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On adjoint variational problems and adjoint extremal surfaces

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The paradigm for the development of the part of the calculus of variations in which the unknown functions depend upon one variable was always the theory of geodetic lines on curved surfaces, which was founded by Gauss and built up by Darboux, in particular. Not only are the investigations of Jacobi and Kneser (whose influence on the theory of geodetic lines is well-known) based upon that theory ([1]), but also the direct method of treating the variational problem, for which one has Hilbert to thank.

Now, as far as that part of the calculus of variations in which the unknown function depends upon two independent variables is concerned, one finds that a special case of it has been treated with great thoroughness in surface theory, namely, minimal surfaces, whose theory was developed in a series of classical works. In that problem, we do not think in terms of the existence questions that are connected with those surfaces (i.e., Plateau's problem), but mainly in terms of theorems that address minimal surface "in the small," so ones that are implied by the vanishing of the first variation.

Now, that suggests the idea of making that part of surface theory useful for the purposes of the calculus of variations, that is, of adapting the results that are obtained in the theory of minimal surfaces to more general variational problems, resp., in analogy with what was already implemented in the theory of geodetic lines. In fact, one can achieve that for variational problems of the form:

$$\delta \iint f\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy = 0 \tag{1}$$

(in which the integral includes only the derivatives of the unknown functions). Naturally, in so doing, one will require a *new conceptual picture*, in analogy with the way that the concept of transversality must be introduced in order to be able to adapt the theory of geodetic lines to the calculus of variations. In the following investigations, it is the concepts of the *adjoint variational*

([1]) Hilbert employed, above all, the example of the geodetic line in order to present his method. ("Über das Dirichletsche Prinzip," Jahresbericht d. deutschen Math. Ver., Bd. **8**, pp. 184.)

problem and the *adjoint extremal surface* that will make it possible to adapt the theorems on minimal surfaces to variational problems of the form above.

In fact, the concept of an adjoint minimal surface that Bonnet introduced plays a fundamental role in the aforementioned theory of minimal surfaces. Of course, that association, by which any minimal surface is adjoint to another minimal surface, is ordinarily presented and studied in such a way that in the known Weierstrass representation of minimal surfaces (which assigns an analytic function to each minimal surface in a well-defined way), one replaces the function of a complex variable that appears in it with the i^{th} power of that function, which is a way of treating things that can hardly be useful for purpose of generalization ([2]). However, when one liberates the concept of an adjoint minimal surface from the Weierstrass representation, i.e., when one examines the adjoint surface in general Gaussian coordinates, one will easily arrive at a form that is capable of being generalized to variational problems of the aforementioned type.

When one associates an adjoint surface to any extremal surface of the variational problem (1) in that way, one will arrive at the jumping-off point for the following investigations: Namely, that adjoint surface is not an extremal surface of the variational problem (1) in the general case, but probably a different variational problem – viz., the adjoint variational problem – that can be derived from the original one in a simple way. In that way, one will obtain an association that associates every variational problem of the form (1) with a problem of the same form. That relationship is involutory, and the problem of the minimal surface is even distinguished by the fact that it is a self-adjoint problem (§ **1**).

If one now examines the map of an extremal surface of (10) to its adjoint surface (§ **2**) then one will arrive at the generalizations of the aforementioned classical theorems of the theory of minimal surfaces. Among the results that one obtains, we would like to emphasize the following ones:

In the theory of minimal surfaces, one has the theorem that any minimal surface can be bent into its adjoint. In our more general sphere of ideas, we have the theorem that one can introduce a non-Euclidian metric on the extremal surfaces and their adjoints that depends upon only the corresponding variational problem in such a way that the map in question is a bending in that sense. One gets the result from this that any variational problem determines a metric on its extremal surfaces in a well-defined way (which coincides with the usual arc-length only in the case of minimal surfaces). That variational integral itself will be the surface area when one bases its definition upon arc-length. When one introduces isothermal parameters into that metric, one will get a system of differential equations for the coordinates of the extremal surface of an especially simple type that is well-suited to the study of those surfaces. Those theorems are the generalizations of the known theorems in the representation of minimal surfaces by potential functions, and (it seems to me) those differential equations also are of great service in the further theory of the variational problem (1).

([2]) Bonnet's original definition of the adjoint minimal surface also employs an entirely-specialized coordinate system on the surface (Comptes rendus, 1853, pp. 532).

§ 1.

Concept and main properties of the adjoint extremal surface and the adjoint variational problem.

1. Notations and conventions. – We shall base our investigations upon a variational problem of the form:

$$\delta \iint f(z_x, z_y)\, dx\, dy = 0 \tag{1}$$

and consider any extremal surface $z = z(x, y)$ of it. We set ([3]):

$$z_x = p, \qquad z_y = q,$$

to abbreviate. In order to arrive at an elegant form, we write the equation of the extremal surface by introducing two parameters u, v in the form:

$$x = \xi(u, v), \quad y = \eta(u, v), \quad z = \zeta(u, v)$$

and set:

$$p_1 = \begin{vmatrix} \eta_u & \zeta_u \\ \eta_v & \zeta_v \end{vmatrix}, \qquad p_2 = \begin{vmatrix} \zeta_u & \xi_u \\ \zeta_v & \xi_v \end{vmatrix}, \qquad p_3 = \begin{vmatrix} \xi_u & \eta_u \\ \xi_v & \eta_v \end{vmatrix}. \tag{2}$$

One then has:

$$\zeta(u, v) = z(\xi(u, v), \eta(u, v)) \qquad \text{and} \qquad p = -\frac{p_1}{p_3}, \qquad q = -\frac{p_2}{p_3},$$

and when one introduces the further notation:

$$F(p_1, p_2, p_3) = p_3\, f\left(-\frac{p_1}{p_3}, -\frac{p_2}{p_3}\right), \tag{3}$$

the given variational problem will go to the following problem:

$$\delta \iint F(p_1, p_2, p_3)\, du\, dv = 0. \tag{I}$$

We assume that the integrands $f(p, q)$ and $F(p_1, p_2, p_3)$ are sufficiently-often differentiable functions for all values of the arguments that come under consideration. In addition, the latter is homogeneous of degree one. As a result:

$$p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = F,$$

and one will have the relations:

([3]) In what follows, we shall denote the derivatives of a function by adding corresponding indices.

(3′) $$F_{p_1} = -f_p\,, \qquad F_{p_2} = -f_q\,, \qquad F_{p_3} = f - p\,f_p - q\,f_q\,,$$

as one easily convinces oneself. Those formulas show that the derivatives F_{p_1}, F_{p_2}, F_{p_3} are homogeneous functions of degree 0 in the p_1, p_2, p_3, and are therefore independent of the choice of the parameters u, v.

2. Definition of the adjoint extremal surface. – The Euler-Lagrange differential equations of the variational problem I) read:

$$\frac{\partial}{\partial u}\frac{\partial F}{\partial \xi_u} + \frac{\partial}{\partial v}\frac{\partial F}{\partial \xi_v} = 0\,, \qquad \frac{\partial}{\partial u}\frac{\partial F}{\partial \eta_u} + \frac{\partial}{\partial v}\frac{\partial F}{\partial \eta_v} = 0\,, \qquad \frac{\partial}{\partial u}\frac{\partial F}{\partial \zeta_u} + \frac{\partial}{\partial v}\frac{\partial F}{\partial \zeta_v} = 0\,.$$

When one recalls the easily-verified relations:

$$\frac{\partial F}{\partial \xi_u} = \begin{vmatrix} \eta_v & \zeta_v \\ F_{p_2} & F_{p_3} \end{vmatrix}, \quad \frac{\partial F}{\partial \eta_u} = \begin{vmatrix} \zeta_v & \xi_v \\ F_{p_3} & F_{p_1} \end{vmatrix}, \quad \frac{\partial F}{\partial \zeta_u} = \begin{vmatrix} \xi_v & \eta_v \\ F_{p_1} & F_{p_2} \end{vmatrix},$$

$$\frac{\partial F}{\partial \xi_v} = -\begin{vmatrix} \eta_u & \zeta_u \\ F_{p_2} & F_{p_3} \end{vmatrix}, \quad \frac{\partial F}{\partial \eta_v} = -\begin{vmatrix} \zeta_u & \xi_u \\ F_{p_3} & F_{p_1} \end{vmatrix}, \quad \frac{\partial F}{\partial \zeta_v} = -\begin{vmatrix} \xi_u & \eta_u \\ F_{p_1} & F_{p_2} \end{vmatrix}$$

and introduces three auxiliary functions:

$$\bar{\xi}(u,v)\,, \qquad \bar{\eta}(u,v)\,, \qquad \bar{\zeta}(u,v)$$

(which are determined uniquely up to an additive constant), one can replace those Euler-Lagrange differential equations with the following first-order system of differential equations ([4]):

(II) $$\bar{\xi}_u = \begin{vmatrix} \eta_u & \zeta_u \\ F_{p_2} & F_{p_3} \end{vmatrix}, \qquad \bar{\eta}_u = \begin{vmatrix} \zeta_u & \xi_u \\ F_{p_3} & F_{p_1} \end{vmatrix}, \qquad \bar{\zeta}_u = \begin{vmatrix} \xi_u & \eta_u \\ F_{p_1} & F_{p_2} \end{vmatrix},$$

$$\bar{\xi}_v = \begin{vmatrix} \eta_v & \zeta_v \\ F_{p_2} & F_{p_3} \end{vmatrix}, \qquad \bar{\eta}_v = \begin{vmatrix} \zeta_v & \xi_v \\ F_{p_3} & F_{p_1} \end{vmatrix}, \qquad \bar{\zeta}_v = \begin{vmatrix} \xi_v & \eta_v \\ F_{p_1} & F_{p_2} \end{vmatrix}.$$

([4]) That is the form of the Euler-Lagrange differential equation that one can derive from the vanishing of the first variation *without assuming the existence of second derivatives of the extremal function*. Cf., my paper “Über die Variation der Doppelintegrale,” Journal für die reine u. angew. Math., Bd. **149**, pp. 1-18. Since I exhibited those differential equations for the case of minimal surfaces, I recognize that the auxiliary functions that appear in them coincide with the coordinates of Bonnet’s adjoint minimal surface, and that fact led me to the conceptual picture of the present investigations.

