"Die Gleichberechtigung gleichförmig beschleunigter Beobachter für die elecktromagnetischen Erscheinungen," Proc. Ned. Akad. Wet. **43** (1940), 1288-1299.

The equivalence of uniformly-accelerated observers for electromagnetic phenomena

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Summary. – The conformal invariance of MAXWELL's equations has been proved by several authors; BATEMAN and CUNNINGHAM ([1]) first did that in the year 1910. In the first paragraph of this paper, it will be shown that formula (1.5) for the calculation of the field of a point-charge is also conformally-invariant. A physical interpretation of conformal invariance will also be given for the case of the special theory of relativity. Let B' be an observer that is in uniformly-accelerated translational motion with respect to a Cartesian system, namely, the system of an observer B. There will then exist a fundamental Euclidian tensor $'g_{hi} = \sigma^2 g_{hi}$ for which B' is likewise a preferred observer, just as B is relative to g_{hi}. The acceleration can then be transformed away by means of a conformal transformation. It will then follow from the conformal invariance of the laws of electromagnetism that the observers B and B' are equivalent. Space-time must not be regarded as a Euclidian space then, but as a conformally-Euclidian space (but generally, only as far as electromagnetic phenomena are concerned). Hence, it is not the group of LORENTZ transformations, but the group of conformal transformations that determines the geometry of space-time.

At the conclusion of this article, it will be shown that B and B' will measure the same electromagnetic field of a charged massive particle when one assumes that m takes on a factor of σ^{-1} under the conformal transformation $g_{hi} \to \sigma^2 g_{hi}$. That brings with it the fact that the dimension $[ML]$ is invariant under conformal transformations, which agrees with the constancy of h (dimension = $[ML^2T^{-1}]$).

§ 1. The conformal invariance of the electromagnetic equations. – It is known that the electromagnetic field F_{ij} ($h, i, j, \ldots = 1, 2, 3, 4$) can be derived from a potential φ_i:

$$F_{ij} = 2\,\partial_{[i}\,\varphi_{i]}\,. \tag{1.1}$$

([1]) E. CUNNINGHAM, Proc. London Math. Soc. **8** (1910), 77.
H. BATEMAN, Proc. London Math. Soc. **8** (1910), 223.

In a space-time with a conformal metric – i.e., a space in which g_{ih} is given only up to an arbitrary (non-constant) factor – there exists a tensor density $\mathfrak{G}_{ih}$ of weight – 1/2 that one obtains from g_{hi} as follows:

$$\mathfrak{G}_{ih} = (-\mathfrak{g})^{-1} g_{ih}, \qquad \mathfrak{g} = \mathrm{Det}\,(g_{ih}). \tag{1.2}$$

Now the electromagnetic field satisfies the following equations:

$$\left.\begin{aligned} &a)\quad \partial_{[j} F_{ih]} = 0, \\ &b)\quad \partial_j \mathfrak{F}^{jh} = -\mathfrak{s}^h, \quad \mathfrak{F}^{hi} = \mathfrak{G}^{hj}\mathfrak{G}^{il} F_{jl}, \\ &c)\quad \partial_j \mathfrak{s}^j = 0, \end{aligned}\right\} \tag{1.3}$$

in which $\mathfrak{s}^h$ stands for the current vector-density of weight + 1. These are MAXWELL's equations in conformally-invariant form ([2]).

In the *special theory of relativity*, the line element is Euclidian, and the reference system can be chosen in such a way that one has:

$$g_{11} = g_{22} = g_{33} = -g_{11} = -1, \quad g_{hi} = 0 \qquad (h \neq i). \tag{1.4}$$

In this case, it is possible to give the potential vector φ_i more specifically as a solution of (1.3*b*). Namely, every point-charge determines a field, and one will obtain the total field by adding (or integrating) those fields. It therefore suffices to give an expression for the field of a single point-charge *e*.

In order to do that, we consider an arbitrary point *P*. Let *Q* be the intersection point of the world-line of the point-charge with the past null-cone of *P*, and let u^h be a tangent vector to that world-line at *Q*. As is known, the value of the field potential φ_i that is produced by the charge *e* at the point *P* is:

$$\varphi_i = -\frac{e}{4\pi}\frac{u_i}{u_h R^h}. \tag{1.5}$$

In this, R^h is the radius vector $\overrightarrow{QP}$, and thus, a null vector ($R^h\,R_h = 0$). The electromagnetic field of the point-charge can then be calculated from (1.5) by means of (1.1).

