LESSONS

ON THE

PROPAGATION OF WAVES

AND THE

EQUATIONS OF HYDRODYNAMICS

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FOREWORD

In the course that I taught at the Collége de France during the years 1898-1899 and 1899-1900 ([1]), whose diverse circumstances retarded the publication of this work, I principally proposed to research how boundary conditions exert an influence on the movement of fluids.

As far as liquids are concerned, this amounts to a problem that is analogous to the Dirichlet problem: the *Neumann Problem* ([2]), which is the object of the first chapter of this work. In latter times, the theory of harmonic functions has been subjected to important refinements, most of which are only appended to my own discussion; in my references to a memoir of Stekloff I have used those refinements that are of direct interest to the Neumann problem.

On the contrary, in the case of a gas one is led to the theory of Hugoniot, which has been afforded much attention for several years, thanks to the lessons on *Hydrodynamique, Elasticité, et Acoustique* of Duhem.

To enjoy all of the benefits that mechanics has to offer, this theory – as well as the development of the memoirs *Sur la propagation du mouvement dans les corps* (Journal de l'Ecole Polytechnique, tome XXXIII, lett. 57-59), in which the notion of compatibility is less clear as it is in the memoir of the Journal de Liouville – seems to me to beg several questions. This is why I have discussed the facts that are of a purely kinematical nature separately from those that depend on the dynamical properties of movement. By means of this distinction, as well as what one would expect, many viewpoints are clarified. Thanks to them, in particular, a geometric representation seems immediate. This, in turn, permits us to make the analogy between waves such as Hugoniot imagined and the ones that are considered in vibrational mechanics more precise.

Finally, there is good reason to approach the theory of Hugoniot from the standpoint of the characteristics of equations with more than two independent variables, which is its analytic expression, and which J. Beudon posed the fundamentals of before his cruelly premature death.

The solution of the Cauchy problem for linear equations, according to the path taken by Kirchhoff, is related in a direct manner to the notion of characteristic, and we naturally follow this path. In all of its details, we have nonetheless developed a theory that, except for the important works that were published since the epoch when this course began, has still not arrived at a definitive form.

Moreover, by its very nature, a discussion such as the one that I have attempted to define as the object of the present work, and which, in the eyes of rigor, comprises all of continuum mechanics, cannot be complete, and I have made no pretense that it is.

I would like to thank M. Gaudet here, an old student at l'Ecole Polytechnique, whose collaboration has been precious to me. In a large part, I owe the editing of the first two chapters to him, in which he likewise corrected certain proofs. Moreover, I am happy to express my recognition

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([1]) Chapters I to IV reasonably correspond to the course in 1898-1899, and chapters V-VII to the one in 1899-1900. I have nevertheless included a discussion of the method of Neumann following Stekloff (nos. **12-16**), the necessary conditions for the minimum of the elastic potential (no. **270**), and the final notes to this edition.

([2]) This is the term I have adopted in the text, at least, with Stekloff. In the recent works relating to harmonic functions that appeared during the printing of this present work this same term is used in a completely different sense. There is therefore reason to modify it, all the more so if Fr. And C. Neumann have recognized the importance of the problem in question; the priority – at least as far as it concerns the present publication – seems to be due to Bjerknes and Dini.

of all of my auditors at the Collége de France, whose kind assistance has greatly facilitated the publication of these lessons.

J. HADAMARD

FIRST CHAPTER

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THE SECOND BOUNDARY-VALUE PROBLEM IN THE THEORY OF HARMONIC FUNCTIONS ([3])

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§ 1. – CLASSICAL PROPERTIES OF HARMONIC FUNCTIONS

1. – One knows that a function V is called *harmonic* in a domain D if for all interior points of D, V and all of its first two derivatives have definite values, and moreover:

1. It satisfies the equation: $\Delta V = 0$;

2. In the case for which the domain D is unbounded, V is *regular* at infinity, i.e., that it behaves like a potential and its derivatives behave like the derivatives of a potential.

Some harmonic functions, outside of attracting masses are:

1. The potentials of spatial distributions:

$$\iiint \frac{U}{r} \, dxdydz,$$

2. The potentials of simple coverings:

$$\iint \frac{U}{r} \, dS,$$

3. The potentials of double coverings:

$$\iint U \frac{d\frac{1}{r}}{dn} \, dS,$$

which are functions of the point M, and in which U denotes a function of the integration element that is called a *density* or *weight.*

With only the condition on the weight U that it be everywhere finite, these potentials are everywhere finite and continuous, as well as their first derivatives, except in the case that there are surface potentials on the surface of the domain; on this surface, there will

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([3]) See, among others: Bjerknes, *Sur le mouvement simultaneé des corps,* Act. Soc. Sc. Christiana, 1868-1871; DINI, *Sull' Equazione* $\Delta^2 u = 0$, Annali di Matematica, series II, tome 5, 1871; BETTI, *Principii dell Idrodinamica razionale*, Mem. Ac. Sc. Bologne, tomes 1-5, 1871-1874; C. NEUMANN, *Untersuchungen über das Logarithmische und Newton'sche Potential*, Leipzig, 1877; Fr. NEUMANN, *Potential und Kugelfunctionen*, edited by C. Neumann, Leipz. 1887; STEKLOFF, C. R. Ac. Sc., passim; *Les Méthodes général pour résoudre les problèmes fondamentaux de la Physique mathématique*, Ann. Fac. Sc., Toulouse, 2nd series, tome II, and another work (in Russian) of the same title, Kharkow; 1901; Poincaré, passim, etc.

be discontinuities in the potentials of the double covering and the derived potential of the simple covering. The integrals that represent either the first of these functions or the normal derivative of the second one experience two abrupt augmentations that are equal to $2\pi U$ when one passes from a point in the neighborhood of the surface on the side of the positive normal to a point on the surface, and when one passes from the latter to a point in a neighborhood on the other side. On the contrary, the tangential derivatives of the potential of the simple covering remain continuous. The normal derivative of the potential of the double covering remain likewise continuous under certain conditions of continuity on the weight *U*, which are, in particular, verified if *U* has first and second derivatives on the surface ([4]).

If *V* is a harmonic function in a domain then, upon denoting the radius vector that issues from an arbitrary point of the boundary surface to the point *A* by *r*, one has:

$$\frac{1}{4\pi}\iint\left(V\frac{d\frac{1}{r}}{dn}-\frac{1}{r}\frac{dV}{dn}\right)dS=\begin{cases}0, & \text{if } A \text{ is exterior to the domain,}\\ V_A, & \text{if } A \text{ is interior to it.}\end{cases}$$

This formula expresses the value of *V* at *A* as a function of its values and those of its normal derivatives on an arbitrary surface surrounding this point, and the fact that it takes the form of a sum of potentials of simple and double coverings.

From this, one concludes that a harmonic function in a domain:

1. is *analytic*, i.e., it is developable in a Taylor series about any point in the interior of the domain.
2. has neither a maximum nor a minimum at an interior point.

The first of these properties may be further stated in the following form: If two harmonic functions V_1 and V_2 are defined in two domains D_1 and D_2, that are exterior to each other, but have a common boundary Σ and their values $\overline{V}_1$ and $\overline{V}_2$, as well as their normal derivatives $\frac{d\overline{V}_1}{dn}$ and $\frac{d\overline{V}_2}{dn}$ (both of which are regarded in the same sense), are the same on Σ, then V_1 and V_2 are analytic continuations of each other.

