigid coordinate system. In this way, the beam axis, which – more generally than before – also can be a space curve and therefore can represent spatially curved beams, becomes the carrier of a one-parameter family of coordinate systems ("trièdres mobiles," to *E.* and *F. Cosserat*), or, otherwise speaking, a continuous sequence of "oriented points." Let the orientation of the coordinate system (points, resp.) in the initial state be determined by constant functions of the curve parameters. The continuum of coordinate systems (oriented points, resp.) is deformed by displacing the initial points and rotating the axes, in which these alterations are also constant functions of the curve parameters. The "deformations," suitably defined, will then be associated with static quantities with the help of the principal of virtual displacement, and the behavior of the deformations on a free interval boundary provides their meaning as static quantities. The differential equations that they satisfy are the necessary conditions that they are in equilibrium on an element. Therefore, for this "one-dimensional (or better yet: one-parameter) *Cosserat* continuum" the static quantities are already determined from the kinematics by means of the principal of virtual displacement.

The extension of this way of thinking to two and three-dimensional regions is simple and leads to the notion of the "*Cosserat* surface," which consists of ∞^2 points, and the "*Cosserat* space," which consists of ∞^3 points. Compared to the possible motions of a rigid body, which generally has six functional degrees of freedom, a continuum of unoriented points, which is a special case of the *Cosserat* continuum, has at most three degrees of freedom, which are all given by the displacement field.

In their seminal monograph [1], *E.* and *F. Cosserat* have systematically treated the mechanics of continuous systems that consist of oriented points, generally in a representation that is very hard to read nowadays. The work of the *Cosserats* cannot, however, be regarded as isolated: Just as in the quest to develop a mechanical model of the ether, similarly, in the discussion of the constitution of anisotropic elastic bodies one is almost unavoidably compelled to consider such mechanics. This was precisely the theme that was of greatest interest to many researchers of the nineteenth century. Here, we mention only the work of *Kelvin* [2], *Poisson* [3], and, particularly that of *Voigt* [4], whose ideas came very close to those of *E.* and *F. Cosserat.* After that, these questions were either regarded as meaningless or as better solved by other methods, and it seemed as if the Cosserat program had only a historical significance and seemed almost forgotten, if one overlooks a few French researchers, such as *Sudria* [5] ([1]).

The objective of this article is to give a modern representation of the *Cosserat* ideas, while restricting to linear deformations, and to develop a complete system of kinematical and static equations in this framework. The motivation for this work was the fact that a kinematical model was used in the "continuum theory of dislocations" that corresponded to an incompatible *Cosserat* continuum ([2]). As the introductory example above shows, there is also a problem in classical rigidity in which the *Cosserat* approach might be useful. The author will elsewhere present a theory of shells in which one will be led in this manner to very self-evident equations, and therefore to an insight into connections, which have been very complicated to understand by the usual arguments up till now. Finally, there are interesting links between the problems of differential geometry, especially non-*Riemannian*, as well as nonholonomic, spaces and dislocation theory that were first observed by *Kondo* [8], and *Bilby*, *Bullough* and *Smith* [9].

2. Kinematics of COSSERAT continua

We use generalized coordinates $q^{(i)}$ (i = 1, 2, 3); the orientation of a given point, which is given by the position vector:

$$\mathfrak{r} = \mathfrak{r}(q^{(i)}) \tag{2.1}$$

in the initial state, is generally chosen in such a way that the local coordinate system is given by the unit vectors:

$$\mathfrak{g}_i = \partial_i \mathfrak{r} = \frac{\partial \mathfrak{r}}{\partial q^i} \,. \tag{2.2}$$

We agree that $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, in that order, shall define an orthogonal system. In the sense of the chosen determination of metric this coordinate system is "parallel," since the covariant derivatives of the unit vectors indeed vanishes. Under a dilatation the origin of the local coordinate system is displaced by:

$$\mathfrak{w} = \mathfrak{w}(q^{(i)}), \tag{2.3}$$

and the system itself is rotated about its origin by:

([1]) Note added in proof: In a remarkable study, *Ericksen* and *Truesdell* have developed a theory of finitely deformed beams and shells on the basis of the *Cosserat* ideas: *J.L. Ericksen & C. Truesdell:* Exact Theory of Stress and Strain in Rods and Shells; Arch. Rational Mechanics and Analysis, Vol. 1 (1958) 4.

([2]) For this, one might confer *Kröner* [6] and *Seeger* [7].

$$\vec{\varphi} = \vec{\varphi}(q^{(i)}). \tag{2.4}$$

(Since the rotation is assumed to be small, it is permissible to represent it by a vector $\vec{\varphi}$.) We describe the deformation state by the deformation vectors:

$$\vec{\varepsilon}_i = \partial_i \mathfrak{w} + \mathfrak{g}_i \times \vec{\varphi}, \tag{2.5}$$

$$\vec{\kappa}_i = \partial_i \vec{\varphi}, \tag{2.6}$$

whose meaning can be clarified by the introductory example: There, one has:

$$q^{(1)} = x, \qquad q^{(2)} = y, \qquad q^{(3)} = z,$$

$$\mathfrak{g}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathfrak{g}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathfrak{g}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathfrak{w} = \begin{bmatrix} u(x) \\ 0 \\ w(x) \end{bmatrix}, \quad \vec{\varphi} = \begin{bmatrix} 0 \\ \varphi(x) \\ 0 \end{bmatrix},$$

such that one then has:

$$\vec{\varepsilon}_1 = \begin{bmatrix} u'(x) \\ 0 \\ w'(x) + \varphi(x) \end{bmatrix}, \quad \vec{\varepsilon}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{\varepsilon}_3 = \begin{bmatrix} -\varphi(x) \\ 0 \\ 0 \end{bmatrix},$$

$$\vec{\kappa}_1 = \begin{bmatrix} 0 \\ \varphi'(x) \\ 0 \end{bmatrix}, \quad \vec{\kappa}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{\kappa}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can summarize these vectors by the matrices:

$$\mathfrak{e} = \begin{bmatrix} \varepsilon_{11} = u'(x) & \varepsilon_{21} = 0 & \varepsilon_{31} = -\varphi(x) \\ \varepsilon_{12} = 0 & \varepsilon_{22} = 0 & \varepsilon_{23} = 0 \\ \varepsilon_{13} = w'(x) + \varphi(x) & \varepsilon_{23} = 0 & \varepsilon_{33} = 0 \end{bmatrix}$$

and:

$$\mathfrak{k} = \begin{bmatrix} \chi_{11} = 0 & \chi_{21} = 0 & \chi_{31} = 0 \\ \chi_{12} = \varphi'(x) & \chi_{22} = 0 & \chi_{23} = 0 \\ \chi_{13} = 0 & \chi_{23} = 0 & \chi_{33} = 0 \end{bmatrix}.$$

As usual, $\varepsilon_{11} = u'(x)$ is the stretching in the x-direction. Under the action of the displacement field $w(x)$, the system of orthogonal coordinates $x = x_1$, $y = x_2$, $z = x_3$, goes to a non-orthogonal system x_1^*, x_2^*, x_3^*, whose x_1^*–axis is tangent to the deformed x_1-curve (Fig. 2), by means of the rotation $\varphi(x)$ in an orthogonal system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. It is:

$$\angle\,(x_1^*, \tilde{x}_3) = \frac{\pi}{2} - (w' + \varphi) = \frac{\pi}{2} - \varepsilon_{13}$$
$$\angle\,(x_3^*, \tilde{x}_1) = \frac{\pi}{2} - (w' - \varphi) = \frac{\pi}{2} - \varepsilon_{31}$$

such that the ε_{ik} are obviously the natural generalizations of the distortion quantities of the point continuum. The meaning of the deformation quantities κ_{ik} is immediate: they describe relative rotations of the local coordinate system within the system.

