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Stress functions and compatibility conditions in continuum mechanics

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With 3 Figures

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Summary: In this paper, an introduction is given to the theory of "stress functions," which were first considered by J. C. Maxwell. The geometrical aspect of these functions leads to remarkable relations between the classical theory of stress fields and the general theory of relativity. On the other hand, in the Lagrangian view of mechanics, stress functions form the "reaction tensor" that is produced by the condition that Euclidian geometry must be unchanged by any deformation of a continuous medium. Thus, the theory of stress functions joins two heterogeneous parts of mathematical physics.

1. Synthetic construction of the tensor of stress functions.

A simply-connected rigid body (Fig. 1) is found to be in equilibrium under the influence of continuously-distributed outer surface forces. Along with the Cartesian coordinates x, y, z, we employ general coordinates $x^{(i)}$, for which the functional determinant of the coordinate transformation will vanish, except for some exceptional points at most; such points will generally be excluded from our considerations.

We next recall some known things: One imposes the outer surface forces $d\mathbf{K}$ as linear vector functions of the normal vector $\mathbf{n}$ of the outer surface (*):

$$dK^i = S^{i\alpha}\, n_\alpha\, df \qquad (1.1)$$

$$S^{i\alpha}\, n_\alpha = \sum_{\alpha=1}^{3} S^{i\alpha} n_\alpha \qquad (i = 1, 2, 3),$$

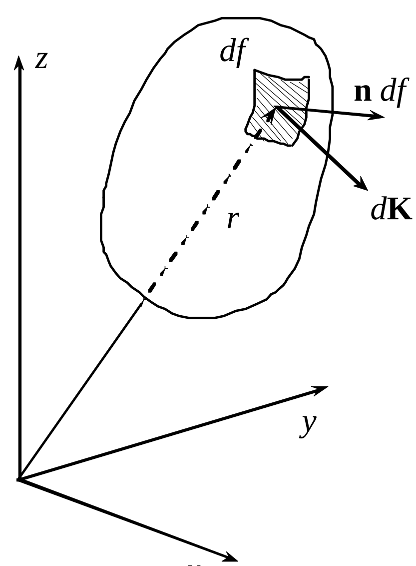

Figure 1.

and formulates the equilibrium conditions with the help of the principle of virtual displacements:

(*) One will sum over Greek indices in the event that they appear twice in a formal group.

$$\iint_{(f)} dK^{\lambda}\delta v_{\lambda} = \iint_{(f)} S^{\lambda\alpha}\delta v_{\lambda} n_{\alpha} df = 0. \tag{1.2}$$

δ is a "field of virtual displacements" that is compatible with the rigidity of the body, but otherwise arbitrary. Gauss's integral theorem then gives:

$$\iiint_{(V)} \nabla_{\alpha}(S^{\lambda\alpha}\delta v_{\lambda})\, dV = 0; \tag{1.3}$$

∇_{α} is the symbol of covariant derivation. (1.3) then decomposes into:

$$\iiint_{(V)} \nabla_{\alpha} S^{\lambda\alpha}\delta v_{\lambda}\, dV = 0, \qquad \iiint_{(V)} S^{\lambda\alpha}\nabla_{\alpha}(\delta v_{\lambda})\, dV = 0. \tag{1.4a, b}$$

It will follow from (1.4*a*) in a well-known way that the *stress tensor* **S** must satisfy:

$$\nabla_{\alpha} S^{i\alpha} = 0; \tag{1.5}$$

i.e., the vectorial divergence of the stress tensor will vanish in the interior of the domain. Due to the rigidity condition, the virtual displacement field will obey *Killing's* equation:

$$\nabla_{\alpha}(\delta v_{\lambda}) = \nabla_{[\alpha}(\delta v_{\lambda]}) = \tfrac{1}{2}[\nabla_{\alpha}(\delta v_{\lambda}) - \nabla_{\lambda}(\delta v_{\alpha})], \tag{1.6}$$

such that (1.4b) will give the symmetry of the stress tensor:

$$S^{[ik]} = 0 \qquad (S^{ik} = S^{(ik)} = \tfrac{1}{2}[S^{ik} + S^{ki}], \text{ resp.}). \tag{1.7}$$