The second property may be generalized, in the sense that not only does a harmonic function have neither a maximum nor a minimum, but if one is given the extreme values L and L' of this function on a closed surface then one may find an upper bound for the difference of the values that it takes at two given points on the interior of that surface that is a definite fraction of $L' - L$. One of the consequences of that remark is Harnack's theorem, which says:

1. A series whose terms are all positive harmonic functions in a domain *D*, as well as the derived series, may not be convergent at an interior point of this domain without being uniformly convergent and harmonic in any domain that is interior to *D*.

([4]) See, for example, Liapounoff: *Sur les potentials de double couche*, Kharkow, 1897; and *Sur certaines questiones qui se rattachent au problème de Dirichlet*, Journal de mathèmatiques, 1898.

2. A series of harmonic functions that is uniformly convergent on the boundary of a domain is, moreover, uniformly convergent and harmonic in the entire domain.

2. – The notion of a harmonic function may be extended to an arbitrary number of variables. In the plane, one knows that any harmonic function is the real part of a complex function of the imaginary variable variable, $z = x + iy$, and conversely. If V is the harmonic function then $V + iW$ is the complex function; W is another harmonic function, which will be called the *conjugate* of V. One has:

$$\frac{dV}{ds} = \frac{dW}{dn},$$

in which ds is an arbitrary element of arc and dn is an element of arc that is normal to the first and positive to the left. From this, one infers that:

$$V = \int \frac{dW}{dn} ds, \qquad -W = \int \frac{dV}{dn} ds.$$

A conjugated function is therefore defined up to a constant.

If V is uniform then W will be uniform only if:

$$\int \frac{dV}{dn} ds = 0,$$

around any bounding contour.

The function that plays the same the same role in the plane as $1/r$ does in space is log $1/r$. It gives rise to potentials (*logarithmic* potentials) that are analogous to the ones that we just discussed in no. **1**, and which are harmonic in any region of the plane that is exterior to attracting lines or surfaces.

A harmonic function in the plane will be called *regular at infinity* if one may find two constants M and C such that this function approaches infinity like:

$$M \log \frac{1}{r} + C.$$

3. **Boundary-value problems**. – One may propose to determine a harmonic function V either by knowing its values on the boundary $\overline{V}$ by the knowing the values of its normal derivative $\frac{d\overline{V}}{dn}$, or by knowing the values of $\frac{d\overline{V}}{dn} - h\overline{V}$, in which h is a positive quantity. The first of these problems is the *Dirichlet problem:* the *interior* problem is when the entire domain is at a finite distance, and the *exterior* problem is when it is extended to infinity in any sense; in the latter case, it is well known that the function must be regular at infinity.

If the Dirichlet problem has a solution then there is only one, except in the case of the exterior problem in the plane, because then the condition that two functions are regular at infinity does not imply that their difference is null there. One must therefore give one of the two constants M and C. One may not always give C arbitrarily, but one may always give M; for example, if one imposes the condition that $M = 0$.

4. – The question concerning harmonic functions whose solution is of the most interest to hydrodynamics is not the Dirichlet problem, but the *second boundary-value problem* or *Neumann problem,* in which one is given the values of the normal derivatives. The second boundary-value problem, whose study is much less advanced than that of the Dirichlet problem, is the one that we shall now address. Like the former, it may be *interior* or *exterior.* In the former case, Gauss's theorem gives a condition of possibility:

(**1**)
$$\begin{cases} \iint \dfrac{d\overline{V}}{ds} dS = 0 & \textit{in space}, \\[2ex] \int \dfrac{d\overline{V}}{ds} ds = 0 & \textit{in the plane}, \end{cases}$$

assuming that the integrals are taken over the set of boundary points for the domain. By contrast, if the problem is possible then an arbitrary additive constant must obviously enter into the solution.

In the case of the exterior problem, there is no possibility condition and no arbitrary additive constant if one is in space because of the regularity condition. On the contrary, in the plane the additive constant C persists. As for the constant M in the term $M \log 1/r$, it is determined by Gauss's theorem:

$$M = \frac{1}{2\pi} \int \frac{d\overline{V}}{dn} ds.$$

5. Generalized problems. – One may further determine a function U by the condition:

(**2**) $$\Delta U = f,$$

in which f is given function, and by boundary-value conditions that are similar to the foregoing.

The problems that are thus posed immediately reduce to the corresponding problems in harmonic functions.

Indeed, set:

$$U = -W + V,$$

in which W is a spatial potential:

$$W = \frac{1}{4\pi} \iiint \frac{f}{r} dx dy dz.$$

One has:

$$\Delta V = 0,$$

with

$$\overline{V} = \overline{U} + \overline{W},$$

in which:

$$\frac{d\overline{V}}{dn} = \frac{d\overline{U}}{dn} + \frac{d\overline{W}}{dn}.$$

An analogous transformation will make not the left-hand side of the partial differential equation, but the boundary-value condition, disappear.

§ 2. – THE SECOND BOUNDARY-VALUE PROBLEM EXISTENCE OF THE SOLUTION

6. – In order to establish the existence of the second boundary-value problem, Lord Kelvin has indicated a method that is analogous to the one that Riemann gave for the Dirichlet problem. Look for the minimum of the integral:

$$I = \iiint \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] dxdydz,$$

for the functions V that satisfy the equation:

$$K = \iint VFdS, \tag{3}$$

in which K is an arbitrary – but non-null – constant; we denote the given values of the normal derivatives on the surface by F.

If one assumes the possibility condition, namely:

$$\iint FdS = 0, \tag{1'}$$

then I is positive for any function V, and is not annulled if equation (**3**) is verified; indeed, I may be annulled only if V is a constant, which will give (by virtue of equation (**1′**)):

$$K = 0.$$

From this, one is led to think that I has a positive minimum for the functions V that satisfy equation (**3**). If one assumes – and this has not been proved – that this minimum is actually attained for a particular function, then one may prove, like Lord Kelvin, that this function V constitutes a solution of the proposed boundary-value problem, up to a constant factor.

Indeed, if we change V into $V + \delta W$ in I then it becomes $I + \delta_1 I + \delta_2 I + \ldots$ in which $\delta_n I$ represents the set of terms in ε^n in the development of I in increasing powers of ε.

One must have:

$$\delta_1 I = 0$$

for any W, provided that this function satisfies:

$$(\mathbf{3}') \qquad \iint W\, F\, dS = 0\,.$$

However:

$$\delta_1 I = 2\varepsilon \iiint \left(\frac{\partial V}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial W}{\partial z} \right) dxdydz$$
$$= 2\varepsilon \left[-\iint W \frac{dV}{dn} dS - \iiint W \Delta V dxdydz \right].$$

It is therefore necessary that the equation:

$$(\mathbf{4}) \qquad \iint W \frac{dV}{dn} dS = \iiint W \Delta V dxdydz,$$

be a consequence of equation $(\mathbf{3}')$. If we first take a function for W that is null on the surface and has the same sign as ΔV everywhere, moreover, then one sees that one must have:

$$\Delta V = 0$$

in this domain. Equation (**4**) then reduces to its first term. This shows that $\frac{dV}{dn}$ must be proportional to F.