The transition to the spacelike continuum proceeds in such a way that one now carries out the previous operations on the individual coordinate surfaces. A new, not generally holonomic, coordinate system arises in a spacelike continuum as a result of the displacement field W and the rotation field $\vec{\varphi}$; the deformations can then be interpreted as they were in the model case.

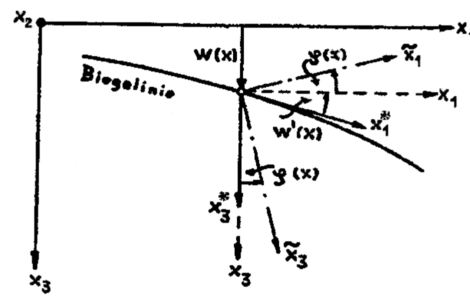

Fig. 2

The 18 deformations $\vec{\varepsilon}_i$ and $\vec{\kappa}_i$ are derived from the 6 dilatations $\mathfrak{w}$ and $\vec{\varphi}$, according to (2.5) and (2.6). Conversely, if one integrates these equations for given deformation then one obtains:

$$\vec{\varphi}(\mathfrak{v}) = \vec{\varphi}(\mathfrak{v}_0) + \int_{\mathrm{v}_0}^{\mathrm{v}} \vec{\kappa}_\alpha \, ds^\alpha, \tag{2.7}$$

$$\mathfrak{w}(\mathfrak{v}) = \mathfrak{w}(\mathfrak{v}_0) + \vec{\varphi}(\mathfrak{v}_0) \times (\mathfrak{v} - \mathfrak{v}_0) + \int_{\mathrm{v}_0}^{\mathrm{v}} [\vec{\varepsilon}_\alpha(\mathrm{s}) + (\mathfrak{s} - \mathfrak{v}) \times \vec{\kappa}_\alpha(\mathrm{s})] \, ds^\alpha; \tag{2.8}$$

in which $\mathfrak{s}$ is the position vector of the integration path, and the summation convention for tensor calculus has been used ([3]). The terms that were integrated out represent rigid motions, which the deformation state certainly cannot alter. In order for the dilatation to be a unique function of position the integrals in (2.7) and (2.8) must be total differentials, and from the conditions:

$$\partial_{[i}\vec{\kappa}_{l]} = \tfrac{1}{2}(\partial_i \vec{\kappa}_l - \partial_l \vec{\kappa}_i) = 0, \tag{2.9}$$

$$\partial_{[i}\vec{\varepsilon}_{l]} + \vec{\mathrm{g}}_{[i} \times K_{l]} = 0, \tag{2.10}$$

one is led to equations that can also be obtained immediately from (2.5) and (2.6) through the elimination of $\mathfrak{w}$ and $\vec{\varphi}$. These are the sufficient conditions for the dilatations to be uniquely determined by the deformations, up to rigid motions. We would like to

([3]) Greek indices are summed from 1 to 3.

put them into another form that will simplify our discussion. For that purpose, we introduce the "permutation tensor" e^{ikl}:

$$e^{ikl} = \mathfrak{g}^i \cdot (\mathfrak{g}^k \times \mathfrak{g}^l) = \begin{cases} +\dfrac{1}{\sqrt{g}} & \text{when } (i,k,l) \text{ is an even permutation of } (1,2,3) \\ -\dfrac{1}{\sqrt{g}} & \text{when } (i,k,l) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{when at least two indices are equal;} \end{cases} \tag{2.11}$$

in which the "contravariant unit vectors" $\mathfrak{g}^k$ define an orthonormal system and are defined by:

$$\mathfrak{g}^k \cdot \mathfrak{g}_i = \delta_i^k = \begin{cases} 0 & \text{for } k \neq i \\ 1 & \text{for } k = i, \end{cases} \tag{2.12}$$

and g is the determinant of the metric tensor:

$$g = \mathrm{Det}(g_{ik}) = \mathrm{Det}(\mathfrak{g}_i \cdot \mathfrak{g}_k). \tag{2.13}$$

We thus obtain the following "matching conditions" (i.e., "compatibility conditions"):

$$\overset{(1)}{\mathrm{J}}{}^k = e^{k\lambda\mu}\, \partial_\lambda \vec{\kappa}_\mu = 0, \tag{2.14}$$

$$\overset{(2)}{\mathrm{J}}{}^k = e^{k\lambda\mu}\, [\partial_\lambda \vec{\varepsilon}_\mu + \mathfrak{g}_\lambda \times \vec{\kappa}_\mu] = 0; \tag{2.15}$$

the "incompatibilities" $\overset{(1)}{\mathrm{J}}{}^k$ and $\overset{(2)}{\mathrm{J}}{}^k$ must vanish. It is obvious that these 18 equations are not independent of each other since their solutions – the 18 deformations – must involve 6 arbitrary functions (the dilatations w and the $\vec{\varphi}$). In fact, there exist 6 differential identities between them, namely the "divergence equations:"

$$\partial_\alpha \left(\sqrt{g}\, \overset{(1)}{\mathrm{J}}{}^\alpha \right) = 0, \tag{2.16}$$

$$\partial_\alpha \left(\sqrt{g}\, \overset{(2)}{\mathrm{J}}{}^\alpha \right) + \mathrm{g}_\alpha \times \left(\sqrt{g}\, \overset{(1)}{\mathrm{J}}{}^\alpha \right) = 0, \tag{2.17}$$

such that only 18 – 6 = 12 independent matching conditions remain, as one must have.