We will now show that formula (1.5) is also conformally-invariant. Here, we mean by that:

If a conformal transformation $'g_{ih} = \sigma^2 g_{ih}$ were performed in such a way that the fundamental tensor remained Euclidian, and if the potential vector:

([2]) J. A. SCHOUTEN and J. HAANTJES, "Ueber die the konforminvariant Gestalt der Maxwellschen Gleichungen und der elektromagnetischen Impulsenergiegleichungen," Physica **1** (1934), 869-872. Cf., also: J. A. SCHOUTEN and J. HAANTJES, "Ueber die konforminvariant Gestalt der relativistischen Bewegungsgleichungen," Proc. Kon. Akad. v. Wetensch., Amsterdam **39** (1936), 1059-1065.

$$\varphi'_{i'} = -\frac{e}{4\pi}\frac{'g_{i'j'}\,u^{j'}}{'g_{h'j'}\,u^{h'}R^{j'}} \tag{1.6}$$

were constructed in reference to a Cartesian coordinate system (h) that belongs to $'g_{ih}$ [i.e., a system for which $'g_{ih}$ is determined by the numbers (1.4)] then $\varphi'_{i'}$ would differ from φ_i only by a gradient vector. $\varphi'_{i'}$ and φ_i would then lead to the same electromagnetic field. In other words: Formulas (1.5) for the calculation of the electromagnetic field are conformally-invariant.

The transformation that takes the Euclidian reference system (h) that belongs to g_{hi} to a Euclidian reference system (h') that belongs to $'g_{hi}$ is a conformal transformation. Now, any conformal transformation can be composed of inversions and LORENTZ transformations ([3]). We then need only to prove the aforementioned invariance under an inversion, since the invariance under LORENTZ transformations is immediately clear from (1.5). In order to do that, we consider the following inversion (which is regarded as a coordinate transformation here):

$$x^{h'} = \frac{x^h}{x^i x_i}\,\delta_h^{h'} . \tag{1.7}$$

Differentiation of (1.7) leads to the transformation coefficients of a vector:

$$A_h^{h'} = \delta_h^{h'}\frac{1}{x^i x_i} - \frac{2x_h\,x^j}{(x^i x_i)^2}\,\delta_j^{h'} . \tag{1.8}$$

g_{ih} is determined by the numbers:

$$g_{i'h'} = A_{i'h'}^{ih}\,g_{ih} \tag{1.9}$$

relative to (h'). A brief calculation will show that:

$$g_{i'h'} = (x^j x_j)^2\,g_{ih}\;\delta_{i'}^{i}\,\delta_{h'}^{h} . \tag{1.10}$$

The transformed fundamental tensor is determined by the numbers (1.4) relative to (h'), so:

$$'g_{i'h'} = g_{ih}\;\delta_{i'}^{i}\,\delta_{h'}^{h} , \tag{1.11}$$

which, in conjunction with (1.10), will yield:

$$'g_{ih} = \sigma^2\,g_{ih} ,\quad \sigma^2 = (x^j x_j)^{-2}. \tag{1.12}$$

([3]) Cf., J. HAANTJES, "Conformal representations of an n-dimensional Euclidian space with a non-definite fundamental form on itself," Proc. Kon. Akad. v. Wetenschapen, Amsterdam **40** (1937), 700-705. Cf., also SCHOUTEN-STRUIK, *Einführung II*, Noordhoff, pp. 210.

We will next calculate the vector $'R^{h'}$. Let the coordinates of P (Q, resp.) be x^h (y^h, resp.). One will then have:

$$R^h = x^h - y^h, \tag{1.13}$$

with

$$R^h R_h = x^h x_h - 2x^h y_h + y^h y_h = 0. \tag{1.14}$$

The numbers that determine $'R^{h'}$ relative to (h') are the coordinate differences between P and Q relative to (h'):

$$'R^{h'} = x^{h'} - y^{h'} = \left(\frac{x^h}{x^i x_i} - \frac{y^h}{y^i h_i}\right)\delta_h^{h'}. \tag{1.15}$$

Now let $y^h = y^h(\tau)$ be the world-line of the point-charge; let τ be an arbitrary parameter. The tangent vector u^h at Q is determined by the numbers:

$$u^{h'} = (A_h^{h'})_Q u^h = \left(\delta_h^{h'} - \frac{2y_h\, y^j}{y^i\, y_i}\delta_j^{h'}\right)\frac{1}{y^i\, y_i}\frac{dy^h}{d\tau} \qquad \left(u^h = \frac{dy^h}{d\tau}\right). \tag{1.16}$$

relative to (h') [cf., (1.8)].