(1.1), (1.5), and (1.6) can be combined into the well-known Ansatz of continuum mechanics:

$$\iiint_{(V)} S^{\lambda\alpha}\delta\varepsilon_{\lambda\alpha}\, dV - \iint_{(f)} dK^{\lambda}\delta v_{\lambda} = 0; \tag{1.8}$$

in this:

$$\delta\varepsilon_{\lambda\alpha} = \delta\varepsilon_{(\lambda\alpha)} = \tfrac{1}{2}[\nabla_{\lambda}(\delta v_{\alpha}) + \nabla_{\alpha}(\delta v_{\lambda})] \tag{1.9}$$

is the *virtual distortion tensor*. The symmetry of the stress tensor will then follow immediately from that of the distortion tensor. The divergence condition (1.5) and the relation (1.1) will be obtained by partially integrating the volume integral in (1.8) once more with the use of (1.9). *Piola* [**1**] gave the mechanical interpretation of the Ansatz (1.8) in the context of Lagrangian mechanics: The Ansatz formulates the equilibrium of the outer surface forces under the auxiliary condition that the virtual displacement field is distortion-free, so the body will remain rigid. The *Lagrangian factors* are included in the stress tensor; i.e., the reactions to this kinematic pressure condition. In order to arrive at the stresses imprinted on the non-rigid (e.g., elastic) continuum from the reaction stresses of the rigid body, one appeals to another principle – viz., the *release principle* – according to which, under a relaxation of the kinematic conditions, the reactions will become imprinted (i.e., measurable) force quantities that primarily depend upon those geometric quantities whose variation was previously forbidden (*Hamel* [**2**]).

The symmetric stress tensor **S** can now be converted into a skew-symmetric *transverse stress tensor* of rank four. Formally, this can happen with the help of the *e-tensor*, which is a covariant (contravariant, resp.) measure for the oriented volumes of the coordinate unit mesh. This third-rank tensor is skew-symmetric in all indices:

$$e^{ikl} = e^{[ikl]} = \frac{1}{3!}[e^{ikl} + e^{kli} + e^{lik} - e^{kil} - e^{ilk} - e^{lki}] \,. \tag{1.10}$$

If we denote the determinant of the components g_{ik} of the metric by g then we will have:

$$e^{ikl} = \mathrm{sgn}\,(ikl) \cdot \frac{1}{\sqrt{g}}\,, \qquad e_{ikl} = \mathrm{sgn}\,(ikl) \cdot \sqrt{g}\,. \tag{1.11}$$

The covariant derivative of **e** is – understandably – zero, since the unit volume, with its proper covariant measure, must have the same value everywhere.

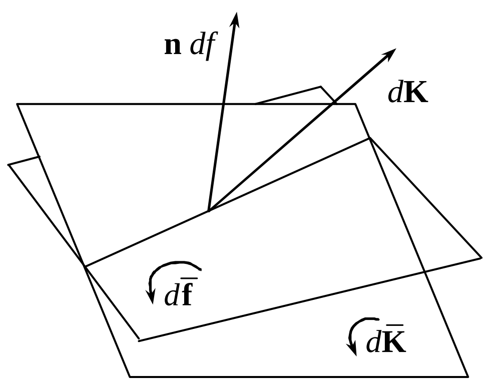

Figure 2

In order to construct the transversal stress tensor **T** intuitively, we introduce oriented surface elements $d\,\bar{\mathbf{f}}$ and $d\,\bar{\mathbf{K}}$, in place of the vectors $\mathbf{n} \cdot df$ and $d\mathbf{K}$ (Fig. 2), which are perpendicular to **n** ($d\mathbf{K}$, resp.) and whose sense of traversal and volume measure the direction and magnitude of these vectors, respectively. $d\,\bar{\mathbf{f}}$ and $d\,\bar{\mathbf{K}}$ are skew-symmetric tensors of rank 2:

$$\begin{aligned} df^{ik} &= df^{[ik]} = \tfrac{1}{2} e^{ik\alpha} n_\alpha\, df, \\ dK_{lm} &= dK_{[lm]} = e_{mn\alpha}\, dK^\alpha ; \end{aligned} \tag{1.12a}$$

the inverses of (1.12a) read:

$$\begin{aligned} n_i\, df &= e_{i\alpha\beta}\, df^{\alpha\beta}, \\ dK^i &= \tfrac{1}{2} e^{i\alpha\beta}\, dK_{\alpha\beta}\,, \end{aligned} \tag{1.12b}$$

resp. In a completely analogous way, we finally describe the field of virtual displacements by a skew-symmetric tensor $\delta\bar{\mathbf{v}}$:

$$\begin{aligned} \delta v^{ik} &= \tfrac{1}{2} e^{ik\alpha}\, \delta v_\alpha , \\ \delta v_i &= e_{i\alpha\beta}\, \delta f^{\alpha\beta} . \end{aligned} \tag{1.13}$$

The linear connection between $d\,\bar{\mathbf{K}}$ and $d\,\bar{\mathbf{f}}$ mediates the transverse stress tensor $\mathbf{T}$:

$$dK_{lm} = T_{\alpha\beta,\,lm}\, df^{\alpha\beta} ; \tag{1.14}$$

i.e., $\mathbf{T}$ rotates and distorts $d\,\bar{\mathbf{f}}$ in such a way that $d\,\bar{\mathbf{K}}$ comes about. Thus, we get:

$$\iint_{(f)} T_{\alpha\beta,\rho\alpha}\, \delta v^{\rho\alpha} df^{\alpha\beta} = 0 \tag{1.15}$$

as the expression of equilibrium in the outer surface forces. An application of Gauss's integral theorem will take (1.15) to:

$$\iiint_{(V)} \nabla_\lambda (T_{\alpha\beta,\rho\sigma}\, \delta v^{\rho\sigma})\, e^{\lambda\alpha\beta} dV = 0. \tag{1.16}$$

