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Analogous systems of shell equations

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1. Introduction. – For a long time, it has been known that one can formulate the field equations for an elastic problem in two different ways: The first, and most common, procedure consists of expressing the material law of all force quantities in terms of the displacements; the equilibrium conditions are then the differential equations for the displacement field. Dual to that, however, one can also derive the equilibrium of the force quantities automatically when one introduces stress functions. They will be determined for the material law in such that a compatible state of deformation will arise. Which of the two possibilities one chooses depends upon the type of problem that one poses, and in particular, upon its boundary conditions. **H. Schaefer** ([1]) has shown that the two groups of equations for the plate/disc system (i.e., the planar surface carrier) will differ essentially only by the sign of the transverse contraction number when one relates the three displacement components to the three stress functions of the system. Naturally, the fact that this analogy exists is no accident, but is based, in the final analysis, upon the existence of a bilinear form in which the force quantities and the virtual deformations are combined into a scalar: namely, the virtual work that is done by the internal forces. If one wishes to pursue such analogies then one must put the principle of virtual displacements at the center of all considerations; as is known, the principle of virtual forces can also enter in place of it in the case of small displacements and deformations. One will indeed also come to the dual variational problems of elastomechanics quite easily by starting from these two principles. – The goal of the present article is to construct systematically a general theory of the bending of weakly-deformed elastic shells in which the aforementioned analogies will emerge clearly. The fact that such analogies exist for shells was pointed out, above all, by the Russian researchers. One will find investigations of it in, e.g., **V.** S. **Vlassov** ([2]) and **V. V. Novoshilov** ([3]), in which further Russian literature is given. Recently, **H. Schaefer** ([4]) has further examined the equations of the right circular cylinder and established the complete analogy between the two groups of equations. One now seeks to not only write down the analogies for arbitrary shells (which is not too difficult after neglecting some things if one uses tensor analysis), but to systematically derive the fact that the statics and kinematics of the shell

([1]) **H. Schaefer**, Abh. Braunschw. Wiss. Ges. **8** (1956), pp. 142.
([2]) **V. S. Vlassov**, *Allgemeine Schalentheorie und ihre Anwendung in der Technik*, Berlin, 1958.
([3]) **V. V. Novoshilov**, *The Theory of Thin Shells*, Gronigen, 1959.
([4]) **H. Schaefer**, Ing.-Arch. **29** (1960), pp. 125.

can be regarded as a so-called **Cosserat** surface, and therefore a surface that consists of nothing but oriented elements. With the help of that shell model, one can formulate all of the kinematic and static equations of the shell and their analogies so simply and fruitfully with the use of the principle of virtual displacements that one can recall them from one's memory with no difficulty each time. By a clever choice of field quantities, one can then also integrate the law of elasticity into the analogies and thus ultimately obtain the desired two representations of the shell problem.

2. Surface-theoretic tools. – We shall begin with a review of some concepts and formulas from the theory of surfaces, for which, we shall refer to the presentation in **A E. Green** and **W. Zerna** ([1]):

If we let $q^{(1)}$ and $q^{(2)}$ be **Gauss**ian parameters on a surface, and let:

$$\mathbf{r} = \mathbf{r}\,(q^{(1)}, q^{(2)}) \tag{2.1}$$

be the position vector from a fixed point O to a non-singular point P on that surface then we can define a local dreibein by the *dimension vectors*:

$$a_i = \partial_i\,\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i} \qquad (i = 1, 2) \tag{2.2}$$

and the unit normal vector:

$$\mathbf{E} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\,\mathbf{a}_1 \times \mathbf{a}_2\,|} \tag{2.3}$$

of the surface (Fig. 1). Along with the dimension vectors $\mathbf{a}_i$, one has the *covariant permutation vectors:*

$$\mathbf{e}_i = \mathbf{E} \times \mathbf{a}_i \qquad (i = 1, 2) \tag{2.4}$$

and the *curvature vectors:*

$$\mathbf{b}_i = -\,\partial_i\,\mathbf{E} \qquad (i = 1, 2), \tag{2.5}$$

which are also tangent to the surface. If one now goes over to new parameters $\bar{q}^{(1)}$ and $\bar{q}^{(2)}$ then one will get the associated vectors $\bar{\mathbf{a}}_i$, $\bar{\mathbf{e}}_i$, and $\bar{\mathbf{b}}_i$ by a *covariant transformation:*

$$\bar{\mathbf{a}}_i = \mathbf{a}_\alpha \frac{\partial q^\alpha}{\partial \bar{q}^i}, \qquad \bar{\mathbf{e}}_i = \mathbf{e}_\alpha \frac{\partial q^\alpha}{\partial \bar{q}^i}, \qquad \bar{\mathbf{b}}_i = \mathbf{b}_\alpha \frac{\partial q^\alpha}{\partial \bar{q}^i}. \tag{2.6}$$

(Greek indices are summed over from 1 to 2!) We shall call vectors whose indices transform in that way *covariantly indexed.* By *covariant projection* – i.e., by scalar multiplication by the dimension vector – one will get:

([1]) **A. E. Green** and **W. Zerna**, *Theoretical Elasticity*, Oxford, 1954.

1. The covariant representation of the *metric tensor* (*first fundamental tensor* of the surface):

$$a_{il} = \mathbf{a}_i \cdot \mathbf{a}_l = a_{li} \,, \tag{2.7}$$

with the determinant:

$$a = a_{11}\, a_{22} - (a_{12})^2. \tag{2.8}$$

2. The covariant representation of the *permutation tensor* of the surface:

$$e_{il} = \mathbf{e}_i \cdot \mathbf{a}_l = -\, e_{li} \,, \tag{2.9}$$

$$|\, e_{il} \,| = \sqrt{a} \qquad \text{for } l \neq i \,, \tag{2.10}$$

and

3. The covariant representation of the *second fundamental tensor* of the surface:

$$b_{il} = \mathbf{b}_i \cdot \mathbf{a}_j = -\,\partial_i \mathbf{E} \cdot \mathbf{a}_l = \mathbf{E} \cdot \partial_i \partial_l\, \mathbf{a}_l = b_{li} \,, \tag{2.11}$$

with the determinant:

$$b = b_{11}\, b_{22} - (b_{12})^2. \tag{2.12}$$

The *third fundamental tensor* of the surface is defined by:

$$c_{il} = \mathbf{b}_i \cdot \mathbf{b}_j = c_{li} \,. \tag{2.13}$$

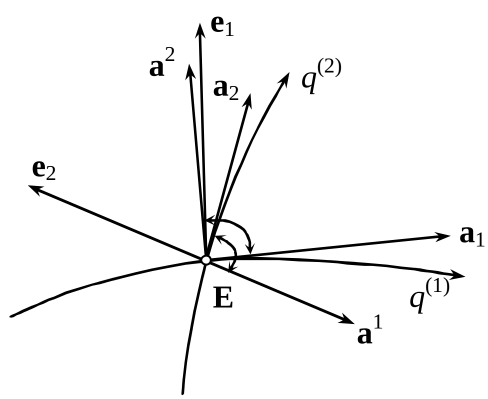

Figure 1. Metric and permutation tensors.

Often, it is preferable to emphasize the symmetry behavior of a second-rank tensor in particular. Let u_{il} be such a tensor; we can then characterize its symmetric part by (†):

$$u_{(il)} = \tfrac{1}{2}(u_{il} + u_{li}), \tag{2.14}$$

and its antisymmetric (i.e., *skew-symmetric*) part by:

(†) Translator: I have taken the liberty of changing the notation for the symmetric and antisymmetric parts of a second-rank tensor to something more currently conventional.

$$u_{[il]} = \tfrac{1}{2}(u_{il} - u_{li}). \tag{2.15}$$

Their sum will give back the original tensor. We will then have:

$$a_{il} = a_{(il)}\,, \qquad e_{il} = e_{[il]}\,, \qquad b_{il} = b_{(il)}\,, \qquad c_{il} = c_{(il)}\,. \tag{2.16}$$

In a known way, we can now further generate the *contravariant dimension vectors* $\mathbf{a}^i$ (i = 1, 2) as solutions of the equations:

$$\mathbf{a}^i \cdot \mathbf{a}_l = a^i_l = \delta^i_l = \begin{cases} 0 & \text{for} \quad l \neq i, \\ 1 & \text{for} \quad l = i. \end{cases} \tag{2.17}$$

Under a parameter transformation, they will go to the new contravariant dimension vectors $\bar{\mathbf{a}}^i$ by the *contravariant transformation:*

$$\bar{\mathbf{a}}^i = \mathbf{a}^\alpha \frac{\partial \bar{q}^i}{\partial q^\alpha}\,. \tag{2.18}$$

The same law of transformation is also true for the *contravariant permutation vectors:*

$$\mathbf{e}^i = \mathbf{E} \times \mathbf{a}^i. \tag{2.19}$$

Contravariant projection yields:

1. The contravariant representation of the metric tensor of the surface:

$$a^{il} = \mathbf{a}^i \cdot \mathbf{a}^l = a^{(il)}, \tag{2.20}$$

$$\text{Det}\,(a^{il}) = \frac{1}{a}. \tag{2.21}$$

2. The contravariant representation of the permutation tensor of the surface:

$$e^{il} = \mathbf{e}^i \cdot \mathbf{a}^l = e^{[il]}, \tag{2.22}$$

$$|\, e^{il} \,| = \frac{1}{\sqrt{a}} \qquad \text{for} \qquad l \neq i. \tag{2.23}$$

3. The “mixed representation” of the second fundamental tensor of the surface:

$$b^i_l = \mathbf{b}_l \cdot \mathbf{a}^i = a^{i\alpha}\, b_{l\alpha}\,, \tag{2.24}$$

from which, one constructs the *mean curvature:*

$$H = \tfrac{1}{2} b^{\alpha}_{\alpha} \tag{2.25}$$

and the ***Gauss**ian curvature:*

$$K = \operatorname{Det}(b^{i}_{l}) = \frac{b}{a}. \tag{2.26}$$

The components of the tensor b^i_l always have the dimension of a reciprocal length. For the *lines of curvature* of the surface, and thus for those parameter curves that are everywhere tangent to the principle directions of the tensor b^i_l (and therefore define an orthogonal net), if R_1 and R_2 are the principle radii of curvature of the surface then for a suitable orientation of the coordinate system, one will have:

$$b^1_1 = -\frac{1}{R_1}, \qquad b^2_2 = -\frac{1}{R_2}; \tag{2.27}$$

in that way, the **Gauss**ian curvature is assumed to be non-negative.

The application of elementary vector operation will lead to the following formulas, which are used occasionally:

$$\mathbf{e}_i \times \mathbf{E} = \mathbf{a}_i, \qquad \mathbf{e}^i \times \mathbf{E} = \mathbf{a}^i, \tag{2.28}$$

$$e_{ik}\, e^{lm} = \delta^l_i\, \delta^m_k - \delta^m_i\, \delta^l_k, \qquad e_{i\alpha}\, e^{l\alpha} = \delta^l_i, \tag{2.29}$$

$$K\delta^l_i - 2H\, b^l_k + c^l_i = 0. \tag{2.30}$$

Now, let ∇_i (…) be the symbol of the covariant differentiation with respect to the surface parameters $q^{(1)}$ and $q^{(2)}$. One will then have:

$$\left.\begin{aligned} \nabla_i \mathbf{E} &= \partial_i \mathbf{E} = -\mathbf{b}_i = -b^{\alpha}_i \mathbf{a}_{\alpha} \\ \nabla_i \mathbf{a}_l &= b_{il}\mathbf{E}, \quad \nabla_i \mathbf{e}_l = \mathbf{a}_l \times \mathbf{b}_i. \end{aligned}\right\} \tag{2.31}$$

It follows from this that:

$$\nabla_i\, a_{lm} = 0, \qquad \nabla_i\, e_{lm} = 0 \tag{2.32}$$

(the metric and permutation tensor are therefore covariantly constant), and furthermore:

$$e^{\alpha\beta}\, \nabla_{\alpha}\, b_{\beta} = 0 \qquad \rightarrow \qquad e^{\alpha\beta}\, \nabla_{\alpha}\, b_{\beta l} = 0 \tag{2.33}$$

are the **Codazzi** equations for the second fundamental tensor of the surface, which must be fulfilled if a single-valued field of normal vectors for the surface is to exist.

Now let a spatial vector field $\mathbf{v} = \mathbf{v}\,(q^{(1)}, q^{(2)})$ be given on the surface that splits into:

$$\left.\begin{aligned} \mathbf{v} &= v_{\alpha}\mathbf{a}^{\alpha} + v\,\mathbf{E} = v^{\alpha}\mathbf{a}_{\alpha} + v\,\mathbf{E}, \\ v_i &= \mathbf{v}\cdot\mathbf{a}_i, \quad v^i = \mathbf{v}\cdot\mathbf{a}^i, \quad v = \mathbf{v}\cdot\mathbf{E}. \end{aligned}\right\} \tag{2.34}$$

$\mathbf{v}$ then combines the vector v_i (v^i, resp.) and the scalar v, in the sense of tensor analysis. Similarly, for the covariantly-indexed spatial vector fields $\mathbf{v}_l = \mathbf{v}_l\,(q^{(1)}, q^{(2)})$, one has:

$$\mathbf{v}_l = v_{l\alpha}\,\mathbf{a}^\alpha + v_i\,\mathbf{E} = v_{l\cdot}^{\alpha}\,\mathbf{a}_\alpha + v_l\,\mathbf{E}, \qquad v_{lm} = \mathbf{v}_l \cdot \mathbf{a}_m\,, \quad v_{l\cdot}^{m} = \mathbf{v}_l \cdot \mathbf{a}^m\,, \quad v_l = \mathbf{v}_l \cdot \mathbf{E}. \tag{2.35}$$

Obviously, one can easily derive rules for multiply-indexed vector fields, such as:

$$v_{lm} = \mathbf{v}_l \times \mathbf{v}_m = v_{[lm]}\,; \tag{2.36}$$

one will then have corresponding decompositions for such fields. From (2.31), the covariant differentiation of vector fields proceeds as follows:

$$\begin{aligned}\nabla_i\,\mathbf{v}\ (= \partial_i\,\mathbf{v}) &= (\nabla_i\,v_\alpha - b_{i\alpha}\,v)\,\mathbf{a}^\alpha + (\nabla_i\,v + b_i^\alpha\,v_\alpha)\,\mathbf{E} \\ &= (\nabla_i\,v^\alpha - b_i^\alpha\,v)\,\mathbf{a}_\alpha + (\nabla_i\,v + b_{i\alpha}\,v^\alpha)\,\mathbf{E},\end{aligned} \tag{2.37}$$

$$\begin{aligned}\nabla_i\,\mathbf{v}_l &= (\nabla_i\,v_{l\alpha} - b_{i\alpha}\,v_l)\,\mathbf{a}^\alpha + (\nabla_i\,v_l + b_i^\alpha\,v_{l\alpha})\,\mathbf{E} \\ &= (\nabla_i\,v_l^\alpha - b_i^\alpha\,v_l)\,\mathbf{a}_\alpha + (\nabla_i\,v_l + b_{i\alpha}\,v_l^\alpha)\,\mathbf{E},\end{aligned} \tag{2.38}$$

with a simple generalization to multiply-indexed vector fields.

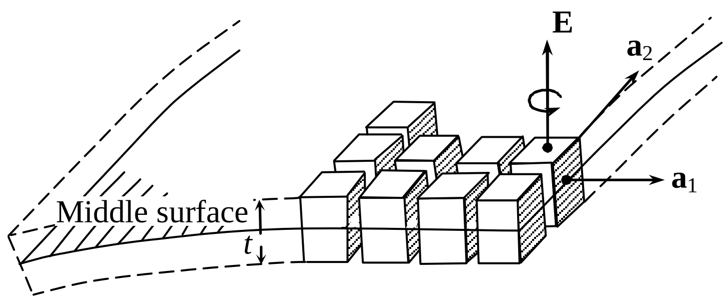

Figure 2. **Cosserat** shell model.

3. Kinematics of the middle surface of the shell. – We imagine (Fig. 2) that the shell has been broken up into an (initially-finite) number of rigid blocks whose middle surfaces will assemble together into the undeformed middle surface of the shell when it is in its undeformed state. They are attached to each other with springs in such a way that they can rotate and displace, and the springs are all relaxed in the initial state. By passing to the limit, we will get a kinematical model for the shell as a surface that coincides with the middle surface of the shell geometrically, but is oriented at each of its points by way of the spatial position of the associated block. Any "point" of that surface can move with six degrees of freedom, and is attached to its neighboring "points" by springs in the manner that is required by the elasticity law for the shell (which will be discussed below). Naturally, instead of a continuum of "oriented points," we can also speak of a continuum of oriented coordinate systems whose origins lie on the middle surface of the shell. Briefly: With our model, we are dealing with a ***Cosserat*** *surface*, which is named for **E.**

and **F. Cosserat** ([1]), who investigated the kinematics and statics of continua that consist of oriented point in a monograph. [The author ([2]) has treated the basic ideas of the **Cosserat** conception of things in more modern notation.] This shell model (which is the logical extension of the model for the engineering theory of the bending of beams, moreover) is anything but new: **E.** and **F. Cosserat** drew upon the shell as an example, but did not, however, strive for an actual theory of shells; the representation, which is currently quite hard to read, is not covariant. **K. Heun** ([3]) employed the symbolism of vector calculus without actually going beyond the scope of the **Cosserat** book, and in that way, the presentation became clearer. Finally, **J. L. Ericksen** and **C. Truesdell** ([4]) developed a theory of finitely-deformed shells on the basis of the **Cosserat** picture, but with a different objective than ours. We will see that it is this model precisely that will lead us to an exceptionally symmetric, and therefore intuitive, theory of shells.