We find the denominator of (1.6) from (1.11), (1.15), and (1.16):

$$\left.\begin{aligned}
'g_{h'i'}\, R^{h'} u^{i'} &= g_{lm}\, \delta_{h'}^l\, \delta_{i'}^m \left(\frac{x^h}{x^j x_j} - \frac{y^h}{y^j y_j}\right)\delta_h^{h'}\left(\delta_i^{i'} - \frac{2y_i\, y^k}{y^j y_j}\delta_h^{h'}\right)\frac{1}{y^j y_j}\frac{dy^i}{d\tau} \\
&= g_{hi}\left(\frac{x^h}{x^j x_j} - \frac{y^h}{y^j y_j}\right)\frac{1}{y^j y_j}\left(\frac{dy^i}{d\tau} - 2y_l\, \frac{y^i}{y^j y_j}\frac{dy^l}{d\tau}\right) \\
&= \frac{1}{(x^j x_j)(y^j y_j)}\left\{x_h \frac{dy^h}{d\tau} - \frac{x^i x_i}{y^j y_j} y_h \frac{dy^h}{d\tau} - \frac{2x_i\, y^i}{y^i y_i} y^h \frac{dy^h}{d\tau} + \frac{2x^i x_i}{y^i y_i} y_h \frac{dy^h}{d\tau}\right\}.
\end{aligned}\right\} \tag{1.17}$$

This expression can be simplified by means of (1.14). We obtain:

$$'g_{h'i'}\, 'R^{h'} u^{i'} = \frac{1}{(x^j x_j)(y^j y_j)}\left(x_h \frac{dy^h}{d\tau} - y_h \frac{dy^h}{d\tau}\right) = \frac{R^h\, x_h}{(x^i x_i)(y^i y_i)}. \tag{1.18}$$

For the calculation of φ_i', the vector $u^{i'}$ must be parallel-displaced (in the sense of $'g_{hi}$) from Q to P. The determining numbers will not change under this pseudo-parallel displacement relative to the system (h'). From (1.6), we need the covariant determining numbers of that pseudo-parallel-displaced vector that are defined by means of $'g_{hi}$. From (1.16), they are:

$$u_{i'}' = 'g_{i'j'}\, u^{j'} = g_{ij}\, \delta_{i'}^i\, \delta_{j'}^j\left(\delta_h^{j'} - \frac{2y_h\, y^m}{y^l\, y_l}\delta_m^{j'}\right)\frac{1}{y^l y_l}\frac{dy^h}{d\tau} = \frac{1}{y^l y_l}\delta_{i'}^i\left(g_{ih} - \frac{2y_h\, y_i}{y^l\, y_l}\right)\frac{dy^h}{d\tau}. \tag{1.19}$$

The determining numbers relative to (h) can now be found with the use of (1.8), (1.14), and (1.19). One will get:

$$'u_i = (A_i^{i'})_P\, 'u_{i'} = \frac{1}{(x^j x_j)(y^j y_j)}\left\{ g_{ih} + \frac{2y_h(x_i - y_i)}{y^j y_j} - \frac{2x_i(x_h - y_h)}{x^j x_j} \right\} \frac{dy^h}{d\tau}. \tag{1.20}$$

Each point P (x^h) belongs to a well-defined point Q on the curve $y^h = y^h(\tau)$, and thus, a well-defined value of τ. One can then consider τ to be a function of x^h. That function will follow from [cf., (1.14)]:

$$x^i x_i + y^i(\tau)\, y_i(\tau) - 2\, x^i y_i(\tau) = 0. \tag{1.21}$$

Differentiating with respect to x^i yields:

$$x_i - y_i - (x_h - y_h)\frac{dy^h}{d\tau}\frac{\partial \tau}{\partial x^i} = 0; \tag{1.22}$$

hence [cf., (1.16)]:

$$\frac{\partial \tau}{\partial x^i} = \frac{x_i - y_i}{R^h u_h}. \tag{1.23}$$

We now return to (1.18) and (1.20) and construct the new potential vector φ_i' that belongs to $'g_{hi}$. We consider the determining numbers of the new potential vector relative to the old reference system (h). From (1.6), (1.18), (1.20), and (1.23), they are:

$$\left.\begin{aligned} \varphi_i' &= -\frac{e}{4\pi}\frac{'u_i}{'g_{h'i'}\,'R^{h'}u^{i'}} \\ &= -\frac{e}{4\pi}\left\{ \frac{u_i}{R^h u_h} + \frac{2y_h}{y^j y_j}\frac{dy^h}{d\tau}\frac{\partial\tau}{\partial x^i} - \frac{2x_h}{x^j x_j} \right\} \\ &= \varphi_i - \frac{e}{4\pi}\partial_i \ln(y^j y_j) + \frac{e}{4\pi}\partial_i \ln(x^j x_j) \\ &= \varphi_i + \frac{e}{8\pi}\{\partial_i \ln(\sigma)_Q - \partial_i \ln(\sigma)_P. \end{aligned}\right\} \tag{1.24}$$

φ_i' and φ_i will then differ by only a gradient vector, from which, it emerges that:

$$F_{ij}' = F_{ij}. \tag{1.25}$$

We have then proved that:

If the electromagnetic field in the special theory of relativity were calculated from a Euclidian fundamental tensor $'g_{ij} = \sigma^2 g_{ij}$, *instead of* g_{ij}, *by means of the same formulas then one would get the same result.*

Upon considering the conformal invariance of MAXWELL's equation, we will obtain the following theorem:

The electromagnetic equations have the same form relative to the Cartesian reference system that belongs to the $'g_{ih}$ *that they have relative to the Cartesian system that belongs to the* g_{ih} *. Those reference systems are therefore all equivalent.*

§ 2. The conformal transformation that belongs to a uniformly-accelerated observer. – We shall now consider the world-line of a point (or an observer) B' that is in uniformly-accelerated translational motion relative to a Cartesian reference system B. Let B' be at rest relative to B at time $t = 0$. We choose the direction of the x^1-axis in such a way that it coincides with the direction of motion of B'. The equation for the world-line of B:

$$a\,\{(x^1)^2 - (x^4)^2\} + 2x^1 = 0, \qquad x^2 = x^3 = 0 \tag{2.1}$$

will then be verified, as in what follows.

That world-line is a hyperbola whose asymptotic directions are null directions. The equation for the hyperbola reads:

$$x^1 = \frac{2}{a(\lambda^2 - 1)}, \qquad x^4 = \frac{2\lambda}{a(\lambda^2 - 1)}, \qquad x^2 = x^3 = 0 \tag{2.2}$$

in parametric form.

It emerges from this that:

$$\frac{dx^1}{ds} = \frac{2\lambda}{\lambda^2 - 1}, \qquad \frac{dx^4}{ds} = \frac{\lambda^2 + 1}{\lambda^2 - 1}, \qquad ds^2 = (dx^4)^2 - (dx^1)^2 = \frac{4}{a^2}\frac{(d\lambda)^2}{(\lambda^2 - 1)^2}, \tag{2.3}$$

so

$$\frac{d^2x^1}{ds^2} = \frac{a(\lambda^2 + 1)}{\lambda^2 - 1}, \qquad \frac{d^2x^4}{ds^2} = \frac{2a\lambda}{\lambda^2 - 1}. \tag{2.4}$$

We find:

$$\left\{\left(\frac{d^2x^1}{ds^2}\right)^2 - \left(\frac{d^2x^4}{ds^2}\right)^2\right\}^{1/2} = |\,a\,| \tag{2.5}$$

for the length of the vector $d^2 x^h / ds^2$.

At every point P of the world-line, the acceleration of B' is then equal to:

$$a\,c^2 \tag{2.6}$$

relative to an observer that moves with constant velocity (relative to B) and has the same velocity as B' at P, so for that observer, one will have $ds^2 = c^2\,dt^2$.

We now ask what the conformal coordinate transformation:

$$x^{h'} = f^{h'}(x^i) \tag{2.7}$$

would be that would make the world-line of B' have the equation:

$$x^{1'} = 0, \qquad x^{2'} = 0, \qquad x^{3'} = 0. \tag{2.8}$$

relative to the system (h').