If one writes those equations in the following form:

$$\text{(III)} \qquad d\bar{\xi} = F_{p_3}\,d\eta - F_{p_2}\,d\zeta, \qquad d\bar{\eta} = F_{p_1}\,d\zeta - F_{p_3}\,d\xi, \qquad d\bar{\zeta} = F_{p_2}\,d\xi - F_{p_1}\,d\eta$$

then one will see immediately (keeping in mind that the expressions for F_{p_1}, F_{p_2}, F_{p_3} are independent of choice of parameters *u*, *v*) that the surface:

$$x = \bar{\xi}(u,v), \quad y = \bar{\eta}(u,v), \quad z = \bar{\zeta}(u,v)$$

is defined uniquely, up to a parallel translation, and in particular, it is independent of the choice of the parameters *u*, *v*. We call that surface the *adjoint surface to the extremal surface in question relative to the given variational problem* ([5]).

Not only is the adjoint surface itself defined by equations (III), but also a map of that surface onto the original extremal surface by which we let points with the same parameter values *u*, *v* correspond to each other.

Finally, for the sake of symmetry, we introduce the following notations:

$$(\bar{2}) \qquad \bar{p}_1 = \begin{vmatrix} \bar{\eta}_u & \bar{\zeta}_u \\ \bar{\eta}_v & \bar{\zeta}_v \end{vmatrix}, \qquad \bar{p}_2 = \begin{vmatrix} \bar{\zeta}_u & \bar{\xi}_u \\ \bar{\zeta}_v & \bar{\xi}_v \end{vmatrix}, \qquad \bar{p}_3 = \begin{vmatrix} \bar{\xi}_u & \bar{\eta}_u \\ \bar{\xi}_v & \bar{\eta}_v \end{vmatrix}.$$

When one recalls equations (II), one will get:

$$\bar{p}_1 = \begin{vmatrix} F_{p_1}\zeta_u - F_{p_3}\xi_u & F_{p_2}\xi_u - F_{p_1}\eta_u \\ F_{p_1}\zeta_v - F_{p_3}\xi_v & F_{p_2}\xi_v - F_{p_1}\eta_v \end{vmatrix} = F_{p_1}^2 \begin{vmatrix} \eta_u & \zeta_u \\ \eta_v & \zeta_v \end{vmatrix} + F_{p_1}F_{p_2} \begin{vmatrix} \zeta_u & \xi_u \\ \zeta_v & \xi_v \end{vmatrix} + F_{p_1}F_{p_3} \begin{vmatrix} \xi_u & \eta_u \\ \xi_v & \eta_v \end{vmatrix}$$

([5]) It is not difficult to see that in the case of minimal surfaces:

$$F = \sqrt{p_1^2 + p_2^2 + p_3^2},$$

so this definition coincides with the usual convention on the adjoint minimal surface. Namely, if one introduces isothermal coordinates on the minimal surface:

$$\xi_u^2 + \eta_u^2 + \zeta_u^2 = \xi_v^2 + \eta_v^2 + \zeta_v^2, \qquad \xi_u\,\xi_v + \eta_u\,\eta_v + \zeta_u\,\zeta_v = 0$$

then equations (II) will go to the following ones:

$$\begin{aligned} \bar{\xi}_u &= -\xi_v, & \bar{\eta}_u &= -\eta_v, & \bar{\zeta}_u &= -\zeta_v, \\ \bar{\xi}_v &= \xi_u, & \bar{\eta}_v &= \eta_u, & \bar{\zeta}_v &= \zeta_u, \end{aligned}$$

i.e., ξ and $\bar{\xi}$, η and $\bar{\eta}$, ζ and $\bar{\zeta}$ are conjugate potential functions. That is just the usual definition of the adjoint minimal surfaces.

$$= F_{p_1}(p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3}) = F F_{p_1} ,$$

and in the same way, one has the equations:

$$\bar{p}_2 = F F_{p_2} , \quad \bar{p}_3 = F F_{p_3} , \tag{4}$$

which lead to the proportion:

$$\bar{p}_1 : \bar{p}_2 : \bar{p}_3 = F_{p_1} : F_{p_2} : F_{p_3} . \tag{5}$$

3. Definition of the adjoint variational problem. – The fact that the *adjoint surfaces* to the extremal surfaces of the given variational problem that were just defined *are the extremal surfaces of a new variational problem that is completely determined by the original variational problem and is independent of the choice of the extremal surface* will be a great importance in what follows. We will characterize that variational problem – viz., the *adjoint variational problem* to the origin one – quite simply and prove the involutory character of that relationship by showing that the adjoint surfaces to the extremal surfaces that relate to the latter variational problem are the extremal surfaces to the original problem.

In order to see that clearly, we would first like to take x, y to be independent variables and assume that the equation of the surface that is adjoint to the extremal surface $z = z(y, y)$:

$$\bar{x} = \bar{\xi}(x, y) , \quad \bar{y} = \bar{\eta}(x, y) , \quad \bar{z} = \bar{\eta}(x, y)$$

can be converted into the form:

$$\bar{z} = \bar{z}(\bar{x}, \bar{y}) .$$

In other words, we consider a piece of the extremal surface for which the determinant:

$$\bar{p}_3 = \begin{vmatrix} \bar{\xi}_x & \bar{\eta}_x \\ \bar{\xi}_y & \bar{\eta}_y \end{vmatrix} = F F_{p_3} = f(f - p f_p - q f_q)$$

is non-zero. When we then introduce:

$$\bar{p} = \frac{\partial \bar{z}}{\partial \bar{x}} , \qquad \bar{q} = \frac{\partial \bar{z}}{\partial \bar{y}} ,$$

to abbreviate, and recall equations $(\bar{2})$, (3′), and (4), we will then have:

$$\bar{p} = -\frac{\bar{p}_1}{\bar{p}_3} = -\frac{F_{p_1}}{F_{p_3}} = \frac{f_p}{f - p f_p - q f_q} ,$$

(6)
$$\bar{q} = -\frac{\bar{p}_2}{\bar{p}_3} = -\frac{F_{p_2}}{F_{p_3}} = \frac{f_q}{f - p\,f_p - q\,f_q}\,.$$

Those equations exhibit the fact [which one can also infer from equations (5)] that the direction of the surface normal $(\bar{p}, \bar{q})$ at a point of the adjoint surface depends upon only the direction of the normal (*p*, *q*) at the corresponding point of the extremal surface. It is not difficult to verify the formula:

$$\frac{\partial(\bar{p}, \bar{q})}{\partial(p, q)} = \frac{f}{(f - p\,f_p - q\,f_q)^3}\,(f_{pp}\,f_{qq} - f_{pq}^2)\,.$$

If that expression is non-zero (which we would like to assume from now on) then we can solve equations (6), which we can combine into the one equation:

(6′)
$$\bar{p}\,dp + \bar{q}\,dq = \frac{df}{f - p\,f_p - q\,f_q}$$

for *p* and *q*. We can get a transparent solution formula that lets the symmetry in the two variable-pairs *p*, *q* and $\bar{p}$, $\bar{q}$ emerge clearly in the following way: We imagine that *p*, *q* are expressed in terms of $\bar{p}$, $\bar{q}$ and define the function $\bar{f}(\bar{p}, \bar{q})$ by the following equation:

(7)
$$\bar{f}(\bar{p}, \bar{q}) = \frac{1}{f - p\,f_p - q\,f_q}\,.$$

In complete analogy with (6), we will then have:

$(\bar{6})$
$$p = \frac{\bar{f}_{\bar{p}}}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}}\,, \quad q = \frac{\bar{f}_{\bar{q}}}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}}\,.$$

In fact, when one recalls (6), one can bring the defining equation (7) into the form:

(8)
$$\bar{f}\,df + f\,d\bar{f} = (\bar{p}\,dp + \bar{q}\,dq) + (p\,d\bar{p} + q\,d\bar{q})\,,$$

and due to (6′) and (7), one will have:

$$p\,d\bar{p} + q\,d\bar{q} = f\,d\bar{f} = f(\bar{f}_{\bar{p}}\,d\bar{p} + \bar{f}_{\bar{q}}\,d\bar{q})\,,$$

i.e.:

(9)
$$p = f\,\bar{f}_{\bar{p}}\,, \qquad q = f\,\bar{f}_{\bar{q}}\,.$$

If one introduces those expressions for p, q into the right-hand side of equation (8) then that will give:

$$(\bar{7}) \qquad f = \frac{1}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}} .$$

One will get the stated equation $(\bar{6})$ by substituting that expression for f in equations (9).