The changes of position of an "oriented point" on the middle surface of the shell are: A rotation, which is so small that one can describe it by a rotation vector and a displacement $\mathbf{v}$; let $|\,\mathbf{v}\,|$ be small in comparison to the shell thickness t. The functions:

$$\boldsymbol{\omega} = \boldsymbol{\omega}(q^{(1)}, q^{(2)}) = \omega^{\alpha}\,\mathbf{a}_{\alpha} + \omega\,\mathbf{E}, \qquad \mathbf{v} = \mathbf{v}\,(q^{(1)}, q^{(2)}) = v_{\alpha}\,\mathbf{a}^{\alpha} + v\,\mathbf{E} \tag{3.1}$$

shall be continuous over the middle surface and continuously-differentiable sufficiently often. Naturally, among them, one also finds the rigid motions:

$$\overset{\circ}{\boldsymbol{\omega}} = \boldsymbol{\omega}_0\,, \qquad \overset{\circ}{\mathbf{v}} = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) \tag{3.2}$$

of a finite part of the shell with constant vectors $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(\mathbf{r}_0)$, $\mathbf{v}_0 = \mathbf{v}\,(\mathbf{r}_0)$. We derive the deformations from the changes in position (3.1) as *relative* changes in rotation and displacement as follows: The *absolute* changes are:

$$(d\boldsymbol{\omega})_{\text{absolute}} = \partial_{\alpha}\boldsymbol{\omega}\,dq^{\alpha} = \nabla_{\alpha}\boldsymbol{\omega}\,dq^{\alpha}, \qquad (d\mathbf{v})_{\text{absolute}} = \partial_{\alpha}\mathbf{v}\;dq^{\alpha} = \nabla_{\alpha}\mathbf{v}\;dq^{\alpha}; \tag{3.3}$$

furthermore, one has:

$$(d\boldsymbol{\omega})_{\text{absolute}} = (d\boldsymbol{\omega}), \qquad (d\mathbf{v})_{\text{absolute}} = (d\mathbf{v}) + \boldsymbol{\omega}\times d\mathbf{r}. \tag{3.4}$$

If we define the relative changes to be linear functions of the differential advance dq^i :

$$(d\boldsymbol{\omega})_{\text{relative}} = \boldsymbol{\chi}_{\alpha}\,dq^{\alpha}\,, \qquad (d\mathbf{v})_{\text{relative}} = \boldsymbol{\varepsilon}_{\alpha}\,dq^{\alpha} \tag{3.5}$$

and observe that from (1.2), we will have:

$$d\mathbf{r} = \boldsymbol{a}_{\alpha}\,dq^{\alpha} \tag{3.6}$$

([1]) **E.** and **F. Cosserat**, *Théorie des corps déformables*, Paris, 1909.

([2]) **W. Günther**, Abh. Braunschw. Wiss. Ges. **10** (1958), pp. 195.

([3]) **K. Heun**, "Ansätze und allgemeine Methoden der Systemmechanik," Enz. d. Math. Wiss. IV, Leipzig, 1914; pp. 2, 11.

([4]) **J. L. Ericksen** and **C. Truesdell**, Arch. Rat. Mech. and Analysis **1**, **4** (1958), pp. 295.

on the middle surface of the shell then we will get:

$$\boldsymbol{\chi}_i = \nabla_i\, \boldsymbol{\omega}, \qquad \boldsymbol{\varepsilon}_i = \nabla_i\, \mathbf{v} + \mathbf{a}_i \times \boldsymbol{\omega}. \tag{3.7}$$

These deformations are obviously covariantly-indexed vector fields. We decompose them into:

$$\boldsymbol{\chi}_i = \chi_{i\cdot}^{\alpha}\, \mathbf{a}_\alpha + \chi_i\, \mathbf{E}, \qquad \boldsymbol{\varepsilon}_i = \varepsilon_{i\alpha}\, \mathbf{a}^\alpha + \varepsilon_i\, \mathbf{E}, \tag{3.8}$$

with

$$\chi_{i\cdot}^{\alpha} = \boldsymbol{\chi}_i \cdot \mathbf{a}^i, \quad \chi_i = \boldsymbol{\chi}_i \cdot \mathbf{E}, \quad \varepsilon_{il} = \boldsymbol{\varepsilon}_i \cdot \mathbf{a}_i, \quad \varepsilon_i = \boldsymbol{\varepsilon}_i \cdot \mathbf{E}. \tag{3.9}$$

Twelve deformation quantities then arise from the six changes of position (3.1) that are derived from the components of the changes of position using (2.37) as follows:

$$\chi_{i\cdot}^{l} = \nabla_i\, \omega^l - b_i^l\, \omega, \qquad \chi_i = \nabla_i\, \omega - b_{i\alpha}\, \omega^\alpha, \tag{3.10}$$

$$\varepsilon_{il} = \nabla_i\, v_l - b_{il}\, v - e_{il}\, \omega, \qquad \varepsilon_i = \nabla_i\, v - b_i^\alpha\, v_\alpha - e_{i\alpha}\, \omega^\alpha. \tag{3.11}$$

The second of equations (3.7) [equations (2.11), resp.] can be solved for the rotation $\boldsymbol{\omega}$ (its components, resp.):

$$\boldsymbol{\omega} = (\nabla_\alpha \mathbf{v} - \boldsymbol{\varepsilon}_\alpha) \cdot \mathfrak{A}^\alpha, \tag{3.12}$$

with the dyadically-represented matrices:

$$\mathfrak{A}^i = -\,\mathbf{E}\,\mathbf{e}^i + \mathbf{e}^i\,\mathbf{E}, \tag{3.13}$$

or

$$\omega^i = e^{i\alpha}\,(\nabla_\alpha \mathbf{v} + b_\alpha^\beta v_\beta - \varepsilon_\alpha), \qquad \omega = e^{\alpha\beta}\,(\nabla_\alpha v_\beta - \varepsilon_{[\alpha\beta]}). \tag{3.14}$$

Hence, only the quantities $\varepsilon_i = \boldsymbol{\varepsilon}_i \cdot \mathbf{E}$ and the antisymmetric part of the deformation tensor actually enter into (3.12).

The geometric meaning of the deformation numbers (3.10) and (3.11) is immediate from the way that they come about: The $\chi_{i\cdot}^{l}$ describe the *distortion* of the shell, and the χ_i are the normal components of the relative rotation. The symmetric part $\varepsilon_{(il)}$ of the tensor ε_{il} describe the deformations of the middle surface, when regarded as a point-continuum, that take place in the tangent plane. Namely, let:

$$\mathbf{a}_i' = \partial_i\,(\mathbf{r} + \mathbf{v}) = \mathbf{a}_i + \nabla_i\, \mathbf{v} \tag{3.15}$$

be the dimension vectors of the deformed middle surface, so when one neglects the products of deformation quantities, the new metric tensor will become:

$$a_{il}' = \mathbf{a}_i' \cdot \mathbf{a}_l' = a_{il} + 2\,\varepsilon_{(il)}\,, \tag{3.16}$$

$$\varepsilon_{(il)} = \tfrac{1}{2}(\nabla_i v_l + \nabla_l v_i) - b_{il}\, v\,; \tag{3.17}$$

however, $\tfrac{1}{2}(a'_{il} - a_{il})$ is precisely the aforementioned deformation. We note that due to the fact that:

$$a'^{i\alpha} a'_{l\alpha} = \delta^i_l = \begin{cases} 0 & \text{for} \quad l \neq i \\ 1 & \text{for} \quad l = i \end{cases} \tag{3.18}$$

the difference between the contravariant dimension vectors will be:

$$a'^{il} - a^{il} = -\,2\varepsilon^{(il)}. \tag{3.19}$$

The skew-symmetric part $\varepsilon_{[il]}$ is equivalent to the scalar angle ψ:

$$\psi = \tfrac{1}{2} e^{\alpha\beta}\, \varepsilon_{[\alpha\beta]}\,, \qquad \varepsilon_{[il]} = e_{il}\;\psi, \tag{3.20}$$

and from (3.11), one will have:

$$\psi = \tfrac{1}{2} e^{\alpha\beta}\, \nabla_\alpha v_\beta - \omega\,. \tag{3.21}$$

That shows that $\varepsilon_{[il]}$ corresponds to the difference between the "mean rotation" that originates in the deformation of the middle surface and the **Cosserat** rotation ω around the normal to the shell.

The interpretation of the quantities ε_i is: The unit normal vector $\mathbf{E}'$ of the deformed middle surface that emerges from $\mathbf{E}$ by a small rotation $\boldsymbol{\vartheta}$, namely:

$$\mathbf{E}' = \mathbf{E} + \boldsymbol{\vartheta} \times \mathbf{E} \tag{3.22}$$

does not, however, generally lie in the direction of the normal to the undeformed middle surface that is carried by the block at that position:

$$\mathbf{E}'' = \mathbf{E} + \boldsymbol{\omega} \times \mathbf{E}. \tag{3.23}$$

It will then follow easily from:

$$\mathbf{E}' \cdot \mathbf{a}'_i = 0 \tag{3.24}$$

and (3.15) that:

$$\mathbf{E} = \mathbf{E} - (\mathbf{E} \cdot \nabla_\alpha \mathbf{v})\; \mathbf{a}^\alpha, \tag{3.25}$$

and furthermore, with (3.7):

$$\mathbf{E}' = \mathbf{E}'' - \varepsilon_\alpha\, \mathbf{a}^\alpha. \tag{3.26}$$

That will determine $\boldsymbol{\vartheta}$ as:

$$\boldsymbol{\vartheta} = \boldsymbol{\omega} - \mathbf{e}^\alpha\, \varepsilon_\alpha\,. \tag{3.27}$$

The deformations ε_i are then a measure of the extent to which normal to the undeformed middle surface that is carried by the deformation has rotated into the normal to the deformed normal surface. One is then dealing with the transverse shear deformations of the shell. Now, if the deformations $\boldsymbol{\chi}_i$ and $\boldsymbol{\varepsilon}_i$ are given as functions of the surface parameters then it will not generally be possible to calculate single-valued changes of position $\boldsymbol{\omega}$ and **v** from them by integrating equations (3.7). In order for that to be possible, the deformations must satisfy certain *compatibility conditions* (naturally, in the **Cosserat** sense), moreover, which can be easily written down from (3.7) using (2.31):

$$e^{\alpha\beta}\,\nabla_\alpha\,\boldsymbol{\chi}_\beta = 0, \qquad e^{\alpha\beta}\,(\nabla_\alpha\,\boldsymbol{\varepsilon}_\beta + \mathbf{a}_\alpha \times \boldsymbol{\chi}_\beta) = 0. \tag{3.28}$$

If they are fulfilled then the integration of (3.7), when extended over a surface curve C (with running position vector **s**) that goes from P_0 (with position vector $\mathbf{r}_0$) to P (with position vector **r**), will produce the changes in position:

$$\left.\begin{aligned} \boldsymbol{\omega}(\mathbf{r}) &= \boldsymbol{\omega}(\mathbf{r}_0) + \int_{P_0}^{P} \boldsymbol{\chi}_\alpha\, dq^\alpha, \\ \mathbf{v}(\mathbf{r}) &= \mathbf{v}(\mathbf{r}_0) + \boldsymbol{\omega}(\mathbf{r}_0) \times \int_{P_0}^{P} [\varepsilon_\alpha - (\mathbf{r}-\mathbf{s}) \times \boldsymbol{\chi}_\alpha]\, dq^\alpha, \end{aligned}\right\} \tag{3.29}$$

which are independent of the special choice of connecting curve; the parts that have been integrated out are the rigid motions (3.2). If we omit them and observe the definition (3.5) of the deformations then we will have:

$$\boldsymbol{\omega}(\mathbf{r}) = \int_{P_0}^{P} (d\boldsymbol{\omega})_{relative}\,, \qquad \mathbf{v}\,(\mathbf{r}) = \int_{P_0}^{P} [(d\mathbf{v})_{relative} - (\mathbf{r} - \mathbf{s}) \times (d\boldsymbol{\omega})_{relative}]; \tag{3.30}$$

the changes in position at P are given, up to a rigid motion, by the kinematics of the relative changes of position between P_0 and P for that starting point.

We now associate each point of the surface curve C (which can be the boundary curve of the middle surface to the shell, in particular) with a dreibein of unit vectors: Let **n** be its unit normal vector that lies in the tangential plane, let $\mathbf{t} = d\mathbf{s} \,/\, ds$ be its unit tangent vector, and let **E** be the unit vector of the surface normal, as before. **n**, **t**, and **E**, in that sequence, shall define a right-handed system such that **n** will be the exterior normal vector when a surface element is circumnavigated in the positive sense. We decompose:

$$(d\boldsymbol{\omega})_{relative} = \boldsymbol{\chi}\, ds, \qquad (d\mathbf{v})_{relative} = \boldsymbol{\varepsilon}\, ds \tag{3.31}$$

along those directions:

$$\left.\begin{aligned} (d\boldsymbol{\omega})_{relative} &= (\chi_N ds)\,\mathbf{n} + (\chi_D ds)\,\mathbf{t} + (\chi_G ds)\,\mathbf{E}, \\ (d\mathbf{v})_{relative} &= (\gamma\, ds)\,\mathbf{n} + (\varepsilon\, ds)\,\mathbf{t} + (\beta\, ds)\,\mathbf{E}. \end{aligned}\right\} \tag{3.32}$$

In this: χ_N is the change in the normal curvature, χ_D is the change in torsion, and χ_G is the change in geodetic curvature of the integration curve as a result of the deformation of the surface, γ is its change in direction in the tangential plane, ε is its rotation, and $\beta = \varepsilon_\alpha\, t^\alpha$ is its change in direction in the **t** – **E**-plane.

We substitute (3.32) in (3.30), perform a partial integration in the second integral and drop the terms that have been integrated, which again represent rigid motions.

What finally remains is:

$$\left.\begin{aligned} \boldsymbol{\omega}(\mathbf{r}) &= \int_{P_0}^{P} (\chi_N \mathbf{n} + \chi_D \mathbf{t} + \chi_G \mathbf{E})\, ds, \\ \mathbf{v}(\mathbf{r}) &= \int_{P_0}^{P} \left\{ \varepsilon \mathbf{t} - (\mathbf{r} - \mathbf{s}) \left[(\chi_N \mathbf{n} + \chi_D \mathbf{t} + \chi_G \mathbf{E}) + \frac{\partial}{\partial s} (\beta \mathbf{n} - \gamma \mathbf{E}) \right] \right\} ds. \end{aligned}\right\} \tag{3.33}$$

Kinematically, this deformation means that the shear deformations β and γ do not appear in the calculation of the displacements from the deformations autonomously, but will be converted into "kinematically-equivalent supplementary distortions." However, we can calculate not only $\mathbf{v}$, but also (and this is important for the boundary-value problems) the change $\partial \mathbf{v} / \partial n$ perpendicular to the surface curve C: It is:

$$\frac{\partial \mathbf{v}}{\partial n} = n^\alpha \nabla_\alpha \mathbf{v} \qquad \text{with} \qquad n^i = \mathbf{n} \cdot \mathbf{a}^i . \tag{3.34}$$

When this is substituted in (3.7), that will give:

$$\mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial n} = \varepsilon_{(\alpha\beta)} n^\alpha n^\beta, \quad \mathbf{t} \cdot \frac{\partial \mathbf{v}}{\partial n} = \varepsilon_{\alpha\beta} n^\alpha t^\beta + \boldsymbol{\omega} \cdot \mathbf{E}, \quad \mathbf{E} \cdot \frac{\partial \mathbf{v}}{\partial n} = \varepsilon_\alpha n^\alpha + \boldsymbol{\omega} \cdot \boldsymbol{t}, \tag{3.35}$$

or when we employ the matrices:

$$\mathfrak{E} = \varepsilon_{\alpha\beta} \mathbf{a}^\alpha \mathbf{a}^\beta + \varepsilon_\alpha \mathbf{a}^\alpha \mathbf{E} \tag{3.36}$$

(viz., the deformation matrix), in dyadic representation, and:

$$\mathfrak{T} = \mathbf{E}\,\mathbf{t} - \mathbf{t}\,\mathbf{E},$$

we will have:

$$\frac{\partial \mathbf{v}}{\partial n} = \mathbf{n} \cdot \mathfrak{E} + \boldsymbol{\omega} \cdot \mathfrak{T}; \tag{3.37}$$

naturally, $\boldsymbol{\omega}$ is deduced from (3.33).