$\delta\bar{\mathbf{v}}$ is the field of a rigid displacement, so it can be split into a rigid translation and a rotation:

$$\delta v^{[ik]} = \delta \underset{0}{v}^{[ik]} + r^{[i}\, \delta \underset{0}{\omega}^{k]} \tag{1.17}$$

with constant quantities $\delta \underset{0}{\bar{\mathbf{v}}}$ and $\delta \underset{0}{\boldsymbol{\omega}}$; $\mathbf{r}$ is the body-fixed vector from the *translation point* to the outer surface point that is being considered. With that, (1.16) decomposes into the condition for force equilibrium:

$$\iiint_{(V)} \nabla_{[\lambda} T_{\alpha\beta]\rho\sigma} \cdot \delta \underset{0}{v}^{\rho\sigma} e^{\lambda\alpha\beta} dV = 0$$

or

$$\boxed{\nabla_{[s} T_{ik]lm} = 0} \tag{1.18}$$

and the condition for moment equilibrium:

$$\iiint_{(V)} \nabla_{[\lambda} (T_{\alpha\beta]\rho\sigma} r^\rho) \delta \underset{0}{\omega}^\sigma e^{\lambda\alpha\beta} dV$$

or

$$\boxed{\nabla_{[s} (T_{ik]\lambda m} r^\lambda) = 0.} \tag{1.19a}$$

Now, one has:

$$\nabla_s\, r^\lambda = \delta_s^\lambda = \begin{cases} 0 & \text{for } \lambda \neq s, \\ 1 & \text{for } \lambda = s. \end{cases} \tag{1.20}$$

Thus, (1.19a) will be equivalent to:

$$\boxed{T_{[ikl]m} = 0.} \tag{1.19b}$$

By definition, **T** is skew-symmetric in the first and last index pairs:

$$T_{ik,\,lm} = T_{[ik],[lm]}, \tag{1.21}$$

but, above and beyond that, it will satisfy a "cyclic symmetry" (1.19b) and a "Bianchi identity" (1.18) – in other words, **T** has the structure of a curvature tensor. Just as it does for a curvature tensor (cf., e.g., Levi-Cività [**3**]), the *even symmetry*:

$$T_{ik,\,lm} = T_{ik,\,lm} \tag{1.22}$$

will follow from the skew symmetry (1.21) and the cyclic symmetry (1.19b). A simple count will yield that **T** has only six algebraically-independent components as a result of its numerous symmetries, which is just as many as **S** (*).

The stress state of a three-dimensional continuum can thus be interpreted geometrically as the "curvature state" of a non-Euclidian "stress space" whose curvature tensor is **T**. A result of this kind was to be expected, moreover. First of all (from the viewpoint of the theory of general relativity), the stress tensor is part of the energy-impulse tensor, and will thus give rise to a spatial curvature, and secondly, stresses can also be easily related to the curvature ratios of the *Airy stress surface* for a planar stress state; cf., Section 2 for this.

We then give the connection between **S** and **T**, which can be deduced from (1.1), (1.12), and (1.14):

$$S^{ik} = \tfrac{1}{4} e^{i\beta}\, e_{k\sigma\mu}\, T_{\alpha\beta,\,\lambda\mu}\,, \qquad\qquad T_{ik,\,lm} = e_{ik\alpha}\, e_{lm\beta}\, S^{\alpha\beta}. \tag{1.23}$$

One recognizes that the symmetry of **S** corresponds to the cyclic symmetry of **T** and the divergence condition for **S** corresponds to the Bianchi identity for **T**. Our next objective will be to find a general representation for **T**. To that end, we regard the equilibrium conditions (1.18) and (1.19a) as the integrability conditions for two systems of partial differential equations. In fact, (1.18) is the necessary and sufficient condition for **T** to be the "rotation" of a tensor **X**:

$$T_{ik,\,lm} = \nabla_i\,(X_{k,\,lm} - X_{k,\,ml}) - \nabla_k\,(X_{i,\,lm} - X_{i,\,ml}), \tag{1.24}$$

in which we have expressly accentuated the skew symmetry in m and l. It follows, correspondingly, from (1.19a) that:

$$T_{ik,\,\lambda m}\, r^{\lambda} = \nabla_i Y_{km} - \nabla_k Y_{im}\,. \tag{1.25}$$

If one combines (1.24) and (1.25) and considers (1.21) then one will get:

(*) In n dimensions, the curvature tensor will have $n^2\,(n^2 - 1) / 12$ algebraically-independent components and the symmetric stress tensor $n\,(n + 1) / 2$.