There are several such transformations, although under certain physically-plausible restrictions, there is just one of them. Namely, we will assume that the transformation satisfies the following requirements:

I. The origin of the reference system is invariant.

II. The transformation does not change when one replaces either:

a) x^2 with $-x^2$, or
b) x^3 with $-x^3$, or
c) x^4 with $-x^4$ (and thus switches past and future).

III. g_{ih} has the same determining numbers at the origin relative to (h') that it has relative to (h), and therefore (– 1, – 1, – 1, + 1). In other words: Let $\sigma = 1$ at the origin.

The most general infinitesimal local conformal transformation that satisfies the requirements I and III is:

$$\left.\begin{aligned}
x^{1'} &= \frac{1}{N}\{\quad x^1 + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + a_{15}\, g_{ij}\, x^i x^j\},\\
x^{2'} &= \frac{1}{N}\{-a_{12}x^1 + x^2 + a_{23}x^3 + a_{24}x^4 + a_{25}\, g_{ij}\, x^i x^j\},\\
x^{3'} &= \frac{1}{N}\{-a_{13}x^1 - a_{23}x^2 + x^3 + a_{34}x^4 + a_{35}\, g_{ij}\, x^i x^j\},\\
x^{4'} &= \frac{1}{N}\{+a_{14}x^1 + a_{24}x^2 + a_{34}x^3 + x^4 + a_{45}\, g_{ij}\, x^i x^j\},
\end{aligned}\right\} \tag{2.9}$$

in which one has:

$$N = 1 - 2\, a_{15}\, x^1 - 2\, a_{25}\, x^2 - 2\, a_{35}\, x^3 - 2\, a_{45}\, x^4, \tag{2.10}$$

and the coefficients $a_{\lambda\kappa}$ (λ, $\kappa = 1, \ldots, 5$) are infinitesimal. The requirement II then leads to the following infinitesimal transformation:

$$\left.\begin{aligned} x^{1'} &= \frac{x^1 + \frac{1}{2} b g_{ij} x^i x^j}{1 - bx^1} &&\text{or} = x^1 + b\{(x^1)^2 + \tfrac{1}{2} g_{ij} x^i x^j\}, \\ x^{2'} &= \frac{x^2}{1 - bx^1} &&\text{or} = x^2 + b\, x^1 x^2, \\ x^{3'} &= \frac{x^3}{1 - bx^1} &&\text{or} = x^3 + b\, x^1 x^3, \\ x^{4'} &= \frac{x^4}{1 - bx^1} &&\text{or} = x^4 + b\, x^1 x^4, \end{aligned}\right\} \tag{2.11}$$

when one neglects b^2. The associated finite transformations can be found as solutions to the system of differential equations:

$$\left.\begin{aligned} &a) \quad \frac{dx^1}{db} = (x^1)^2 + \tfrac{1}{2} g_{ij} x^i x^j, \\ &b) \quad \frac{dx^a}{db} = x^1 x^a \quad (a = 2, 3, 4). \end{aligned}\right\} \tag{2.12}$$

It emerges from (2.12) that:

$$\frac{d(g_{ij}\, x^i x^j)}{db} = 2x^1 \,\{(x^2)^2 - (x^3)^2 - (x^2)^2 - (x^1)^2 - \tfrac{1}{2} g_{ij}\, x^i\, x^j\} = x^1\, g_{ij}\, x^i\, x^j\,; \tag{2.13}$$

hence [with (2.12a)]:

$$\frac{d^2 x^1}{db^2} = 2x^1 \frac{dx^1}{db} + \tfrac{1}{2} g_{ij}\, x^i\, x^j = 3x^1 \frac{dx^1}{db} - (x^1)^3. \tag{2.14}$$

The solution to this differential equation will be obtained most simply when one first substitutes $x^1 = -\dfrac{d}{db} \ln y$. That substitution leads to $y''' = 0$, from which, it emerges that:

$$x^1 = \frac{a^1 - \beta b}{1 - a^1 b + \frac{1}{2} \beta b^2}, \tag{2.15}$$

in which a^1 and β are constants. Substituting this expression in (2.12*b*) will yield:

$$x^a = \frac{a^a}{1 - a^1 b + \frac{1}{2} \beta b^2} \qquad (a = 2, 3, 4). \tag{2.16}$$