We shall now return to the compatibility conditions (3.28) and specify its tensor representation:

$$e^{\alpha\beta} (\nabla_\alpha \chi^l_{\beta\cdot} - b^l_\alpha \chi_\beta) = 0, \qquad e^{\alpha\beta} (\nabla_\alpha \chi_\beta + b_{\alpha\lambda}\, \chi^\lambda_{\beta\cdot}) = 0, \tag{3.38}$$

$$e^{\alpha\beta} (\nabla_\alpha \varepsilon_{\beta l} - b_{\alpha l}\, \varepsilon_\beta - e_{\alpha l}\, \chi_\beta) = 0, \qquad e^{\alpha\beta} (\nabla_\alpha \varepsilon_\beta + b^\lambda_\alpha \varepsilon_{\beta\lambda} + e_{\alpha\lambda} \chi^\lambda_{\beta\cdot}) = 0. \tag{3.39}$$

In place of the distortions $\boldsymbol{\chi}_i$, (for the sake of later developments) we would now like introduce other distortion quantities:

$$\boldsymbol{\kappa}^i = \kappa^{i\alpha} \mathbf{a}_\alpha + \kappa^i \mathbf{E} \tag{3.40}$$

or

$$\boldsymbol{\rho}_i = \rho_{i\alpha}\,\mathbf{a}^{\alpha} + \rho_i\,\mathbf{E}, \tag{3.41}$$

resp., by way of:

$$\boldsymbol{\kappa}^i = e^{\,i\alpha}\,\boldsymbol{\chi}_\alpha, \quad \boldsymbol{\chi}_i = e_{\alpha i}\,\boldsymbol{\kappa}^{\alpha} \tag{3.42}$$

or

$$\boldsymbol{\rho}_i = \boldsymbol{\chi}_i \times \mathbf{E} + (\boldsymbol{\chi}_i \cdot \mathbf{E})\,\mathbf{E}, \qquad \boldsymbol{\chi}_i = \mathbf{E} \times \boldsymbol{\rho}_i + (\mathbf{E} \cdot \boldsymbol{\rho}_i)\,\mathbf{E}, \tag{3.43}$$

resp. In terms of components, one will have:

$$\kappa^{il} = e^{i\alpha}\chi_{\alpha\cdot}^{\ l}, \quad \chi_{i\cdot}^{\ l} = e_{\alpha i}\,\kappa^{\alpha l}, \ \ \kappa^i = e^{i\alpha}\chi_\alpha, \quad \chi_i = e_{\alpha i}\,\kappa^{\alpha} \tag{3.44}$$

or

$$\rho_{il} = e_{l\alpha}\chi_{\alpha\cdot}^{\ l}, \quad \chi_{i\cdot}^{\ l} = e^{\alpha l}\,\rho_{i\alpha}, \quad \rho_i = \kappa_i, \quad \chi_i = \rho_i, \tag{3.45}$$

resp. The components of the distortions $\boldsymbol{\kappa}^i$ and $\boldsymbol{\rho}_i$ are then connected by:

$$\kappa^{il} = -\,e^{i\alpha}\,e^{l\beta}\,\rho_{\alpha\beta}, \qquad \rho_{il} = -\,e_{\alpha i}\,e_{\beta l}\,\kappa^{\alpha\beta}, \qquad \kappa^i = e^{i\alpha}\rho_\alpha, \quad \rho_i = e_{\alpha i}\,\kappa^{\alpha}. \tag{3.46}$$

[Our $\boldsymbol{\kappa}^i$ correspond to **Schaefer**'s $(-\boldsymbol{\kappa}^i)$ ([1]).]

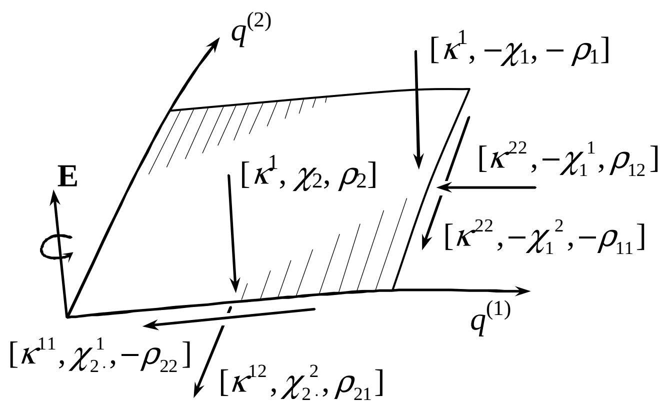

Figure 3. Schema of the distortions.

A *schematic* glimpse of the mutual relationships between the quantities $\boldsymbol{\chi}_i$, $\boldsymbol{\kappa}^i$ and $\boldsymbol{\rho}_i$ is given in Fig. 3; one sees that ρ_{il} is the distortion tensor that is usually employed in the theory of shells. [Our ρ_{il} corresponds approximately to ω_{il} in **Green** and **Zerna** ([2]) and to κ_{il} in **W. Flügge** ([3]).] We shall now summarize the kinematical equations for the quantities $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\kappa}_i$ ($\boldsymbol{\rho}_i$, resp.) that will be interesting to us in what follows. With the use of the $\boldsymbol{\kappa}_i$, that will be the total system of kinematical equations; that is already a part of our ultimate system of analogue shell equations (we shall characterize it by an asterisk):

$$\boldsymbol{\kappa}_i = e^{i\alpha}\,\nabla_\alpha\,\boldsymbol{\omega}, \qquad \boldsymbol{\varepsilon}_i = \nabla_i\,\mathbf{v} + \mathbf{a}_i \times \boldsymbol{\omega} \tag{3.47*}$$

or

([1]) See footnote 4 on Page 1.
([2]) See footnote 1 on page 2.
([3]) **W. Flügge**, *Statik und Dynamik der Schalen*, 2nd ed., Berlin-Göttingen-Heidelberg, 1957.

$$\kappa^{il} = e^{i\alpha}(\nabla_\alpha \omega^l - b_\alpha^l \omega), \qquad \kappa^i = e^{i\alpha}(\nabla_\alpha \omega + b_{\alpha\beta}\omega^\beta), \tag{3.48}^*$$

$$\varepsilon_{il} = \nabla_i v_l - b_{il} v - e_{il}\omega, \qquad \varepsilon_i = \nabla_i \mathbf{v} + b_i^\alpha v_\alpha + e_{i\alpha}\omega^\alpha, \tag{3.49}^*$$

resp., and the compatibility conditions:

$$\nabla_\alpha \boldsymbol{\kappa}^\alpha = 0, \qquad e^{\alpha\beta}\nabla_\alpha \boldsymbol{\kappa}_\beta + \mathbf{a}_\alpha \times \boldsymbol{\kappa}^\alpha = 0 \tag{3.50}^*$$

or

$$\nabla_\alpha \kappa^{\alpha l} - b_\alpha^l \boldsymbol{\kappa}^\alpha = 0, \qquad \nabla_\alpha \kappa^\alpha + b_{\alpha\beta}\kappa^{\alpha\beta} = 0, \tag{3.51}^*$$

$$e^{\alpha\beta}(\nabla_\alpha \varepsilon_{\beta l} - b_{\alpha l}\varepsilon_\beta) - \varepsilon_{\alpha l}\kappa^\alpha = 0, \qquad e^{\alpha\beta}(\nabla_\alpha \varepsilon_\beta + b_\alpha^\lambda \varepsilon_{\alpha\lambda}) + \varepsilon_{\alpha l}\kappa^{[\alpha\beta]} = 0, \tag{3.52}^*$$

resp. The first of equations (3.52) can be solved for the κ^l :

$$\kappa^l = e^{l\lambda} e^{\alpha\beta}(\nabla_\alpha \varepsilon_{\beta l} - b_{\alpha l}\varepsilon_\beta) . \tag{3.52}^*$$

These quantities are then established for a compatible state of deformation by the deformations ε_i and can, if so desired, be eliminated completely from the compatibility conditions.

We shall give the representation of the conventional distortion tensor ρ_{il} in terms of changes of position:

$$\rho_{il} = e_{l\alpha}(\nabla_i \omega^\alpha - b_i^\alpha \omega) \tag{3.54}$$

and further convert the second of the compatibility conditions (2.39) from them:

$$e^{\alpha\beta}(\rho_{\alpha\beta} - b_\alpha^\lambda \varepsilon_{\beta\lambda} - \nabla_\alpha \varepsilon_\beta) = 0. \tag{3.55}$$

That shows that the tensor:

$$\tilde{\rho}_{il} = \rho_{il} - b_i^\alpha \varepsilon_{l\alpha} - \nabla_i \varepsilon_l \tag{3.56}$$

is symmetric for a compatible state of deformation. The basis for that is easy to see: Namely, if we express the right-hand side of (3.56) in terms of changes of position using (3.49) and (3.14) then we will find that:

$$\tilde{\rho}_{il} = -\nabla_i\nabla_l v - b_i^\alpha \nabla_l v_\alpha - b_l^\alpha \nabla_i v_\alpha - \nabla_l b_i^\alpha \cdot v_\alpha + \varepsilon_{il} v, \tag{3.57}$$

and as a minor calculation will show, that is identical to:

$$\tilde{\rho}_{il} = -\nabla_i\nabla_l v \cdot \mathbf{E}, \tag{3.58}$$

with an obvious symmetry in the two indices. The fact that (3.58) goes to the tensor of plate curvatures when the shell degenerates into a planar plate also prompts an intuitive interpretation for the shell. In order to do that, we recall the representation (3.15) and (3.25) for the dimension vectors and the unit normal vector of the deformed shell and calculate the second fundamental tensor of the deformed middle surface:

$$b'_{il} = - \partial_i \mathbf{E}' \cdot \mathbf{a}'_l = - \nabla_i \, [\mathbf{E} - (\mathbf{E} \cdot \nabla_\alpha \mathbf{v}) \, \mathbf{a}^\alpha] \cdot [\mathbf{a}_l + \nabla_l \, \mathbf{v}]; \tag{3.59}$$

if we again neglect the products of displacements then a brief calculation will yield:

$$b'_{il} = b_{il} + \nabla_i \nabla_l \, \mathbf{v} \cdot \mathbf{E} = b_{il} - \tilde{\rho}_{il} \,. \tag{3.60}$$

The symmetric distortion tensor is then equal to the negative change in the second fundamental tensor of the middle surface as a result of its deformation.

4. Statics of the stress quantities. – We reduce the external loads on the shell (volume forces, forces on the soffits of the shell) for each element of the shell to forces **p** *df* and moments **q** *df* on its middle surface *df*. Along the boundary of the shell (we assume that the shell is simply bounded), we might also apply forces $d\mathbf{K} = \mathbf{K}\, ds$ and moments $d\mathbf{M} = \mathbf{M}\, ds$ to the boundary elements *ds* of the middle surface. Let the boundary be oriented in the sense that is given by (2.3). From the principle of virtual displacements, the negative virtual work that is done by internal forces and moments (hence, the stress quantities, here) during the deformation is equal to the virtual work that is done by the external forces and moments in the case of equilibrium. Naturally, the virtual changes of position here are those of our kinematical shell model. We shall now define *virtual deformations* by:

$$\delta\boldsymbol{\chi}_i = \nabla_i \, (\delta\boldsymbol{\omega}), \qquad \delta\boldsymbol{\varepsilon}_i = \nabla_i \, (\delta\mathbf{v}) + \mathbf{a}_i \times \delta\boldsymbol{\omega}, \tag{4.1}$$

and thus assume that the field of virtual changes of position is differentiable, and express the principle of the virtual displacements as:

$$\begin{aligned} -\,\delta A^{(i)} &= -\iint e^{\alpha\beta} \, (\mathbf{K}_\alpha \cdot \delta\boldsymbol{\varepsilon}_i + \mathbf{M}_\alpha \cdot \delta\boldsymbol{\chi}_\beta) \, df \\ &= \iint (\mathbf{p} \cdot \delta\mathbf{v} + \mathbf{q} \cdot \delta\boldsymbol{\omega}) \, df + \oint d\mathbf{K} \cdot \delta\mathbf{v} + d\mathbf{M} \cdot \delta\boldsymbol{\omega}, \end{aligned} \tag{4.2}$$

which introduces the *force stresses* $\mathbf{N}_i$ and *moment stresses* $\mathbf{M}_i$ into our kinematics. If we substitute (4.1) into (4.2) and convert it with **Stokes**’s theorem then that will give:

$$\begin{aligned} -\iint \{ (e^{\alpha\beta} \, \nabla_\alpha \mathbf{K}_\beta + \mathbf{p}) \cdot \delta\mathbf{v} + [e^{\alpha\beta} \, (\nabla_\alpha \mathbf{M}_\beta + \mathbf{a}_\alpha \times \mathbf{K}_b)] \cdot \delta\boldsymbol{\omega} \} \, df \\ = \oint \{ (d\mathbf{K} - \mathbf{K}_\alpha dq^\alpha) \cdot \delta\mathbf{v} + (d\mathbf{M} - \mathbf{M}_\alpha dq^\alpha) \cdot \delta\boldsymbol{\omega} \} . \end{aligned} \tag{4.3}$$

Since the virtual changes of position $\delta\mathbf{v}$ and $\delta\boldsymbol{\omega}$ can be chosen to have all of their components independent of each other for a **Cosserat** continuum, the equilibrium conditions that are valid inside the region will follow from that:

$$e^{\alpha\beta} \, \nabla_\alpha \mathbf{K}_\beta = -\,\mathbf{p}, \qquad e^{\alpha\beta} \, (\nabla_\alpha \mathbf{M}_\beta + \mathbf{a}_\alpha \times \mathbf{K}_\beta) = -\,\mathbf{q}, \tag{4.4}$$

and the relations:

$$d\mathbf{K} = \mathbf{K}_\alpha \, dq^\alpha, \qquad d\mathbf{M} = \mathbf{M}_\alpha \, dq^\alpha, \tag{4.5}$$

which are valid on the boundary of the region. If one takes a region that is bounded by parameter curves then one can read off the static interpretation of the stress quantities that we chose from (4.5):

$$\mathbf{K}_i = K_{i\cdot}^{\ \alpha}\mathbf{a}_\alpha + K_i\,\mathbf{E} \tag{4.6}$$

and

$$\mathbf{M}_i = M_{i\alpha}\mathbf{a}^\alpha + M_i\,\mathbf{E}. \tag{4.7}$$

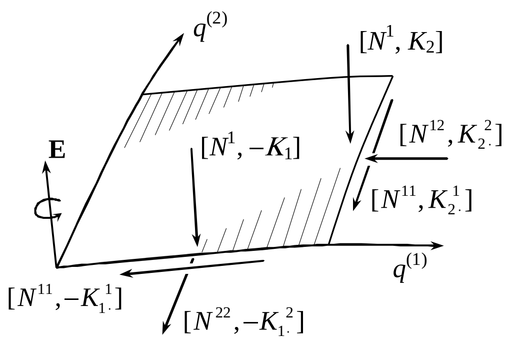

Figure 4. Schema of the forces on the shell.

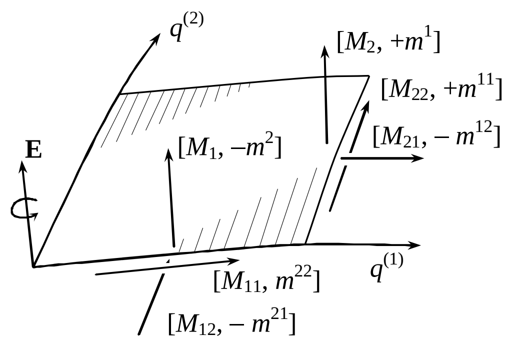

Figure 5. Schema of the moments on the shell.

(Cf., Figs. 4 and 5). In essence: The $K_{i\cdot}^{\ i}$ are the shear stresses on the membrane, the $K_{i\cdot}^{\ l}$ ($l \neq i$) are the longitudinal stresses on the membrane, and the K_i are the transverse force-stresses, while the M_{ii} are the bending moments, the M_{il} ($l \neq i$) are the drilling moments, and the M_i are the moments around the normal to the shell. The six-fold functional static indeterminacy of the equilibrium problem of the shell is expressed, on the one hand, by the fact that statically only the six equilibrium conditions (4.4) are compatible for the twelve stresses (4.6) and (4.7), and on the other hand, by the fact that the decomposition (4.5) of the boundary forces and moments is not uniquely determined, since the homogeneous equations:

$$\mathbf{K}_\alpha\, dq^\alpha = 0, \qquad \mathbf{M}_\alpha\, dq^\alpha = 0 \tag{4.8}$$

have three arbitrary scalar solutions. We shall now restrict ourselves to the case in which the shell is free of loads that are due to surface forces and moments, and therefore to only an *equilibrium system* of forces and moments; one can always get back to the general case by splitting off a particular solution. The equilibrium conditions:

$$e^{\alpha\beta}\,\nabla_\alpha \mathbf{K}_\beta = 0, \qquad e^{\alpha\beta}\,[\nabla_\alpha \mathbf{M}_\beta + \mathbf{a}_\alpha \times \mathbf{K}_\beta] = 0 \tag{4.9}$$

then have the form of the compatibility conditions (3.28), in which the relative changes in rotation $\boldsymbol{\chi}_i$ correspond to the force-stress $\mathbf{K}_i$, and the relative changes in displacement correspond to the moment-stresses $\mathbf{M}_i$. It will then follow that a system of stresses that satisfies the equilibrium conditions (4.9) can be represented in a manner that is analogous to (3.7) by vectorial stress functions $\boldsymbol{\Omega}$ and $\boldsymbol{\Phi}$ that correspond to the changes in position $\boldsymbol{\omega}$ and $\mathbf{v}$:

$$\mathbf{K}_i = \nabla_i\,\boldsymbol{\Omega}, \qquad \mathbf{M}_i = \nabla_i\,\boldsymbol{\Phi} + \mathbf{a}_i \times \boldsymbol{\Omega}\,. \tag{4.10}$$

[Equations (4.9) and (4.10) were derived in a different way by the author ([1]) in a non-covariant form.] The second of these equations can, in turn, be solved for **Ω** (its components, resp.) [cf., (3.12) to (3.14)]:

$$\boldsymbol{\Omega} = (\nabla_\alpha \boldsymbol{\Phi} - \mathbf{M}_\alpha) \cdot \mathfrak{A}^\alpha \tag{4.11}$$

or

$$\left.\begin{aligned} \Omega^i &= e^{i\alpha}[\nabla_\alpha \Phi + b_\alpha^\beta \Phi_\beta - M_\alpha], \\ \Omega &= \tfrac{1}{2} e^{\alpha\beta}[\nabla_\alpha \Phi_\beta - M_{[\alpha\beta]}], \end{aligned}\right\} \tag{4.12}$$

resp.; the symmetric part of the moment tensor does not actually enter into (4.11).