Indeed, we shall see that one has:

$$\frac{\iint UFdS}{\iint W \frac{dV}{dn} dS} = \lambda,$$

for any function U in which λ is a well-defined number. It obviously suffices to show that this ratio is the same for two arbitrary functions U and U'. Now, for any U and U' one may always find a constant μ such that when $U + \mu U'$ is substituted for W in equation $(\mathbf{3}')$ satisfies this equation.

It must therefore also satisfy the equation:

$$\iint W \frac{dV}{dn} dS = 0,$$

which suffices to prove the proposition. One infers from this that:

$$\iint U\left(F - \lambda \frac{dV}{dn}\right) dS = 0,$$

for any function U; it is therefore necessary that all of the integration elements are separately null, namely:

(**5**) $$F = \lambda \frac{dV}{dn},$$

Q.E.D.

Conversely, a harmonic function V that satisfies equation (**5**) will satisfy equation (**3**) for a conveniently-chosen K, and of all of the functions that satisfy equation (**3**) it is the one that will give a minimum for the integral I.

Moreover, since it is clear that the previous reasons raise the same objections as the analogous reasons of Riemann, nothing permits us to confirm the existence of the minimum considered.

7. – In reality, not only is one uncertain, *a priori*, that an arbitrary problem in the calculus of variations has a solution, but it is easy to see that the case in which the solution exists must be in no way considered as more general than the contrary case.

For example, consider the integral:

$$\int \sqrt{dx^2 + dy^2}.$$

If one looks for the minimum of that integral over all the different arcs of the curves that join two points of the plane A and B then one sees that this minimum exists and is given by the line segment AB.

Now look for the minimum of the same integral when it is taken, no longer over all the arcs of curves joining A to B, but only over the ones that admit given tangents at A and B. It is easy to confirm that this minimum is not effectively attained. Indeed, there exist lines (for example, arcs of conics that are asymptotically flat) that admit the given tangents at A and B, and whose length differs from that of the straight line AB by as little as one likes. This latter length is therefore the desired minimum; however, it does not correspond to any line that satisfies the conditions of the problem.

We therefore have two problems in the calculus of variations such that the first one admits a solution, but the second one does not. Now, there is no reason, *a priori*, to pose one of these problems more often than the other, and it is the second one that will be the most natural to envision if the unknown function in the proposed integral is involved by way of not only its first derivatives, but also its second derivatives.

In general, if we consider the integral:

$$\int_{x_0}^{x_1} F(x, y, y', \cdots, y^{(\mu)})\, dx,$$

and if we are given the values of $y, y', \cdots, y^{(\nu)}$ for $x = x_0$ and $x = x_1$ then the classical methods of the calculus of variations may be used to find the minimum of the integral if $\nu \leq \mu - 1$. On the contrary, if $\nu \geq \mu$ then the minimum is not effectively attained (except for particular values of the givens).

8. – The theory of harmonic functions itself easily provides analogous examples. Indeed, look for the minimum of the integral:

$$I = \iiint_T \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right] dxdydz.$$

The function V is subject to the double condition that V *and* its normal derivative take the given values $\overline{V}$ and $\overline{V}'_n$ on the boundary surface S of T. If such a minimum is attained it will necessarily correspond to a harmonic function, whereas nothing that verifies both of the given boundary conditions exists.

The minimum is, moreover, furnished by the harmonic function V_0 that takes the given values V on S. This is what one will confirm (at least in the case for which this function has finite derivatives in a neighborhood of S) upon considering functions of the form:

$$V = \frac{FV_0 + \lambda\varphi}{F + \lambda},$$

in which λ is a positive constant; φ is an arbitrary function that satisfies the conditions $\varphi = \overline{V}$, $\dfrac{d\varphi}{dn} = \overline{V}'_n$ on S. The equation of S is $F = 0$, in which F is positive on the interior of T, and the derivatives $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$ are not null on S. One will have:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{\partial \varphi}{\partial x} + \frac{F}{F+\lambda}\frac{\partial (V_0 - \varphi)}{\partial x} + (V_0 - \varphi)\frac{\partial}{\partial x}\left(\frac{F}{F+\lambda}\right) \\
\frac{\partial V}{\partial y} &= \frac{\partial \varphi}{\partial y} + \frac{F}{F+\lambda}\frac{\partial (V_0 - \varphi)}{\partial y} + (V_0 - \varphi)\frac{\partial}{\partial y}\left(\frac{F}{F+\lambda}\right) \\
\frac{\partial V}{\partial z} &= \frac{\partial \varphi}{\partial z} + \frac{F}{F+\lambda}\frac{\partial (V_0 - \varphi)}{\partial z} + (V_0 - \varphi)\frac{\partial}{\partial z}\left(\frac{F}{F+\lambda}\right).
\end{aligned}$$

One sees that the function V satisfies the given boundary conditions for any V since $V_0 - \varphi$ is unique on S. Moreover, by virtue of the hypotheses made on V_0 and F, the quotient $\dfrac{V_0 - \varphi}{F}$ does not exceed a certain limit K; from this, it results that the absolute values of the derivatives of V are everywhere less than a limit *that is independent of* λ.

From this, for very small λ the integral I tends towards the quantity:

$$I_0 = \iiint_T \left[\left(\frac{\partial V_0}{\partial x} \right)^2 + \left(\frac{\partial V_0}{\partial y} \right)^2 + \left(\frac{\partial V_0}{\partial z} \right)^2 \right] dxdydz,$$

as one sees upon dividing the integral into two parts. One relates to the region where $F < 0$ and goes to zero with ε (for any λ) because the volume of integration is infinitesimal; the other relates to the region in which $F > \varepsilon$ and (ε has a definite, but arbitrary, value) the differences:

$$\frac{\partial V}{\partial x} - \frac{\partial V_0}{\partial x}, \qquad \frac{\partial V}{\partial y} - \frac{\partial V_0}{\partial y}, \qquad \frac{\partial V}{\partial z} - \frac{\partial V_0}{\partial z},$$

are infinitesimal with λ.

The desired minimum is therefore I_0, and it is not attained by functions that satisfy the imposed conditions ([5]).

§ 3. – CASE OF THE PLANE

9. – In the plane (Dini, *loc. cit.*), the second problem immediately reduces to the first.

By virtue of the equation:

$$\frac{dV}{dn} = -\frac{dW}{ds},$$

in which W is the conjugate function of the desired V, one is obviously led to the following operations:

1. Quadratures to determine W along the boundary contours.
2. Solution of the first boundary-value problem for W.
3. Derivation of W and quadrature:

$$\int \frac{dW}{dn} ds = \int \frac{\partial W}{\partial x} dy - \frac{\partial W}{\partial y} dx = \int \frac{dW}{dn} ds = V.$$

10. – Discussion of the solution. – 1. *Interior problem.* – This pertains to whenever the possibility condition is verified, namely:

$$\textbf{(1)} \qquad \int \frac{dV}{dn} ds = 0,$$

([5]) Quite recently, Hilbert has arrived at a modification of the reasoning of Riemann in such a way as to prove the existence of the solution for the Dirichlet problem and, more generally, for an arbitrary problem of the calculus of variations, by means of obviously restrictive conditions whose necessity results from what we just said.

in which the integral is taken along the set of contours of exterior or interiors.