We can summarize our formulas as follows:

If one subjects the quantities p, q to the conditions:

$$(10) \qquad f \neq 0 , \qquad f - p\,f_p - q\,f_q \neq 0, \qquad f_{pp}\,f_{qq} - f_{pq}^2 \neq 0$$

and introduces the quantities $\bar{p}$, $\bar{q}$ *by means of the equations:*

$$(6) \qquad \bar{p} = \frac{f_p}{f - p\,f_p - q\,f_q} , \qquad \bar{q} = \frac{f_q}{f - p\,f_p - q\,f_q} ,$$

but introduces the function $\bar{f}(\bar{p},\bar{q})$ *by the equation:*

$$(7) \qquad \bar{f}(\bar{p},\bar{q}) = \frac{1}{f - p\,f_p - q\,f_q} ,$$

or

$$(8) \qquad f(p, q)\,\bar{f}(\bar{p},\bar{q}) = 1 + p\,\bar{p} + q\,\bar{q}$$

(which is equivalent to that) then one will have:

$$(\bar{6}) \qquad p = \frac{\bar{f}_{\bar{p}}}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}} , \qquad q = \frac{\bar{f}_{\bar{q}}}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}} ,$$

and furthermore:

$$(\bar{7}) \qquad f = \frac{1}{\bar{f} - \bar{p}\,\bar{f}_{\bar{p}} - \bar{q}\,\bar{f}_{\bar{q}}}$$

and

$$(9) \qquad \bar{f}_{\bar{p}} = \frac{p}{f} , \qquad \bar{f}_{\bar{q}} = \frac{q}{f} ,$$

$$(\bar{9}) \qquad f_p = \frac{\bar{p}}{\bar{f}} , \qquad f_q = \frac{\bar{q}}{\bar{f}} .$$

Therefore, that connection exists between the derivatives $\partial z / \partial x = p$, $\partial z / \partial y = q$ of an extremal surface of the variational problem:

$$\delta \iint f(z_x, z_y)\, dx\, dy = 0 \tag{1}$$

and the derivatives $\bar{z}_{\bar{x}} = \bar{p}$, $\bar{z}_{\bar{y}} = \bar{q}$ of its adjoint surface $\bar{z} = \bar{z}(\bar{x}, \bar{y})$. We will show that this function is an extremal function of the variational problem:

$$\delta \iint \bar{f}(\bar{z}_{\bar{x}}, \bar{z}_{\bar{y}})\, d\bar{x}\, d\bar{y} = 0\,, \tag{$\bar{1}$}$$

which we would like to refer to as the *adjoint variational problem* to (1).

4. Continuation. – One can geometrically characterize the connection between the integrands $f(p, q)$ and $\bar{f}(\bar{p}, \bar{q})$ of the original and adjoint variational problem in an elegant way. To that end, we consider the second-order surface:

$$Z^2 = X^2 + Y^2 + 1$$

in a space whose rectangular coordinates are denoted by X, Y, Z, and imagine that the integrand of the original variational problem is realized by the surface:

$$Z = f(X, Y)\,. \tag{11}$$

One infers immediately from the elementary formulas of analytic geometry that the pole of the tangent plane that is drawn at the point X, Y, Z of the latter surface relative to the second-order surface above possesses the following coordinates:

$$\bar{X} = \frac{f_X}{f - X\, f_X - Y\, f_Y}\,, \qquad \bar{Y} = \frac{f_Y}{f - X\, f_X - Y\, f_Y}\,, \qquad \bar{Z} = \frac{1}{f - X\, f_X - Y\, f_Y}\,.$$

Upon eliminating X, Y from those equations, we get:

$$\bar{Z} = \bar{f}(\bar{X}, \bar{Y})\,, \tag{12}$$

in which $\bar{f}(\bar{X}, \bar{Y})$ is just the integrand of the adjoint variational problem. That is, *the surfaces* (11) *and* (12) *that are defined by the integrands of the two variational problems* (viz., the original and adjoint ones) *are polar-reciprocal surfaces relative to the second-order surface considered.*

One sees *the involutory character of adjointness* from this immediately [which already follows from the complete symmetry of formulas (6), (7), $(\bar{6})$, $(\bar{7})$, moreover]; i.e., when one defines the adjoint variational problem to the variational problem $(\overline{1})$, one will get back to the original one (1)([6]).

Our formulas will become more elegant when we once more introduce general parameters *u*, *v* as independent variables in place of the *x*, *y* that we have used up to now, which comes down to the same thing (with the notations in **1**.) as taking the function $F(p_1, p_2, p_3) = p_3\, f\left(-\frac{p_1}{p_3}, -\frac{p_2}{p_3}\right)$ to be the integrand of the variational problem, instead of $f(p, q)$. In a corresponding way, we set the integrand of the adjoint problem equal to:

$$(\overline{3}) \qquad \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3) = \bar{p}_3\, \bar{f}\left(-\frac{\bar{p}_1}{\bar{p}_3}, -\frac{\bar{p}_2}{\bar{p}_3}\right).$$

The relations (6.), (7), and $(\overline{6})$, $(\overline{7})$ then take on the following simple form [when one recalls (3)]:

$$(13) \qquad \bar{p}_1 = \bar{F}\,F_{p_1}, \qquad \bar{p}_2 = \bar{F}\,F_{p_2}, \qquad \bar{p}_3 = \bar{F}\,F_{p_3},$$

$$(\overline{13}) \qquad p_1 = F\,\bar{F}_{\bar{p}_1}, \qquad p_2 = F\,\bar{F}_{\bar{p}_2}, \qquad p_3 = F\,\bar{F}_{\bar{p}_3},$$

which one can naturally also establish independently of the foregoing as follows: Since F_{p_1}, F_{p_2}, F_{p_3} are, in fact, homogeneous functions or degree 0, one can express the quantities p_1 / p_3 and p_2 / p_3 in terms of $\bar{p}_1 / \bar{p}_3$ and $\bar{p}_2 / \bar{p}_3$ by means of the equations in the first row (the proportion $\bar{p}_1 : \bar{p}_2 : \bar{p}_3 = F_{p_1} : F_{p_2} : F_{p_3}$, resp.). Thus, say, the third equation, will imply $\bar{F}$ as a homogeneous function of degree one in $\bar{p}_1 : \bar{p}_2 : \bar{p}_3$. ([7]) If one gives equations (13), which one can also write in the form:

$$\bar{p}_1\, dp_1 + \bar{p}_2\, dp_2 + \bar{p}_3\, dp_3 = \bar{F}\, dF,$$

the components p_1, p_2, p_3, resp., then that will give:

$$p_1\, \bar{p}_1 + p_2\, \bar{p}_2 + p_3\, \bar{p}_3 = F\,\bar{F},$$

and from that:

$$\bar{F}\, dF + F\, d\bar{F} = (\bar{p}_1\, dp_1 + \bar{p}_2\, dp_2 + \bar{p}_3\, dp_3) + (p_1\, d\bar{p}_1 + p_2\, d\bar{p}_2 + p_3\, d\bar{p}_3).$$

([6]) The inequalities (10) are also constructed symmetrically, because one easily verifies the relation:

$$(f_{pp}\, f_{qq} - f_{pq}^2)(\bar{f}_{\bar{p}\bar{p}}\, \bar{f}_{\bar{q}\bar{q}} - \bar{f}_{\bar{p}\bar{q}}^2) = \left(\frac{1}{f\,\bar{f}}\right)^4,$$

such that [when one recalls (7)] those inequalities are equivalent to the statement that $f(p, q)$ and $\bar{f}(\bar{p}, \bar{q})$ are finite and non-zero.

([7]) One should confer another simple calculation of $\bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ in **6**.

As a result, one will have:

$$F\,d\bar{F} = p_1\,d\bar{p}_1 + p_2\,d\bar{p}_2 + p_3\,d\bar{p}_3\ ,$$

and that is the basic content of equations $(\overline{13})$.

If we then denote the sub-determinants of the matrix:

$$\begin{pmatrix} \xi_u & \eta_u & \zeta_u \\ \xi_v & \eta_v & \zeta_v \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} \bar{\xi}_u & \bar{\eta}_u & \bar{\zeta}_u \\ \bar{\xi}_v & \bar{\eta}_v & \bar{\zeta}_v \end{pmatrix}, \qquad \text{resp.,}$$

that is defined by the original extremal surface (its adjoint surface, resp.) by p_1, p_2, p_3 ($\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$, resp.), as in **2.**, then from (5), we will have the proportion $\bar{p}_1 : \bar{p}_2 : \bar{p}_3 = F_{p_1} : F_{p_2} : F_{p_3}$. If we then introduce the function $\bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ in the way that was just given then, from $(\overline{13})$, we will have the analogous proportion:

$$(\bar{5}) \qquad p_1 : p_2 : p_3 = \bar{F}_{\bar{p}_1} : \bar{F}_{\bar{p}_2} : \bar{F}_{\bar{p}_3}\ .$$

Naturally, one assumes in this that one restricts oneself to a region on the original extremal surface such that the determination of $\bar{F}$ from equations (13) is possible, i.e., that one can express p_1 / p_3 and p_2 / p_3 in terms of $\bar{p}_1 / \bar{p}_3$ and $\bar{p}_2 / \bar{p}_3$ by using the proportion that arises from it. The conditions for that to be possible were given in (10) [the remark ([6]), resp.].