Both systems (2.14) and (2.15) may be combined. We then solve (2.15) for the $\vec{\kappa}_\mu$ (in which, for the sake of generality, we would also like to consider incompatible deformations); after a long intermediate computation, we obtain:

$$\vec{\kappa}_\mu = e^{\alpha\beta\sigma}\Big[(\partial_\alpha\vec{\varepsilon}_\beta\cdot\mathfrak{g}_\mu)\,\mathfrak{g}_\sigma - \tfrac{1}{2}\Big[(\partial_\alpha\vec{\varepsilon}_\beta\cdot\mathfrak{g}_\sigma)\,\mathfrak{g}_\mu\Big] =$$
$$-\Big[\Big(\overset{(2)}{\mathrm{J}}{}^{\sigma}\cdot\mathfrak{g}_\mu\Big)\,\mathfrak{g}_\sigma - \tfrac{1}{2}\Big[(\overset{(2)}{\mathrm{J}}{}^{\sigma}\cdot\mathfrak{g}_\sigma)\,\mathfrak{g}_\mu\Big].$$

One substitutes this into (2.14), and thus obtains:

$$e^{k\lambda\mu}e^{\alpha\beta\sigma}\partial_\lambda\Big[(\partial_\alpha\vec{\varepsilon}_\beta\cdot\mathfrak{g}_\mu)\,\mathfrak{g}_\sigma - \tfrac{1}{2}\Big[(\partial_\alpha\vec{\varepsilon}_\beta\cdot\mathfrak{g}_\sigma)\,\mathfrak{g}_\mu\Big] = \tag{2.19}$$
$$= \overset{(1)}{\mathrm{J}}{}^{k} + e^{k\lambda\mu}\,\partial_\lambda\Big[\Big(\overset{(2)}{\mathrm{J}}{}^{\sigma}\cdot\mathfrak{g}_\mu\Big)\,\mathfrak{g}_\sigma - \tfrac{1}{2}\Big[(\overset{(2)}{\mathrm{J}}{}^{\sigma}\cdot\mathfrak{g}_\sigma)\,\mathfrak{g}_\mu\Big].$$

Before we analyze this system, it is convenient to convert to pure tensor notation, since the computations are already extensive, and then use that opportunity to also represent the most important of the previous kinematical equations in pure tensor form. We set:

$$\begin{array}{lll}
\mathfrak{w} = w_\alpha\,\mathfrak{g}^\alpha & \text{with} & w_\alpha = \mathfrak{w}\bullet\mathfrak{g}_\alpha, \\
\vec{\varphi} = \varphi^\alpha\,\mathfrak{g}_\alpha & \text{with} & \varphi^\alpha = \vec{\varphi}\bullet\mathfrak{g}_\alpha, \\
\vec{\varepsilon}_i = \varepsilon_{i\alpha}\,\mathfrak{g}^\alpha & \text{with} & \varepsilon_{\square\alpha} = \vec{\varepsilon}_i\bullet\mathfrak{g}_\alpha, \\
\vec{\kappa}_i = \kappa_i^{\alpha}\,\mathfrak{g}_\alpha & \text{with} & \kappa_{i.}^{\ \alpha} = \vec{\kappa}_i\bullet\mathfrak{g}^\alpha, \\
\overset{(1)}{\mathrm{J}}{}^{k} = \overset{(1)}{I}{}^{k\alpha}\,\mathfrak{g}_\alpha & \text{with} & \overset{(1)}{I}{}^{k\alpha} = \overset{(1)}{\mathrm{J}}{}^{k}\bullet\mathfrak{g}^\alpha, \\
\overset{(2)}{\mathrm{J}}{}^{k} = \overset{(2)}{I}{}^{k\cdot}_{\ \cdot\alpha}\,\mathfrak{g}_\alpha & \text{with} & \overset{(2)}{I}{}^{k\cdot}_{\ \cdot\alpha} = \overset{(2)}{\mathrm{J}}{}^{k}\bullet\mathfrak{g}_\alpha,
\end{array} \tag{2.20}$$

(one can show in any case that the tensor quantities thus defined are of rank 1 or 2, resp.), and we further replace the ordinary derivatives with the covariant derivatives.

One then has, since $e_{il\alpha} = \mathfrak{g}_i\cdot(\mathfrak{g}_l\times\mathfrak{g}_\alpha)$, that the equations:

$$\varepsilon_{il} = \nabla_i w_l - e_{il\alpha}\,\varphi^\alpha \tag{2.5a}$$
$$\kappa_{i.}^{\ l} = \nabla_i\,\varphi^l \tag{2.5b}$$

become the definition of the deformation quantities; the matching conditions then look like:

$$\nabla_{[i}\,\kappa_{l\,].}^{\ \ m} = 0, \tag{2.9a}$$
$$\nabla_{[i}\,\varepsilon_{l\,]m} + e_{\alpha m\,[i}\,\kappa_{l\,].}^{\ \ \alpha} = 0, \tag{2.10a}$$

or:

$$\overset{(1)}{I}{}^{kl} = e^{k\lambda\mu}\nabla_l\,\kappa_{\mu.}^{\ l} = 0, \tag{2.14a}$$
$$\overset{(2)}{I}{}^{k\cdot}_{\ \cdot\alpha} = e^{k\lambda\mu}\nabla_l\,\varepsilon_{\mu l} + \delta_l^k\,\kappa_{a.}^{\ a} - \kappa_{l.}^{\ k} = 0, \tag{2.15a}$$

resp., with the following differential identities:

$$\nabla_\alpha\,\overset{(1)}{I}{}^{\alpha l} = 0, \tag{2.16a}$$
$$\nabla_\alpha\,\overset{(2)}{I}{}^{\alpha\cdot}_{\ \cdot l} + e_{\alpha\beta\,l}\,\overset{(1)}{I}{}^{\alpha l} = 0, \tag{2.17a}$$

between them. Solving (2.15a) for the deformations $\kappa_{m.}{}^{l}$ gives:

$$\kappa_{m.}{}^{l} = e^{\alpha\beta\sigma}\left[\,\delta_{\sigma}^{l}\nabla_{\alpha}\varepsilon_{\beta\mu} - \tfrac{1}{2}\,\delta_{\mu}^{l}\nabla_{\alpha}\varepsilon_{\beta\sigma}\right] - \left[\,\overset{(2)}{I}{}^{l\cdot}_{\cdot\mu} - \tfrac{1}{2}\,\delta_{\mu}^{l}\,\overset{(2)}{I}{}^{\alpha\cdot}_{\cdot\alpha}\right], \tag{2.18a}$$

and then, after substituting in (2.15a), one has:

$$\left[e^{k\alpha\lambda}\,e^{l\beta\mu} + \tfrac{1}{2}e^{kl\alpha}\,e^{\beta\mu\lambda}\right]\nabla_{\alpha}\nabla_{\beta}\,\varepsilon_{\mu\lambda} =$$
$$= \left[\,\overset{(1)}{I}{}^{kl} + e^{k\alpha\lambda}\,\nabla_{\alpha}\overset{(2)}{I}{}^{l\cdot}_{\cdot\lambda} + \tfrac{1}{2}e^{kl\alpha}\,\nabla_{\alpha}\overset{(2)}{I}{}^{\lambda\cdot}_{\cdot\lambda}\right] = \overset{*}{I}{}^{kl}. \tag{2.19a}$$