If the given area has just one contour then this condition expresses that the function $\overline{W}$ is uniform along this contour, since one has:

$$\int \frac{dV}{dn} ds = \int \frac{dW}{ds} ds.$$

From this, the function W is well defined over the entire area, and the integral $\int \frac{\partial W}{\partial x} dy - \frac{\partial W}{\partial y} dx,$ which is uniform in the same area, gives us the desired function.

If there are n boundary contours then the integral $\int \frac{dV}{dn} ds$ will not be null on each of them, in general, and consequently the function $\overline{W}$ will have periods on these contours. However, $n - 1$ of these periods will disappear (and, as a result, the n^{th} one, by virtue of equation (**1**)) after subtracting conveniently chosen logarithmic functions from the function $V + iW$, for example, functions of the form:

$$\lambda_h \log(x + iy - \alpha_h) \qquad (h = 1, 2, \ldots, n - 1),$$

in which $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the names of points that are situated in the interiors of the $n - 1$ contours and the λ's are real constants. The terms thus introduced modify V only by a uniform quantity, moreover.

Having taken this precaution, one will know the values of W on the boundaries, but only *up to an additive constant on each of them.*

When $n = 1$, the unique constant that is thus added to the values of W on the contour is similarly added to the function W over the entire area; it has no influence on the value of V.

However, for $n > 1$ only one of the additive constants $c_1, c_2, \ldots, c_n$ that correspond to the different contours of the area $C_1, C_2, \ldots, C_n$, respectively, may be considered to be insignificant. The other $n - 1$ constants (or rather, the differences $c_1 - c_n$, $c_2 - c_n$, …, $c_{n-1} - c_n$) influence the function W in an essential manner.

On the other hand, in order for the function V to be uniform in the area considered, it is necessary that one have:

$$\int \frac{dW}{dn} ds = 0$$

on each of the n contours, and the n equations thus obtained reduce to $n - 1$, moreover, since the function W is harmonic. The question is therefore one of determining $c_1, c_2, \ldots, c_n$ (upon supposing $c_n = 0$, as one has the right to do, from the foregoing) in such a way as to satisfy the conditions that we just wrote.

Let φ_i be the harmonic function that takes the constant value 1 on C_i, and the value zero on each of the $n - 1$ other given contours; let γ_j^i be the integral $\int \frac{d\varphi_i}{dn} ds$ taken along the contour C_j. The addition of the constant c_i to the values of W on the contour C_i will obviously add the term $c_i\varphi_i$ to W and the term $c_i\gamma_j^i$ to $\int_{C_j}\frac{dW}{dn}ds$. The equations that the c_i must satisfy will therefore be of the form:

$$\sum_{i=1}^{n-1} c_i\gamma_j^i = \alpha_j, \qquad (j = 1, 2, \dots, n),$$

in which the α_i are given quantities.

These equations, which reduce to $n - 1$ distinct ones, determine the c_i , at least when the determinant $\sum\pm\gamma_1^1\gamma_2^2\cdots\gamma_{n-1}^{n-1}$ is non-null.

However, this last hypothesis might not be realized, which amounts to saying that one might never satisfy the equations:

$$\sum_{i=1}^{n-1} c_i\gamma_j^i = \alpha_j, \qquad (j = 1, 2, \dots, n\text{-}1),$$

with values of c_i that are all non-null.

Indeed, if φ_i is always taken between 0 and 1 then the quantity γ_i^i is certainly negative and the quantity γ_j^i $(i \neq j)$ is positive ([6]). In particular, one has $\gamma_n^i > 0$ $(i = 1, 2, \dots, n - 1)$, and, as a consequence, the identity $\sum_{i=1}^{n}\gamma_j^i = 0$ (which results from the obvious identity $\varphi_1 + \varphi_2 + \dots + \varphi_n = 1$), gives:

$$|\gamma_j^i| > \sum_{i=1}^{n-1}{}' \gamma_j^i,$$

in which the sign $\sum'$ denotes a summation over all indices i that are different from j. Therefore, the equation that corresponds to the index j will not be satisfied if c_j has the largest absolute value of the quantities $c_1, c_2, \dots, c_{n-1}$.

2. *Exterior problem.* – As before, one subtracts logarithmic terms such that the conjugated function W does not have periodic points on boundary contour from V; one agrees to determine the function W in such a way that the constant M (no. **2**) is null. With these conditions, things happen exactly as they did for the interior problem: the function φ_i will be the harmonic function that is equal to one on the contour of index i and zero on all of the other contours, and is regular at infinity with a constant M equal to zero.

([6]) The quantities γ_j^i will not have to be null if $C_1, C_2, \dots, C_{n-1}$ denote the interior contours; indeed, for this to be true, it is necessary that $d\varphi_i / dn$ be everywhere null on C_j. However, in this case the function φ_i and the function that equals 0 (for $i \neq j$) or 1 (for $i = j$) in all of the interior of C_j will be (no. **1**) analytic continuations of each other and φ_i will be constant, which is absurd.

§ 4. – CASE OF SPACE – APPLICATION OF NEUMANN'S METHOD

11. – The question in the case of space is not as simple as it was in the planar case. Therefore, it will not suffice to solve the Dirichlet problem by an arbitrary method in order to solve the second boundary-value problem.

By contrast, one may deduce this solution from the solution of the Dirichlet problem *by the Neumann method*. Indeed, one knows that the latter gives the desired solution in the form of a potential of two sheets.

Having said this, suppose that we are concerned with the exterior problem, and take a potential of one sheet with density $F/2\pi$, in which the normal is positive towards the interior of the domain (i.e., to the exterior of the given boundary surface S), and F denotes the given value of the normal derivative. Let W be that potential and let U be its value on the surface. Then, determine a two-sheeted potential that is defined in the *interior* of S, and takes the values U on the points that infinitely close to S and situated in the interior. Let W' be that potential, which is obtained by Neumann's method, and whose normal derivative is continuous upon crossing S.

The desired function is:

$$V = W - W'.$$

Indeed, this function, being the difference of two potentials, is harmonic in the domain considered. Moreover, its normal derivative, when taken on the boundary surface, has the value:

$$\frac{dV}{dn} = \frac{dW}{dn} - \frac{dW'}{dn} = \frac{dW}{dn} - \frac{dU}{dn} = 2\pi \frac{F}{2\pi} = F.$$

In the case of the interior problem, there is a possibility condition:

$$\iint F\,dS = 0.$$

Moreover, one follows exactly the same path as for the exterior problem. The possibility condition expresses the idea that the constant C that must be added in the Neumann method to the two-sheeted potential in the Dirichlet problem will be null. The difference of the normal derivatives is calculated as in the preceding case.

12. – Nevertheless, two types of objections may be raised against the use of the Neumann method:

1. Neumann proved the legitimacy of his method in the case of a convex surface that is not doubly starlike. Poincaré ([7]) lifted that restriction by proving the convergence of Neumann's developments in the most extended case.

2. It is not obvious that the function so obtained always has a normal derivative, and the study of the existence conditions for that derivative is quite delicate (see Liapounoff, *Journal de Mathèmatiques pures et appliquées* 1898).

([7]) *Acta Mathematica*, t. 20; 1896.