Now, if an equilibrium system of stresses in a boundary-loaded shell is given then, corresponding to (3.29), the associated vectorial stress functions can be calculated uniquely by integrating (4.10) when one is given their initial values:

$$\left.\begin{aligned} \boldsymbol{\Omega}(\mathbf{r}) &= \boldsymbol{\Omega}(\mathbf{r}_0) + \int_{P_0}^{P} \mathbf{K}_\alpha dq^\alpha, \\ \boldsymbol{\Phi}(\mathbf{r}) &= \boldsymbol{\Phi}(\mathbf{r}_0) + \boldsymbol{\Omega}(\mathbf{r}_0) \times (\mathbf{r} - \mathbf{r}_0) + \int_{P_0}^{P} [\mathbf{M}_\alpha - (\mathbf{r} - \mathbf{s}) \times \mathbf{K}_\alpha]\, dq^\alpha. \end{aligned}\right\} \tag{4.13}$$

The terms that were integrated out:

$$\overset{\circ}{\boldsymbol{\Omega}} = \boldsymbol{\Omega}(\mathbf{r}_0), \qquad \overset{\circ}{\boldsymbol{\Phi}} = \boldsymbol{\Phi}(\mathbf{r}_0) + \boldsymbol{\Omega}(\mathbf{r}_0) \times (\mathbf{r} - \mathbf{r}_0), \tag{4.14}$$

which correspond to the rigid motions (3.2), are the *zero-stress functions* of shell statics, since when they are substituted into (4.9), they will obviously create a stress-free state in the shell. We can also ascertain the boundary-values of the stress functions from boundary loads with the help of (4.10) and (4.5), moreover: If we start at one of its points then we can take the boundary curve to the path of integration and get:

$$\boldsymbol{\Omega}\,(\mathbf{r}_{\text{bdy.}}) = \int_{P_0}^{P} d\mathbf{K}, \qquad \boldsymbol{\Phi}\,(\mathbf{r}_{\text{bdy.}}) = \int_{P_0}^{P} [d\mathbf{M} - (\mathbf{r} - \mathbf{s}) \times d\mathbf{K}], \tag{4.15}$$

when we drop the zero-stress functions. These boundary values are then given by the dyname that is calculated for the reference point P and consists of the boundary forces and moments that act between P_0 and P. [Compare that with (3.30).] We now once more decompose things along the orthogonal unit vectors **n**, **t**, **E**:

$$\left.\begin{aligned} d\mathbf{K} &= \mathbf{K}\, ds = (L\, ds)\,\mathbf{n} + (T\, ds)\,\mathbf{t} + (Q\, ds)\,\mathbf{E}, \\ d\mathbf{M} &= \mathbf{M}\, ds = (M_D\, ds)\,\mathbf{n} + (M_B\, ds)\,\mathbf{t} + (M_N\, ds)\,\mathbf{E}; \end{aligned}\right\} \tag{4.16}$$

([1]) **W. Günther**, Abh. Braunschw. Wiss. Ges. **8** (1956), pp. 111.

In this, one has, when referred to a unit of curve length on the boundary: L is the longitudinal force on the membrane, T is the shear force on the membrane, Q is the transverse force, M_D is the drill moment, M_B is the bending moment, and M_N is the moment around the normal to the shell. Just as in (3.33), one will then get:

$$\left.\begin{aligned} \mathbf{\Omega}(\mathbf{r}_{bdy}) &= \int_{P_0}^{P} (L\,\mathbf{n} + T\,\mathbf{t} + Q\,\mathbf{E})\,ds, \\ \mathbf{\Phi}(\mathbf{r}_{bdy}) &= \int_{P_0}^{P} \{M_B\mathbf{t} - (\mathbf{r}-\mathbf{s})\times[(L\,\mathbf{n} + T\,\mathbf{t} + Q\,\mathbf{E}) + \frac{\partial}{\partial s}(M_N\mathbf{n} - M_D\mathbf{E})]\}\,ds, \end{aligned}\right\} \tag{4.17}$$

up to trivial zero-stress functions.

Statically, that conversion means that the moment contributions M_N and M_D will not appear by themselves in the calculation of the boundary values of the stress function from a given equilibrium system of boundary loads, but will be converted into a statically-equivalent *supplementary force*. In the theory of plates ($M_N \equiv 0$, $\partial\mathbf{E} / \partial s \equiv 0$), that is ***Kirchoff****'s supplementary force* $Q^* = -\,\partial M_D / \partial s$. Furthermore, we can calculate the normal derivative $\partial\mathbf{\Phi} / \partial n$, say, for the boundary curve. We will get:

$$\left.\begin{aligned} \mathbf{n}\cdot\frac{\partial\Phi}{\partial n} &= M_{(\alpha\beta)}\,n^\alpha n^\beta, \\ \mathbf{t}\cdot\frac{\partial\Phi}{\partial n} &= M_{\alpha\beta}\,n^\alpha t^\beta + \mathbf{\Omega}\cdot\mathbf{E}, \\ \mathbf{E}\cdot\frac{\partial\Phi}{\partial n} &= M_\alpha\,n^\alpha - \mathbf{\Omega}\cdot\mathbf{t}, \end{aligned}\right\} \tag{4.18}$$

or, in dyadic matrix notation:

$$\mathfrak{M} = M_{\alpha\beta}\,a^\alpha a^\beta + M_\alpha\,a^\alpha\,\mathbf{E}, \tag{4.19}$$

$$\frac{\partial\Phi}{\partial n} = \mathbf{n}\cdot\mathfrak{M} + \mathbf{\Omega}\cdot\mathfrak{T}, \tag{4.20}$$

resp., in which $\mathbf{\Omega}$ is inferred from (4.17). [Confer formulas (3.35) to (3.37) for this.]

Our next step consists of transforming the force-stresses $\mathbf{K}_i$ (4.6) by way of:

$$\mathbf{N}^i = e^{i\alpha}\,\mathbf{K}_\alpha, \qquad\qquad \mathbf{K}_i = e_{\alpha i}\,\mathbf{N}^\alpha, \tag{4.21}$$

or, in terms of components:

$$\left.\begin{aligned} N^{il} &= e^{i\alpha}K_{\alpha\cdot}^{\;l}, \quad & K_{i\cdot}^{\;l} &= e_{\alpha i}N^{\alpha l}, \\ N^{i} &= e^{i\alpha}K_{\alpha}, \quad & K_{i} &= e_{\alpha i}N^{\alpha}, \end{aligned}\right\} \tag{4.22}$$

corresponding to (3.42). With that, we have introduced the shell forces N^{il} (membrane stresses) and N^i (transverse stress), which are useful in the theory of shells. We then get the following static equations, which correspond to the kinematical relations (3.47) to (3.52), and belong with them, along with our system of analogous shell equations:

$$N^i = e^{i\alpha}\,\nabla_\alpha \mathbf{\Omega}, \qquad M_i = \nabla_i\,\mathbf{\Phi} + \mathbf{a}_i \times \mathbf{\Omega}, \tag{4.23}^*$$

or

$$\left.\begin{aligned} N^{il} &= e^{i\alpha}(\nabla_\alpha \Omega^l - b_\alpha^l\,\Omega), \\ N^i &= e^{i\alpha}(\nabla_\alpha \Omega + b_{\alpha\beta}\,\Omega^\beta), \end{aligned}\right\} \tag{4.24}^*$$

$$\left.\begin{aligned} M_{il} &= \nabla_{\alpha i}\,\Omega_l - b_{il}\,\Phi - e_{il}\,\Omega), \\ M_i &= \nabla_l\,\Phi + b_i^\alpha \Phi_\alpha + e_{i\alpha}\,\Omega^\alpha), \end{aligned}\right\} \tag{4.25}^*$$

$$\nabla_\alpha \mathbf{N}^\alpha = 0, \qquad e^{\alpha\beta}\,\nabla_\alpha \mathbf{M}_\beta + \mathbf{a}_\alpha \times \mathbf{N}^\alpha = 0, \tag{4.26}^*,$$

resp. In components:

$$\nabla_\alpha N^{\alpha l} - b_\alpha^l\; N^\alpha = 0, \qquad \nabla_\alpha N^\alpha + b_{\alpha\beta}\, N^{\alpha\beta} = 0, \tag{4.27}^*$$

$$\left.\begin{aligned} e^{\alpha\beta}[\nabla_\alpha M_{\beta l} - b_{\alpha l} M_\beta] - e_{\alpha l} N^\alpha &= 0, \\ e^{\alpha\beta}[\nabla_\alpha M_\beta + b_\alpha^\lambda M_{\beta\lambda}] + e_{\alpha\beta} N_{[\alpha\beta]} &= 0. \end{aligned}\right\} \tag{4.28}^*$$

The first of equations (4.28) can be solved for the transverse forces N^i :

$$N^i = e^{l\lambda}\, e^{\alpha\beta}\,[\nabla_\alpha M^{\beta\lambda} - b_{\alpha\lambda}\, M_\beta], \tag{4.29}^*$$

so those quantities will already be established by the moments M_i in the equilibrium case, and can, if so desired, be eliminated from the equilibrium conditions completely.

For the ultimate connection to the conventional representation of shell theory (in which, analogies are still known only very little), we also convert the shell moments $\mathbf{M}_i$ into new moments $\mathbf{m}_i$ [cf., (3.44)]:

$$\left.\begin{aligned} \mathbf{m}^i &= e^{i\alpha}[\mathbf{M}_\alpha \times \mathbf{E} + (\mathbf{M}_\alpha \cdot \mathbf{E})\mathbf{E}], \\ \mathbf{M}_i &= e_{\alpha i}[\mathbf{E} \times \mathbf{m}^\alpha + (\mathbf{E} \cdot \mathbf{m}^\alpha)\mathbf{E}], \end{aligned}\right\} \tag{4.30}$$

or, in components:

$$\left.\begin{aligned} m^{il} &= \mathbf{m}^i \cdot \mathbf{a}^l = e^{i\alpha} e^{l\beta} M_{\alpha\beta}, \\ m^i &= \mathbf{m}^i \cdot \mathbf{E} = e^{i\alpha} M_\alpha, \\ M_{il} &= \mathbf{M}_i \cdot \mathbf{a}_l = e_{\alpha i}\, e_{\beta l}\, m^{\alpha\beta}, \\ M_i &= \mathbf{M}_i \cdot \mathbf{E} = e_{\alpha i}\;\; M^\alpha. \end{aligned}\right\} \tag{4.31}$$

The equilibrium conditions (4.26) are then converted into:

$$\nabla_\alpha N^{\alpha l} - b_\alpha^l\; N^\alpha = 0, \qquad \nabla_\alpha N^\alpha + b_{\alpha\beta}\, N^{\alpha\beta} = 0, \tag{4.32}$$

$$\left.\begin{aligned} \nabla_\alpha m^{\alpha l} - N^l + e^{\alpha l} b_{\alpha\beta}\, m^\beta &= 0, \\ \nabla_\alpha m^\alpha + e_{\alpha\beta}[N^{\alpha\beta} + b_\lambda^\beta m^{\lambda\alpha}] &= 0. \end{aligned}\right\} \tag{4.33}$$

which is how they are given in, e.g., **Green** and **Zerna**, but without the moments m^i. Figs. 4 and 5 show *schematically* the mutual arrangement of the stress quantities that were defined in this section.

Finally, we shall give two easily-verified forms for the negative virtual work that is done by stress quantities:

$$\left.\begin{aligned} -\delta A^{(i)} &= \iint (\mathbf{N}^\alpha \cdot \delta\boldsymbol{\varepsilon}_\alpha - \mathbf{M}_\alpha \cdot \delta\boldsymbol{\kappa}^\alpha)\, df\,, \\ -\delta A^{(i)} &= \iint (\mathbf{N}^\alpha \cdot \delta\boldsymbol{\varepsilon}_\alpha + \mathbf{m}^\alpha \cdot \delta\boldsymbol{\rho}_\alpha)\, df\,. \end{aligned}\right\} \tag{4.34}$$

Naturally, we can also arrive at the equilibrium conditions that correspond to these expressions.

5. Theory of embeddings: geometry, statics, kinematics. – The equations that were found up to now still do not encompass the entire shell problem; they must be extended by a material law. However, we know such a material law only for the spatial point continuum – namely, **Hooke**'s law, in the linear elastic case. We must then express the spatial deformations in terms of the middle surface and condense the spatial stresses into the stress quantities, which is only meaningful for thin shells. In order to do that, we introduce a coordinate $q^{(3)}$ that is perpendicular to the middle surface of the shell and associate each point of the shell space with the position vector:

$$\mathbf{R}\,(q^{(1)}, q^{(2)}\,; z) = \mathbf{r}\,(q^{(1)}, q^{(2)}) + z\,\mathbf{E}\,(q^{(1)}, q^{(2)}), \qquad -\frac{t}{2} \le z \le +\frac{t}{2}\,. \tag{5.1}$$

(We shall once more adopt the notations that were employed by **Green** and **Zerna**.) The dimension vectors of this spatial coordinate system (as usual, the indices $i, l, \ldots; \alpha, \beta, \ldots$ assume the values 1 and 2) are:

$$\left.\begin{aligned} \mathbf{g}_i &= \partial_i \mathbf{R} = \partial_i \mathbf{r} + z\,\partial_i \mathbf{E} = \mathbf{a}_i - z\,\mathbf{b}_i\,, \\ \mathbf{g}_3 &= \partial_3 \mathbf{R} = \mathbf{E}, \end{aligned}\right\} \tag{5.2}$$

which implies that the covariant spatial metric tensor is:

$$\left.\begin{aligned} g_{il} &= \mathbf{g}_i \cdot \mathbf{g}_l = a_{il} - 2z\,b_{il} + z^2 c_{il} = g_{(il)}\,, \\ g_{i3} &= \mathbf{g}_i \cdot \mathbf{E} = 0, \\ g_{33} &= \mathbf{E} \cdot \mathbf{E} = 1. \end{aligned}\right\} \tag{5.3}$$

The contravariant spatial dimension vectors $\mathbf{g}^l$, as the solutions to the equations:

$$\mathbf{g}^l \cdot \mathbf{g}_l = \delta_i^l\,, \tag{5.4}$$

will become a power series in z:

$$\mathbf{g}^l = \mathbf{a}^l + z\,\mathbf{b}^l + z^2 b_\alpha^l\;\mathbf{b}^\alpha + \ldots \tag{5.5}$$

Moreover, one has:

$$\mathbf{g}^3 = \mathbf{E}, \tag{5.6}$$

such that the contravariant spatial metric tensor will have the components:

$$\left.\begin{aligned} g^{il} &= \mathbf{g}^i \cdot \mathbf{g}^l = a^{il} + 2z\, b^{il} + 3z^2 c^{il} + \cdots, \\ g^{i3} &= \mathbf{g}^i \cdot \mathbf{E} = 0, \\ g^{33} &= \mathbf{E} \cdot \mathbf{E} = 1. \end{aligned}\right\} \tag{5.7}$$

If:

$$g = g_{11}\, g_{22} - (g_{12})^2 \tag{5.8}$$

is the determinant of the spatial metric tensor then one will have:

$$h = \sqrt{\frac{g}{a}} = 1 - 2Hz + Kz^2 + \ldots; \tag{5.9}$$

h is the ratio of the volume:

$$df\,(q^{(1)}, q^{(2)}\,; z) = (\mathbf{g}_1\, dq^1 \times \mathbf{g}_2\, dq^2) \cdot \mathbf{E} \tag{5.10}$$

of a surface element that is parallel to the middle surface to the volume:

$$df\,(q^{(1)}, q^{(2)}\,; 0) = (\mathbf{a}_1\, dq^1 \times \mathbf{a}_2\, dq^2) \cdot \mathbf{E} \tag{5.11}$$

of the corresponding element of the middle surface.