For the resulting incompatibility tensor $\overset{*}{I}{}^{kl}$, we now have: on the basis of (2.16a), it is divergence-free, i.e., one has:

$$\nabla_{\alpha}\overset{*}{I}{}^{kl} = 0, \tag{2.21}$$

and on the basis of (2.17a), it is symmetric, i.e.:

$$\overset{*}{I}{}^{kl} = \overset{*}{I}{}^{lk}. \tag{2.22}$$

The same must also be true for the left-hand side of (2.19a), and it can be easily established that the divergence-free character follows by immediate computation, whereas the symmetry follows from the following argument: We separate the deformation tensor $\varepsilon_{\lambda\mu}$ into its symmetric part:

$$\overset{S}{\varepsilon}_{\mu\lambda} = \varepsilon_{(\lambda\mu)} = \tfrac{1}{2}(\varepsilon_{\mu\lambda} + \varepsilon_{\lambda\mu}) \tag{2.23}$$

and its anti-symmetric part:

$$\overset{A}{\varepsilon}_{\mu\lambda} = \varepsilon_{[\lambda\mu]} = \tfrac{1}{2}(\varepsilon_{\mu\lambda} - \varepsilon_{\lambda\mu}). \tag{2.24}$$

In three dimensions, one can always replace an anti-symmetric tensor with a vector by the following prescription:

$$\overset{A}{\varepsilon}_{\mu\lambda} = e_{\mu\lambda\sigma}\,\varepsilon^{\sigma}, \qquad \varepsilon^{\sigma} = \tfrac{1}{2}e^{\mu\lambda\sigma}\,\overset{A}{\varepsilon}_{\mu\lambda}. \tag{2.25}$$

One then has:

$$\varepsilon_{\mu\lambda} = \overset{S}{\varepsilon}_{\mu\lambda} + e_{\mu\lambda\sigma}\,\varepsilon^{\sigma}. \tag{2.26}$$

When this is substituted into (2.19a), this gives:

$$e^{k\alpha\lambda}\,e^{l\beta\mu}\,\nabla_{\alpha}\nabla_{\beta}\,\overset{S}{\varepsilon}_{\mu\lambda} = \overset{*}{I}{}^{kl}, \tag{2.27}$$

if one ignores the terms that include ε^{σ}. The left-hand side is symmetric in the lower index pairs (α, β) and (λ, μ), and therefore also in the corresponding upper pairs, which then implies symmetry in the index pair (k, l).

In conclusion, we would like to consider the kinematical parts in the special case of an ordinary continuum of points. It is noteworthy that in such continua the rotation $\vec{\varphi}$ is already determined by the displacement field $\mathfrak{w}$ ("trièdre cachèe in [1]), namely, through the "mean rotation:"

$$\vec{\varphi} = \tfrac{1}{2} \operatorname{rot} \mathfrak{w} \tag{2.28}$$

or:

$$\varphi^i = \tfrac{1}{2} e^{i\alpha\beta} \nabla_\alpha w_\beta \,. \tag{2.29}$$

From (2.5a), the deformation tensor ε is therefore symmetric:

$$\varepsilon_{il} = \tfrac{1}{2}(\nabla_i w_l + \nabla_i w_l) = \varepsilon_{il}^S ; \tag{2.30}$$

conversely, by (2.5a), it also follows from $\varepsilon_{il} = \varepsilon_{il}^S$ that $\vec{\varphi}$ is the mean rotation (2.28) of the displacement field. The symmetry of the deformation tensor ε is therefore characteristic of an ordinary continuum of points, and its compatibility is determined by the equations:

$$\overset{*}{I}{}^{kl} = e^{k\alpha\lambda} e^{l\beta\mu} \nabla_\alpha \nabla_\beta \varepsilon_{\mu\lambda}^S = 0. \tag{2.31}$$

If they are satisfied then, from (2.8), and taking into account the relation:

$$\vec{\kappa}_\alpha = \operatorname{rot} \vec{\varepsilon}_\alpha^{\,S} , \tag{2.32}$$

which is easily derived (2.18a) in this case, one can compute the displacement w from the deformations $\vec{\varepsilon}_i$ in the following way:

$$\mathfrak{w}(\mathfrak{r}) = \mathfrak{w}(\mathfrak{r}_0) + \left[\tfrac{1}{2}(\operatorname{rot} \mathfrak{w})_{\mathfrak{r}=\mathfrak{r}0}\right] \times (\mathfrak{r} - \mathfrak{r}_0) + \\ + \int_{s_0}^{\mathfrak{s}} [\vec{\varepsilon}_\alpha^{\,S}(s) + (s - \mathfrak{s}) \times \operatorname{rot} \vec{\varepsilon}_\alpha^{\,S}(s)]\, ds^\alpha. \tag{2.33}$$

3. Statics of COSSERAT continua

In order to find those static quantities that are associated with the deformations in a spacelike continuum of oriented points, we go back to the principal of virtual displacement. Let a volume element dV be loaded with an external force $\mathfrak{k}\, dV$ and an external moment $\mathfrak{m}\, dV$, a bounding surface element df with the external force $\mathfrak{q}\, df$ and the external moment $\mathfrak{p}\, df$. The assumption that the forces are independent of the moments is characteristic of the statics of *Cosserat* continua; *Kröner* [6] and *Rieder* [10] have given a physical interpretation for this in the context of stresses in ferromagnetic crystals.

This system of external forces and moments is in equilibrium on a rigid body that is bounded by F when one has:

$$-\iiint\limits_{(V)} [\mathfrak{k} \bullet \delta\mathfrak{w} + \mathfrak{m} \bullet \delta\vec{\varphi}]\, dV - \iint\limits_{(F)} [\mathfrak{q} \bullet \delta\mathfrak{w} + \mathfrak{p} \bullet \delta\vec{\varphi}]\, df = 0. \tag{3.1}$$

We introduce the rigidity conditions:

$$\delta\vec{\varepsilon}_\alpha = \partial_\alpha(\delta\mathfrak{w}) + \mathfrak{g}_\alpha \times \delta\vec{\varphi} = 0, \tag{3.2}$$

$$\delta\vec{\kappa}_\alpha = \partial_\alpha(\delta\vec{\varphi}) = 0, \tag{3.3}$$

in the integral (3.1) by means of *Lagrange* multipliers $\mathfrak{S}^\alpha dV$ and $\mathfrak{T}^\alpha dV$ and obtain:

$$\iiint\limits_{(V)} [\mathfrak{S}^\alpha \bullet \delta\vec{\varepsilon}_\alpha + \mathfrak{T}^\alpha \bullet \delta\vec{\kappa}_\alpha - \mathfrak{k} \bullet \delta\mathfrak{w} - \mathfrak{m} \bullet \delta\vec{\varphi}]\, dV -$$
$$- \iint\limits_{(F)} [\mathfrak{q} \bullet \delta\mathfrak{w} + \mathfrak{p} \bullet \delta\vec{\varphi}]\, df = 0, \tag{3.4}$$

in which it is now permissible for the displacement fields $\delta\mathfrak{w}$ and $\delta\vec{\varphi}$, which are compatible with the geometrical requirements, to be non-rigid. If one introduces the virtual dilatations into this by way of (3.2) and (3.3) then *Gauss's* integral theorem, with $dV = \sqrt{g}\, dq^1\, dq^2\, dq^3$, gives:

$$-\iiint\limits_{(V)} \left\{ \left[\frac{1}{\sqrt{g}} \partial_\alpha(\sqrt{g}\,\mathfrak{S}^\alpha) + \mathfrak{q} \right] \cdot \delta\mathfrak{w} + \left[\frac{1}{\sqrt{g}} \partial_\alpha(\sqrt{g}\,\mathfrak{T}^\alpha) + \mathfrak{g}_\alpha \times \mathfrak{T}^\alpha + \mathfrak{p} \right] \cdot \delta\vec{\varphi} \right\} dV +$$
$$+ \iint\limits_{(F)} \left\{ [\mathfrak{S}^\alpha n_\alpha - \mathfrak{q}] \bullet \delta\mathfrak{w} + [\mathfrak{T}^\alpha n_\alpha - \mathfrak{p}] \bullet \delta\vec{\varphi} \right\} df = 0. \tag{3.5}$$