$$X_{k,\,im} - X_{k,\,mi} - X_{i,\,km} + X_{i,\,mk} = -\nabla_i F_{km} + \nabla_k F_{im} \tag{1.26}$$

with

$$F_{km} = Y_{km} - r^{\lambda}\,(X_{k,\,\lambda m} - X_{k,\,m\lambda}).$$

One now adds the equations that arise from (1.26) by switching k (i, resp.) with m to (1.26). This will lead to:

$$X_{m,\,ik} - X_{m,\,ki} = -\nabla_i F_{(km)} + \nabla_k F_{[im]} + \nabla_m F_{[ik]}\,, \tag{1.27}$$

such that the desired representation of **T** will be found to be:

$$\boxed{T_{ik,lm} = \nabla_i \nabla_m F_{(kl)} + \nabla_k \nabla_l F_{(im)} - \nabla_i \nabla_l F_{(km)} - \nabla_k \nabla_m F_{(il)}.} \tag{1.28}$$

This representation of the stress field **T** by a symmetric *stress function tensor* **F** is necessary and sufficient for equilibrium to prevail in the stress field.

We defer the further examination of (1.28) to the next section and next concern ourselves with the construction of the symmetric stress tensor **S** from the stress function tensor **F**: With a simple change in the index notations, it will follow from (1.23) and (1.28) that:

$$S^{ik} = e^{i\alpha\lambda}\,e^{k\beta\mu}\,T_{\alpha\lambda,\,\beta\mu} = e^{i\alpha\lambda}\,e^{k\beta\mu}\,\nabla_\alpha\nabla_\beta F_{(\lambda\mu)} = \nabla_\alpha\nabla_\beta\,(e^{i\alpha\lambda}\,e^{k\beta\mu}\,F_{(\lambda\mu)}) = \nabla_\alpha\nabla_\beta\,U^{i\alpha,\,k\beta}. \tag{1.29}$$

and we note that we can express **S** in terms of **F**, as well as in terms of a skew-symmetric stress function tensor **U** (*Finzi* [**4**]). As one can see from its definition (1.29), **U** has the same algebraic symmetries as **T**, and therefore just as many independent components as **F** – namely, six. However, the representation of **S** in terms of **U** is valid for an arbitrary dimension n, even n = 2, when **U** has only one component (which is just the Airy stress function). In the representation (1.29) of **S** in terms of **F**, we can liberate ourselves of the e-tensor. Namely, since one has:

$$e^{i\alpha\lambda}\,e^{k\beta\mu} = (g^{ik}\,g^{\alpha\beta} - g^{\alpha k}\,g^{i\beta})\,g^{\lambda\mu} + (g^{i\beta}\,g^{\alpha\mu} - g^{\alpha\beta}\,g^{i\mu})\,g^{\lambda k} + (g^{i\mu}\,g^{\alpha k} - g^{\alpha\mu}\,g^{ik})\,g^{\lambda\beta}, \tag{1.30}$$

it will emerge from (1.29) that:

$$S_{(ik)} = T_{(ik)} - \tfrac{1}{2}\,g_{ik}\,T \tag{1.31}$$

with

$$T_{(ik)} = g^{\alpha\beta}\,T_{\alpha i,\,k\beta}\,, \qquad T = g^{\lambda\mu}\,T_{\lambda\mu}\,. \tag{1.32}$$

S thus corresponds to the *Einstein* tensor, which is the energy-impulse tensor of the general theory of relativity, whose vectorial divergence vanishes identically. The symmetry of T_{ik}, and thus that of **S**, will follow from the cyclic symmetry of **T**. **S** will then be expressed in terms of **F** in the following way:

$$\boxed{S_{ik} = \Delta_2 F_{(ik)} - \nabla_i \nabla^\alpha F_{(\alpha\beta)} + \nabla_i \nabla_k F + g_{ik}(\nabla^\alpha \nabla^\beta F_{(\alpha\beta)} - \Delta_2 F),} \tag{1.33}$$

with $F = g^{\alpha\beta} F_{\alpha\beta}$. $\Delta_2 = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the *second Beltrami operator*, which is the covariant analogue of the Laplacian operator.

In many cases, it is more convenient to introduce a stress function tensor **Φ**, in place of **F**, by way of:

$$\Phi_{(ik)} = F_{(ik)} - \tfrac{1}{2} g_{ik} F. \tag{1.34}$$

One will then have:

$$\boxed{S_{ik} = \Delta_2 \Phi_{(ik)} - \nabla_i \nabla^\alpha \Phi_{(\alpha k)} - \nabla_k \nabla^\alpha \Phi_{(\alpha i)} + g_{ik} \nabla^\alpha \nabla^\beta \Phi_{(\alpha\beta)}.} \tag{1.35}$$

This representation is valid for all dimensions, while (1.33) will fail for $n = 1$ and $n = 2$.