We now begin with the aforementioned reduction and concentrate the spatial stresses τ^{il}, τ^{i3} (we assume that τ^{33} vanishes everywhere) to the stress quantities of the shell: The sectional force that acts upon an element $\overset{l}{df} = \sqrt{g_{ii}}\; dq^i\, dz$ ($i \neq l$) is:

$$d\overset{l}{\mathbf{K}} = (\tau^{\alpha\beta}\, \mathbf{g}_\beta + \tau^{\alpha 3}\, \mathbf{E})\; \overset{l}{n}_\alpha\, \overset{l}{df}\,; \tag{5.12}$$

in this:

$$\overset{l}{n}_\alpha = \overset{l}{\mathbf{n}} \cdot \mathbf{g}_\alpha$$

is the spatial (!) decomposition of the unit normal vector $\overset{l}{\mathbf{n}}$ to the surface $\overset{l}{df}$, when referred to the section:

$$\overset{1}{\mathbf{n}} = \frac{g^1}{\sqrt{g^{11}}} = \frac{\sqrt{g}\,\mathbf{g}^1}{\sqrt{g_{22}}}, \qquad \overset{2}{\mathbf{n}} = -\frac{g^2}{\sqrt{g^{22}}} = -\frac{\sqrt{g}\,\mathbf{g}^2}{\sqrt{g_{11}}}, \tag{5.13}$$

which can be summarized in:

$$\overset{l}{\mathbf{n}}_\alpha = \frac{e_{\lambda i}}{\sqrt{a}}\frac{\sqrt{g}\,\mathbf{g}^\lambda}{\sqrt{g_{ii}}} = e_{\lambda i}\,h\,\frac{g^\lambda}{\sqrt{g_{ii}}} \qquad (i \neq l). \tag{5.14}$$

Therefore, one will have:

$$\overset{l}{n}_\alpha = e_{\alpha i}\,h\,\frac{1}{\sqrt{g_{ii}}} \tag{5.15}$$

and

$$d\overset{l}{\mathbf{K}} = e_{\alpha i}\,(h\,\tau^{\alpha\beta}\,\mathbf{g}_\beta + h\,\tau^{\alpha 3}\,\mathbf{E})\,dq^i\,dz. \tag{5.16}$$

With **Green** and **Zerna**, we let:

$$\tau^{\alpha\beta}\,\mathbf{g}_\beta = \sigma^{\alpha\beta}\,\mathbf{a}_\beta, \qquad \sigma^{\alpha\beta} = \tau^{\alpha\lambda}\,\mathbf{g}_\lambda \cdot \mathbf{a}^\beta = \tau^{\alpha\beta} - z\,b_\lambda^\beta\,\tau^{\alpha\lambda} \tag{5.17}$$

define a (generally asymmetric) *reduced stress tensor* σ^{il} ; we will then have:

$$d\overset{l}{\mathbf{K}} = e_{\alpha i}\,(h\,\sigma^{\alpha\beta}\,\mathbf{a}_\beta + h\,\tau^{\alpha 3}\,\mathbf{E})\,dq^i\,dz. \tag{5.18}$$

From (4.5), we will have:

$$d\overset{i}{s} = \sqrt{a_{ii}}\;dq^i \tag{5.19}$$

on the sectional element of the middle surface:

$$\mathbf{K}_i\,dq^i = \int_{(z)} d\overset{l}{\mathbf{K}} \qquad (i \neq l), \tag{5.20}$$

so

$$\mathbf{K}_i = e_{\alpha i}\left(\int_{(z)} h\,\sigma^{\alpha\beta}\,dz\right)\mathbf{a}_\beta + e_{\alpha i}\left(\int_{(z)} h\,\tau^{\alpha 3}\,dz\right)\mathbf{E}. \tag{5.21}$$

Upon comparing this to (4.6), (4.21), we will get:

$$\mathbf{N}^i = \left(\int_{(z)} h\,\sigma^{i\alpha}\,dz\right)\mathbf{a}_\alpha + \left(\int_{(z)} h\,\tau^{i3}\,dz\right)\mathbf{E}\,; \tag{5.22}$$

in terms of components, that is:

$$N^{il} = \int_{(z)} h\,\sigma^{il}\,dz\,, \qquad N^i = \int_{(z)} h\,\tau^{i3}\,dz\,. \tag{5.23}$$

One similarly finds that:

$$d\overset{l}{\mathbf{M}} = z\,\mathbf{E} \times d\overset{l}{\mathbf{K}} = e_{i\alpha}\, e_{\lambda\beta}\, z\, h\, \sigma^{\alpha\beta}\, \mathbf{a}^{\lambda}\, dq^i\, dz \qquad (i \neq l). \tag{5.24}$$

With:

$$\mathbf{M}_i\, dq^i = \int_{(z)} d\overset{l}{\mathbf{M}} \qquad (i \neq l), \tag{5.25}$$

it will then follow that:

$$\mathbf{M}_i = e_{i\alpha}\, e_{\lambda\beta} \left(\int_{(z)} z\, h\, \sigma^{\alpha\beta}\, dz \right) \mathbf{a}^{\lambda}, \tag{5.26}$$

or in components:

$$M_{il} = e_{i\alpha}\, e_{\lambda\beta} \int_{(z)} z\, h\, \sigma^{\alpha\beta}\, dz, \qquad M_i = 0. \tag{5.27}$$

If one compares this with (4.31) then one will get:

$$m^{il} = \int_{(z)} z\, h\, \sigma^{il}\, dz \tag{5.28}$$

for the conventional moment tensor that is induced by the stresses. The moments M_i around the normal to the shell vanish in this theory of stresses that is derived from the (nonsingular!) stresses in the spatial point-continuum, and naturally the same thing will be true for their equivalent moments m^i. The second of the equilibrium conditions (4.28) will then degenerate into an algebraic relation:

$$e_{\alpha\beta}\, N^{[\alpha\beta]} = e^{\alpha\beta}\, b^{\lambda}_{\beta}\, M_{\alpha\lambda}, \tag*{(5.29)*}$$

and the second of the equilibrium conditions (4.33) will degenerate into the known equation:

$$2\, M^{[il]} = N^{il} - N^{li} = b^i_{\alpha}\, m^{\alpha l} - b^l_{\alpha}\, m^{\alpha i}. \tag{5.30}$$

Furthermore, the surface components Ω^i of the stress function $\boldsymbol{\Omega}$ now lose their autonomy. From (4.12), they will then be dependent upon the stress function $\boldsymbol{\Phi}$:

$$\Omega^i = e^{i\alpha}\, [\nabla_{\alpha}\Phi + b^{\beta}_{\alpha}\, \Phi_{\beta}], \tag*{(5.31)*}$$

and form that, one will have:

$$\boldsymbol{\Omega} \cdot \mathbf{t} = -\frac{\partial \Phi}{\partial n} \cdot \mathbf{E}, \tag*{(5.32)*}$$

which can also be inferred from (4.18).

Finally, with (4.31), (4.29) will go to the equation:

$$N^{l} = e^{l\lambda}\, e^{\alpha\beta}\, \nabla_{\alpha} M_{\beta\lambda} = \nabla_{\alpha}\, m^{\alpha l}, \tag*{(5.33)*}$$

which formally agrees with the corresponding plate equation.

We shall now turn to the spatial deformations, which we would like to return to under the assumption on the deformation of the middle surface of the shell that its transverse shear part ε_i vanishes [that assumption is necessary only if one is to establish the analogy with equation (5.27)], with which, the lateral forces N^i will become reaction forces. It will then follow from $\varepsilon_i = 0$, as we can adopt with no further calculation from the corresponding static equations, that:

$$e_{\alpha\beta}\,\kappa^{[\alpha\beta]} = e^{\alpha\beta}\,b_\beta^\lambda\,\varepsilon_{\alpha\lambda}\,; \tag{5.34}^*$$

this algebraic compatibility condition corresponds to the algebraic equilibrium condition (5.29). Moreover, in analogy to (5.31), one will have:

$$\omega^i = e^{i\alpha}\,[\nabla_\alpha v + b_\alpha^\beta\; v_\alpha]. \tag{5.35}^*$$

The rotations around the tangents to the coordinate lines to the middle surface will then depend upon the displacement $\boldsymbol{\upsilon}$. Finally, in agreement with (5.32), one will have:

$$\boldsymbol{\omega}\cdot\mathbf{t} = -\frac{\partial v}{\partial n}\cdot\mathbf{E} \tag{5.36}^*$$

for a curve on the surface. The static relation (5.33) corresponds to the kinematical one:

$$\kappa^l = e^{l\lambda}\,e^{\alpha\beta}\,\nabla_\alpha\,\varepsilon_{\beta\lambda}\,, \tag{5.37}^*$$

and the symmetric distortion tensor $\tilde{\rho}_{il}$ that was introduced in (3.56) will finally reduce to:

$$\tilde{\rho}_{il} = \rho_{il} - b_i^\alpha\,\varepsilon_{l\alpha}\,; \tag{5.38}$$

it follows from this that:

$$2\,\rho_{[il]} = \rho_{il} - \rho_{li} = b_i^\alpha\,\varepsilon_{l\alpha} - b_l^\alpha\,\varepsilon_{i\alpha} \tag{5.39}$$

[which one might compare with (5.30)], and that:

$$\tilde{\rho}_{il} = \rho_{(il)} - \tfrac{1}{2}(b_i^\alpha\,\varepsilon_{l\alpha} + b_l^\alpha\,\varepsilon_{i\alpha}). \tag{5.40}$$

The spatial deformations γ_{il} are now defined by:

$$\gamma_{il} = \gamma_{[il]} = \tfrac{1}{2}(g'_{il} - g_{il})\,, \tag{5.41}$$

in which g'_{il} is the spatial metric tensor after the deformation. Together with (5.3), that will imply:

$$\gamma_{il} = \tfrac{1}{2}(a'_{il} - a_{il}) - z(b'_{il} - b_{il}) + z^2\cdot\tfrac{1}{2}(c'_{il} - c_{il})\,. \tag{5.42}$$

If we recall (3.16) and (3.60) then what will next arise is:

$$\gamma_{il} = \varepsilon_{(il)} + z\,\tilde{\rho}_{il} + z^2 \cdot \tfrac{1}{2}(c'_{il} - c_{il})\,. \tag{5.43}$$

Now, we have:

$$c'_{il} = b_i^{\beta'} b'_{l\beta} = a^{\alpha\beta'} b'_{i\alpha} b'_{l\beta} = (a^{\alpha\beta} - 2\varepsilon^{[\alpha\beta]})(b_{i\alpha} - \tilde{\rho}_{i\alpha})(b_{l\beta} - \tilde{\rho}_{l\beta}), \tag{5.44}$$

in which (3.19) was employed. On the basis of (5.38), one will then have, approximately:

$$c'_{il} - c_{il} = -\,(b_i^\alpha \rho_{l\alpha} + b_l^\alpha \rho_{i\alpha}) \tag{5.45}$$

so:

$$\gamma_{il} = \varepsilon_{(il)} + z\,\tilde{\rho}_{il} - z^2 \cdot \tfrac{1}{2}(b_i^\alpha \rho_{l\alpha} + b_l^\alpha \rho_{i\alpha})\,. \tag{5.46}$$

The weighted mean values of γ_{il} will be:

$$\left.\begin{aligned} \tilde{\varepsilon}_{il} &= \frac{1}{t}\int_{(z)} \gamma_{il}\,dz = \varepsilon_{(il)} - \frac{t^2}{12}\cdot\frac{1}{2}(b_i^\alpha \rho_{l\alpha} + b_l^\alpha \rho_{i\alpha}), \\ \tilde{\rho}_{il} &= \frac{12}{t^3}\int_{(z)} z\,\gamma_{il}\,dz = \rho_{(il)} - \frac{1}{2}(b_i^\alpha \varepsilon_{l\alpha} + b_l^\alpha \varepsilon_{i\alpha}). \end{aligned}\right\} \tag{5.47}$$

If one considers (5.39) and (5.40) then this result will suggest that one might set the antisymmetric part $\varepsilon_{[il]}$ of the tensor ε_{il} equal to:

$$2\,\varepsilon_{[il]} = \varepsilon_{il} - \varepsilon_{il} = \frac{t^2}{12}(b_i^\alpha \rho_{l\alpha} - b_l^\alpha \rho_{i\alpha})\,, \tag{5.48}$$

or when written terms of $\boldsymbol{\kappa}^i$:

$$e^{\alpha\beta}\,\varepsilon_{[\alpha\beta]} = \frac{t^2}{12} e_{\alpha\beta}\, b_\lambda^\alpha\, \kappa^{\lambda\beta}\,, \tag{5.49*}$$

resp., if we assume that this assumption does contradict the law of elasticity that we still need to present. (As we shall show, that is not the case.) With the assumption (5.49), the rotation ω around the normal to the shell will also become dependent upon the displacement vector **v** now; a lengthy calculation will show that:

$$\omega = \frac{1}{1 - \dfrac{t^2}{12}K}\cdot\frac{1}{2} e^{\alpha\beta}\nabla_\alpha\left[v_\beta - \frac{t^2}{12} b_\beta^\lambda(\nabla_\lambda v + b_\lambda^\mu\, v_\mu\right]. \tag{5.50}$$

For thin shells, it is always permissible to ignore quantities of the form $t \cdot b_i^l$ as being so small in comparison to 1 that one needs to include them at most to the first power. One will then have:

$$\omega \approx \tfrac{1}{2} e^{\alpha\beta} \nabla_\alpha \left(v_\beta - \frac{t^2}{12} b_\beta^\lambda \nabla_\lambda v \right). \tag{5.51}$$

It is simpler, and also probably necessary for practical calculations, as a rule, to assume that the tensor ε_{il} is symmetric:

$$\varepsilon_{il} = \varepsilon_{(il)} \qquad (\varepsilon_{[il]} = 0, \text{ resp.}). \tag{5.52}$$

That leads directly to:

$$\omega = \tfrac{1}{2}\, e^{\alpha\beta} \nabla_\alpha v_\beta\,; \tag{5.53}$$

ω will then be the *mean rotation* that exists in the deformation of the membrane. With $\varepsilon_i = 0$ and (5.52), our **Cosserat** continuum will ultimately degenerate into an ordinary point-continuum, in which only the self-sufficient meaning of the displacement vector **v** will still be relevant.

We easily find from (3.26) that the spatial shear deformations γ_{i3} are:

$$\gamma_{i3} = \varepsilon_\alpha\, \mathbf{a}^\alpha \cdot \mathbf{g}^i\,. \tag{5.54}$$

They therefore vanish at the same time as the ε_i . Ultimately, it follows from (3.26) that the deformation:

$$\gamma_{i3} = \mathbf{g}_3' \cdot \mathbf{g}_3' - 1 \tag{5.55}$$

will vanish in our model in any case.

6. The complete shell equations. – The law of elasticity of an isotropic material in the linear-elastic domain reads:

$$\tau^{il} = \frac{E}{1-\nu^2} [(1-\nu)\, g^{i\alpha} g^{l\beta} + \nu\, g^{\alpha\beta} g^{il}]\, \gamma_{\alpha\beta}\,, \tag{6.1}$$

if we assume, as usual, that the stress state is planar, and thus ignore the fact that this contradicts the kinematical equation $\gamma_{33} = 0$. We easily obtain the following formula for the reduced stress tensor σ^{il} from (6.1), along with (5.17):

$$h\, \sigma^{il} = \frac{E}{1-\nu^2} [(1-\nu)\, g^{i\alpha} (h\, \mathbf{g}^\beta \cdot \mathbf{a}^l) + \nu\, g^{\alpha\beta} (h\, \mathbf{g}^i \cdot \mathbf{a}^l)]\, \gamma_{\alpha\beta}\,, \tag{6.2}$$

in which we now develop everything on the right in powers of z and once more keep terms of the form $z b_i^l$ only up to the first power. We then remark that the asymmetry (5.48) of the tensor ε_{il} of membrane deformations will no longer be regarded in that approximation, such that it can be assumed to be symmetric. After integrating over the shell thickness t according to (5.23) and (5.28), the following equations will represent the material law for the shell:

$$N^{il} = \frac{1-\nu^2}{E\,t}\Big\{(1-\nu)\,a^{i\alpha}a^{l\beta}\left[\varepsilon_{(\alpha\beta)} + \frac{t^2}{12}\left(\frac{3}{2}b^\lambda_\alpha\rho_{\lambda\beta} + \frac{1}{2}b^\lambda_\beta\rho_{\lambda\alpha} - 2H\,\rho_{\alpha\beta}\right)\right]$$
$$+\ \nu\left[a^{il}\varepsilon + \frac{t^2}{12}[a^{il}b^{\alpha\beta}\rho_{\alpha\beta} + b^{il}\rho - 2Ha^{il}\rho)\right]\Big\}, \tag{6.3}$$

$$m^{il} = \frac{E\,t^3}{12(1-\nu^2)}\Big\{(1-\nu)\,a^{i\alpha}a^{l\beta}\left[\rho_{\alpha\beta} + \left(b^\lambda_\alpha\varepsilon_{(\lambda\beta)} + b^\lambda_\beta\varepsilon_{(\lambda\alpha)} - 2H\,\varepsilon_{(\alpha\beta)}\right)\right]$$
$$+\ \nu\left[a^{il}\rho + [a^{il}b^{\alpha\beta}\varepsilon_{(\alpha\beta)} + b^{il}\varepsilon - 2Ha^{il}\varepsilon)\right]\Big\}, \tag{6.4}$$

with

$$\varepsilon = a^{\alpha\beta}\,\varepsilon_{\alpha\beta}\,, \qquad \rho = a^{\alpha\beta}\,\rho_{\alpha\beta}\,. \tag{6.5}$$

Now, since the antisymmetric part of the longitudinal force tensor is already given by the shell moments from (5.30), we construct the symmetric part of (6.3):

$$N^{[il]} = \frac{E\,t}{1-\nu^2}\Big\{(1-\nu)\,a^{i\alpha}a^{l\beta}\left[\varepsilon_{(\alpha\beta)} + \frac{t^2}{12}\left(b^\lambda_\alpha\rho_{\lambda\beta} + b^\lambda_\beta\rho_{\lambda\alpha} - 2H\,\rho_{\alpha\beta}\right)\right]$$
$$+\ \nu\left[a^{il}\varepsilon + \frac{t^2}{12}[a^{il}b^{\alpha\beta}\rho_{\alpha\beta} + b^{il}\rho - 2Ha^{il}\rho)\right]\Big\}. \tag{6.6}$$

If we then introduce the geometric tensors:

$$P^{il,\,rs} = P^{(il),\,rs} = (b^{ir} - H\,a^{ir})\,a^{ls} + (b^{lr} - H\,a^{lr})\,a^{is} = \overset{*}{b}{}^{ir}a^{ls} + \overset{*}{b}{}^{lr}a^{is} \tag{6.7}$$

and

$$Q^{il,\,rs} = Q^{(il),\,(rs)} = (b^{il} - H\,a^{il})\,a^{rs} + (b^{rs} - H\,a^{rs})\,a^{il} = \overset{*}{b}{}^{il}a^{rs} + \overset{*}{b}{}^{rs}a^{il} \tag{6.8}$$

then we will have:

$$N^{(il)} = \frac{E\,t}{1-\nu^2}\left\{(1-\nu)\left[a^{i\alpha}a^{l\beta}\varepsilon_{(\alpha\beta)} + \frac{t^2}{12}P^{il,\alpha\beta}\rho_{\alpha\beta}\right] + \nu\left[a^{il}\varepsilon + \frac{t^2}{12}Q^{il,\alpha\beta}\rho_{\alpha\beta}\right]\right\}, \tag{6.9}$$

$$m^{il} = \frac{E\,t^3}{12(1-\nu^2)}\left\{(1-\nu)\left[a^{i\alpha}a^{l\beta}\rho_{\alpha\beta} + P^{il,\alpha\beta}\varepsilon_{(\alpha\beta)}\right] + \nu\left[a^{il}\rho + Q^{il,\alpha\beta}\varepsilon_{(\alpha\beta)}\right]\right\}. \tag{6.10}$$