In this, the quantities:

$$n_\alpha = \mathfrak{n} \bullet \mathfrak{g}_\alpha \tag{3.6}$$

are the covariant components of the external normal vector for the bounding surface. From the vanishing of the surface integrals, it follows that:

$$\mathfrak{S}^\alpha n_\alpha = \mathfrak{q}, \qquad \mathsf{T}^\alpha n_\alpha = \mathfrak{p}, \tag{3.7}$$

and this gives the mechanical interpretation for the *Lagrange* multipliers $\mathfrak{S}^\iota$ and $\mathfrak{T}^\iota$: for example, let the bounding surface element be a piece of a coordinate surface $\mathfrak{q}^{(1)}$ = const.; the associated normal vector then becomes:

$$\mathfrak{n}^{(1)} = \frac{\mathrm{g}^1}{\sqrt{\mathrm{g}^1 \cdot \mathrm{g}^1}} = \frac{\mathrm{g}^1}{\sqrt{\mathrm{g}^{11}}}, \tag{3.8}$$

and thus:

$$n_1^{(1)} = \frac{1}{\sqrt{\mathrm{g}^{11}}}, \qquad n_1^{(2)} = n_1^{(3)} = 0. \tag{3.9}$$

The partial force $\mathfrak{q}^{(1)}$ of the surface traction is coupled with the "stress vector" $\mathfrak{S}^1$ by the relation:

$$\mathfrak{S}^1 = \mathfrak{q}^{(1)}\sqrt{\mathrm{g}^{11}}\,. \tag{3.10}$$

In the same way, the remaining $\mathfrak{S}^k$ may be interpreted as "force stresses" and the $\mathfrak{T}^k$ as "moment stresses." In the interior of the body, it follows from the vanishing of the volume integral in (3.5) that they satisfy the equilibrium conditions:

$$\partial_\alpha(\sqrt{g}\,\mathfrak{S}^\alpha) + (\sqrt{g}\,\mathfrak{k}) = 0, \tag{3.11}$$

$$\partial_\alpha(\sqrt{g}\,\mathfrak{T}^\alpha) + \mathfrak{g}_\alpha \times (\sqrt{g}\,\mathfrak{S}^\alpha) + (\sqrt{g}\,\mathfrak{m}) = 0. \tag{3.12}$$

These are 6 equilibrium conditions for the 18 stresses $\mathfrak{S}^k$ and $\mathfrak{T}^k$, so the static problem is 12-fold functionally undetermined.

We shall treat only the case of vanishing bulk forces $\mathfrak{k}$ and bulk moments $\mathfrak{m}$, from which the general case can be recovered by adding a particular solution. One thus has:

$$\partial_\alpha(\sqrt{g}\,\mathfrak{S}^\alpha) = 0, \quad \partial_\alpha(\sqrt{g}\,\mathfrak{T}^\alpha) + \mathfrak{g}_\alpha \times (\sqrt{g}\,\mathfrak{S}^\alpha) = 0 \qquad (3.13)(3.14)$$

in the domain of definition, and, unchanged from (3.7):

$$\mathfrak{S}^\alpha n_\alpha = \mathfrak{q}, \qquad \mathfrak{T}^\alpha n_\alpha = \mathfrak{p}, \tag{3.7}$$

on the bounding surface of the region. The equilibrium conditions (3.13) and (3.14) for the stresses $\mathfrak{S}^k$ and $\mathfrak{T}^k$ now have the same form as the divergence conditions (2.16) and (2.17) for the incompatibilities $\overset{(1)}{\mathrm{I}}{}^k$ and $\overset{(2)}{\mathrm{I}}{}^k$ of the deformation field. When we combine this with (2.14) and (2.15) we can therefore represent the stresses in such a way that these equilibrium conditions are satisfied identically:

$$\mathfrak{S}^k = e^{k\lambda\mu}\,\partial_\lambda\,\mathfrak{F}_\mu, \qquad \mathfrak{T}^k = e^{k\lambda\mu}\,[\partial_\lambda\,\mathfrak{G}_\mu, + \mathfrak{g}_\alpha \times \mathfrak{F}_\mu] \quad (3.15)(3.16)$$

with the help of 18 arbitrary "stress functions" $\mathfrak{F}_\mu$ and $\mathfrak{G}_\mu$, which correspond to the incompatible deformations $\vec{\kappa}_\mu$ and $\vec{\varepsilon}_\mu$. On the other hand, the static problem is 12-fold functionally undetermined, so its general solution (3.15) and (3.16) has 6 functions too many. However, one remarks that, corresponding to (2.5) and (2.6), there are stress functions $\bar{\mathfrak{F}}_\mu$ and $\bar{\mathfrak{G}}_\mu$ that produce the "null" stress state:

$$\bar{\mathfrak{F}}_\mu = \partial_\mu \vec{\boldsymbol{\Phi}}, \qquad \bar{\mathfrak{G}}_\mu = \partial_\mu\,\mathfrak{W} + \mathfrak{g}_\mu \times \vec{\boldsymbol{\Phi}} \qquad (3.17)(3.18)$$

with arbitrary vector fields $\vec{\Phi}$ and $\mathfrak{W}$; they correspond to the position changes $\vec{\varphi}$ and $\mathfrak{w}$, whereas the "null stress functions" $\bar{\mathfrak{F}}_\mu$ and $\bar{\mathfrak{G}}_\mu$ correspond to compatible deformation fields. From the well-known analogy in the theory of ordinary point continua (*Weber* [11]), these are also valid in *Cosserat* continua. Thus, there is the possibility of constructing null stress functions $\bar{\mathfrak{F}}_\mu$ and $\bar{\mathfrak{G}}_\mu$ for a suitable choice of 6 functions $\vec{\Phi}$ and $\mathfrak{W}$ by setting 6 of the 18 stress functions $\mathfrak{F}_\mu$ and $\mathfrak{G}_\mu$ to zero ([4]), or, what is more practical, by subjecting the stress function field to the "divergence conditions:"

$$\partial_\mu(\sqrt{g}\,\mathfrak{F}^\mu) = 0, \qquad \partial_\mu(\sqrt{g}\,\mathfrak{G}^\mu) + \mathfrak{g}_\mu \times (\sqrt{g}\,\mathfrak{F}^\mu) = 0. \tag{3.19)(3.20}$$

Naturally, the functions $\vec{\Phi}$ and $\mathfrak{W}$, which are sources of the null stress functions, are only determined up to "rigid motions:"

$$\vec{\Phi} = \vec{\Phi}_0 \tag{3.21}$$

and:

$$\mathfrak{W} = \mathfrak{W}_0 + \vec{\Phi}_0 \times (\mathfrak{r} - \mathfrak{r}_0) \tag{3.22}$$

with constant vectors $\vec{\Phi}_0$ and $\mathfrak{W}_0$!