5. The adjoint surface as the extremal surface of the adjoint variational problem. – In order to exhibit the connection between adjoint surface and adjoint variational problems, we return to the defining equations for those surfaces, which we establish in the forms:

$$\text{(III)} \qquad d\bar{\xi} = F_{p_3}\,d\eta - F_{p_2}\,d\zeta\ , \qquad d\bar{\eta} = F_{p_1}\,d\zeta - F_{p_3}\,d\xi\ , \qquad d\bar{\zeta} = F_{p_2}\,d\xi - F_{p_1}\,d\eta\ ,$$

and calculate the analogously-constructed expressions:

$$\bar{F}_{\bar{p}_3}\,d\bar{\eta} - \bar{F}_{\bar{p}_2}\,d\bar{\zeta}\ , \qquad \bar{F}_{\bar{p}_1}\,d\bar{\zeta} - \bar{F}_{\bar{p}_3}\,d\bar{\xi}\ , \qquad \bar{F}_{\bar{p}_2}\,d\bar{\xi} - \bar{F}_{\bar{p}_1}\,d\bar{\eta}\ .$$

When one recalls (III) and $(\overline{13})$, a simple calculation will give the equations:

$$\bar{F}_{\bar{p}_3}\,d\bar{\eta} - \bar{F}_{\bar{p}_2}\,d\bar{\zeta} = \frac{1}{F}\,[\,p_3(F_{p_1}\,d\zeta - F_{p_3}\,d\xi) - p_2(F_{p_2}\,d\xi - F_{p_1}\,d\eta)]$$

$$= \frac{1}{F}\,[-(p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3})\,d\xi + F_{p_1}(p_1\,d\xi + p_2\,d\eta + p_3\,d\zeta)] = -\,d\xi\,,$$

and in the same way:

$$(\overline{\text{III}}) \qquad \bar F_{\bar p_1}\,d\bar\zeta - \bar F_{\bar p_3}\,d\bar\xi = -\,d\eta\,, \qquad \bar F_{\bar p_2}\,d\bar\xi - \bar F_{\bar p_1}\,d\bar\eta = -\,d\zeta\,.$$

When written out in detail, those equations will read as follows:

$$(\text{II}) \qquad \begin{aligned} -\,\xi_u &= \begin{vmatrix} \bar\eta_u & \bar\zeta_u \\ \bar F_{\bar p_2} & \bar F_{\bar p_3} \end{vmatrix}, \quad -\,\eta_u = \begin{vmatrix} \bar\zeta_u & \bar\xi_u \\ \bar F_{\bar p_2} & \bar F_{\bar p_1} \end{vmatrix}, \quad -\,\zeta_u = \begin{vmatrix} \bar\xi_u & \bar\eta_u \\ \bar F_{\bar p_1} & \bar F_{\bar p_2} \end{vmatrix}, \\ -\,\xi_v &= \begin{vmatrix} \bar\eta_v & \bar\zeta_v \\ \bar F_{\bar p_2} & \bar F_{\bar p_3} \end{vmatrix}, \quad -\,\eta_v = \begin{vmatrix} \bar\zeta_v & \bar\xi_v \\ \bar F_{\bar p_2} & \bar F_{\bar p_1} \end{vmatrix}, \quad -\,\zeta_v = \begin{vmatrix} \bar\xi_v & \bar\eta_v \\ \bar F_{\bar p_1} & \bar F_{\bar p_2} \end{vmatrix}. \end{aligned}$$

However, upon eliminating ξ, η, ζ from those equations, that will imply the equations:

$$\begin{aligned} \frac{\partial}{\partial u}\begin{vmatrix} \bar\eta_v & \bar\zeta_v \\ \bar F_{\bar p_2} & \bar F_{\bar p_3} \end{vmatrix} - \frac{\partial}{\partial v}\begin{vmatrix} \bar\eta_u & \bar\zeta_u \\ \bar F_{\bar p_2} & \bar F_{\bar p_3} \end{vmatrix} &= \frac{\partial}{\partial u}\begin{vmatrix} \bar\zeta_v & \bar\xi_v \\ \bar F_{\bar p_2} & \bar F_{\bar p_1} \end{vmatrix} - \frac{\partial}{\partial v}\begin{vmatrix} \bar\zeta_u & \bar\xi_u \\ \bar F_{\bar p_2} & \bar F_{\bar p_1} \end{vmatrix} \\ &= \frac{\partial}{\partial u}\begin{vmatrix} \bar\xi_v & \bar\eta_v \\ \bar F_{\bar p_1} & \bar F_{\bar p_2} \end{vmatrix} - \frac{\partial}{\partial v}\begin{vmatrix} \bar\xi_u & \bar\eta_u \\ \bar F_{\bar p_1} & \bar F_{\bar p_2} \end{vmatrix} = 0\,, \end{aligned}$$

which coincide with the Euler-Lagrange differential equations for the adjoint variational problems:

$$(\bar{\text{I}}) \qquad \delta \iint \bar F(\bar p_1, \bar p_2, \bar p_3)\,du\,dv = 0\,.$$

Therefore, the adjoint surfaces are, in fact, solutions to the adjoint variational problem.

However, equations $(\overline{\text{II}})$ show that the adjoint surface to the extremal surface $x = \bar\xi(u,v)$, $y = \bar\eta(u,v)$, $z = \bar\zeta(u,v)$ relative to the variational problem that was just described is the surface whose equations are:

$$x = -\,\xi(u, v)\,, \qquad y = -\,\eta(u, v)\,, \qquad z = -\,\zeta(u, v)\,,$$

i.e., the mirror image of the original extremal surface relative to the coordinate origin.

We then arrive at the following result:

For any variational problem:

(I) $$\delta\iint F(p_1, p_2, p_3)\,du\,dv = 0,$$

the adjoint surface to any extremal surface of that problem is an extremal surface to the adjoint variational problem:

($\bar{\text{I}}$) $$\delta\iint \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)\,du\,dv = 0.$$

If one defines the adjoint surface to that surface relative to the variational problem then one will get the mirror image of the original extremal surface relative to the coordinate origin. Obviously, that surface is likewise an extremal surface of the original variational problem.

§ 2.

On the map from an extremal surface to its adjoint surface.

6. Invariance of the variational integral. – As was remarked before in **2**., the equations:

(III) $$d\bar{\xi} = \bar{F}_{\bar{p}_3}\,d\eta - \bar{F}_{\bar{p}_2}\,d\zeta, \qquad d\bar{\eta} = \bar{F}_{\bar{p}_1}\,d\zeta - \bar{F}_{\bar{p}_3}\,d\xi, \qquad d\bar{\zeta} = \bar{F}_{\bar{p}_2}\,d\xi - \bar{F}_{\bar{p}_1}\,d\eta,$$

which serve as the definition of adjoint surfaces, simultaneously imply a map between those surfaces that makes points of both surfaces that belong to the same parameter values correspond to each other. We shall address that map in what follows.

In **2**., we inferred the following relations from the definition of the adjoint surface above:

(4) $$\bar{p}_1 = F(p_1, p_2, p_3)\,F_{p_1}, \qquad \bar{p}_2 = F(p_1, p_2, p_3)\,F_{p_2}, \qquad \bar{p}_3 = F(p_1, p_2, p_3)\,F_{p_3}.$$

On the other hand, based upon the definition of the adjoint variational problem, we have the equations:

(13) $$\bar{p}_1 = \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)\,F_{p_1}, \qquad \bar{p}_2 = \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)\,F_{p_2}, \qquad \bar{p}_3 = \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)\,F_{p_3}.$$

As a result, we have:

(14) $$F(p_1, p_2, p_3) = \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3),$$

or:

$$\iint_G F(p_1, p_2, p_3)\,du\,dv = \iint_G \bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)\,du\,dv$$

(for any region G), i.e., *the integral of the original variational problem, when extended over a piece of an extremal surface for it, is equal to the integral of the adjoint variational problem, when extended over the correspond piece of the adjoint surface.*

Equation (14) gives a convenient means for representing the adjoint variational problem. Namely, one gets the integrand $\bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ when one expresses p_1, p_2, p_3 in terms of $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ and substitutes those expressions for the arguments in the function $F(p_1, p_2, p_3)$.

7. Corresponding directions. – The defining equation (III) for the adjoint surface immediately implies the relation:

$$d\xi\,d\bar{\xi} + d\eta\,d\bar{\eta} + d\zeta\,d\bar{\zeta} = 0 \,,$$

i.e., *any line element on an extremal surface corresponds to a line element that is perpendicular to it.*

8. Invariance of a non-Euclidian arc-length. – In order to arrive at an essentially-deeper property of the map in question, we recall the known fact that any minimal surface can be bent into its adjoint surface. Naturally, we would not expect that in general variational problems. However, we can probably give a non-Euclidian metric on the extremal surface (the adjoint surface, resp.) that depends upon only the corresponding variational problem and is such that our map becomes a bending in the sense of that metric.