One observes that the tensor $\overset{*}{b}{}^{il}$ that was just introduced is the deviator of the second fundamental tensor in the coordinates of the lines of principle curvature:

$$\overset{*}{b}{}^{11} = a^{11} \cdot \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right), \qquad \overset{*}{b}{}^{22} = a^{22} \cdot \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \qquad \overset{*}{b}{}^{12} = \overset{*}{b}{}^{21} = 0. \tag{6.11}$$

The problem of the inversion of the material law can be solved easily starting from (6.9), (6.10). Namely, as one can confirm by substitution, within the scope of what we have neglected:

$$\rho_{il} = \frac{12}{E t^3}\left\{(1+\nu)\left[a_{i\alpha}a_{l\beta}m^{\alpha\beta} - \frac{t^2}{12}P_{il,\alpha\beta}N^{(\alpha\beta)}\right] - \nu\left[a_{il}m - \frac{t^2}{12}Q_{il,\alpha\beta}N^{(\alpha\beta)}\right]\right\}, \tag{6.12}$$

$$\varepsilon_{(il)} = \frac{1}{E t}\left\{(1+\nu)\left[a_{i\alpha}a_{l\beta}\,N^{(\alpha\beta)} - P_{il,\alpha\beta}m^{\alpha\beta}\right] - \nu\left[a_{il}\,N - Q_{il,\alpha\beta}m^{\alpha\beta}\right]\right\}, \tag{6.13}$$

with

$$m = a_{\alpha\beta}\,m^{\alpha\beta}, \qquad N = a_{\alpha\beta}\,N^{\alpha\beta}. \tag{6.14}$$

Here, we might comment upon the asymmetry of the moment tensor m^{il} : From (6.10), one has:

$$2\,m^{[il]} = m^{il} - m^{li} = \frac{t^2}{12}(b^i_\alpha N^{\alpha l} - b^l_\alpha N^{\alpha i}), \tag{6.16}$$

which exhibits the analogy to the algebraic equilibrium condition (5.30), as well as to the kinematic equations (5.39) and (5.48). The scalar part of the stress function $\mathbf{\Omega}$ will also be established by (6.16): In analogy with (5.51), one will have:

$$\Omega = \frac{1}{2}e^{\alpha\beta}\nabla_\alpha\left(\Phi_\beta - \frac{t^2}{12}b^\lambda_\beta\,\nabla_\lambda\Phi\right); \tag{6.17}$$

moreover, $\mathbf{\Omega}$ is determined completely by $\mathbf{\Phi}$, along with (5.31). One sees that in order to evaluate the symmetry behavior of the moment tensor, we must add a law of elasticity and a compatibility condition. That connection gets blurred for a planar plate: For it, the symmetry of the moment tensor can be inferred from the symmetry of the spatial shear stresses alone, which can, in their own right, be generally based upon only the special kinematics of the spatial point-continuum. [**D. Rüdiger** ([1]) has discussed the asymmetry of the moment tensor in a somewhat different context.] The usual simplification:

$$m^{[il]} = 0 \tag{6.18}$$

corresponds to the assumption (4.52). It has the equation for the scalar part of Ω:

$$\Omega = \tfrac{1}{2}e^{\alpha\beta}\,\nabla_\alpha\,\Phi_\beta \tag{6.19}$$

([1]) **D. Rüdiger**, Ing.-Arch. **28** (1959), pp. 281.

as a consequence, in analogy to (5.53).

We shall proceed with our treatment of the equations of elasticity and convert them into moments M_{il} and distortions κ^{il}, although we would like to spare ourselves the details of the calculation. With the geometric tensors:

$$\left.\begin{aligned} U^{il}_{\dots rs} &= \delta^i_r \overset{*}{b}{}^l_r + \delta^l_r \overset{*}{b}{}^i_r - 2\overset{*}{b}{}^{il} a_{rs}, \\ V^{il}_{\dots rs} &= a^{il} \overset{*}{b}_{rs} - \overset{*}{b}{}^{il} a_{rs}, \\ \tilde{U}^{il}_{\dots rs} &= \overset{*}{b}{}^i_r\, \delta^l_r + \overset{*}{b}{}^l_r\, \delta^i_r - 2a^{il} \overset{*}{b}_{rs}, \\ \tilde{V}^{il}_{\dots rs} &= \overset{*}{b}{}^i_r\, a_{rs} - a^{il} \overset{*}{b}_{rs} = -V^{il}_{\dots rs}, \end{aligned}\right\} \tag{6.20}$$

if we symbolically introduce the matrices:

$$\mathfrak{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \overset{*}{\mathfrak{B}} = \begin{pmatrix} \overset{*}{b}{}^1_1 & \overset{*}{b}{}^2_1 \\ \overset{*}{b}{}^1_2 & \overset{*}{b}{}^2_2 \end{pmatrix} \tag{6.21}$$

for the mixed forms of the first fundamental tensor and the deviator of the second fundamental tensor, resp., then the dualities:

$$\tilde{U}^{il}_{\dots rs}(\mathfrak{D}, \mathfrak{B}) = U^{il}_{\dots rs}(\mathfrak{D}, \mathfrak{B}), \qquad \tilde{V}^{il}_{\dots rs}(\mathfrak{D}, \mathfrak{B}) = V^{il}_{\dots rs}(\mathfrak{D}, \mathfrak{B}), \tag{6.22}$$

will exist, and the law of elasticity will now read:

$$\left.\begin{aligned} N^{(il)} &= \frac{Et}{1-\nu^2}\left[(1-\nu)\left(a^{i\alpha}a^{l\beta}\varepsilon_{(\alpha\beta)} + \frac{t^2}{12}U^{il}_{\dots\alpha\beta}\,\kappa^{\alpha\beta}\right) + \nu\left(a^{il}\varepsilon + \frac{t^2}{12}V^{il}_{\dots\alpha\beta}\,\kappa^{\alpha\beta}\right)\right], \\ M_{il} &= -\frac{Et^3}{12(1-\nu^2)}\left[(1-\nu)\left(a_{i\alpha}\,a_{l\beta}\,\kappa^{\alpha\beta} + \tilde{U}_{il\dots}^{\ \ \alpha\beta}\,\varepsilon_{(\alpha\beta)}\right) + \nu\left(a_{il}\,\kappa + \frac{t^2}{12}V_{il\dots}^{\ \ \alpha\beta}\,\varepsilon_{(\alpha\beta)}\right)\right]; \end{aligned}\right\} \tag{6.23*}$$

$$\left.\begin{aligned} \kappa^{il} &= -\frac{12}{Et^3}\left[(1+\nu)\left(a^{i\alpha}a^{l\beta}M_{\alpha\beta} + \frac{t^2}{12}\tilde{U}^{il}_{\dots\alpha\beta}\,N^{(\alpha\beta)}\right) - \nu\left(a^{il}M + \frac{t^2}{12}\tilde{V}^{il}_{\dots\alpha\beta}\,N^{[\alpha\beta]}\right)\right], \\ \varepsilon_{(il)} &= \frac{1}{Et}\left[(1+\nu)\left(a_{i\alpha}\,a_{l\beta}\,N^{[\alpha\beta]} + U_{il\dots}^{\ \ \alpha\beta}\,M_{\alpha\beta}\right) - \nu\left(a_{il}\,N + V_{il\dots}^{\ \ \alpha\beta}\,M_{\alpha\beta}\right)\right], \end{aligned}\right\} \tag{6.24*}$$

with:

$$\kappa = a_{\alpha\beta}\,\kappa^{\alpha\beta}, \qquad M = a^{\alpha\beta}\,M_{\alpha\beta}. \tag{6.25}$$

These get combined with the equilibrium conditions:

$$\left.\begin{aligned} \nabla_\alpha N^{\alpha l} - e^{\alpha\lambda} e^{\beta\mu} \nabla_\beta M_{\mu\lambda} &= 0, \\ e^{\alpha\lambda} e^{\beta\mu} \nabla_\alpha \nabla_\beta M_{\mu\lambda} + b_{\alpha\beta} N^{\alpha\beta} &= 0, \\ e_{\alpha\beta} N^{(\alpha\beta)} + e^{\alpha\beta}\, b_\alpha^\lambda M_{\beta\lambda} &= 0, \end{aligned}\right\} \qquad (6.26)^*$$

whose solutions can be derived from a stress function **Φ**:

$$\left.\begin{aligned} N^{il} &= e^{i\alpha} e^{l\beta} [\nabla_\alpha \nabla_\beta \Phi + \nabla_\alpha (b_\beta^\lambda \Phi_\lambda)] - \tfrac{1}{2} e^{i\alpha} e^{\lambda\beta} b_\alpha^l \nabla_\lambda \Phi_\beta, \\ M_{il} &= \tfrac{1}{2}(\nabla_i \Phi_l + \nabla_l \Phi_i) - b_{il}\,\Phi + \underline{\frac{t^2}{12}\cdot\frac{1}{2}(b_l^\alpha \nabla_i - b_i^\alpha \nabla_l)\nabla_\alpha \Phi}. \end{aligned}\right\} \qquad (6.27)^*$$

The underlined term will drop away when the moment tensor is assumed to be symmetric. Similarly, the deformations, which satisfy the compatibility conditions:

$$\left.\begin{aligned} \nabla_\alpha \kappa^{\alpha\beta} - e^{\alpha\lambda} e^{\beta\mu} b_\alpha^l \nabla_\beta \varepsilon_{\mu\lambda} &= 0, \\ e^{\alpha\lambda} e^{\beta\mu} \nabla_\alpha \nabla_\beta \varepsilon_{\mu\lambda} + b_{\alpha\beta} \kappa^{\alpha\beta} &= 0, \\ e_{\alpha\beta} \kappa^{[\alpha\beta]} + e^{\alpha\beta} b_\alpha^\lambda \nabla_\beta \varepsilon_{\beta\lambda} &= 0, \end{aligned}\right\} \qquad (6.28)^*$$

can be derived from a displacement vector *v*:

$$\left.\begin{aligned} \kappa^{il} &= e^{i\alpha} e^{l\beta} [\nabla_\alpha \nabla_\beta v + \nabla_\alpha (b_\beta^\lambda v_\lambda)] - \tfrac{1}{2} e^{i\alpha} e^{\lambda\beta} b_\alpha^l \nabla_\lambda v_\beta, \\ \varepsilon_{il} &= \tfrac{1}{2}(\nabla_i v_l + \nabla_l v_i) - b_{il}\, v + \underline{\frac{t^2}{12}\cdot\frac{1}{2}(b_l^\alpha \nabla_i - b_i^\alpha \nabla_l)\nabla_\alpha v}. \end{aligned}\right\} \qquad (6.27)^*$$

The underlined term will drop away when the tensor ε_{il} is assumed to be symmetric.

The entire longitudinal force tensor is also determined completely by the symmetric part of ε_{il} and the moment tensor. One calculates it from:

$$N^{il} = \frac{E\,t}{1-v^2}\left[(1-v)\left(a^{i\alpha} a^{l\beta} \varepsilon_{(\alpha\beta)} + \frac{t^2}{12} W^{il}_{\ldots\alpha\beta}\,\kappa^{\alpha\beta}\right) + v\left(a^{il}\varepsilon + \frac{t^2}{12} V^{il}_{\ldots\alpha\beta}\,\kappa^{\alpha\beta}\right)\right], \qquad (6.30)$$

with

$$W^{il}_{\ldots\alpha\beta} = U^{il}_{\ldots\alpha\beta} + \tfrac{1}{2}(b_s^i\,\delta_r^l - b_s^l\,\delta_r^i)\,. \qquad (6.31)$$

A formal analogy can also be exhibited for the distortions κ^{il} ; however, that is not required, since the law of elasticity will produce the full tensor directly here.

The elastic energy can also be given very easily now. The deformation energy is:

$$\Pi\,[\varepsilon_{(il)},\, \kappa^{(rs)}] = \frac{1}{2}\iint \frac{E\,t}{1-v^2}[(1-v)\,a^{\alpha\lambda} a^{\beta\mu} + v\,a^{\alpha\beta} a^{\lambda\mu}]\,\varepsilon_{(\alpha\beta)}\,\varepsilon_{(\lambda\mu)}\,df$$

$$+\ \frac{1}{2}\iint \frac{E\,t^3}{12(1-\nu^2)}\{[(1-\nu)\,a_{\alpha\lambda}a_{\beta\mu}+\nu\,a_{\alpha\beta}\,a_{\lambda\mu}]\,\kappa^{(\alpha\beta)}\kappa^{(\lambda\mu)}$$
$$+\ 2[(1-\nu)U^{\alpha\beta}_{\dots\lambda\mu}+\nu V^{\alpha\beta}_{\dots\lambda\mu}]\,\varepsilon_{(\alpha\beta)}\,\kappa^{(\lambda\mu)}\}\,df\ , \tag{6.32*}$$

and the stress energy is:

$$\Pi[\bar{M}_{il},\bar{N}^{rs}] = \frac{1}{2}\iint \frac{12}{E\,t^3}[(1+\nu)\,a^{\alpha\lambda}a^{\beta\mu}-\nu\,a^{\alpha\beta}a^{\lambda\mu}]\,M_{(\alpha\beta)}\,M_{(\lambda\mu)}\,df$$
$$+\ \frac{1}{2}\iint \frac{1}{E\,t}\{[(1+\nu)\,a_{\alpha\lambda}a_{\beta\mu}-\nu\,a_{\alpha\beta}\,a_{\lambda\mu}]\,N^{(\alpha\beta)}N^{(\lambda\mu)}$$
$$+\ 2[(1+\nu)\tilde{U}^{\alpha\beta}_{\dots\lambda\mu}-\nu\tilde{V}^{\alpha\beta}_{\dots\lambda\mu}]\,M_{(\alpha\beta)}\,N^{(\lambda\mu)}\}\,df\ . \tag{6.33*}$$

In conclusion, we would like to discuss the question of boundary conditions. The virtual change of the deformation energy is equal to the negative virtual work done by the stress quantities, such that we will get from (4.2) that:

$$\delta\Pi\,[\mathbf{v},\,\boldsymbol{\omega}] = \delta\Pi\,[\varepsilon_{il}\,(\mathbf{v},\,\boldsymbol{\omega}),\ \kappa^{rs}\,(\boldsymbol{\omega})] = -\,\delta A^{(i)}$$
$$= \iint e^{\alpha\beta}\{\mathbf{K}_\beta\cdot[\nabla_\alpha(\delta\mathbf{v})+\mathbf{a}_\alpha\times\delta\boldsymbol{\omega}]+\mathbf{M}_\beta\cdot\nabla_\alpha(\delta\boldsymbol{\omega})\}\,df\ . \tag{6.34}$$

If the equilibrium conditions are fulfilled for the stress quantities then only one boundary expression will remain:

$$\delta\Pi\,[\mathbf{v},\,\boldsymbol{\omega}] = \delta\Pi\,[\mathbf{v},\,\boldsymbol{\omega}]_{\text{bdy}} = \oint e^{\alpha\beta}(\mathbf{K}_\beta\cdot\delta\mathbf{v}+\mathbf{M}_\beta\cdot\delta\boldsymbol{\omega})\,n_\alpha ds\,, \tag{6.35}$$

$$\delta\Pi\,[\mathbf{v},\,\boldsymbol{\omega}] = \oint(\mathbf{K}_\alpha\,t^\alpha\cdot\delta\mathbf{v}+\mathbf{M}_\alpha\,t^\alpha\cdot\delta\boldsymbol{\omega})\,ds\,. \tag{6.36}$$

Now, one has:

$$t^{\,i} = \frac{dq^i}{ds}\,, \tag{6.37}$$

and since:

$$\mathbf{M}_i\cdot\mathbf{E} = 0,\qquad \boldsymbol{\varepsilon}_i\cdot\mathbf{E} = 0,$$

(6.36) can be converted into:

$$\delta\Pi\,[\mathbf{v}] = \oint[\mathbf{K}\cdot\delta\mathbf{v}+\mathbf{M}\cdot\delta\boldsymbol{\omega}(\mathbf{v})]\,ds = \oint[\mathbf{K}\cdot\delta\mathbf{v}-(\mathbf{e}^\alpha\cdot\mathbf{M})(\nabla_\alpha(\delta\mathbf{v})\cdot\mathbf{E}]\,ds\,, \tag{6.38}$$

with the help of (3.12) and (4.16). Moreover, one has:

$$\mathbf{e}^\alpha = n^\alpha\,\mathbf{t} - t^\alpha\,\mathbf{n}. \tag{6.39}$$

That implies:

$$\delta\Pi\,[\mathbf{v}] = \oint\left\{[\mathbf{K}+(M_D\mathbf{E})]\cdot\frac{\partial(\delta\mathbf{v})}{\partial s}-(M_B\mathbf{E})\cdot\frac{\partial(\delta\mathbf{v})}{\partial n}\right\}ds\,, \tag{6.40}$$

and finally, after a partial integration:

$$\delta\Pi\,[\mathbf{v}] = \oint\left\{\left[\mathbf{K} - \frac{\partial}{\partial s}(M_D\mathbf{E})\right]\cdot\delta\mathbf{v} - (M_B\mathbf{E})\cdot\frac{\partial(\delta\mathbf{v})}{\partial n}\right\}ds\,, \tag{6.41*}$$

with the supplementary force in the integrand that has been known since (4.17). In addition, we have again introduced the stress quantities; they are:

$$M_D = \mathbf{M}\cdot\mathbf{n} = M_{\alpha\beta}\,t^\alpha\,n^\beta, \qquad M_B = \mathbf{M}\cdot\mathbf{t} = M_{(\alpha\beta)}\,t^\alpha\,t^\beta, \tag{6.42}$$

and furthermore, from (4.5), (4.16), and (4.21):

$$\mathbf{K} = \mathbf{K}_\alpha\,t^\alpha = \mathbf{N}^\alpha\,n_\alpha = (N^{\alpha\beta}\,\mathbf{a}_\beta + N^\alpha\,\mathbf{E})\,n_\alpha\,. \tag{6.43}$$