2. Null stress functions and compatibility conditions. Geometric and mechanical interpretation of the stress functions.

The stress functions that are associated with a given equilibrium system of stresses can be found by integrating the system of differential equations (1.28). This integration problem was solved by *Schaefer* [**5**] by a method that *Einstein* had developed for the treatment of weak gravitational fields. The six equations of the homogeneous system of differential equations that belongs to (1.28):

$$\nabla_i \nabla_m \overset{0}{F}_{(kl)} + \nabla_k \nabla_l \overset{0}{F}_{(im)} - \nabla_i \nabla_l \overset{0}{F}_{(km)} - \nabla_k \nabla_m \overset{0}{F}_{(il)} = 0 \tag{2.1}$$

are indeed algebraic, but not completely independent of each other, since they are coupled with each other by the three equations of the Bianchi identity. Under suitable regularity assumptions, three of them will be a consequence of the remaining three. However, this connection cannot be formulated in a covariant manner (*Bach* [**6**]). From the existence of this connection, we can, however, draw the conclusion that the general solution of (2.1) will contain three arbitrary functions. We would like to determine this solution, and at the same time, prove that there can be no other solutions of (2.1). If, for the moment, we write (2.1) as:

$$\nabla_{[i} \overset{0}{\gamma}_{k]lm} = 0, \tag{2.2}$$

with

$$\overset{0}{\gamma}_{klm} = \nabla_m \overset{0}{F}_{(kl)} - \nabla_k \overset{0}{F}_{(lm)} - \nabla_l \overset{0}{F}_{(km)}, \tag{2.3}$$

then (2.2) will be the integrability condition for:

$$\nabla_m \overset{0}{F}_{(kl)} - \nabla_l \overset{0}{F}_{(km)} = \nabla_k \omega_{[ml]}, \tag{2.4}$$

where ω is a yet-to-be-determined skew-symmetric tensor. If one switches k and l in (2.4) and subtracts the result from (2.4) then one will find that:

$$\nabla_k \{\overset{0}{F}_{(lm)} + \omega_{[lm]}\} - \nabla_l \{\overset{0}{F}_{(km)} + \omega_{[km]}\} = 0, \tag{2.5}$$

and that will, once more, be necessary and sufficient for one to have:

$$\overset{0}{F}_{(lm)} + \omega_{[lm]} = \nabla_l\, v_m\,. \tag{2.6}$$

If one finally switches l with m in (2.6) and adds (subtracts, resp.) then one will get the general solution of (2.1):

$$\overset{0}{F}_{(lm)} = \tfrac{1}{2}(\nabla_l\, v_m + \nabla_m\, v_l) = \nabla_{(l}\, v_{m)}\,. \tag{2.7}$$

$\overset{0}{\mathbf{F}}$ is the symmetric gradient tensor, and $\boldsymbol{\omega}$ is the rotation of an arbitrary vector field **v** that contains three arbitrary functions in its own right. $\overset{0}{\mathbf{F}}$ is called the *null stress function tensor,* since it will make no contribution to the stress field when it is substituted into (1.28). It can be used, for example, to make three of the six components of **F** equal to zero with no change in the associated stress state (and no rotation of the coordinate system!), and one will thus come to the Ansätzen of *Maxwell* [**7**] or *Morera* [**8**], or also perhaps to subject the tensor **F** (1.34) to a divergence condition, and thus facilitate the integration problem that mentioned to begin with (*Schaefer*, *loc. cit.*).

A glimpse at (2.7) and (1.9) shows that each of the kinematically possible distortion fields is a field of null stress functions, and conversely (*Weber* [**9**]). Equations (2.1) are, as a consequence, nothing but the well-known *compatibility conditions* that are necessary and sufficient conditions for the equations (1.9) to be integrable, if they are regarded as differential equations for the displacement vector. One can also recognize the fact that the conditions (2.1) are necessary geometrically. Namely, if we think of the metric tensor g_{ik} of Euclidian space as being changed infinitesimally into:

$$\bar{g}_{ik} = g_{ik} + 2\, F_{(ik)}\, \delta t \tag{2.9}$$

(δt is an infinitesimal constant whose higher powers can be neglected) and we calculate the curvature tensor that belongs to $\bar{g}_{ik}$ then we will find that it will be equal to **T** (1.28), up to the factor δt. The F_{ik} then emerge as the *stress potentials* (*) of a spatial warp, which is how we have already interpreted the stress state. **T** will indeed be zero when the F_{ik} change the metric, but not the Euclidian character, of space (and conversely). That will certainly be the case when the change in the metric comes about as a result of an infinitesimal distortion of the continuum.