In order to arrive at that metric, we proceed as follows: We let a, b, c initially denote arbitrary quantities to be determined later, and consider the matrix:

$$\begin{pmatrix} a\,d\xi & b\,d\eta & c\,d\zeta \\ a\,F_{p_1} & b\,F_{p_2} & c\,F_{p_3} \end{pmatrix} .$$

Upon applying the known Laplace identity, while recalling (III), we will get:

$$(a^2\,d\xi^2 + b^2\,d\eta^2 + c^2\,d\zeta^2)(a^2\,F_{p_1}^2 + b^2\,F_{p_2}^2 + c^2\,F_{p_3}^2) - (a^2\,F_{p_1}\,d\xi + b^2\,F_{p_2}\,d\iota + c^2\,F_{p_3}\,d\zeta)^2$$

$$= \begin{vmatrix} b\,d\eta & c\,d\zeta \\ b\,F_{p_2} & c\,F_{p_3} \end{vmatrix}^2 + \begin{vmatrix} c\,d\zeta & a\,d\xi \\ c\,F_{p_3} & a\,F_{p_1} \end{vmatrix}^2 + \begin{vmatrix} a\,d\xi & b\,d\eta \\ a\,F_{p_1} & b\,F_{p_2} \end{vmatrix}^2$$

$$= b^2\,c^2\,d\bar{\xi}^2 + c^2\,a^2\,d\bar{\eta}^2 + a^2\,b^2\,d\bar{\zeta}^2 \,.$$

We make the quadratic term on the left-hand side vanish by choosing $a^2 F_{p_1}$, $b^2 F_{p_2}$, $c^2 F_{p_3}$ to be equal to p_1, p_2, p_3, resp., i.e., by choosing:

$$a^2 = \frac{p_1}{F_{p_1}}, \qquad b^2 = \frac{p_2}{F_{p_2}}, \qquad c^2 = \frac{p_3}{F_{p_3}}.$$

Since we will have:

$$a^2 F_{p_1}^2 + b^2 F_{p_2}^2 + c^2 F_{p_3}^2 = p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = F$$

for that choice of a, b, c, the Laplace identity above will yield:

$$F\left(\frac{p_1}{F_{p_1}}\,d\xi^2 + \frac{p_2}{F_{p_2}}\,d\eta^2 + \frac{p_3}{F_{p_3}}\,d\zeta^2\right) = \frac{p_1\,p_2\,p_3}{F_{p_1}\,F_{p_2}\,F_{p_3}}\left(\frac{F_{p_1}}{p_1}\,d\bar\xi^2 + \frac{F_{p_2}}{p_2}\,d\bar\eta^2 + \frac{F_{p_3}}{p_3}\,d\bar\zeta^2\right).$$

However, the relations (13), $(\overline{13})$, in conjunction with equation (14), imply the connection:

$$(13') \qquad \frac{p_1}{F_{p_1}} = F^2\,\frac{\bar F_{\bar p_1}}{\bar p_1}, \qquad \frac{p_2}{F_{p_2}} = F^2\,\frac{\bar F_{\bar p_2}}{\bar p_2}, \qquad \frac{p_3}{F_{p_3}} = F^2\,\frac{\bar F_{\bar p_3}}{\bar p_3}.$$

We can then put the quadratic form on the right-hand side into the following form:

$$F\sqrt{\frac{p_1\,p_2\,p_3}{F_{p_1}\,F_{p_2}\,F_{p_3}}}\sqrt{\frac{\bar F_{\bar p_1}\,\bar F_{\bar p_2}\,\bar F_{\bar p_3}}{\bar p_1\,\bar p_2\,\bar p_3}}\left(\frac{\bar p_1}{\bar F_{\bar p_1}}\,d\bar\xi^2 + \frac{\bar p_2}{\bar F_{\bar p_2}}\,d\bar\eta^2 + \frac{\bar p_3}{\bar F_{\bar p_3}}\,d\bar\zeta^2\right).$$

However, with that, we get the relation:

$$\sqrt{\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}}\left(\frac{p_1}{F_{p_1}}\,d\xi^2 + \frac{p_2}{\bar F_{p_2}}\,d\eta^2 + \frac{p_3}{F_{p_3}}\,d\zeta^2\right)$$
$$= \sqrt{\frac{\bar F\,\bar F_{\bar p_1}\,\bar F_{\bar p_2}\,\bar F_{\bar p_3}}{\bar p_1\,\bar p_2\,\bar p_3}}\left(\frac{\bar p_1}{\bar F_{\tilde p_1}}\,d\bar\xi^2 + \frac{\bar p_2}{\bar F_{\tilde p_2}}\,d\bar\eta^2 + \frac{\bar p_3}{\bar F_{\bar p_3}}\,d\bar\zeta^2\right),$$

and that implies the desired conclusion.

Namely, we imagine a non-Euclidian metric has been introduced onto the original extremal surface such that the differential of arc-length will be:

$$ds^2 = \sqrt{\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}}\left(\frac{p_1}{F_{p_1}}\,d\xi^2 + \frac{p_2}{\bar F_{p_2}}\,d\eta^2 + \frac{p_3}{F_{p_3}}\,d\zeta^2\right)$$

and in the same way, define the differential of arc-length on the adjoint surface to be:

$$d\bar{s}^2 = \sqrt{\frac{\bar{F}\,\bar{F}_{\bar{p}_1}\,\bar{F}_{\bar{p}_2}\,\bar{F}_{\bar{p}_3}}{\bar{p}_1\,\bar{p}_2\,\bar{p}_3}}\left(\frac{\bar{p}_1}{\bar{F}_{\bar{p}_1}}\,d\bar{\xi}^2 + \frac{\bar{p}_2}{\bar{F}_{\bar{p}_2}}\,d\bar{\eta}^2 + \frac{\bar{p}_3}{\bar{F}_{\bar{p}_3}}\,d\bar{\zeta}^2\right).$$

Those two definitions are completely *symmetric*, since the metric on the extremal surface is constructed from the equation of that surface and the original variational problem in the same way that the metric on the adjoint surface is constructed from that surface (the adjoint variational problem, resp.). We have then arrived at the following theorem:

Based upon that non-Euclidian metric, the map of the extremal surface to the adjoint surface is a bending.

9. The variational integral as a non-Euclidian surface area. – The two differential forms ds^2 and $d\bar{s}^2$ are simultaneously real (imaginary, resp.). That is because, with the help of the equation:

$$\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}\;\frac{\bar{F}\,\bar{F}_{\bar{p}_1}\,\bar{F}_{\bar{p}_2}\,\bar{F}_{\bar{p}_3}}{\bar{p}_1\,\bar{p}_2\,\bar{p}_3} = \frac{1}{F^4} > 0\,,$$

which can be easily verified on the basis of the relations (13), $(\overline{13})$, and (14), it will follow that the two roots that appear in ds^2 ($d\bar{s}^2$, resp.) are simultaneously real (imaginary, resp.). In the latter case, we would like to take the definition of arc-length above, multiplied by *i*, to be our metric in order to get real arc-lengths in any case. We correspondingly set:

$$\lambda_1 = \sqrt{\pm\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}\,\frac{p_1}{F_{p_1}}}\,,\quad \lambda_2 = \sqrt{\pm\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}\,\frac{p_2}{F_{p_2}}}\,,\quad \lambda_3 = \sqrt{\pm\frac{F\,F_{p_1}\,F_{p_2}\,F_{p_3}}{p_1\,p_2\,p_3}\,\frac{p_3}{F_{p_3}}}\,,$$

$$\bar{\lambda}_1 = \sqrt{\pm\frac{\bar{F}\,\bar{F}_{\bar{p}_1}\,\bar{F}_{\bar{p}_2}\,\bar{F}_{\bar{p}_3}}{\bar{p}_1\,\bar{p}_2\,\bar{p}_3}\,\frac{\bar{p}_1}{\bar{F}_{\bar{p}_1}}}\,,\quad \bar{\lambda}_2 = \sqrt{\pm\frac{\bar{F}\,\bar{F}_{\bar{p}_1}\,\bar{F}_{\bar{p}_2}\,\bar{F}_{\bar{p}_3}}{\bar{p}_1\,\bar{p}_2\,\bar{p}_3}\,\frac{\bar{p}_2}{\bar{F}_{\bar{p}_2}}}\,,\quad \bar{\lambda}_3 = \sqrt{\pm\frac{\bar{F}\,\bar{F}_{\bar{p}_1}\,\bar{F}_{\bar{p}_2}\,\bar{F}_{\bar{p}_3}}{\bar{p}_1\,\bar{p}_2\,\bar{p}_3}\,\frac{\bar{p}_3}{\bar{F}_{\bar{p}_3}}}\,, \tag{15}$$

in which the positive (negative, resp.) sign is taken *everywhere*. We take the signs that give real values to λ_1, λ_2, λ_3, $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$. The quantities λ_1, λ_2, λ_3 are then determined uniquely, up to sign. (We will add more to that later.) Furthermore, since (13′) implies that:

$$\lambda_1\,\bar{\lambda}_1 = \lambda_2\,\bar{\lambda}_2 = \lambda_3\,\bar{\lambda}_3 = F^2\sqrt{\frac{1}{F^4}}\,,$$

we would like to determine the signs of $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ in such a way that:

$$\lambda_1 \bar{\lambda}_1 = \lambda_2 \bar{\lambda}_2 = \lambda_3 \bar{\lambda}_3 = 1 . \tag{16}$$