It will then follow from (2.12*a*), (2.15), and (2.16) that:

$$\beta = -\tfrac{1}{2} g_{ij}\, a^i\, a^j. \tag{2.17}$$

The constants a^i will be determined from the initial values of x^h. That will yield $a^h = (x^h)_0$. If we again call the initial values x^h and the transformed determining numbers $x^{h'}$ then the *finite* conformal transformation will read (b is no longer infinitesimal here):

$$\left.\begin{aligned} x^{1'} &= \frac{x^1 + \frac{1}{2} b\, g_{ij} x^i x^j}{1 - b x^1 - \frac{1}{4} b^2\, g_{ij} x^i x^j}, \\ x^{a'} &= \frac{x^a}{1 - b x^1 - \frac{1}{4} b^2\, g_{ij} x^i x^j} \quad (a = 2, 3, 4). \end{aligned}\right\} \tag{2.18}$$

One gets the inverse transformation when one replaces the parameter b with $-b$ in (2.18).

The world-line (2.1) of B' has the equation:

$$(a + b)\,\{(x^{1'})^2 - (x^{4'})^2\} + 2x^{1'} = 0, \qquad x^{2'} = x^{3'} = 0 \tag{2.19}$$

relative to (h'). If we then choose $b = -a$ then that world-line will take on the desired form:

$$x^{1'} = x^{2'} = x^{3'} = 0. \tag{2.20}$$

The transformation (2.18) *with b = – a is then the only conformal transformation that satisfies the three aforementioned requirements and leads to equation* (2.20).

In regard to the transformations (2.18), we can make the following remarks:

1. The transformation (2.18) defines a one-parameter group with b as its parameter. If we denote the transformation that is associated with the parameter b by T_b then the multiplication law can be written as:

$$T_b \cdot T_b = T_{b+c}\,, \;\; T_0 = I \quad \text{(identity)}. \tag{2.21}$$

We will omit the proof of this theorem here.

2. The following conformal transformation is derived in the paper ([4]) that was cited before:

$$x^{h'} = \frac{x^h - \frac{1}{2} b^h x^i x_i}{1 - b^i x_i}, \qquad b^i\, b_i = 0. \tag{2.22}$$

Now, the transformation (2.18) is the product of two such transformations – the one, by $b^h\left(-\dfrac{b}{2}, 0, 0, -\dfrac{b}{2}\right)$, and the other by $b^h\left(-\dfrac{b}{2}, 0, 0, \dfrac{b}{2}\right)$.

3. The determining numbers of the fundamental tensors g_{ij} relative to (h) can be calculated by means of the formal $g_{h'i'} = A^{hi}_{h'i'}\, g_{hi}$. That will yield:

([4]) *Loc. cit.* ([3]), pp. 704.

$$g_{1'1'} = g_{2'2'} = g_{3'3'} = g_{4'4'} = -\,(1 - b\,x^1 - \tfrac{1}{4} b^2\, g_{ij}\, x^j\, x^i)^2, \qquad g_{h'i'} = 0 \quad (h' \neq i'). \tag{2.23}$$

§ 3. The physical consequence. – The *special theory of relativity* is based upon the following postulate: If a coordinate system B is chosen such that the laws of physics take on their simplest form relative to it (i.e., a preferred coordinate system) then *the same* laws will also be true relative to any other coordinate system B' that moves with a *uniform translation* relative to B. There are preferred observers, and every preferred observer is associated with a certain Cartesian coordinate system in space-time. These observers have constant velocities relative to each other, while the associated reference systems emerge from each other by means of LORENTZ transformations. The vacuum speed of light is equal to c for every observer, and thus constant. The preferred observers are all *equivalent.*

However, we can now go somewhat further (at least, as far as the electromagnetic laws of physics are concerned) and extend the class of preferred observers. We can also consider the observers that relate to Cartesian systems in *uniformly-accelerated translational motion*. They are then the observers whose world-lines are generalized circles. Namely, we showed in the second paragraph that one could transform away the acceleration by means of a conformal transformation, and in the first paragraph we showed that the laws of electromagnetism are conformally-invariant; i.e., they do not change under conformal transformations.