From the standpoint of system mechanics, this is closely related to another interpretation for the stress functions: We write the compatibility conditions for $\delta\varepsilon$ using (1.33) in the symmetric form:

$$\Delta_2(\delta\varepsilon_{ik}) - \nabla_i\nabla^\alpha(\delta\varepsilon_{\alpha k}) - \nabla_k\nabla^\alpha(\delta\varepsilon_{\alpha i}) + \nabla_i\nabla_k(\delta\varepsilon) + g_{ik}\,[\nabla^\alpha\nabla^\beta(\delta\varepsilon_{\alpha\beta}) - \Delta_2(\delta\varepsilon)] = 0. \tag{2.10}$$

(*) They correspond to the *gravitational potentials* of the theory of relativity.

(The left-hand side of (2.10) is, moreover, a self-adjoint differential expression!). We can then also formulate the Ansatz (1.8) of continuum mechanics in the following way. (For the sake of simplicity, we have ignored the outer surface forces, which are of no interest here, and we have also made all of the boundary integrals that will appear in what follows vanish by a suitable assumption about the state of distortion.):

$$\iiint_{(V)} \left\{ S^{\alpha\beta} \delta\varepsilon_{\alpha\beta} - F^{\alpha\beta} [\Delta_2 (\delta\varepsilon_{\alpha\beta}) - \cdots] \right\} dV = 0. \tag{2.11}$$

$F^{\alpha\beta}$ is a symmetric *Lagrangian tensor*, and when (2.10) is multiplied by it, that will be added to (1.8) as an auxiliary condition; (1.9) is then resolved. One will obtain (1.33) immediately by two partial integrations when one observes that the $\delta\varepsilon$ are now arbitrary. The Lagrangian tensor **F** is then the tensor of the stress functions, and appears to be the system of reactions to the Euclidian condition (2.10) here, and thus to the fact that it is forbidden for the continuum to be the result of an infinitesimal distortion of Euclidian space. One can go still further, and also add the Bianchi identity that is valid for the compatibility conditions as an auxiliary condition to (2.10); the associated Lagrangian tensor then proves to be the tensor $\overset{0}{\mathbf{F}}$ of null stress functions. That shall not be pursued any further here. We shall also refrain from giving an analogous derivation of the skew-symmetric tensor **U** of stress functions (1.29) with the help of the compatibility conditions (2.1), since that would yield no new insights.

We turn once more to the non-Euclidian interpretation of the stress field. Euclidian space will become a non-Euclidian space whose Einstein tensor (1.33) is **S** under the infinitesimal transformation (2.9). The projection of the force $d\mathbf{K}$ onto the normal vector of the associated outer surface element df will be:

$$dK^{\alpha} n_{\alpha} = S^{\alpha\beta} n_{\alpha} n_{\beta} \, df \, . \tag{2.12}$$

According to Herglotz [**10**], the quantity $S^{\alpha\beta} n_{\alpha} n_{\beta}$ is the Gaussian curvature of the geodetic surface that is perpendicular to the normal vector at the point of application of the force in stress space. In general, one has for $n \geq 3$: The normal stress at the point P is equal to the sum of the Gaussian curvatures of those $\binom{n-1}{2}$ geodetic surfaces that are perpendicular to the normal vector at P in stress space at the location P. This remarkable theorem is the generalization of a theorem that is true for plane stress states: If one imagines the boundary curve of a plane continuum as being projected perpendicular to that plane onto the (infinitesimal) Airy stress surface then at any point of the boundary curve, the normal stress will be equal to the normal curvature of the projected point at the corresponding point. It is worthwhile to consider the case of n = 2 somewhat closer, since special aspects of that situation exist. As is known, the Einstein tensor is identically zero in two dimensions, and is thus unneeded for a representation of the stress tensor. The *second fundamental tensor* **h** of the infinitesimal Airy stress surface, which characterizes the curvature properties of the surface as the building block of the curvature tensors, will enter in place of it. If the stress surface is given by:

$$x^{(3)} = \Phi(x^{(1)}, x^{(2)}) \cdot \delta t \tag{2.13}$$

(Φ is the Airy stress function) then one will have, up to higher-order quantities:

$$h_{ik} = \nabla_i \nabla_j \Phi \cdot \delta t \tag{2.14}$$

and

$$S_{ik} = - h_{ik} + g_{ik} h, \tag{2.15}$$

with $h = g^{\alpha\beta} h_{\alpha\beta}$. The normal stress will then become:

$$S_{\alpha\beta} n^\alpha n^\beta = - h_{\alpha\beta} n^\alpha n^\beta + h. \tag{2.16}$$

The normal vector **n** arises from the tangent vector **t** to the boundary curve by a rotation through 90°:

$$n^\alpha = e^{\alpha\lambda} t_\lambda \tag{2.17}$$

$e^{ik} = e^{[ik]}$ is the analogue of the two-dimensional e-tensor that is defined by (1.10), (1.11). If one introduces this into (2.16) and observes that:

$$e^{\alpha\lambda} e^{\beta\mu} = g^{\alpha\beta} g^{\lambda\mu} - g^{\alpha\lambda} g^{\beta\mu} \tag{2.18}$$

then it will follow that:

$$S_{\alpha\beta} n^\alpha n^\beta = h_{\alpha\beta} t^\alpha t^\beta \tag{2.19}$$

for the normal stress, which is known to be the normal curvature of the surface curve that goes through **t**.