The sufficient conditions that were obtained in this regard by Liapounoff are of a relatively complicated form, and there is nothing to say that these conditions are satisfied by the successive functions that are formed in the Neumann method.

This second objection may be likewise lifted. Thanks to the work of Stekloff and Korn, we shall show, as Stekloff did in his previously-cited works, that it is sufficient to establish the legitimacy of the Neumann method as one habitually studies it in order to apply it to the problem we are concerned with.

Furthermore, as we shall see, the Neumann method thus applied amounts to the method that was given by Robin to find an electric distribution in equilibrium.

We let M denote an arbitrary definite point, and let r denote its distance from a variable point M' of the surface. (All of the quantities that relate to the latter point will be indicated by accented letters.) This is why we let dS' denote an element of the surface S that surrounds the point M' when we integrate over all positions of that point on the surface, and let dn' denote a normal element at M' when dn denotes a normal element at M, and let V' denote the value of an arbitrary function V at M'.

Moreover, an arbitrary function V may have different expressions depending on whether one is exterior or interior to S; according to the usage, the expression will be denoted by V_e in the former case and V_i in the latter. This is why we have to distinguish the two normal derivatives $\frac{dV_i}{dn}$ and $\frac{dV_e}{dn}$ at a point of the surface for a potential of one sheet, for example.

In addition, we consider the integral:

$$\iint \rho' \frac{d\frac{1}{r}}{dn} dS' = \iint \frac{\rho'}{r^2} \cos(r, n) dS',$$

(ρ is the density) – in other words, the value of $\frac{dV}{dn}$ that one obtains upon differentiating the $\iint$ sign, when this differentiation is legitimate, and when one knows the mean of the two derivatives $\frac{dV_i}{dn}$ and $\frac{dV_e}{dn}$.

We denote this latter quantity by the notation $\left(\frac{dV}{dn}\right)$.

Let ρ_0 be a continuous function on the surface S, and consider the sequence of quantities:

$$
(\mathbf{6}) \qquad \begin{cases}
\rho_1 = \dfrac{1}{2\pi}\displaystyle\iint \rho_0' \frac{d\frac{1}{r}}{dn} dS', \\
\rho_2 = \dfrac{1}{2\pi}\displaystyle\iint \rho_1' \frac{d\frac{1}{r}}{dn} dS', \\
\quad\cdots \\
\rho_k = \dfrac{1}{2\pi}\displaystyle\iint \rho_k' \frac{d\frac{1}{r}}{dn} dS',
\end{cases}
$$

which are precisely the quantities that appear in Robin's method ([8]). These quantities may be expressed with the aid of one-sheeted potentials. Indeed, if one sets:

$$
(\mathbf{7}) \qquad \begin{cases}
V_1 = -\dfrac{1}{2\pi}\displaystyle\iint \frac{\rho_0'}{r} dS', \\
V_2 = -\dfrac{1}{2\pi}\displaystyle\iint \left(\frac{dV_1}{dn}\right)' \frac{dS'}{r}, \\
\quad\cdots \\
V_k = -\dfrac{1}{2\pi}\displaystyle\iint \left(\frac{dV_{k-1}}{dn}\right)' \frac{dS'}{r},
\end{cases}
$$

then equations (**6**) will obviously give:

$$
\left(\frac{dV_1}{dn}\right) = \rho_1, \qquad \left(\frac{dV_2}{dn}\right) = \rho_2, \quad \left(\frac{dV_k}{dn}\right) = \rho_k.
$$

From known properties of the one-sheeted potentials, all of the integrals in (**6**) and (**7**) will exist and will be continuous when the function ρ_0 possesses this property.

On the other hand, since V_k is harmonic in the interior of our domain and possesses normal derivatives on the boundary, one has:

$$
V_k = -\frac{1}{2\pi}\iint \left(V_k' \frac{d\frac{1}{r}}{dn'} - \frac{1}{r}\frac{dV_{ki}'}{dn'} \right)' \frac{dS'}{r},
$$

but V_k' is a one-sheeted potential with density $-\dfrac{1}{2\pi}\left(\dfrac{dV_{k-1}}{dn}\right)$. One therefore has:

([8]) Nevertheless, in the latter the function ρ_0 is taken in such a way that $\iint \rho_0 dS \neq 0$, whereas, on the contrary, we shall suppose that the same integral is null.

$$\frac{dV'_{ki}}{dn'} = \left(\frac{dV'_k}{dn'}\right) - \left(\frac{dV'_{k-1}}{dn'}\right).$$

If we substitute this in the preceding equation then the term $-\frac{1}{2\pi}\iint \frac{1}{r}\left(\frac{dV_{k-1}}{dn}\right) dS'$ will disappear from the left-hand side, and we will get:

$$-\frac{1}{2\pi}\iint \left(\frac{dV'_k}{dn'}\right)\frac{dS'}{r} = V_{k+1} = -\frac{1}{2\pi}\iint V'_k \frac{d\frac{1}{r}}{dn'} dS'.$$

However, these equations are precisely the ones that one writes in Neumann's method, which apply when one starts with the function:

$$(7') \qquad V_1 = -\frac{1}{2\pi}\iint \frac{\rho'_0}{r} dS'.$$

Nevertheless, we thus obtain the successive Neumann functions only on the same surface. However, it is easy to deduce the expression for these same functions in the interior and exterior domains. Indeed, let v_k be the two-sheeted potential of weight $-\frac{1}{2\pi}V_{k-1}$, i.e., one of the desired functions. On the surface (since the aforementioned two-sheeted potential has the value V_k on that surface itself), one will have:

$$v_{ki} = V_k + V_{k-1},$$

and this equation is true in the entire interior domain, moreover. Similarly, one has:

$$v_{ke} = V_k - V_{k-1},$$

in the exterior domain.

It results from this that the potentials v_{ki} and v_{ke} admit normal derivatives, since the V_k admit them. Moreover, if one takes the values of the normal derivatives of V_k into account then one gets:

$$\frac{dv_{ki}}{dn} = \frac{dv_{ke}}{dn} = \left(\frac{dV_k}{dn}\right) - \left(\frac{dV_{k-2}}{dn}\right).$$

We have therefore proved that v_k admits normal derivatives on either side of S, and that these normal derivatives are equal to each other.

13. – To extend the same conclusion to the sum of the series formed from the v_k as indicated by Neumann, it is necessary to invoke, with Liapounoff, two lemmas, the first of which relates to mode of continuity of $\left(\frac{dV}{dn}\right)$ on the surface, and the second of which relates to the way in which the normal derivatives tend to their limits in a neighborhood of that surface. One supposes that the surface S is everywhere regular (at least in a neighborhood of the points considered), and, in particular:

1. that it admit a definite tangent plane at each point;
2. that there exist a sufficiently small length D such that a parallel to the normal at an arbitrary point of S does not cut S in two points in the interior of the sphere that has this point for its center and D for its radius;
3. that planes that are tangent to two of its points and situated a distance r from each other form an angle less than Kr, where K denotes a number that one may assign once and for all ([9]).

13 (cont.). – It easily results from the foregoing that each of these tangent planes makes an angle less than Kr (K denotes a constant) with the chord that joins the two points ([10]).