Upon borrowing from the theory of surfaces if one now writes:

$$\begin{gathered} E^* = \lambda_1 \xi_u^2 + \lambda_2 \eta_u^2 + \lambda_3 \zeta_u^2 , \quad G^* = \lambda_1 \xi_v^2 + \lambda_2 \eta_v^2 + \lambda_3 \zeta_v^2 , \\ F^* = \lambda_1 \xi_u \xi_v + \lambda_2 \eta_u \eta_v + \lambda_3 \zeta_u \zeta_v , \end{gathered} \tag{17}$$

and correspondingly:

$$\begin{gathered} \bar{E}^* = \bar{\lambda}_1 \bar{\xi}_u^2 + \bar{\lambda}_2 \bar{\eta}_u^2 + \bar{\lambda}_3 \bar{\zeta}_u^2 , \quad \bar{G}^* = \bar{\lambda}_1 \bar{\xi}_v^2 + \bar{\lambda}_2 \bar{\eta}_v^2 + \bar{\lambda}_3 \bar{\zeta}_v^2 , \\ \bar{F}^* = \bar{\lambda}_1 \bar{\xi}_u \bar{\xi}_v + \bar{\lambda}_2 \bar{\eta}_u \bar{\eta}_v + \bar{\lambda}_3 \bar{\zeta}_u \bar{\zeta}_v \end{gathered} \tag{$\overline{17}$}$$

for the adjoint surface, then one can consider the integral:

$$\iint \left| \sqrt{E^* G^* - F^{*2}} \right| du\, dv , \qquad \iint \left| \sqrt{\bar{E}^* \bar{G}^* - \bar{F}^{*2}} \right| du\, dv , \qquad \text{resp.,}$$

when based upon the metric that was given above, *to ne the non-Euclidian surface area* of the extremal surface (the adjoint surface, resp.). One gets a remarkable expression for the integrand. In fact, when one applies the Laplace identity to the matrix:

$$\begin{pmatrix} \sqrt{\lambda_1}\, \xi_u & \sqrt{\lambda_2}\, \eta_u & \sqrt{\lambda_3}\, \zeta_u \\ \sqrt{\lambda_1}\, \xi_v & \sqrt{\lambda_2}\, \eta_v & \sqrt{\lambda_3}\, \zeta_v \end{pmatrix} ,$$

that will give:

$$| E^* G^* - F^{*2} | = | \lambda_1 \lambda_2 p_3^2 + \lambda_2 \lambda_3 p_1^2 + \lambda_3 \lambda_1 p_2^2 | = F^2 ,$$

and one will get the following result:

$$\iint \left| \sqrt{E^* G^* - F^{*2}} \right| du\, dv = \iint F\, du\, dv ,$$

i.e., *the given variational integral, when extended over a piece of the extremal surface, is equal to the non-Euclidian surface area of that surface patch, when it is based upon the metric that was given above.*

We now point out that one can write the quantities λ_1, λ_2, λ_3 in the following forms:

$$\lambda_1 = \sqrt{\pm \frac{p_1}{p_2\, p_3} : \frac{\bar{p}_1}{\bar{p}_2\, \bar{p}_3}} , \qquad \lambda_2 = \sqrt{\pm \frac{p_2}{p_3\, p_1} : \frac{\bar{p}_2}{\bar{p}_3\, \bar{p}_1}} , \qquad \lambda_3 = \sqrt{\pm \frac{p_3}{p_1\, p_2} : \frac{\bar{p}_3}{\bar{p}_1\, \bar{p}_2}} ,$$

with corresponding expressions for the quantities $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, which points to the symmetric character of those formulas.

10. Isothermal parameters. – A glimpse at formulas (15) will show that the quantities λ_1, λ_2, λ_3 are homogeneous functions of degree zero in the p_1, p_2, p_3. Since the ratios of the functional determinants p_1, p_2, p_3 will remain unchanged with the introduction of new variables for the u, v, this has the consequence that λ_1, λ_2, λ_3 will mean functions on the given extremal surface that are independent of the parameters u, v.

We would now like to introduce new parameters α, β:

$$\alpha = \alpha(u, v), \quad \beta = \beta(u, v),$$

in such a way that the two relations:

$$\lambda_1 \xi_\alpha^2 + \lambda_2 \eta_\alpha^2 + \lambda_3 \zeta_\alpha^2 = \lambda_1 \xi_\beta^2 + \lambda_2 \eta_\beta^2 + \lambda_3 \zeta_\beta^2,$$
$$\lambda_1 \xi_\alpha \xi_\beta + \lambda_2 \eta_\alpha \eta_\beta + \lambda_3 \zeta_\alpha \zeta_\beta = 0$$

will be true. That is always possible, because a classical theorem from the theory of differential forms says that when $\alpha(u, v)$ and $\beta(u, v)$ mean solutions to the systems of differential equations:

$$\beta_u = \frac{F^*\alpha_u - E^*\alpha_v}{\sqrt{E^*G^* - F^{*2}}}, \quad \beta_v = \frac{G^*\alpha_u - F^*\alpha_v}{\sqrt{E^*G^* - F^{*2}}},$$

the quadratic differential form:

$$\lambda_1 d\xi^2 + \lambda_2 d\eta^2 + \lambda_3 d\zeta^2 = E^* du^2 + 2F^* du\, dv + G^* dv^2$$

will go to:

$$\theta(\alpha, \beta)(d\alpha^2 + d\beta^2),$$

in which $\theta(\alpha, \beta)$ means a function of α, β. If we introduce such α, β as new parameters then will have:

$$E^* = G^*, \qquad F^* = 0.$$

We refer to such parameters as *isothermal parameters with our non-Euclidian metric* on the given extremal surface and refer to the curves α = const. (β = const., resp.) as *isothermal curves*.

The differential equations of our variational problem assume an especially simple form in isothermal parameters (in that sense) that greatly simplifies the study of those surfaces. In that way, we are dealing with an analogue of the theorem by Weierstrass that the equations of a minimal surface are potential functions in isothermal parameters (in the ordinary sense).

If we set:

$$X = \sqrt{\lambda_1}\,(\xi_\alpha + i\,\xi_\beta)\,, \quad Y = \sqrt{\lambda_2}\,(\eta_\alpha + i\,\eta_\beta)\,, \quad Z = \sqrt{\lambda_3}\,(\zeta_\alpha + i\,\zeta_\beta)\,,$$

to abbreviate, then we can combine the equations $E^* = G^*$, $F^* = 0$, which are characteristic of the choice of parameters α, β, into the one equation:

$$X^2 + Y^2 + Z^2 = 0\,. \tag{18}$$

Furthermore, since we also have, at the same time, $\bar{E}^* = \bar{G}^*$, $\bar{F}^* = 0$, due to the invariance of our arc-length, when we introduce the corresponding abbreviations:

$$\bar{X} = \sqrt{\bar{\lambda}_1}\,(\bar{\xi}_\alpha + i\,\bar{\xi}_\beta)\,, \qquad \bar{Y} = \sqrt{\bar{\lambda}_2}\,(\bar{\eta}_\alpha + i\,\bar{\eta}_\beta)\,, \qquad \bar{Z} = \sqrt{\bar{\lambda}_3}\,(\bar{\zeta}_\alpha + i\,\bar{\zeta}_\beta)\,,$$

we will have, in the same way:

$$\bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 = 0\,. \tag{18}$$

Finally, the equation in no. **7**:

$$d\xi\,d\bar{\xi} + d\eta\,d\bar{\eta} + d\zeta\,d\bar{\zeta} = 0\,,$$

in conjunction with the relations $\lambda_1\bar{\lambda}_1 = \lambda_2\bar{\lambda}_2 = \lambda_3\bar{\lambda}_3 = 1$, will imply the equation:

$$X\,\bar{X} + Y\,\bar{Y} + Z\,\bar{Z} = 0\,. \tag{20}$$

However, it follows from equations (18), (19), (20) with no difficulty that the quantities *X*, *Y*, *Z* are proportional to the quantities $\bar{X}$, $\bar{Y}$, $\bar{Z}$, i.e., when *t* means the proportionality factor, one will have:

$$\bar{X} = t\,X\,, \qquad \bar{Y} = t\,Y\,, \qquad \bar{Z} = t\,Z\,.$$

In order to determine the factor *t*, we add equations (II) to that, which we will write in the following form:

$$\frac{\bar{X}}{\sqrt{\bar{\lambda}_1}} = t\sqrt{\lambda_1}\,X = F_{p_3}\frac{Y}{\sqrt{\lambda_2}} - F_{p_3}\frac{Z}{\sqrt{\lambda_2}}\,, \qquad \frac{\bar{Y}}{\sqrt{\bar{\lambda}_2}} = t\sqrt{\lambda_2}\,Y = F_{p_1}\frac{Z}{\sqrt{\lambda_2}} - F_{p_2}\frac{X}{\sqrt{\lambda_1}}\,,$$

$$\frac{\bar{Z}}{\sqrt{\bar{\lambda}_3}} = t\sqrt{\lambda_3}\,Z = F_{p_2}\frac{X}{\sqrt{\lambda_1}} - F_{p_1}\frac{Y}{\sqrt{\lambda_2}}\,.$$