We can develop the boundary formula for the virtual change of the stress energy (6.33) from (4.34) in an entirely analogous way. If the stress quantities define an equilibrium system then the result will be:

$$\delta\Pi\,[M_{il}(\mathbf{\Phi}),\,N^{rs}(\mathbf{\Phi})] = -\oint\left\{\left[\boldsymbol{\kappa} - \frac{\partial}{\partial s}(\boldsymbol{\gamma}\mathbf{E})\right]\cdot\delta\mathbf{\Phi} - (\varepsilon\,\mathbf{E})\cdot\frac{\partial(\delta\mathbf{\Phi})}{\partial n}\right\}ds\,, \tag{6.44*}$$

with

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}^\alpha\,n_\alpha = (\kappa^{\alpha\beta}\,\mathbf{a}_\beta + \kappa^\alpha\,\mathbf{E})\,n_\alpha\,, \tag{6.45}$$

and from (3.31), (3.32), (3.5), we will have:

$$\gamma = \boldsymbol{\varepsilon}\cdot\mathbf{n} = \varepsilon_{\alpha\beta}\,t^\alpha\,n^\beta, \qquad \varepsilon = \boldsymbol{\varepsilon}\cdot\mathbf{t} = \varepsilon_{(\alpha\beta)}\,t^\alpha\,t^\beta. \tag{6.46}$$

7. The shell equations in the coordinates of the lines of curvature. Cylindrical and spherical shell. Membrane stress state. – If it is even possible, we choose the parameter curves in the middle surface to be the lines of curvature; in particular, we would do that when the boundary of the shell is a line of curvature or consists of them piecewise. Since we are dealing with an orthogonal net, we can simplify our notation somewhat; we set:

$$a_{11} = \alpha_1\,, \qquad a_{22} = \alpha_2\,,$$

and that will make:

$$a^{11} = \frac{1}{\alpha_1}\,, \qquad a^{22} = \frac{1}{\alpha_2}\,, \qquad a = \alpha_1\,\alpha_2\,. \tag{7.1}$$

The three-index symbols that are required for covariant differentiation are:

$$\left.\begin{aligned}
&\Gamma_{11}^{1} = \frac{1}{2}\frac{1}{\alpha_1}\partial_1\alpha_1, \quad \Gamma_{12}^{1} = \frac{1}{2}\frac{1}{\alpha_1}\partial_2\alpha_1, \quad \Gamma_{22}^{1} = -\frac{1}{2}\frac{1}{\alpha_1}\partial_1\alpha_2, \\
&\Gamma_{11}^{2} = -\frac{1}{2}\frac{1}{\alpha_2}\partial_2\alpha_1, \quad \Gamma_{12}^{2} = \frac{1}{2}\frac{1}{\alpha_2}\partial_1\alpha_2, \quad \Gamma_{22}^{2} = \frac{1}{2}\frac{1}{\alpha_2}\partial_2\alpha_2.
\end{aligned}\right\} \tag{7.2}$$

In what follows, the geometric tensors (6.20) and (6.31) will be further composed, in which the signs will be chosen according to (2.27):

$$\left.\begin{aligned}
&U^{11}_{\dots 11} = U^{11}_{\dots 12} = U^{11}_{\dots 21} = 0, \\
&U^{11}_{\dots 22} = \frac{\alpha_2}{\alpha_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right); \\
&U^{12}_{\dots 12} = U^{12}_{\dots 22} = 0, \\
&U^{12}_{\dots 12} = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \quad U^{12}_{\dots 21} = -2\left(\frac{1}{R_1} - \frac{1}{R_2}\right); \\
&U^{22}_{\dots 11} = -\frac{\alpha_1}{\alpha_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \\
&U^{22}_{\dots 12} = U^{22}_{\dots 21} = U^{22}_{\dots 22} = 0; \\
&V^{11}_{\dots 11} = V^{11}_{\dots 12} = V^{11}_{\dots 21} = 0; \\
&V^{11}_{\dots 22} = \frac{\alpha_2}{\alpha_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \\
&V^{12}_{\dots rs} = 0, \\
&V^{22}_{\dots 11} = -\frac{\alpha_1}{\alpha_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \\
&V^{22}_{\dots 12} = V^{22}_{\dots 21} = V^{22}_{\dots 22} = 0. \\
&\tilde{U}^{il}_{\dots rs} = -U^{il}_{\dots rs}, \quad \tilde{V}^{il}_{\dots rs} = -V^{il}_{\dots rs}; \\
&\tilde{U}_{il\dots}^{rs} = \frac{\alpha_i\,\alpha_l}{\alpha_r\,\alpha_s}U^{il}_{\dots rs}, \\
&\qquad \text{(corresponding expressions for the remaining quantities)} \\
&W^{12}_{\dots 12} = \frac{1}{2R_1}, \qquad W^{12}_{\dots 21} = -\frac{1}{2}\left(\frac{2}{R_1} - \frac{1}{R_2}\right), \\
&W^{21}_{\dots 11} = \frac{1}{2}\left(\frac{1}{R_1} - \frac{2}{R_2}\right), \quad W^{21}_{\dots 21} = \frac{1}{2R_2}; \\
&\text{otherwise:} \\
&W^{il}_{\dots rs} = U^{il}_{\dots rs}.
\end{aligned}\right\} \tag{7.3}$$

It will then follow that:

$$\left.\begin{aligned}
N^{11} &= \frac{E\,t}{1-v^2}\frac{1}{\alpha_1\,\alpha_2}\left[\varepsilon_{11}+v\frac{\alpha_1}{\alpha_2}\varepsilon_{22}+\frac{t^2}{12}\alpha_1\alpha_2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\kappa^{22}\right],\\
N^{(12)} &= \frac{E\,t}{1-v^2}(1-v)\frac{1}{\alpha_1\,\alpha_2}\left[\varepsilon_{(12)}+\frac{t^2}{12}\alpha_1\alpha_2\,\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(\kappa^{12}-\kappa^{21})\right]\approx\frac{E\,t}{1-v^2}(1-v)\frac{1}{\alpha_1\,\alpha_2}\varepsilon_{(12)},\\
N^{22} &= \frac{E\,t}{1-v^2}\frac{1}{\alpha_1\,\alpha_2}\left[\varepsilon_{22}+v\frac{\alpha_2}{\alpha_1}\varepsilon_{11}-\frac{t^2}{12}\alpha_1\alpha_2\,\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\kappa^{11}\right];
\end{aligned}\right\}\quad(7.4)^*$$

$$\left.\begin{aligned}
M_{11} &= -\frac{E\,t^3}{12(1-v^2)}\alpha_1\,\alpha_2\left[\kappa^{11}+v\frac{\alpha_2}{\alpha_1}\kappa^{22}-\frac{1}{\alpha_1\alpha_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\varepsilon_{22}\right],\\
M_{12} &= -\frac{E\,t^3}{12(1-v^2)}(1-v)\alpha_1\,\alpha_2\kappa^{12},\qquad M_{21}=-\frac{E\,t^3}{12(1-v^2)}(1-v)\alpha_1\,\alpha_2\kappa^{21},\\
M_{22} &= -\frac{E\,t^3}{12(1-v^2)}\alpha_1\,\alpha_2\left[\kappa^{22}+v\frac{\alpha_1}{\alpha_2}\kappa^{11}-\frac{1}{\alpha_1\alpha_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\varepsilon_{11}\right].
\end{aligned}\right\}\quad(7.5)^*$$

with the following inversion:

$$\left.\begin{aligned}
\kappa^{11} &= -\frac{12}{E\,t^3}\frac{1}{\alpha_1\,\alpha_2}\left[M_{11}-v\frac{\alpha_1}{\alpha_2}M_{22}-\frac{t^2}{12}\alpha_1\alpha_2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)N^{22}\right],\\
\kappa^{12} &= -\frac{12}{E\,t^3}(1+v)\frac{1}{\alpha_1\,\alpha_2}M_{12},\qquad \kappa^{21}=-\frac{12}{E\,t^3}(1+v)\frac{1}{\alpha_1\,\alpha_2}M_{21},\\
\kappa^{22} &= -\frac{12}{E\,t^3}\frac{1}{\alpha_1\,\alpha_2}\left[M_{22}-v\frac{\alpha_2}{\alpha_1}M_{11}+\frac{t^2}{12}\alpha_1\alpha_2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)N^{11}\right];
\end{aligned}\right\}\quad(7.6)^*$$

$$\left.\begin{aligned}
\varepsilon_{11} &= \frac{1}{E\,t}\alpha_1\,\alpha_2\left[N^{11}-v\frac{\alpha_2}{\alpha_1}N^{22}+\frac{1}{\alpha_1\alpha_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)M_{22}\right],\\
\varepsilon_{(12)} &= \frac{1}{E\,t}(1+v)\,\alpha_1\,\alpha_2\left[N^{(12)}+\frac{1}{\alpha_1\alpha_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(M_{12}-M_{21})\right]\approx\frac{1}{E\,t}(1+v)\,\alpha_1\,\alpha_2N^{(12)},\\
\varepsilon_{22} &= \frac{1}{E\,t}\alpha_1\,\alpha_2\left[N^{22}-v\frac{\alpha_1}{\alpha_2}N^{11}-\frac{1}{\alpha_1\alpha_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)M_{11}\right].
\end{aligned}\right\}\quad(7.7)^*$$

Furthermore, one has:

$$\left.\begin{aligned}
N^{12} &= \frac{E\,t}{1-v^2}(1-v)\frac{1}{\alpha_1\,\alpha_2}\left\{\varepsilon_{(12)}+\frac{t^2}{12}\alpha_1\alpha_2\cdot\frac{1}{2}\left[\frac{1}{R_1}\kappa^{12}-\left(\frac{2}{R_1}-\frac{1}{R_2}\right)\kappa^{21}\right]\right\},\\
N^{21} &= \frac{E\,t}{1-v^2}(1-v)\frac{1}{\alpha_1\,\alpha_2}\left\{\varepsilon_{(12)}+\frac{t^2}{12}\alpha_1\alpha_2\,\frac{1}{2}\left[\frac{1}{R_2}\kappa^{21}+\left(\frac{1}{R_1}-\frac{2}{R_2}\right)\kappa^{12}\right]\right\}.
\end{aligned}\right\}\quad(7.8)$$

Since we can neglect the antisymmetric part $\kappa^{[il]}$ of κ^{il} in this context, we can write (7.8) somewhat more intuitively as:

$$\left.\begin{aligned} N^{12} &= \frac{E\,t}{1-\nu^2}(1-\nu)\frac{1}{\alpha_1\,\alpha_2}\left[\varepsilon_{(12)} - \frac{t^2}{12}\alpha_1\alpha_2\cdot\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\kappa^{12}\right],\\ N^{21} &= \frac{E\,t}{1-\nu^2}(1-\nu)\frac{1}{\alpha_1\,\alpha_2}\left[\varepsilon_{(12)} + \frac{t^2}{12}\alpha_1\alpha_2\,\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\kappa^{21}\right]. \end{aligned}\right\} \tag{7.9}$$

One can split the equilibrium condition (6.26) in the best way using (4.27), (4.28), since the formulas would become too opaque otherwise:

$$\left.\begin{aligned} &\partial_1 N^{11} + \partial_2 N^{21} + 2\,\Gamma^1_{11}N^{11} + \Gamma^1_{12}(N^{12}+2N^{21})\\ &\qquad + \Gamma^1_{22}N^{22} + \Gamma^2_{12}N^{11} + \Gamma^2_{22}N^{21} + \frac{1}{R_1}N^1 = 0,\\ &\partial_1 N^{12} + \partial_2 N^{22} + 2\,\Gamma^2_{22}N^{22} + \Gamma^2_{12}(N^{21}+2N^{11})\\ &\qquad + \Gamma^2_{11}N^{11} + \Gamma^1_{12}N^{22} + \Gamma^1_{11}N^{12} + \frac{1}{R_2}N^2 = 0; \end{aligned}\right\} \tag{7.10}$$

$$\left.\begin{aligned} N^1 &= \frac{1}{\alpha_1\alpha_2}(\partial_1 M_{22} - \partial_2 M_{12} + \Gamma^1_{11}M_{11} + \Gamma^2_{22}M_{12} - \Gamma^1_{12}M_{21} - \Gamma^2_{12}M_{22}),\\ N^2 &= \frac{1}{\alpha_1\alpha_2}(\partial_2 M_{11} - \partial_1 M_{21} + \Gamma^2_{22}M_{22} + \Gamma^1_{11}M_{21} - \Gamma^2_{12}M_{12} - \Gamma^1_{12}M_{11}); \end{aligned}\right\} \tag{7.11}$$

$$\partial_1 N^1 + \partial_2 N^2 + (\Gamma^1_{11}+\Gamma^2_{12})N^1 + (\Gamma^1_{12}+\Gamma^2_{22})N^2 - \frac{1}{R_1}N^{11} - \frac{1}{R_2}N^{22} = 0. \tag{7.12}$$

Finally, from (6.29), one has:

$$\left.\begin{aligned} \kappa^{11} &= \frac{1}{\alpha_1\,\alpha_2}\left[\partial_2\partial_2 v + \Gamma^2_{22}\left(\frac{1}{R_2}v_2 - \partial_2 v\right) - \Gamma^2_{22}\,\partial_2 v - \partial_2\left(\frac{1}{R_2}v_2\right)\right],\\ \kappa^{12} &= -\frac{1}{\alpha_1\,\alpha_2}\left[\partial_1\partial_2 v + \Gamma^1_{12}\left(\frac{1}{R_2}v_1 - \partial_1 v\right) - \Gamma^2_{12}\,\partial_2 v - \frac{1}{R_2}\cdot\frac{1}{2}(\partial_1 v_2 - \partial_2 v_1) - \partial_2\left(\frac{1}{R_2}v_2\right)\right],\\ \kappa^{21} &= -\frac{1}{\alpha_1\,\alpha_2}\left[\partial_1\partial_2 v + \Gamma^2_{12}\left(\frac{1}{R_2}v_2 - \partial_2 v\right) - \Gamma^1_{12}\,\partial_1 v - \frac{1}{R_1}\cdot\frac{1}{2}(\partial_2 v_1 - \partial_1 v_2) - \partial_1\left(\frac{1}{R_2}v_2\right)\right],\\ \kappa^{22} &= \frac{1}{\alpha_1\,\alpha_2}\left[\partial_1\partial_1 v + \Gamma^1_{11}\left(\frac{1}{R_1}v_1 - \partial_1 v\right) - \Gamma^2_{11}\,\partial_2 v - \partial_1\left(\frac{1}{R_1}v_1\right)\right]; \end{aligned}\right\} \tag{7.13}$$

$$\left.\begin{aligned}
\varepsilon_{11} &= \partial_1 v_1 - \Gamma^1_{11} v_1 - \Gamma^1_{11} v_1 + \frac{\alpha_1}{R_1} v, \\
\varepsilon_{(12)} &= \tfrac{1}{2}(\partial_1 v_2 + \partial_2 v_1) - \Gamma^1_{12} v_1 - \Gamma^2_{12} v_2, \\
\varepsilon_{22} &= \partial_2 v_2 - \Gamma^1_{22} v_1 - \Gamma^2_{22} v_2 + \frac{\alpha_2}{R_1} v.
\end{aligned}\right\} \tag{7.14}$$

If one now substitutes the deformations, when expressed in terms of the displacements (7.13), (7.14), in the equations (7.4) [(7.9), resp.] for the material law then the stress quantities N^{il} and M^{il} , and from (7.11), the lateral forces, as well, will be known as functions of the displacement field of the middle surface. The equilibrium conditions (7.10) and (7.12) will then yield three differential equations for the components v_1, v_2 of the displacement vector field **v**. One proceeds analogously in order to derive the differential equation of the dual problem for the components Φ_1, Φ_2, and Φ of the stress function vector $\mathbf{\Phi}$. That shall not be pursued further here, since it does not lead to either any special difficulties or any new insights.