We now use boundary surface conditions (3.7) and compute the totals:

$$\mathfrak{K} = \iint_{(f)} \mathfrak{q}\, df, \qquad \mathfrak{M}_0 = \iint_{(f)} [\mathfrak{p} + \mathfrak{r} \times \mathfrak{q}]\, df \tag{3.23}$$

of the bounding surface loads, which act on a bounding surface f that is encircled by a curve C. From (3.7) and (3.15), (3.16), one has:

$$\mathfrak{q} = e^{\alpha\lambda\mu}\, \partial_\lambda \mathfrak{F}_\mu\, n_\alpha, \qquad \mathfrak{p} = e^{\alpha\lambda\mu}\, [\partial_\lambda \mathsf{G}_\mu + \mathfrak{g}_\lambda \times \mathfrak{F}_\mu]\, n_\alpha. \tag{3.24)(3.25}$$

If one substitutes this in (3.23) and applies the *Stokes* integral theorem then one obtains:

$$\mathfrak{K} = \int_C \mathfrak{F}_\mu\, ds^\mu, \qquad \mathfrak{M}_0 = \int_C [\mathfrak{G}_\mu + \mathfrak{r} \times \mathfrak{F}_\mu]\, ds^\mu. \tag{3.26)(3.27}$$

In these expressions, the stress functions appear as "edge forces" and "edge moments" that are exerted on the surface boundary; a piece of the boundary is therefore subjected to the force:

$$\delta\mathsf{K} = \mathfrak{F}_\mu\, ds^\mu, \tag{3.28}$$

and the moment:

$$\delta\mathfrak{M} = \mathfrak{G}_\mu\, ds^\mu. \tag{3.29}$$

([4]) This is not generally possible in an arbitrary way.

In a complete theory (i.e., one that is extended by the matter law), these formulas can serve to formulate the boundary value problem for the stress functions. In the rest, the totals are "null" when one substitutes the null stress functions $\bar{\mathfrak{F}}$ and $\bar{\mathfrak{G}}$ in (3.26) and (3.27); one can regard this as a check on the calculations that we did up till now.

We return to the equilibrium conditions (3.13), (3.14), whose first group obviously represents force equilibrium for a volume element, and whose second group describes the moment equilibrium. We introduce the "force stress tensor" by way of:

$$\mathfrak{S}^{\alpha} = S^{\alpha\beta}\,\mathfrak{g}_{\beta}, \qquad S^{\alpha\beta} = \mathfrak{S}^{\alpha} \bullet \mathfrak{g}_{\beta}, \tag{3.30}$$

and the corresponding "moment stress tensor" by way of:

$$\mathfrak{T}^{\alpha} = T^{\alpha}_{.\beta}\,\mathfrak{g}^{\beta}, \qquad T^{\alpha}_{.\beta} = \mathfrak{T}^{\alpha} \bullet \mathfrak{g}_{\beta}. \tag{3.31}$$

The tensor representation of the equilibrium conditions (cf. (2.16a), (2.17a)) then takes the form:

$$\nabla_{\alpha} S^{\alpha i} = 0, \qquad \nabla_{\alpha} T^{\alpha}_{.\beta} + e_{\alpha\beta l}\, S^{\alpha\beta} = 0 \tag{3.13a)(3.14a}$$

from which the second group can also be converted into the form:

$$e^{ik\beta}\;\nabla_{\alpha} T^{\alpha}_{.\beta} + S^{ik} - S^{ki} = 0. \tag{3.14b}$$

From this, it emerges that in the absence of the volume element the force stress tensor is symmetric when and only when the moment stress tensor is divergence-free, and therefore, in particular, when this tensor vanishes, hence, in the case of ordinary point continua, in which the symmetry of the stress tensor S^{ik} is known as the "*Boltzmann* axiom" (*Hamel* [12]).

The further treatment of the equilibrium conditions proceeds in a manner that is completely analogous to what was done with the kinematical part: From (3.16) and (3.17), we can eliminate the stress functions and obtain, corresponding to (2.18):

$$\mathfrak{F}_{\mu} = e^{\alpha\beta\sigma}\left[(\partial_{\alpha}\mathfrak{G}_{\beta} \bullet \mathfrak{g}_{\mu}) - \tfrac{1}{2}(\partial_{\alpha}\mathfrak{G}_{\beta} \bullet \mathfrak{g}_{\sigma})\mathfrak{g}_{\mu}\right] - \left[(\mathfrak{T}^{\sigma} \bullet \mathfrak{g}_{\mu})\,\mathfrak{g}_{\sigma} - \tfrac{1}{2}(\mathfrak{T}^{\sigma} \bullet \mathfrak{g}_{\sigma})\mathfrak{g}_{\mu}\right] \tag{3.32}$$

and corresponding to (2.18a):

$$F_{\mu.}^{\;\;l} = e^{\alpha\beta\sigma}\left[\,\delta^{l}_{\sigma}\;\nabla_{\alpha}\,G_{\beta\mu} - \tfrac{1}{2}\,\delta^{l}_{\mu}\;\nabla_{\alpha}\,G_{\mu\beta}\right] - \left[T^{l}_{.\mu} - \tfrac{1}{2}\,\delta^{l}_{\mu}\,T^{\alpha}_{.\alpha}\right]. \tag{3.32a}$$

Furthermore, corresponding to (2.14a), (2.15a), we have:

$$S^{kl} = e^{k\lambda\mu}\,\nabla_{\lambda}\,F_{\mu.}^{\;\;l}, \qquad T^{k}_{.l} = e^{k\lambda\mu}\,\nabla_{\lambda}\,G_{\mu l} + \delta^{k}_{l}\,F_{\alpha.}^{\;\;\alpha} - F_{l.}^{\;\;k}, \tag{3.15a)(3.16a}$$

and by combining (3.15a) and (3.32a) we then have, corresponding to (2.19a):

$$e^{\alpha\beta\sigma}\left[e^{k\alpha\lambda}\,e^{l\beta\mu}+\tfrac{1}{2}\,e^{kl\alpha}\,e^{\beta\mu\lambda}\right]\nabla_\alpha\nabla_\beta\,G_{\beta\mu}=$$
$$=\left[S^{kl}+e^{k\alpha\lambda}\,\nabla_\alpha T^{l}{}_{.\lambda}+\tfrac{1}{2}\,e^{kl\alpha}\,\nabla_\alpha T^{\lambda}{}_{.\lambda}\right]=\overset{*}{S}{}^{kl}. \tag{3.33}$$