14. – With these conditions, we let V be the one-sheeted potential of density ρ that is defined on the surface S, and look for the order of the magnitude of the difference between the values that $\left(\frac{dV}{dn}\right)$ takes at two neighboring points M and M_1 of S as a function of the distance $MM_1 = \delta$, namely, the quantity:

$$\iint \rho'\left(\frac{\cos\psi}{r^2} - \frac{\cos\psi_1}{r_1^2}\right) dS',$$

in which r, r_1 denote the distances from the points M, M_1 to an arbitrary point M' of the surface, which is the center of the element dS', and ψ, ψ_1 denote the angles that $M'M$ and $M'M_1$ make with the normals Mn and M_1n_1 at M and M_1 (*fig.* 1).

We divide S into two subsets, one of which – s – consists of points whose distance to M is less than $\mu\delta$ (μ is a definite number that is larger than 1), and the other of which – s_0 – consists of the rest of S.

([9]) Liapounoff supposed only that this angle is less than kr^{χ}; it is easy to adapt the reasoning to what follows from this new hypothesis.

([10]) We let K denote any of various positive numbers that are dependent on the surface, but independent of the choice of points M, M_1, M' and the form of the function ρ.

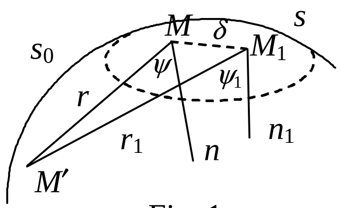

Fig. 1.

In the first subset the integral $\int \frac{\rho' \cos\psi}{r^2} ds'$ will be less than $KA\int \frac{ds}{r}$ (upon denoting the maximum of $|\rho|$ on S by A), a quantity that, by means of the various hypotheses that we have made, will be less than $KA\delta$. A completely similar argument applies to $\int \frac{\rho' \cos\psi_1}{r_1^2} ds'$.

In the subset s_0 the ratios $\frac{r}{\delta}, \frac{r_1}{\delta}$ are less than $\mu - 1$, and the ratio $\frac{r}{r_1}$ falls between $\frac{\mu}{\mu+1}$ and $\frac{\mu}{\mu-1}$. On the other hand, the angle $M'M_1M$, or its supplement, is less than Kr (see no. **13** (cont.)), and, as a consequence, the angle $MM'M_1$ is less than $K\delta$; the same is true for $|\cos\psi - \cos\psi_1|$, by virtue of the inequalities:

$$\tfrac{1}{2}|\cos\psi - \cos\psi_1| < |\psi - \psi_1| < (Mn, M_1n_1) + (M'M, M'M_1).$$

On the other hand, the inequality $|r_1 - r| < \delta$ easily shows that $\left|\frac{1}{r_1^2} - \frac{1}{r^2}\right|$ is less than $\frac{K\delta}{r^3}$, and, as a consequence, that the quantity:

$$\rho\left(\frac{\cos\psi}{r^2} - \frac{\cos\psi_1}{r_1^2}\right) = \rho\left[\cos\psi\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right) + \frac{\cos\psi - \cos\psi_1}{r_1^2}\right],$$

seems to be less than $\frac{K\delta A}{r^2}$.

Now, the integral $\iint \frac{dS'}{r^2}$, when taken over the portion of the surface S for which $r > R$, is less than $K\,|\log K\,|$.

Therefore, we finally conclude that the difference considered is less than $KA\delta \log \delta$. We remark that the upper bound so obtained does not assume that ρ is continuous; it suffices that this function be continuous.

14 (cont.). – In the second case, let M_2 be a neighboring point to M that is situated on a definite side of S – for example, the interior – in such a manner that MM_2 is not tangent to the surface, and makes an angle that is greater than a definite limit with it; let r_2 be the distance from M_2 to an arbitrary point M' of S, which, under these conditions, has a ratio with MM_2 that remains greater than a definite limit; let ψ_2 be the angle that $M'M_2$ makes with the normal Mn at M; let φ, φ_2 be the angles that $M'M, M'M_2$ make with normal $M'n'$ at M (*fig.* 2). We seek to evaluate the difference that exists between the integral $\iint \frac{\rho' \cos\psi_2}{r_2^2} dS'$ and the analogous integral $\iint \frac{\rho' \cos\psi}{r^2} dS'$ taken over M.

This time, in addition to the preceding hypotheses that we made on the form of the surface S, we will make another one on the values of the function ρ; we suppose that the value of ρ at M' differs from the value ρ_0 that this function takes at the point M by a quantity that is less than Lr^{α}, if we let α denote a definite exponent (which is obviously equal to 1 if M is arbitrary and r is not constant), and let L denote a constant.

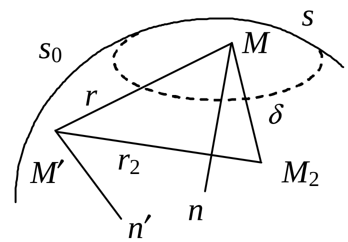

Fig. 2

We therefore remark that the integral $\iint \frac{\rho_0 \cos\varphi}{r^2} dS'$ is equal to $2\pi\rho_0$ and that the integral $\iint \rho_0 \frac{\cos\varphi_2}{r_2^2} dS'$ is equal to $2\pi\rho_0$, and we write the desired difference in the form:

$$\iint \rho\left(\frac{\cos\psi}{r^2} - \frac{\cos\psi_2}{r_2^2}\right) dS' = I_1 + I_2 + I_3,$$

$$I_1 = \iint \rho_0 \frac{\cos\psi}{r^2} dS' - \iint \rho_0 \frac{\cos\psi_2}{r_2^2} dS',$$

$$I_2 = \iint (\rho' - \rho_0)\left(\frac{\cos\varphi}{r^2} - \frac{\cos\varphi_2}{r_2^2}\right) dS',$$

$$I_3 = \iint \rho'\left(\frac{\cos\psi - \cos\varphi}{r^2} - \frac{\cos\psi_2 - \cos\varphi_2}{r_2^2}\right) dS'.$$

We know the quantity I_1. In order to evaluate I_2, we once more decompose S into two parts s and s_0 that are formed from points whose distance to the point M is less than or greater than $\mu\delta$, respectively, where this time d denotes the distance MM_2, and μ denotes a fixed number (greater than 1).

In s, the two integrals:

$$\iint (\rho' - \rho_0) \frac{\cos\varphi}{r^2} dS' \qquad \text{and} \qquad \iint (\rho' - \rho_0) \frac{\cos\varphi_2}{r_2^2} dS',$$

are less than $KL\delta^{\alpha}$, by virtue of the hypothesis that was made on $\rho' - \rho_0$ and the fact that r^2 has a finite ratio with δ.

In s_0, it will suffice to remark that:

$$|\cos\varphi - \cos\varphi_2| < 2\,|\varphi - \varphi_2| < |K \sin(\varphi - \varphi_2)| < \frac{K\delta}{r},$$

and that $\left|\frac{1}{r^2}-\frac{1}{r_2^2}\right|<\frac{K\delta}{r^3}$, in order to see that the integral:

$$\iint_{S_0}(\rho'-\rho_0)\left(\frac{\cos\varphi}{r^2}-\frac{\cos\varphi_2}{r_2^2}\right)dS'$$

is less than $KL\delta\iint_{S_0}\frac{dS_0'}{r^{3-\alpha}}$. However, the integral $\iint\frac{dS'}{r^{3-\alpha}}$, which is taken over a portion of the surface for which $r > R$, is less than $\frac{K}{R^{1-\alpha}}$. Hence, $I_2 < KL\delta^\alpha$.