Our equations will become especially simple for the right circular cylinder (radius R), upon which we introduce the coordinates $q^{(1)} = s_1$ in the direction of the generators, $q^{(2)} = s_2$ (= $R\varphi$) in the circumferential direction, and $q^{(3)} = z$ in the direction of the exterior normal, such that:

$$\alpha_1 = \alpha_2 = 1, \quad R_1 \to \infty, \qquad R_2 = R\,. \tag{7.15}$$

All of the three-index symbols vanish, and it is no longer necessary to distinguish between the covariant and contravariant indices; for the sake of simplicity, we shall now write all indices as superscripts. With $c^2 = t^2 / 12\, R^2$, we will get:

$$\left.\begin{aligned}
N_{11} &= \frac{E\,t}{1-v^2}(\varepsilon_{11} + v\,\varepsilon_{22} - c^2 R\,\kappa_{22}), \\
N_{12} &= \frac{E\,t}{1-v^2}(1-v)\left(\varepsilon_{(12)} + \tfrac{1}{2} c^2 R\,\kappa_{12}\right), \\
N_{12} &= \frac{E\,t}{1-v^2}(1-v)\left(\varepsilon_{(12)} - \tfrac{1}{2} c^2 R\,\kappa_{21}\right), \\
N_{22} &= \frac{E\,t}{1-v^2}(\varepsilon_{22} + v\,\varepsilon_{11} + c^2 R\,\kappa_{11});
\end{aligned}\right\} \tag{7.16}$$

$$\left.\begin{aligned}
M_{11} &= -\frac{E\,t^3}{12(1-\nu^2)}(\kappa_{11}+\nu\,\kappa_{22}+\frac{1}{R}\varepsilon_{22}),\\
M_{12} &= -\frac{E\,t^3}{12(1-\nu^2)}(1-\nu)\,\kappa_{12},\\
M_{12} &= -\frac{E\,t^3}{12(1-\nu^2)}(1-\nu)\,\kappa_{21},\\
M_{22} &= -\frac{E\,t^3}{12(1-\nu^2)}(\kappa_{22}+\nu\,\kappa_{11}-\frac{1}{R}\varepsilon_{11});
\end{aligned}\right\} \tag{7.17}$$

$$N_1 = \partial_1 M_{22} - \partial_1 M_{22}\,, \qquad N_2 = \partial_2 M_{11} - \partial_1 M_{21}\,; \tag{7.18}$$

$$\left.\begin{aligned}
\kappa_{11} &= \partial_2\,\partial_2 v - \frac{1}{R}\partial_2\,v_2,\\
\kappa_{12} &= -\partial_1\,\partial_2 v - \frac{1}{R}\cdot\frac{1}{2}(\partial_1\,v_2 - \partial_2\,v_1),\\
\kappa_{21} &= -\partial_1\,\partial_2 v - \frac{1}{R}\partial_1\,v_2,\\
\kappa_{22} &= \partial_1\,\partial_1 v,\\
\varepsilon_{11} &= \partial_1 v_1,\\
\varepsilon_{12} &= \frac{1}{2}(\partial_1\,v_2 + \partial_2\,v_1),\\
\varepsilon_{22} &= \partial_2 v_2 + \frac{1}{R}v.
\end{aligned}\right\} \tag{7.19}$$

$$\left.\begin{aligned}
\partial_1 N_{11} + \partial_2 N_{21} &= 0,\\
\partial_1 N_{12} + \partial_2 N_{22} + \frac{1}{R}N_2 &= 0,\\
\partial_1 N_1 + \partial_2 N_2 - \frac{1}{R}N_{22} &= 0.
\end{aligned}\right\} \tag{7.20}$$

$$\left.\begin{aligned}
&\left(\partial_1\,\partial_1 + \frac{1-\nu}{2}\partial_2\,\partial_2\right)v_1 + \left(\frac{1+\nu}{2}\partial_1\,\partial_2\right)v_2 + \left[-c^2 R\,\partial_1\left(\partial_1\,\partial_1 - \frac{1-\nu}{2}\partial_2\,\partial_2\right) + \frac{\nu}{R}\partial_1\right]v = 0,\\
&\left(\frac{1+\nu}{2}\partial_1\,\partial_2\right)v_1 + \left(\partial_2\,\partial_2 + \frac{1-\nu}{2}\partial_1\,\partial_1\right)v_2 + \left(-c^2 R\frac{3-\nu}{2}\partial_1\,\partial_1\,\partial_2 + \frac{1}{R}\partial_2\right)v = 0,\\
&\left[-c^2 R\,\partial_1\left(\partial_1\,\partial_1 - \frac{1-\nu}{2}\partial_2\,\partial_2\right) + \frac{\nu}{R}\partial_1\right]v_1 + \left(-c^2 R\frac{3-\nu}{2}\partial_1\,\partial_1\,\partial_2 + \frac{1}{R}\partial_2\right)v_2\\
&\qquad + \left[c^2 R^2\left(\Delta\Delta + \frac{2}{R^2}\partial_2\,\partial_2 + \frac{1}{R^4}\right) + \frac{1}{R^2}\right]v = 0,
\end{aligned}\right\} \tag{7.21}$$

with the matrix of differential operators.

These equations are also found in **Vlassov** (1). Since other deformations and stress quantities are used there, this agreement in the results might be regarded as a welcome check on the calculations. If one now works out the dual problem for the stress functions Φ_1, Φ_2, and Φ then one will come to the following final equations (they are the compatibility conditions, when written in terms of the stress functions):

$$\left.\begin{aligned}
&\left(\partial_1\partial_1+\frac{1-\nu}{2}\partial_2\partial_2\right)\Phi_1+\left(\frac{1-\nu}{2}\partial_1\partial_2\right)\Phi_2+\left[c^2R\,\partial_1\left(\partial_1\partial_1+\frac{1+\nu}{2}\partial_2\partial_2\right)-\frac{\nu}{R}\partial_1\right]\Phi=0,\\
&\left(\frac{1-\nu}{2}\partial_1\partial_2\right)\Phi_1+\left(\partial_2\partial_2+\frac{1+\nu}{2}\partial_1\partial_1\right)\Phi_2+\left[-c^2R\,\partial_2\left(\frac{3+\nu}{2}\partial_1\partial_1+2\,\partial_2\partial_2\right)+\frac{1}{R}\partial_2\right]\Phi=0,\\
&\left[c^2R\,\partial_1\left(\partial_1\partial_1+\frac{1+\nu}{2}\partial_2\partial_2\right)-\frac{\nu}{R}\partial_1\right]\Phi_1+\left[-c^2R\,\partial_2\left(\frac{3+\nu}{2}\partial_1\partial_1+2\,\partial_2\partial_2\right)+\frac{1}{R}\partial_2\right]\Phi_2\\
&\qquad+\left[c^2R^2\left(\Delta\Delta-\frac{2}{R^2}\partial_2\partial_2+\frac{1}{R^4}\right)+\frac{1}{R^2}\right]\Phi=0,
\end{aligned}\right\} \tag{7.22}$$

with a likewise symmetric matrix of differential operators. The Ansätze:

$$\left.\begin{aligned}
v_1&=A_1\,e^{\lambda s_1/R}e^{ims_2/R}, &\quad \Phi_1&=B_1\,e^{\lambda s_1/R}e^{ims_2/R},\\
v_2&=A_2\,e^{\lambda s_1/R}e^{ims_2/R}, &\quad \Phi_2&=B_2\,e^{\lambda s_1/R}e^{ims_2/R},\\
v&=A\,e^{\lambda s_1/R}e^{ims_2/R}, &\quad \Phi&=B\,e^{\lambda s_1/R}e^{ims_2/R}
\end{aligned}\right\} \tag{7.23}$$

for the solution of the differential equations (7.21) [(7.22), resp.] lead to the *characteristic equation:*

$$\begin{aligned}
&\lambda^8-2\,(2\,m^2-\nu)\,\lambda^6+\left[\frac{1-\nu^2}{c^2}+6m^2(m^2-1)+1\right]\lambda^4\\
&\qquad-2m^2\,[2m^4+(4-\nu)\,m^2+2]\,\lambda^2+m^4\,(m^2-1)=0,
\end{aligned} \tag{7.24}$$

in both cases. Physically, that means: Displacements and stress functions show the same decay towards the boundary, which is certainly plausible, but the differential equations (7.21), (7.22) are not to be considered with no further assumptions. (7.24) differs from the characteristic equations that **W. Flügge** (2) and **C. B. Biezeno** and **R. Grammel** (3) [**K. Girkmann** (4)] have found. [If one simplifies the entire problem by going over to the law of elasticity of planar surface carriers then, as **H. Schaefer** (5) showed, one will get differential equations that differ from each other only by the sign of ν, such that the

(1) See footnote 2 on page 1.
(2) See footnote 3 on page 12.
(3) **C. B. Biezeno** and **R. Grammel**, *Technische Dynamik*, Bd. 1, 2nd ed., Berlin-Göttingen-Heidelberg, 1953.
(4) **K. Girkmann**, *Flächentragwerke*, 5th ed., Vienna, 1959.
(5) See footnote 4 on page 1.

analogy will emerge more strongly; v enters into the characteristic equation itself, which coincides again, only by the combination $(1 - v^2)$.]

One also gets extreme simplifications for the spherical shell. If its radius is R then the second fundamental tensor will be:

$$b_{il} = -\frac{1}{R}\, a_{il}\,, \tag{7.25}$$

and thus covariantly constant in any case. As a result, from (6.27), the moment tensor will be symmetric, and therefore the distortion tensor, as well. The symmetry of the longitudinal force tensor will then follow further from the algebraic equilibrium condition. Briefly: All field quantities on the shell will be described by symmetric tensors, so:

$$\left.\begin{aligned} N^{il} &= N^{(il)} = \frac{E\,t}{1-v^2}[(1-v)\,a^{i\alpha}a^{l\beta} + v\,a^{il}a^{\alpha\beta}]\,\varepsilon_{(\alpha\beta)}, \\ M_{il} &= M_{(il)} = -\frac{E\,t^3}{12(1-v^2)}[(1-v)\,a_{i\alpha}\,a_{l\beta} + v\,a_{il}\,a_{\alpha\beta}]\,\kappa^{(\alpha\beta)}; \end{aligned}\right\} \tag{7.26}$$

$$\left.\begin{aligned} \kappa^{il} &= \kappa^{(il)} = -\frac{12}{E\,t^3}[(1+v)\,a^{i\alpha}a^{l\beta} - v\,a^{il}a^{\alpha\beta}]\,M_{(\alpha\beta)}, \\ \varepsilon_{il} &= \varepsilon_{(il)} = \frac{1}{E\,t}[(1+v)\,a_{i\alpha}\,a_{l\beta} - v\,a_{il}\,a_{\alpha\beta}]\,N^{(\alpha\beta)}; \end{aligned}\right\} \tag{7.27}$$

$$\nabla_\alpha\left(N^{(\alpha l)} + e^{\alpha\lambda}e^{l\mu}M_{(\lambda\mu)}\right) = 0, \qquad e^{\alpha\lambda}e^{l\mu}\nabla_\alpha\nabla_\beta M_{(\lambda\mu)} - \frac{1}{R}\,a_{\alpha\beta}N^{(\alpha l)} = 0; \tag{7.28}$$

(Equilibrium):

$$\nabla_\alpha\left(\kappa^{(\alpha l)} + e^{\alpha\lambda}e^{l\mu}\,\frac{1}{R}\,\varepsilon_{(\lambda\mu)}\right) = 0, \qquad e^{\alpha\lambda}e^{l\mu}\nabla_\alpha\nabla_\beta\,\varepsilon_{(\lambda\mu)} - \frac{1}{R}\,a_{\alpha\beta}\,\kappa^{(\alpha l)} = 0; \tag{7.28}$$

(Compatibility):

$$\left.\begin{aligned} N^{(il)} &= e^{i\alpha}e^{l\beta}\,\nabla_\alpha\nabla_\beta\Phi - \frac{1}{R}\Big[a^{i\alpha}a^{l\beta} - \tfrac{1}{2}(a^{i\alpha}a^{l\beta} + a^{l\alpha}a^{i\beta})\Big]\nabla_\alpha\Phi_\beta, \\ M_{(il)} &= \tfrac{1}{2}(\nabla_i\Phi_l + \nabla_l\Phi_i) + \frac{1}{R}\,a_{il}\,\Phi; \end{aligned}\right\} \tag{7.30}$$

$$\left.\begin{aligned} \kappa^{(il)} &= e^{i\alpha}e^{l\beta}\,\nabla_\alpha\nabla_\beta v - \frac{1}{R}\Big[a^{i\alpha}a^{l\beta} - \tfrac{1}{2}(a^{i\alpha}a^{l\beta} + a^{l\alpha}a^{i\beta})\Big]\nabla_\alpha v_\beta, \\ \varepsilon_{(il)} &= \tfrac{1}{2}(\nabla_i v_l + \nabla_l v_i) + \frac{1}{R}\,a_{il}\,v; \end{aligned}\right\} \tag{7.31}$$

If the sphere degenerates to a planar surface carrier as $R \to \infty$ then equations (7.26), (7.31) will reduce to the known analogue system for the plate/disc; $\Phi(q(1), q(2))$ will become the **Airy** stress function [**H. Schaefer** ([1])].

It is obvious that the equations of the membrane stresses of the shell must be included in our Ansätze. **M. Lagally** ([2]) has given an interesting treatment to the question of membrane stress state; the author ([3]) gave a non-covariant derivation starting from the general equations of the shell. A covariant presentation should be given here: In the moment-free state of the shell, the longitudinal force N^{l} will vanish from (4.29), such that, from (4.24), one will have:

$$b_{\alpha\beta}\,\Omega^{\beta} = -\nabla_{\alpha}\,\Omega. \tag{7.32}$$

We shall now assume that the **Gauss**ian curvature K of the shell does not vanish. We can then define the symmetric tensor:

$$B^{il} = e^{i\lambda}\,e^{l\mu}\,\frac{1}{K}\,b_{\lambda\mu}\,, \tag{7.33}$$

whose matrix is the reciprocal of the matrix of the tensor b_{il} :

$$B^{i\alpha}\,b_{l\alpha} = e^{i\lambda}\,e^{\alpha\mu}\cdot\frac{1}{K}\,b_{\lambda\mu}\,b_{l\alpha} = (a^{i\alpha}\,a^{\lambda\mu} - a^{i\mu}\,a^{\lambda\alpha})\cdot\frac{1}{K}\,b_{\lambda\mu}\,b_{l\alpha} = \frac{1}{K}(2H\,b_l^i - c_l^i)$$

$$= \frac{1}{K}\,K\,\delta_l^i = \delta_l^i\,, \tag{7.34}$$

from (2.30). We contract (7.32) with $B^{\alpha l}$ and observe (5.34); we will then get:

$$\Omega^{l} = -\,e^{\alpha\lambda}\,e^{l\mu}\cdot\frac{1}{K}\,b_{\lambda\mu}\,\nabla_{\alpha}\Omega. \tag{7.35}$$

However, as a result of the algebraic equilibrium condition (4.38), the longitudinal tensor will be symmetric:

$$e_{\alpha\beta}\,N^{[\alpha\beta]} = 0 \tag{7.36}$$

in the moment-free state, and it will follow from this, using (4.24), that:

$$\nabla_{\beta}\,\Omega^{\beta} - 2\,H\,\Omega = 0. \tag{7.37}$$

If one then employs the **Codazzi** equations (2.33) then that will yield the second-order differential equation for $\Omega\,(q^{(1)}, q^{(2)})$:

$$e^{\alpha\lambda}\,e^{\beta\mu}\cdot b_{\lambda\mu}\,\nabla_{\alpha}\left(\frac{1}{K}\nabla_{\beta}\,\Omega\right) + 2\,H\,\Omega = 0. \tag{7.38}$$

([1]) See footnote 1 on page 1.
([2]) **M. Lagally**, Z. angew. Math. Mech. **4** (1924), pp. 377.
([3]) See footnote 1 on page 16.

Lagally referred to it as the *characteristic differential equation* for the problem. We shall show that the equilibrium condition:

$$b_{\alpha\beta} N^{\alpha\beta} = 0 \tag{7.39}$$

is an identity; namely, it will follows from it, in succession, that:

$$\left.\begin{aligned} b_{\alpha\beta}\, e^{\alpha\lambda} (\nabla_\lambda \Omega^\beta - b_\lambda^\beta\, \Omega) &= 0, \\ b_{\alpha\beta}\, e^{\alpha\lambda} \nabla_\lambda \Omega^\beta - e^{[\alpha\lambda]} c_{(\alpha\lambda}) \Omega \equiv b_{\alpha\beta}\, e^{\alpha\lambda} \nabla_\lambda \Omega^\beta &= 0 \\ b_{\alpha\beta}\, e^{\alpha\lambda} \nabla_\lambda \left(\frac{1}{K} e^{\alpha\lambda} e^{\alpha\lambda} b_{\sigma\tau} \nabla_\rho \Omega \right) &= 0, \\ e^{\alpha\lambda} \nabla_\lambda \left[\frac{1}{K} (2H\, b_{\alpha\beta} - c_{\alpha\beta}) \nabla^\beta \Omega \right] &= 0, \\ e^{\alpha\lambda} \nabla_\lambda \left(\frac{1}{K} K\, a_{\alpha\beta} \nabla^\beta \Omega \right) \equiv e^{\alpha\lambda} \nabla_\lambda \nabla_\alpha \Omega &= 0, \end{aligned}\right\} \tag{7.40}$$

and the last equation will be the desired identity.

Naturally, based upon our analogy, everything that we said before about the stress state of the membrane can be adapted word-for-word to the case of "pure bending," in which the middle surface is free of distortions, which is a case that was preferable to consider in the early days of shell theory, since the possibility of a differential-geometric representation had still not reached the point that one could treat the general case in a reasonably transparent form.

It still remains for us to discuss the question of whether the stress state of the membrane, which is indeed statically-determined, is also kinematically possible for a given material law. It is known that this is, however, indeed not the case for the spherical shell of constant wall thickness whose deformations are coupled with the stress quantities by (4.27). Since the moments M_{il} vanish, the κ^{il}, and from (3.51), the κ^i as well, must be zero, which has:

$$\omega^j = 0, \qquad \omega = 0 \tag{7.41}$$

as a consequence, from (3.48). Since $\varepsilon_i = 0$, the second of equations (3.49) will then give:

$$v_i = R\, \nabla_i\, v, \tag{7.42}$$

such that the tensor of membrane deformations will be:

$$\varepsilon_{il} = R \left(\nabla_i \nabla_l\, v + \frac{1}{R^2} a_{il}\, v \right), \tag{7.43}$$

which is then symmetric, as it must be. The state of deformation is then determined by the normal displacement v alone. If one now expresses the longitudinal force tensor N^{il}

in terms of ε_{il} according to (7.26) then one can get two differential equations for v from the still-remaining equilibrium conditions:

$$\nabla_{\alpha} N^{\alpha l} = 0. \tag{7.44}$$

We shall skip the laborious calculation and just write down the result:

$$\nabla_{l}\left(\Delta v + \frac{2}{R^2} v\right) = 0, \tag{7.45}$$

from which, it will follow that:

$$\Delta v + \frac{2}{R^2} v = \text{const.}, \tag{7.46}$$

and that is the desired differential equation, while the characteristic differential equation of the sphere will read:

$$\Delta\Omega + \frac{2}{R^2}\Omega = 0. \tag{7.47}$$

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