That homogeneous system of equations in *X*, *Y*, *Z* implies the vanishing of the determinant:

$$\begin{vmatrix} -t\sqrt{\lambda_1} & \dfrac{F_{p_3}}{\sqrt{\lambda_2}} & -\dfrac{F_{p_2}}{\sqrt{\lambda_3}} \\ -\dfrac{F_{p_2}}{\sqrt{\lambda_1}} & -t\sqrt{\lambda_2} & \dfrac{F_{p_1}}{\sqrt{\lambda_3}} \\ \dfrac{F_{p_2}}{\sqrt{\lambda_1}} & -\dfrac{F_{p_1}}{\sqrt{\lambda_2}} & -t\sqrt{\lambda_3} \end{vmatrix}$$

$$= - \sqrt{\lambda_1 \lambda_2 \lambda_3}\, t^3 - \left(\sqrt{\frac{\lambda_1}{\lambda_2 \lambda_3}} F_{p_1}^2 + \sqrt{\frac{\lambda_2}{\lambda_3 \lambda_1}} F_{p_2}^2 + \sqrt{\frac{\lambda_3}{\lambda_1 \lambda_2}} F_{p_3}^2 \right) t \ .$$

However, the two coefficients in the equation for t that was just obtained ([8]):

$$\lambda_1 \lambda_2 \lambda_3 t^2 + \left(\lambda_1 F_{p_1}^2 + \lambda_2 F_{p_2}^2 + \lambda_3 F_{p_3}^2 \right) = 0$$

are equal to each other, since one finds with no effort that:

$$\lambda_1 F_{p_1}^2 + \lambda_2 F_{p_2}^2 + \lambda_3 F_{p_3}^2 = \sqrt{\pm \frac{F\, F_{p_1} F_{p_2} F_{p_3}}{p_1\, p_2\, p_3}} (p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3}) = \lambda_1 \lambda_2 \lambda_3 \, .$$

One then has:

$$t^2 + 1 = 0 \, , \qquad \text{i.e.,} \qquad t = \pm i \, ,$$

and

$$\bar{X} = \pm i\, X \, , \qquad \bar{Y} = \pm i\, Y \, , \qquad \bar{Z} = \pm i\, Z \, .$$

The equations:

$$\bar{\xi}_\alpha + i\, \bar{\xi}_\beta = \pm i\, \lambda_1 (\xi_\alpha + i\, \xi_\beta) \, , \qquad \bar{\eta}_\alpha + i\, \bar{\eta}_\beta = \pm i\, \lambda_2 (\eta_\alpha + i\, \eta_\beta) \, ,$$

$$\bar{\zeta}_\alpha + i\, \bar{\zeta}_\beta = \pm i\, \lambda_3 (\zeta_\alpha + i\, \zeta_\beta) \, ,$$

which are equivalent to (16), then show that with a suitable choice of sign for λ_1, λ_2, λ_3 (since one is free to do that), one can arrange that the upper sign is valid in those equations. Since λ_1, λ_2, λ_3 are real, one will then have:

$$\text{(IV)} \qquad \begin{aligned} \bar{\xi}_\beta &= \lambda_1 \xi_\alpha \, , & \bar{\eta}_\beta &= \lambda_2 \eta_\alpha \, , & \bar{\zeta}_\beta &= \lambda_3 \zeta_\alpha \, , \\ \bar{\xi}_\alpha &= -\lambda_1 \xi_\beta \, , & \bar{\eta}_\alpha &= -\lambda_2 \eta_\beta \, , & \bar{\zeta}_\alpha &= -\lambda_3 \zeta_\beta \, . \end{aligned}$$

One will then arrive at the following analogue of the aforementioned theorem by Weierstrass:

([8]) Obviously, t must be non-zero, since the assumption that $t = 0$ leads to $\bar{\xi}$ = const., $\bar{\eta}$ = const., $\bar{\zeta}$ = const.

If α, β are isothermal parameters that are based upon the non-Euclidian metric then the differential equations of our variational problem will assume the simple form (IV), *in which* λ_1, λ_2, λ_3 *are the expressions that are given in* (15).

11. Converse of the foregoing theorem. – *If we understand* λ_1, λ_2, λ_3 *to mean the expressions that were given in* (15), *so that* $F(p_1, p_2, p_3)$ *means a homogeneous function of degree one, but* p_1, p_2, p_3 *mean the sub-determinants of the matrix:*

$$\begin{pmatrix} \xi_\alpha & \eta_\alpha & \zeta_\alpha \\ \xi_\beta & \eta_\beta & \zeta_\beta \end{pmatrix},$$

and if those function ξ, η, ζ *are solutions of the system of differential equations:*

$$\text{(IV)} \qquad \begin{aligned} \bar{\xi}_\beta &= \lambda_1 \xi_\alpha, & \bar{\eta}_\beta &= \lambda_2 \eta_\alpha, & \bar{\zeta}_\beta &= \lambda_3 \zeta_\alpha, \\ \bar{\xi}_\alpha &= -\lambda_1 \xi_\beta, & \bar{\eta}_\alpha &= -\lambda_2 \eta_\beta, & \bar{\zeta}_\alpha &= -\lambda_3 \zeta_\beta, \end{aligned}$$

for which:

$$\text{(21)} \qquad \begin{gathered} \lambda_1 \xi_\alpha^2 + \lambda_2 \eta_\alpha^2 + \lambda_3 \zeta_\alpha^2 = \lambda_1 \xi_\beta^2 + \lambda_2 \eta_\beta^2 + \lambda_3 \zeta_\beta^2, \\ \lambda_1 \xi_\alpha \xi_\beta + \lambda_2 \eta_\alpha \eta_\beta + \lambda_3 \zeta_\alpha \zeta_\beta = 0 \end{gathered}$$

then $x = \xi(\alpha, \beta)$, $y = \eta(\alpha, \beta)$, $z = \zeta(\alpha, \beta)$ *will be an extremal surface of the variational problem* $\delta \iint F(p_1, p_2, p_3)\, d\alpha\, d\beta = 0$.

In fact, if one sets:

$$P_1 = \sqrt{\lambda_2 \lambda_3}\, \frac{p_1}{F}, \qquad P_2 = \sqrt{\lambda_3 \lambda_1}\, \frac{p_2}{F}, \qquad P_3 = \sqrt{\lambda_1 \lambda_2}\, \frac{p_3}{F},$$

for the moment, then one will see immediately that the following equations are true:

$$P_1 \sqrt{\lambda_1}\, \xi_\alpha + P_2 \sqrt{\lambda_2}\, \eta_\alpha + P_3 \sqrt{\lambda_3}\, \zeta_\alpha = 0,$$

$$P_1 \sqrt{\lambda_1}\, \xi_\beta + P_2 \sqrt{\lambda_2}\, \eta_\beta + P_3 \sqrt{\lambda_3}\, \zeta_\beta = 0,$$

$$\pm (P_1^2 + P_2^2 + P_3^2) = 1.$$

The vectorial meaning of those equations consists of saying that the vector of unit magnitude $(\varepsilon P_1, \varepsilon P_2, \varepsilon P_3)$, where $\varepsilon = 1$ or $\varepsilon = i$, is perpendicular to the vectors $(\sqrt{\lambda_1}\, \xi_\alpha, \sqrt{\lambda_2}\, \eta_\alpha, \sqrt{\lambda_3}\, \zeta_\alpha)$

and $(\sqrt{\lambda_1}\,\xi_\beta, \sqrt{\lambda_2}\,\eta_\beta, \sqrt{\lambda_3}\,\zeta_\beta)$, which have equal length and are mutually perpendicular, from (21). However, it follows from this that the vector product of the unit vector above with each of the latter vectors will coincide with the other of the latter vectors, up to sign, i.e.:

$$\begin{aligned}
\sqrt{\lambda_1}\,\xi_\alpha &= \pm\varepsilon(P_3\sqrt{\lambda_2}\,\eta_\beta - P_2\sqrt{\lambda_3}\;\zeta_\beta), & \sqrt{\lambda_1}\,\xi_\beta &= \mp\varepsilon(P_3\sqrt{\lambda_2}\,\eta_\alpha - P_2\sqrt{\lambda_3}\;\zeta_\alpha),\\
\sqrt{\lambda_2}\,\eta_\alpha &= \pm\varepsilon(P_1\sqrt{\lambda_3}\,\zeta_\beta - P_3\sqrt{\lambda_1}\;\xi_\beta), & \sqrt{\lambda_2}\,\eta_\beta &= \mp\varepsilon(P_1\sqrt{\lambda_3}\,\zeta_\alpha - P_3\sqrt{\lambda_1}\;\xi_\alpha),\\
\sqrt{\lambda_3}\,\zeta_\alpha &= \pm\varepsilon(P_2\sqrt{\lambda_1}\,\xi_\beta - P_1\sqrt{\lambda_2}\;\eta_\beta), & \sqrt{\lambda_3}\,\zeta_\beta &= \mp\varepsilon(P_2\sqrt{\lambda_1}\,\xi_\alpha - P_1\sqrt{\lambda_2}\;\eta_\alpha),
\end{aligned}$$

in which either the upper sign is chosen everywhere or the lower one. However, if one observes the relations:

$$P_1\sqrt{\lambda_2\,\lambda_3} = \pm F_{p_1}, \qquad P_2\sqrt{\lambda_3\,\lambda_1} = \pm F_{p_2}, \qquad P_3\sqrt{\lambda_1\,\lambda_2} = \pm F_{p_3},$$

which are easy to verify on the basis of (15), then when the last equations are multiplied by $\sqrt{\lambda_1}$ ($\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$, resp.), in conjunction with basic system of differential equations, that will give:

$$\begin{aligned}
\bar{\xi}_\beta &= \pm\varepsilon(F_{p_3}\,\eta_\beta - F_{p_2}\,\zeta_\beta), & \bar{\xi}_\alpha &= \pm\varepsilon(F_{p_3}\,\eta_\alpha - F_{p_2}\,\zeta_\alpha),\\
\bar{\eta}_\beta &= \pm\varepsilon(F_{p_1}\,\zeta_\beta - F_{p_3}\,\xi_\beta), & \bar{\eta}_\alpha &= \pm\varepsilon(F_{p_1}\,\zeta_\alpha - F_{p_3}\,\xi_\alpha),\\
\bar{\zeta}_\beta &= \pm\varepsilon(F_{p_2}\,\xi_\beta - F_{p_1}\,\eta_\beta), & \bar{\zeta}_\alpha &= \pm\varepsilon(F_{p_2}\,\xi_\alpha - F_{p_1}\,\eta_\alpha).
\end{aligned}$$