As for I_3, it is less than $KA\delta$ in s, as one sees immediately upon remarking that $|\cos\psi - \cos\varphi|$ and $|\cos\psi_2 - \cos\varphi_2|$ are less than Kr. In s_0, one remarks that $r\cos\psi - r_2\cos\psi_2$ and $r\cos\varphi - r_2\cos\varphi_2$ are equal to $\delta\cos\theta$ and $\delta\cos\theta'$, if we let θ and θ' be the angles that MM_2 makes with the normals at M and M' – angles whose difference is less than Kr.

The quantity $\frac{\cos\psi-\cos\varphi}{r^2}-\frac{\cos\psi_2-\cos\varphi_2}{r_2^2}$ will therefore be sum of two terms:

$$\frac{r\cos\psi-r_2\cos\psi_2-(r\cos\varphi-r_2\cos\varphi_2)}{r_2^3}=\frac{\delta(\cos\theta-\cos\theta')}{r_2^3}$$

and

$$r(\cos\psi-\cos\varphi)\left(\frac{1}{r^3}-\frac{1}{r_2^3}\right),$$

each of which is less than $\frac{K\delta}{r^2}$. The integral I_3 (which is taken over s_0) is therefore less than $KA\delta\iint_{S_0}\frac{dS'}{r^2}$, i.e., less than $KA\delta\log\delta$.

Finally, this implies that the desired difference is less than $KA\delta^\alpha$ for all $\alpha < 1$ and less than $K(A + L)\,\delta\log\delta$ when $\alpha = 1$.

Nevertheless, our reasoning assumes that the point O_2 approaches M in such a way that the angle MM_2 with the surface is not infinitesimal. However, it is easy for this condition to be satisfied. When it is not realized, it suffices to consider a point M_1 that is situated on S and tends towards M along with M_2, in such a manner that the angle MM_2 makes with the surface is always greater than a definite limit. One will then compare the two integrals $\iint\frac{\cos\psi}{r^2}dS'$ and $\iint\frac{\cos\psi_2}{r_2^2}dS'$ to the analogous integral $\iint\frac{\cos\psi_1}{r_1^2}dS$ that relates to M_1; by virtue of what we just proved and what we previously established (no. **13**), we have upper bounds for the two differences thus obtained that are of the same form as the ones that we just indicated for the case in which MM_2 makes a finite angle with the surface. The conclusion is therefore completely general.

15. – We have a further remark to make concerning the normal derivative of the two-sheeted potential. The previously-imposed conditions on the function r do not permit us to conclude that this derivative has a limit on the surface or even that it is finite; however, the upper bound that we shall find for this derivative, which increases indefinitely as one approaches the surface, will suffice for the rest of the reasoning that follows.

We will obtain this upper limit upon remarking that the derivative of the expression $\frac{\cos\varphi}{r^2}$ in an arbitrary direction (where r is the distance from the variable point M to an arbitrary, but definite, point of the surface M', and φ is the angle between MM' and the normal at M') is less than $\frac{KA}{r^3}$. The derivative of the two-sheeted potential will therefore be less than $KA\iint\frac{dS'}{r^3}$ (where A is the maximum of $|\rho|$). Now, it is easy to insure that this expression is less than $\frac{KA}{\delta}$ (for example, by comparing it to the analogous integral that is taken upon replacing the surface with its tangent plane) if δ is the distance from the point M to the surface.

16. – Having said this, recall the function V_k. Suppose that we have proved that it tends to a constant L (as one is led to do in the Neumann method), with the difference being uniformly less than the k^{th} term of a geometric progression of ratio λ. We first seek to deduce a limit for $\rho_k = \left(\frac{dV_k}{dn}\right)$ from this assumption.

To that effect, if M is a point of the surface and M_2 is a point taken in the interior on the normal at M at a distance δ from the point M then we apply the inequalities that we just found in no. **14** and **14** (cont.) to the function ρ_k. If R_k denotes the maximum of $|\rho_k|$ on S then we first see that the difference of the two values of ρ_{k+1} corresponding to two points of S that are situated at a distance d from each other is less than $KR_k\,d^\alpha$ (α is less than 1, but also as small as one pleases). From this, it results that the value of $\frac{dV_{k+2}}{dn}$ at M_2 satisfies the inequality:

$$\text{(9)}\qquad \left|\left(\frac{dV_{k+2}}{dn}\right)_{M_2} - (\rho_{k+2} - \rho_{k+1})\right| < K(R_k + R_{k+1})\delta^\alpha.$$

Now, V_{k+2} may be put into the form of a two-sheeted potential. Indeed, we have seen that the two-sheeted weight V_k has $V_k + V_{k+1}$ for interior potential. Therefore, up to a constant (once one is given the convergence hypotheses for V), V_k will be a two-sheeted potential whose weight is:

$$W_k = V_k - V_{k+1} + V_{k+2} - V_{k+3} + \dots$$

and is, as a consequence, less than $C\lambda^k$ (C is a definite constant). The inequality that we just found for the normal derivative of the two-sheeted potential thus permits us to put the inequality (**9**) into the form:

$$|\rho_{k+2} - \rho_{k+1}| < K\left((R_k + R_{k+1})\delta^\alpha + \frac{C\lambda^k}{\delta}\right);$$

we set $\delta = \lambda^{\frac{1}{2}}$, $\lambda^{\frac{\alpha}{2}} = \lambda'$, and this becomes ([11]):

(**10**) $$|\rho_{k+2} - \rho_{k+1}| < (K(R_k + R_{k+1}) + C)\lambda'^k.$$

In particular, one will have:

$$R_{k+2} < R_{k+1} + \lambda'^k[K(R_k + R_{k+1}) + C].$$

One will obtain an upper bound for R_k by replacing the successive inequalities thus obtained from the corresponding inequalities. One will then have $R_{k+1} > R_k$, and, as a consequence:

$$R_{k+2} < R_{k+1} + \lambda'^k(2KR_{k+1} + C),$$

or:

$$R_{k+2} + \frac{C}{2K} < (1 + 2K\lambda'^k)(R_{k+1} + \frac{C}{2K}).$$

This inequality shows us that $R_k + \frac{C}{2K}$ converges like an infinite product, and, as a consequence, R_k is a finite quantity.

It results from the inequality (**10**) that the series $\sum(\rho_{k+1} - \rho_k)$ converges uniformly in the manner of a geometric progression. Therefore, at each point of the surface the function ρ_k tends to a limit ρ that (from the defining equations (**6**)) satisfies the equation:

$$\rho = \frac{1}{2\pi}\iint \rho' \frac{d\frac{1}{r}}{dn} dS'.$$

If the integral $\iint \rho_0 dS$ is different from zero then the function ρ is not identically zero (since one has $\iint \rho dS = \iint \rho_0 dS$). It is the distribution of an electric charge density in equilibrium (the surface S is that of a conductor that is isolated and free from all influence), and the method by which it was just obtained is due to Robin.

([11]) K always denotes any of various positive numbers which depend only on the nature of the surface S, and C will always denote any of various positive numbers that depend only on S and the form of the function ρ_0.