Also, it further suffices here to choose the symmetric part of the stress function tensor $G_{\mu\lambda}$, and $\overset{*}{S}{}^{kl}=\overset{*}{S}{}^{lk}$ is a symmetric tensor. Thus, we can also write:

$$e^{k\alpha\lambda}\,e^{l\beta\mu}\,\nabla_\alpha\nabla_\beta\,\overset{S}{G}_{\lambda\mu}=\overset{*}{S}{}^{kl}, \tag{3.33a}$$

which corresponds to formula (2.27). For the ordinary point continuum, one has $\overset{*}{S}{}^{kl}=\overset{*}{S}{}^{lk}$ since the moment stresses vanishes, and we obtain the familiar representation:

$$S^{kl}=e^{k\alpha\lambda}\,e^{l\beta\mu}\,\nabla_\alpha\nabla_\beta\,\overset{S}{G}_{\lambda\mu}, \tag{3.34}$$

of the stress functions in terms of a symmetric tensor $\overset{S}{G}_{\lambda\mu}$. In this case, one can express the total surface tractions in the following way by use of the stress function tensor $\overset{S}{G}_{\lambda\mu}$:

$$\mathfrak{K}=\oint_C e^{\alpha\beta\lambda}\,\nabla_\alpha\overset{S}{G}_{\beta\mu}\;\mathfrak{g}_\lambda\,ds^\mu, \tag{3.35}$$

$$\mathfrak{M}_0=\oint_C\left[\overset{S}{G}_{\lambda\mu}+r^\alpha(\nabla_\lambda\overset{S}{G}_{\alpha\mu}-\nabla_\alpha\overset{S}{G}_{\lambda\mu})\right]\mathfrak{g}^\lambda\,ds^\mu, \tag{3.36}$$

with:

$$r^\alpha=\mathfrak{r}\bullet\mathfrak{g}^\alpha. \tag{3.37}$$

With (3.35) and (3.36), we have rediscovered the representation for the total surface loads, that had already been given by the author (in a somewhat different notation) in [13], and with whose help *Schaefer* [14] has investigated the stress functions of a singular total.

4. Relationship with dislocation theory

In the Introduction, it was already claimed that the kinematical model for dislocation theory is an incompatible *Cosserat* continuum. Without having discussed the kinematical aspects of dislocation theory in detail (cf., [6], in which a thorough analysis is given), we shall only consider the formal relations that lead from the equations of the *Cosserat* continuum to those of dislocation theory.

We begin with a decomposition of the deformation tensor ε_{il} that is due to *Kröner* [6] in which it will be assumed that it vanishes at infinity sufficiently strongly or satisfies suitable boundary conditions in the finite case.

There is then a unique decomposition:

$$\varepsilon_{il}=\nabla_i\,w_l+e_{i\alpha\lambda}\,\nabla^\alpha b^\lambda{}_{.l}, \tag{4.1}$$

in which ε is split into the gradient of a vector field $\mathfrak{w}$ and the curl of a tensor field $\mathfrak{b}$. Likewise, we decompose $\mathfrak{b}$:

$$b^{\lambda}_{.l} = \nabla_l\, c^{\lambda} + e_{l\beta\mu}\, \nabla^{\beta} d^{\mu\lambda}, \tag{4.2}$$

Together with (4.1), this yields:

$$\varepsilon_{il} = \nabla_i\, w_l + \nabla_l\, (e_{i\alpha\lambda}\, \nabla^{\alpha} c^{\lambda}) + e_{i\alpha\lambda}\, e_{l\beta\mu}\, \nabla^{\alpha} \nabla^{\beta} d^{\mu\lambda}. \tag{4.3}$$

We set:

$$e_{i\alpha\lambda}\, \nabla^{\alpha} c^{\lambda} = u_i\, ; \tag{4.4}$$

since u is the curl of the vector field $\mathfrak{c}$, its divergence vanishes identically:

$$\nabla^{\alpha} u_{\alpha} = 0. \tag{4.5}$$

Furthermore, we decompose $d^{\mu\lambda}$ into its symmetric and anti-symmetric parts:

$$d^{\mu\lambda} = d_S^{\mu\lambda} + d_A^{\mu\lambda} = d^{(\lambda\mu)} + e^{\mathrm{r}\mu\nu}\, h_{\mathrm{r}}, \tag{4.6}$$

in which:

$$h_{\mathrm{r}} = \tfrac{1}{2}\, e_{\mathrm{r}\mu\nu}\, d^{\mu\lambda}\, .$$

Finally, we set:

$$\nabla^{\mathrm{r}}\, h_{\mathrm{r}} = p \tag{4.8}$$

and obtain, in summary:

$$\varepsilon_{il} = \nabla_i\, (w_l + u_l) + (e_{i\alpha\lambda}\, c^{\lambda}) + e_{i\alpha\lambda}\;\; e_{l\beta\mu}\, \nabla^{\alpha} \nabla^{\beta} d^{\mu\lambda} - e_{il\alpha}\, [e^{\rho\sigma\alpha}\, \nabla_{\rho}\, u_{\sigma} + \nabla^{\alpha} p], \tag{4.9}$$

in which one must take (4.5) into account.

The second term in (4.9) is, as we discussed previously, symmetric in i and l, and represents the incompatible part $\overset{*}{\varepsilon}_{il}$, in the *Cosserat* sense, of the deformation tensor ε_{il}; namely, if we set:

$$w_l + u_l = w_l', \tag{4.10}$$

and:

$$e^{\rho\sigma\alpha}\, \nabla_{\rho}\, u_{\alpha} + \nabla^{\alpha} p = \varphi^{\alpha}, \tag{4.11}$$

then the deformation tensor ε_{il} takes on the form:

$$\varepsilon_{il} = \nabla_i\, w_l' - e_{il\alpha}\, \varphi^{\alpha} + \overset{*}{\varepsilon}_{(il)}\, . \tag{4.12}$$

Furthermore, we define, as before, the *Cosserat* curvature tensor κ by way of:

$$\kappa_{i.}^{\;l} = \nabla_i\, \varphi^l, \tag{4.13}$$

and recognize that we have rediscovered the *Cosserat* deformations in the form of (4.12) and (4.13), but generally supplemented by the incompatible part $\overset{*}{\varepsilon}_{il}$. From (2.14a) and (2.15a), one thus has the equations:

$$e^{k\lambda\mu}\,\nabla_\lambda \kappa_{\mu..}^{\;\;l} = 0, \tag{4.14}$$

$$e^{k\lambda\mu}\,\nabla_\lambda \varepsilon_{\mu l} + \delta_l^k\,\kappa_{\lambda..}^{\;\;\lambda} - \kappa_{l..}^{\;\;k} = \overset{(2)}{I}{}^{k}_{.l}\,; \tag{4.15}$$

i.e., in the *Cosserat* sense, the curvatures $\kappa_{l.}^{\;l}$ are always compatible, since, from (4.11), a unique rotation vector $\vec{\varphi}$ exists. We shall come back to this.