However, (from the results of **1**.) those equations say that the functions ξ, η, ζ do in fact represent an extremal surface of the variational problem in question. Moreover, from the reality of the functions that appear, it will follow that $\varepsilon = 1$. Therefore, $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ are the coordinates of the adjoint surface or the mirror image of that surface, relative to the coordinate origin.

12. Analogue of a theorem by Darboux. – It is known that Darboux proved the beautiful theorem that if two surfaces can be bent into each other in such a way that the corresponding line elements are mutually perpendicular then those surfaces will be adjoint minimal surfaces. In our sphere of ideas, that theorem possesses the following analogue:

Let F (p_1, p_2, p_3) and $\bar{F}(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ be the integrands in adjoint variational problems. We introduce non-Euclidian metrics on the given surfaces:

$$x = \xi(u, v), \quad y = \eta(u, v), \quad z = \zeta(u, v),$$

and

$$x = \bar{\xi}(u, v), \quad y = \bar{\eta}(u, v), \quad z = \bar{\zeta}(u, v),$$

resp., by defining the arc-lengths using the quadratic differential forms:

$$ds^2 = \lambda_1 \, d\xi^2 + \lambda_2 \, d\eta^2 + \lambda_3 \, d\zeta^2$$

and

$$d\bar{s}^2 = \bar{\lambda}_1 \, d\bar{\xi}^2 + \bar{\lambda}_2 \, d\bar{\eta}^2 + \bar{\lambda}_3 \, d\bar{\zeta}^2 ,$$

resp., in which λ_1, λ_2, λ_3 ($\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, *resp.*) *are the expressions that were given in* (15) ($\lambda_1 \bar{\lambda}_1 = \lambda_2 \bar{\lambda}_2 = \lambda_3 \bar{\lambda}_3 = 1$). *If the given map between two surfaces is a bending, based upon those metrics, and if the corresponding line elements on both surfaces are always orthogonal to each other, moreover, then the given surfaces will be extremal surfaces for the corresponding variational problems, and one of the surfaces will be either the adjoint surface to the other one or its mirror image relative to the coordinate origin.*

In fact, if we introduce isothermal parameters α, β for the first of the given surfaces (as in **9.**) relative to the established metric, such that:

$$E^* = \lambda_1 \, \xi_\alpha^2 + \lambda_2 \, \eta_\alpha^2 + \lambda_3 \, \zeta_\alpha^2 = \lambda_1 \, \xi_\beta^2 + \lambda_2 \, \eta_\beta^2 + \lambda_3 \, \zeta_\beta^2 = G^* ,$$

$$F^* = \lambda_1 \, \xi_\alpha \, \xi_\beta + \lambda_2 \, \eta_\alpha \, \eta_\beta + \lambda_3 \, \zeta_\alpha \, \zeta_\beta = 0 ,$$

then, as a result of our assumptions, we will also have, at the same time:

$$\bar{E}^* = \bar{\lambda}_1 \, \bar{\xi}_\alpha^2 + \bar{\lambda}_2 \, \bar{\eta}_\alpha^2 + \bar{\lambda}_3 \, \bar{\zeta}_\alpha^2 = \bar{\lambda}_1 \, \bar{\xi}_\beta^2 + \bar{\lambda}_2 \, \bar{\eta}_\beta^2 + \bar{\lambda}_3 \, \bar{\zeta}_\beta^2 = \bar{G}^* ,$$

$$\bar{F}^* = \bar{\lambda}_1 \, \bar{\xi}_\alpha \, \bar{\xi}_\beta + \bar{\lambda}_2 \, \bar{\eta}_\alpha \, \bar{\eta}_\beta + \bar{\lambda}_3 \, \bar{\zeta}_\alpha \, \tilde{\zeta}_\beta = 0 .$$

If we, in turn, set:

$$\sqrt{\lambda_1} \, (\xi_\alpha + i \, \xi_\beta) = X , \quad \sqrt{\lambda_2} \, (\eta_\alpha + i \, \eta_\beta) = Y , \quad \sqrt{\lambda_3} \, (\zeta_\alpha + i \, \zeta_\beta) = Z ,$$

$$\sqrt{\bar{\lambda}_1} \, (\bar{\xi}_\alpha + i \, \bar{\xi}_\beta) = \bar{X} , \quad \sqrt{\bar{\lambda}_2} \, (\bar{\eta}_\alpha + i \, \bar{\eta}_\beta) = \bar{Y} , \quad \sqrt{\bar{\lambda}_3} \, (\bar{\zeta}_\alpha + i \, \bar{\zeta}_\beta) = \bar{Z} ,$$

to abbreviate, then the equations above will read simply:

$$X^2 + Y^2 + Z^2 = 0 , \qquad \bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 = 0 .$$

In addition, as a result of the assumed orthogonality of the corresponding line element:

$$X \, \bar{X} + Y \, \bar{Y} + Z \, \bar{Z} = 0 .$$

It follows from this, in turn (as in **9**.), that:

$$\bar{X} = t\,X\,, \qquad \bar{Y} = t\,Y\,, \qquad \bar{Z} = t\,Z\,, \tag{22}$$

in which the proportionality factor t is yet-to-be-determined. It next follows from the last relations upon squaring them and adding that:

$$\bar{E}^* + \bar{G}^* = |\,t^2\,|\,(E^* + G^*)\;.$$

Therefore:

$$|\,t\,| = 1\,, \qquad t = \cos\,\tau + i\,\sin\,\tau\,,$$

in which τ is real. When one recalls the reality of λ_1, λ_2, λ_3, equations (22) will then imply the relations:

$$\bar{\xi}_\alpha = \lambda_1\,(\xi_\alpha \cos\,\tau - \xi_\beta \sin\,\tau)\,,$$
$$\bar{\eta}_\alpha = \lambda_2\,(\eta_\alpha \cos\,\tau - \eta_\beta \sin\,\tau)\,,$$
$$\bar{\zeta}_\alpha = \lambda_3\,(\zeta_\alpha \cos\,\tau - \zeta_\beta \sin\,\tau)\,.$$

If one contracts those equations with ξ_α, η_α, ζ_α, resp., then due to the orthogonality of the corresponding line elements, the left-hand sides will give zero, so:

$$0 = E^* \cos\,\tau - F^* \sin\,\tau = E^* \cos\,\tau\,,$$

i.e., one has $\cos\,\tau = 0$, so $t = \pm\,i$, and we will arrive from (22) at the following system of equations:

$$\bar{\xi}_\beta = \pm\,\lambda_1\,\xi_\alpha,\quad \bar{\eta}_\beta = \pm\,\lambda_2\,\eta_\alpha,\quad \bar{\zeta}_\beta = \pm\,\lambda_3\,\zeta_\alpha\,,$$
$$\bar{\xi}_\alpha = \mp\,\lambda_1\,\xi_\beta,\quad \bar{\eta}_\alpha = \mp\,\lambda_2\,\eta_\beta,\quad \bar{\zeta}_\alpha = \mp\,\lambda_3\,\zeta_\beta\,.$$

Furthermore, since $E^* = G^*$ and $F^* = 0$, the theorem that was proved in **11**. says that the surface that is represented by the functions $\xi\,(\alpha, \beta)$, $\eta\,(\alpha, \beta)$, $\zeta\,(\alpha, \beta)$ will be an extremal surface of the variational problem:

$$\delta \iint F\,(p_1, p_2, p_3)\,d\alpha\,d\beta = 0\,,$$

and the surface that is represented $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ will be either its adjoint surface or the mirror image of the latter relative to the origin.

In the foregoing, we adapted the theorems from the theory of minimal surfaces to our basic variational problem, in which only the first derivatives of the extremal surfaces appear. In fact, our theorems did not require the existence of the second derivatives of the extremal function, as was emphasized in the remark ([4]). However, it would also be interesting to adapt those properties of minimal surfaces in which the second derivatives do appear. For example, we might investigate how the generalization of the known theorem of Bonnet that says that the adjoint map takes lines of curvature on a minimal surface to the asymptotic curves on the adjoint surface would read in our sphere of ideas.

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