We will treat some other extensions of the behavior for $n = 2$ in Section 3.

We conclude with a new derivation of the "Plücker analogy," which is a duality principle that is often used in dynamics.

Let a surface element df be given, along with the stress state that prevails in it. Thus, the force d**K** and the moment d**M** of that force will also be given there, where the moment will be given by:

$$dM_i = e_{i\alpha\beta} r^\alpha dK^\beta = r^\alpha dK_{i\alpha}; \tag{2.20}$$

r is the position vector from the moment reference point to df. One now imagines **r** to be displaced parallel to a boundary point of df. (That will be a single-valued operation in Euclidian space.) In that way, space will be endowed with the non-Euclidian metric of stress space according to (2.9); the curvature tensor will be **T**. Now, **r** will be parallel-displaced along the boundary curve of df in the sense of this new metric; its change after one traversal will be ([1]):

$$\Delta r_i = T_{\alpha\beta,\, i\lambda} r^\lambda df^{\alpha\beta}. \tag{2.21}$$

Since the length of a vector does not change under parallel displacement, **r** + Δ**r** must emerge from **r** by a rotation; let its rotation tensor be $\Delta\boldsymbol{\omega}$.

([1]) This "traversal formula" is frequently used as the definition of the curvature tensor.

$$\Delta r_i = \Delta\omega_{i\rho}\, r^\rho. \tag{2.22}$$

If we recall (1.14) and (2.20) then we can deduce from (2.22) that:

$$dK_{lm} = \Delta\omega_{lm}\,, \qquad\qquad dM_i = \Delta r_i\,. \tag{2.23}$$

The position vector will suffer a rotation in stress space under the "orbital shift" (2) that was written down above, and the "angular velocity" of the rotation will be measured by the skew-symmetric force tensor, while the velocity of the change will itself be measured by the moment vector.

3. Integration of the outer surface forces and their moments. Special spatial stress functions.

The stress field is source-free, so it will be a (tensorial) vortex field. It is therefore easy to arrive at a representation of the forces that act upon a finite piece of the outer surface and their moments as line integrals (i.e., *circulations*) that are taken over the boundary curve of the outer surface piece. The case is similar to the one that pertains to a planar stress state: There, the resultants of the forces that act along a segment AB and their resultant moment are "point-pair functions" that depend only upon the positions of the points A and B, but not on the course of the curve segment, except for complications that can appear for multiply-connected domains. According to the program, we then consider a simply-connected surface patch O_1 that is bounded by a closed, oriented surface curve C; the orientation of the surface curve will then emerge as it did in Sect. 3. Corresponding to (2.2), (2.3), we assume that:

$$T_{ik,\,lm} = 2\,\nabla_{[i}\gamma_{k]\,lm} \tag{3.1}$$

with

$$\gamma_{klm} = \nabla_m F_{(kl)} - \nabla_m F_{(kl)} - \nabla_m F_{(kl)}\,; \tag{3.2}$$

γ is, moreover, the change in the Christoffel three-index symbol (up to sign) that comes from the change (2.9) in the metric. We now introduce the fixed unit vectors **i**, **j**, **k** of a Cartesian coordinate system for which the covariant derivative of its metric will vanish in any coordinate system. The projections of the force vector onto these unit vectors:

$$dK_x = dK^\lambda\, i_\lambda\,, \ \text{etc.,} \tag{3.3}$$

will bee the orthogonal components of the force vector in the (x, y, z) system.

(2) Cartan [**11**].

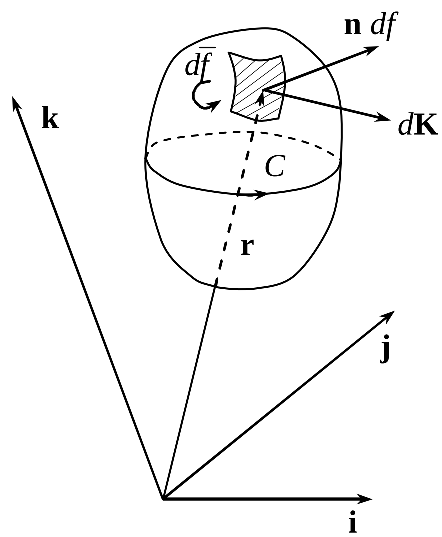

Figure 3.