We can make that result somewhat more precise. In order to do that, we consider the observer B' that has the following world-line [cf., (2.1)]:

$$a\,\{(x^1)^2 - (x^4)^2\} + 2x^1 = 0, \qquad x^2 = x^3 = 0 \tag{3.1}$$

relative to a Cartesian system (h), namely, the system of the observer B. B' is then in uniformly-accelerated motion relative to B with acceleration $a\,c^2$. As one can show, the preferred reference system that belongs to B' is obtained from (h) by means of the transformation [cf., (2.18)]:

$$\left.\begin{aligned} x^{1'} &= \frac{x^1 - \frac{1}{2}a\,g_{ij}x^i x^j}{1 + ax^1 - \frac{1}{4}a^2\,g_{ij}x^i x^j}, \\ x^{a'} &= \frac{x^a}{1 + ax^1 - \frac{1}{4}a^2\,g_{ij}x^i x^j}. \end{aligned}\right\} \tag{3.2}$$

The world-line of B' is:

$$x^{1'} = x^{2'} = x^{3'} = 0 \tag{3.3}$$

relative to (h').

In the first paragraph, we showed that the electromagnetic equations have the same form relative to (h') for B'; (h') is the preferred system for them then. The vacuum speed of light is also equal to c for B'. However, B' does not use the same fundamental tensor as B. Its fundamental tensor has the same determining numbers relative to (h') that g_{ih} has relative to (h). One obtains the fundamental tensor of B' from the fundamental of B

by multiplying it by a factor [cf., (2.23)]. Hence, as far as electromagnetic phenomena are concerned, B and B' are also equivalent then.

That brings with it the idea that we must alter our conception of space-time somewhat. *As far as electromagnetic phenomena are concerned, space-time is not a Euclidian space, but a conformally-Euclidian space.* B and B' are then equivalent, and therefore so are the two associated fundamental tensors. *The transformations* (3.2), *together with the* LORENTZ *transformations, define the conformal group, which determines the geometry of space-time.*

Up to now, no measurements have been taken into consideration. However, the field F_{ij} is measured with the help of a charged massive particle – e.g., by the acceleration of particles that the field provokes. In order to do that, one must first measure the acceleration of the particle with the field and once without it. One then addresses the *change* in the acceleration that is caused by the field F_{ij} . We can now prove that *the two observers B and B′ will obtain the same resulting measurement in that way when we assume that the mass m takes on a factor of* σ^{-1} *under the transformation* $g_{ih} \to \sigma^2\, g_{ih}$. That brings with it the idea that $m\ (-\mathfrak{g})^{1/2}$ must not change under conformal transformations.

Namely, the equation of motion of a charged particle for the observer B:

$$m\,\frac{d^2x^h}{d\tau^2} = \ldots \tag{3.4}$$

will include the term:

$$\frac{e}{c}\frac{dx^i}{d\tau}F_{ij}\,g^{hj}. \tag{3.5}$$

In this, m is the mass for B, while τ is determined by means of $g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 1$. That expression has the form:

$$\frac{e}{c}\frac{dx^{i'}}{d\tau}F_{i'j'}g^{h'j'} = \sigma^3\ \frac{e}{c}\frac{dx^{i'}}{d\tau'}F_{i'j'}{}'g^{h'j'}, \tag{3.6}$$

relative to a preferred system (h') of the observer B'. The left-hand side of the equation of motion (3.4) reads:

$$m\,\frac{d^2x^{h'}}{d\tau^2} + \ldots = m\,\sigma^2\frac{d^2x^{h'}}{(d\tau')^2} + \ldots, \qquad (d\tau')^2 = {}'g_{hi}\,dx^i\,dx^j = \sigma^2\,(d\tau)^2 \tag{3.7}$$

relative to (h'), in which only the terms that contain second derivatives are written down. Once more, as far as measurements are concerned, the observers are equivalent when:

$$m' = \sigma^{-1}\,m\,, \tag{3.8}$$

in which the mass for the observer B' is denoted by m'. Naturally, B and B' will not measure the same acceleration in this case, but the same change in acceleration.

Formula (3.8) means that the mass m is different for two observers that are accelerated relative to each other. m takes on a factor of σ^{-1} under the conformal transformation $g_{ih} \rightarrow \sigma^2 g_{ih}$. A transformation of lengths by σ must then imply a transformation of masses by σ^{-1}. The dimension $[ML]$ is then invariant, which one could also conclude from the constancy of PLANCK's constant h (dimension = $ML^2 T^{-1}$) ([5]).

([5]) Cf., J. A. SCHOUTEN and J. HAANTJES, "Ueber die konforminvariante Gestalt der relativistichen Bewegungsgleichungen," Proc. Kon. Akad. v. Wetensch., Amsterdam **39** (1936), 1063.