The basic formula for dislocation theory now reads:

$$\nabla \times \mathfrak{e} = \mathfrak{v}, \tag{4.16}$$

in which $\mathfrak{v}$ is the "dislocation density" tensor. One can clarify its intuitive meaning by looking at Fig. 3, which shows a "step dislocation," and Fig. 4, which shows a "screw dislocation." In Fig. 3, a lattice plane has been inserted into a regular lattice in which the (strictly removed) dislocation line ends. If a surface x_3 = const. envelops this line once then it intersects the glide plane that is spanned by the "Burgers vector" (glide direction) and the dislocation line; one can think of there being a jump in the displacement field there. The – in this case, singular – tensor component $v^3_{.3}$ measures the magnitude of the jump. In Fig. 4, the screw axis was defined as the dislocation line; here, the glide and dislocation directions agree with each other, and the – likewise singular – tensor component $v^3_{.3}$ again measures the magnitude of the jump. In the continuum theory of dislocations, one goes over to continuous distributions of dislocations; the tensor v then measures the magnitude and type of dislocation line that intersects a given surface element.

From (4.16), one has:

$$e^{k\lambda\mu}\,\nabla_\lambda\,\varepsilon_{\mu\lambda} = v^k_{.l}, \tag{4.16a}$$

and thus, from (4.15):

$$\overset{(2)}{I}{}^{k}_{.l} = v^k_{.l} + \delta_l^k\,\kappa_{\lambda.}^{\;\lambda} - \kappa_l^k, \tag{4.17}$$

whereas, from (4.12):

$$v^k_{.l} = e^{k\lambda\mu}\,\nabla_\lambda \overset{*}{\varepsilon}_{(\mu l)} - \delta_l^k\,\kappa_{\lambda.}^{\;\lambda} + \kappa_l^k, \tag{4.18}$$

hence:

$$\overset{(2)}{I}{}^{k}_{.l} = e^{k\lambda\mu}\,\nabla_\lambda \overset{*}{\varepsilon}_{(\mu l)}\,. \tag{4.19}$$

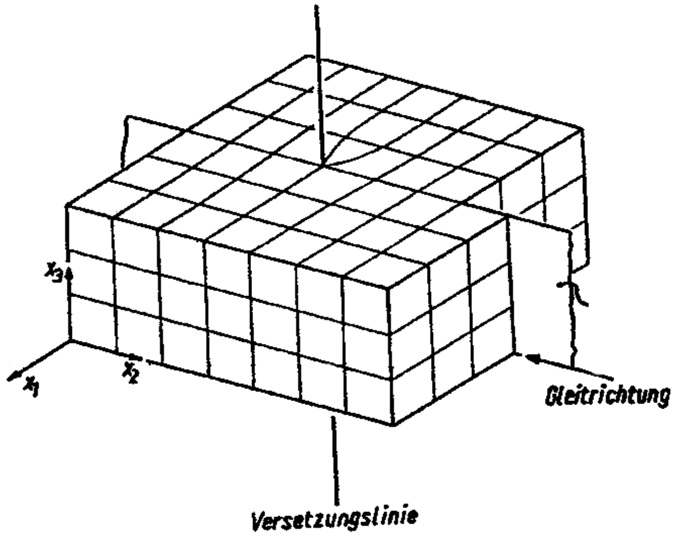

Fig. 3.

The incompatibility tensor $\overset{(2)}{I}{}^{k}_{.l}$ thus has a simple relationship with not only the dislocation density tensor, but also the incompatible part of the deformation tensor. If this part is null then from (4.17), we obtain:

$$v^{k}_{.l} = \kappa_l^k - \delta_l^k \kappa_{\lambda.}^{\ \lambda}, \tag{4.20}$$

a relation that was first derived by *Nye* [15] (here, we are treating the case of the "stress-free structure curvature," cf. [6]). If we proceed further as in sec. 2, in which we use the tensor k from (4.15), and substitute in (4.14) then we find a small discrepancy, in which one must observe that, from (4.16a), the trace $v^{\lambda}_{.\lambda}$ of the dislocation density tensor vanishes:

$$e^{k\alpha\lambda} e^{l\beta\mu} \nabla_\alpha \nabla_\beta \overset{*}{\varepsilon}_{(\mu\lambda)} = \overset{(2)}{I}{}^{kl} = e^{k\alpha\lambda} \nabla_\alpha v^{l}_{.\lambda}; \tag{4.21}$$

in *Kröner's* notation:

$$Ink \overset{*}{\varepsilon} = \mathfrak{v} \times \nabla, \tag{4.22}$$

and the right-hand side vanishes, from (4.16a).

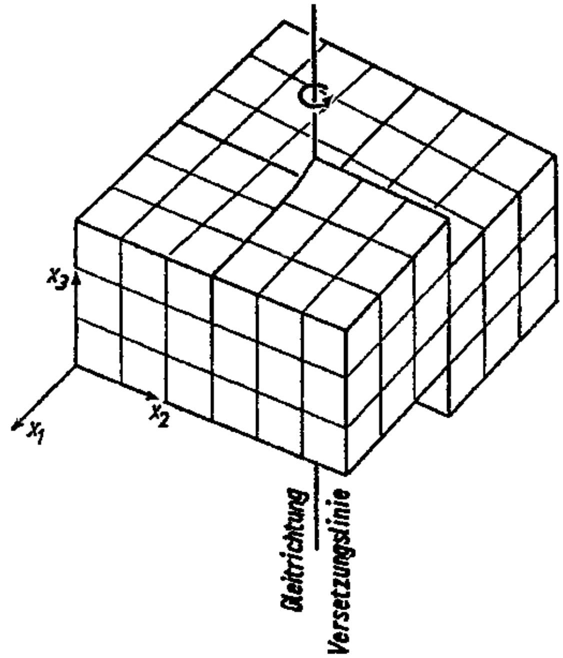

Fig. 4.

We shall return to the question of whether the rotations are incompatible. That is a matter of definition: as we showed, this is always true in the *Cosserat* sense (when one ignores singularities that are not everywhere dense). However, if one calls a tensor field ξ incompatible when *Ink* $\xi \neq 0$, so one has:

$$e^{k\alpha\lambda}\, e^{l\beta\mu}\, \nabla_\alpha \nabla_\beta\, \xi_{\lambda\mu} \neq 0$$

then the rotation will be incompatible, as one recognizes from the following argument:

We decompose the deformation tensor (4.12) into its symmetric and anti-symmetric parts, for which the following expressions are valid:

$$\varepsilon_{il}^{S} = \tfrac{1}{2}\,[\nabla_i(w_l + u_l) + \nabla_l(w_i + u_i)] + \overset{*}{\varepsilon}_{(il)}, \tag{4.23}$$

$$\varepsilon_{il}^{A} = \tfrac{1}{2}\,[\nabla_i(w_l - u_l) - \nabla_l(w_i - u_i)] - e_{il\alpha}\nabla^{\alpha} p, \tag{4.24}$$

and now if ε_{il}^{A} is an incompatible rotation field then we would have:

$$e^{k\alpha\lambda}\, e^{l\beta\mu}\, \nabla_\alpha \nabla_\beta\, \varepsilon_{il}^{A} = -\, e^{il\alpha}\nabla_\alpha(\Delta p). \tag{4.25}$$

Only the scalar field p contributes to the incompatibility (in the latter sense) of the rotation field.

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