One will then have that:

$$dK_x = \tfrac{1}{2} e^{\lambda\rho\sigma} i_\lambda \, dK_{\rho\sigma} = \tfrac{1}{2} e^{\lambda\rho\sigma} i_\lambda \, dT_{\alpha\beta,\,\rho\sigma} \, df^{\alpha\beta} = e^{\lambda\rho\sigma} i_\lambda \nabla_{[\alpha} \gamma_{\beta]\,\rho\sigma} \, df^{\alpha\beta}$$
$$= \nabla_{[\alpha} (\gamma_{\beta]\,\rho\sigma} e^{\lambda\rho\sigma} i_\lambda) \, df^{\alpha\beta} \tag{3.4}$$

is an "outer surface total differential," and therefore, according to Stokes's theorem, the x-component of the force sum that is taken over O_1, namely:

$$K_x = \iint_{O_1} dK_x ,$$

will be converted into the line integral:

$$K_x = \tfrac{1}{2} \oint_C \gamma_{\alpha\rho\sigma} e^{\lambda\rho\sigma} i_\lambda dx^\alpha , \tag{3.5}$$

and correspondingly for the components K_y and K_z . If one substitutes (3.2) into (3.5) then that will give:

$$\boxed{K_x = \oint_C \nabla_\rho F_{(\sigma\alpha)} e^{\lambda\rho\sigma} i_\lambda dx^\alpha .} \tag{3.6}$$

The moment of the forces that act upon O_1 relative to the origin of the Cartesian coordinate system will be calculated using (2.18). We shall skip the intermediate computations and give just the ultimate line integral for the x-component of the resultant moment:

$$M_x = \oint_C r^\lambda \gamma_{\lambda\alpha\rho} i^\beta dx^\alpha \tag{3.7}$$

or also

$$\boxed{M_x = \oint_C [-F_{(\alpha\beta)} + 2r^\lambda \nabla_{[\lambda} F_{(\alpha]\beta)} i^\alpha]\, dx^\beta.} \tag{3.8}$$

One recognizes a clear analogy between (3.6) and (3.8) and the formulas of the plane stress state:

$$K_x = [\nabla_\alpha \Phi \cdot e^{\alpha\lambda} i_\lambda]_A^B, \quad K_y = [\nabla_\alpha \Phi \cdot e^{\alpha\lambda} j_\lambda]_A^B, \quad M = [-\Phi + r^\lambda \nabla_\lambda \Phi]_A^B, \tag{3.9}$$

from which, one can calculate the resultants of the forces that act along the segment *AB* and their resultant moment. (Φ is once more the Airy stress function.) Otherwise, one easily calculates that the integrands of (3.6) and (3.8) will be total differentials when one replaces **F** with the tensor $\overset{0}{\mathbf{F}}$ of null stress functions, such that line integral will vanish, as it must. Conversely, the vanishing of K_x, K_y, and K_z for any arbitrary closed outer surface curve will be necessary for one to have $\mathbf{F} = \overset{0}{\mathbf{F}}$.

To conclude our investigations, we will examine two more special cases of the spatial stress states that were previously examined by *Maxwell*, and later in a monograph of *Klein-Wieghardt* [**12**]. They are spatial stress states that depend upon only the stress function Φ, for which one will, in fact, have:

$$(a) \qquad F_{ik} = g_{ik} \cdot \Phi, \qquad\qquad (b) \qquad F_{ik} = \tfrac{1}{2} \nabla_i \Phi \cdot \tfrac{1}{2} \nabla_j \Phi\,. \tag{3.10}$$

The geometric meaning of these Ansätze in reference to (2.9) is easy to recognize:

For the Ansatz (3.10*a*), the metric in stress space will be:

$$\overline{g}_{ik} = g_{ik} \cdot [1 + 2\Phi \cdot \delta t] = u\,(x^{(1)}, x^{(2)}, x^{(3)}) \cdot g_{ik}\,. \tag{3.11}$$

The stress space can thus be mapped *conformally* to Euclidian space, so it will be "conformal Euclidian."

The Ansatz (3.10*b*), by contrast, will yield:

$$d\overline{s}^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + (d\Phi \cdot \sqrt{\delta t}\,)^2 \tag{3.12}$$

for the arc-length element in stress space, so stress space will be a hypersurface:

$$x^{(1)} = \Phi(x^{(1)}, x^{(2)}, x^{(3)}) \cdot \sqrt{\delta t} \tag{3.13}$$

in four-dimensional Euclidian space, and thus of "class one." Therefore, the two cases in (3.10) will be the natural generalizations of the planar case: In fact, any Airy stress surface $x^{(3)} = \Phi(x^{(1)}, x^{(2)})$ is:

a) Conformally mappable to the Euclidian plane (and thus conformal Euclidian).

b) Potentially contained in three-dimensional Euclidian space (and thus of class one).

Summary

The equilibrium state of a continuous, spatial force system can be geometrized by the introduction of a stress function tensor, and in that way, it will admit an intuitive interpretation. This will yield insightful relationships with the trains of thought in the theory of relativity and special problems that are posed in differential geometry.

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