On the contrary, if $\iint \rho dS = \iint \rho_0 dS = 0$ then the function ρ is identically null. Indeed, this amounts to saying that an electric charge distribution that is in equilibrium with itself and has a null total quantity is in the neutral state throughout; the proof of this proposition is well known.

$|\rho_k|$ is therefore less than $C\lambda'^k$ (in which C denotes a constant). From no. **14**, it results that the difference of the values of ρ_k at two points M, M_1 of the surface is less than $C\lambda'^k \cdot \overline{MM_1}^{\alpha}$. By virtue of no. **14** (cont.), it then results that one has the inequality:

$$\textbf{(11)} \qquad \left| \left(\frac{dV_{ki}}{dn} \right)_{M_2} - (\rho_k - \rho_{k-1}) \right| < C\lambda'^k \cdot \overline{MM_2}^{\alpha},$$

and finally, from what we saw in no. **12**, we have the inequality:

$$\textbf{(12)} \qquad \left| \left(\frac{dv_k}{dn} \right)_{M_2} - (\rho_k - \rho_{k-2}) \right| < C\lambda'^k \cdot \overline{MM_2}^{\alpha}.$$

This latter inequality is true, moreover, whether the point M_2 is interior to the surface ($v_k = v_{ki}$) or exterior ($v_k = v_{ke}$). It proves that the conclusion that relates normal derivatives of the functions v_k extends to the Neumann series that has these functions for its terms, since the coefficient of $\overline{MM_1}^{\alpha}$ in the left-hand side is the general term of an absolutely convergent series; as a consequence, it gives us the solution to the problem.

If $\iint \rho_0 dS = \iint \rho dS \neq 0$ then we must occupy ourselves with the solution of the *exterior* hydrodynamic problem, and, as a consequence, we must consider that the series by which Neumann solved the Dirichlet problem is *interior*. Now, this derivative is alternating, in such a way that this time it will suffice to establish inequalities that are analogous to (**11**) and (**12**) upon replacing V_k, v_k, ρ_k, by $V_k - V_{k-1}$, $v_k - v_{k-1}$, $\rho_k - \rho_{k-1}$. Now, one will obtain parallel inequalities upon replacing ρ_k by $\rho_k - \rho_{k-1}$ in the reasoning that led up to formulas (**11**) and (**12**), which is less than $C\lambda'^k$ under either hypothesis (that $\iint \rho_0 dS$ is or is not null).

The conclusion that we demanded is thus established in any case. It lets us know the solution of the Neumann problem, with ρ_0 taken to be equal to the given values F of the normal derivative, and the function V_1, which is defined by equation (**7′**), is none other than the potential W of no. **11**, up to sign.

17. – If, instead of being given the values of a harmonic function V on the bounding surface S of a volume exactly, one is given only an upper bound for the modulus of V then one may, as we recalled above, assign upper bounds to the modulus of V and its derivatives at an arbitrary interior point.

Similarly, one may propose to find analogous inequalities when one is given not only an upper bound to $|V|$, but also an upper bound for $\left|\frac{dV}{dn}\right|$ on the surface. Here, the issue might not be that of obtaining an upper bound for $|V|$ at a given point, since V is determined only up to a constant. One may only assign a bound to the difference of the values of V at two given arbitrary interior points. The preceding method permits to arrive at this easily ([12]).

Indeed, let α be an upper bound for the modulus of:

$$F = \overline{\frac{dV}{dn}}: \qquad |F| < \alpha.$$

The potential W that was defined above:

$$-V_1 = W = \iint \frac{F}{2\pi r} dS ,$$

will be such that one will have:

$$|W| < K\alpha,$$

in which K is a constant that depends only upon the form of the surface. From this, if $[W]$ represents the oscillation that is possible for W, one deduces that:

$$[W] < 2\,K\alpha.$$

As a result, in the Neumann method, one determines:

V_2,	a two-sheeted potential of weight	$\frac{W}{2\pi}$,
V_3,	" " "	$\frac{V_2}{2\pi}$,
	…	
V_i,	" " "	$\frac{V_{i-1}}{2\pi}$,

and, from Neumann, one has, with λ being a positive constant that is less than one:

$$[V_2] < \lambda\,[W],$$
$$[V_3] < \lambda\,[V_2],$$
$$[V_i] < \lambda\,[V_{i-1}].$$

From this, one deduces:

$$[V_{i+1}] < 2\,\lambda^i K\alpha$$

([12]) POINCARÉ. – *Sur les équations de la Physique Matématique, Rendic. del Circolo matematico di Palermo,* tome 8, pp. 114-115; 1894.

$$[V] = [W + \Sigma V_i]\ [\,[W] + \Sigma[V_i] < \frac{1}{1-\lambda} 2\, K\alpha,$$

or

$$[V] < h\alpha,$$

in which h is a definite constant.

18. – Direct search for inequalities. – Up to a certain point, one may obtain directly analogous conclusions without passing to the Neumann method by employing some of the means by which Poincaré ([13]) established the legitimacy of that method, and first making use of the inequality of Schwartz.

One knows that this inequality results from considering the integral:

$$H = S\,[(A_1 + \lambda B_1)^2 + (A_2 + \lambda B_2)^2 + \ldots + (A_p + \lambda B_p)^2\,]\, d\sigma,$$

in which S is the symbol for a simple or multiple integration taken over the multiplicity σ, and $d\sigma$ is the differential element of σ; A_1, A_2, …, B_1, B_2, … are well-defined functions, and λ is an arbitrary constant. The integral can be written:

(**13**) $$H = I + 2\,\lambda K + \lambda^2 J,$$

upon setting:

$$\begin{aligned} I &= S(A_1^\alpha + A_2^\alpha + \cdots)d\sigma, \\ K &= S(A_1 B_1 + A_2 B_2 + \cdots)d\sigma, \\ J &= S(B_1^\alpha + B_2^\alpha + \cdots)d\sigma. \end{aligned}$$

Since the quadratic form (**13**) is positive for any λ one must have:

(**14**) $$\Delta = IJ - K^2 > 0,$$

and this is what constitutes the Schwartz inequality.

The minimum of H as λ varies is attained when $\lambda = -K/J$ and has the value:

$$H = \frac{\Delta}{J}.$$

Finally, the inequality (**14**) gives an expression for Δ itself in the form of a multiple integral:

$$\Delta = \frac{1}{2} SS\left[\sum_{i,j} (A_i B'_j - A'_j B_i)^2\right] d\sigma d\sigma',$$

in which the elements $d\sigma$, $d\sigma'$ describe the multiplicity σ independently of each other, and A', B' are the values of the functions A, B at a point of $d\sigma'$.

([13]) *Acta Math.*, loc. cit.

We apply the preceding inequality to the case in which σ is a closed surface S. Let V be a harmonic function in the interior of S. First consider the surface integral:

$$H = \iint (V + \lambda)^2 \, dS.$$

It results from the foregoing that the minimum of that integral is Δ/S if we let Δ denote the quadruple integral:

$$\Delta = \frac{1}{2} \int \iiint (V - V')^2 \, dS dS'.$$

In the second, consider the volume integral:

$$G' = \iiint_D \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] d\tau,$$

(in which $d\tau$ is the volume element, $dx\ dy\ dz$), taken over the domain D that is bounded by S.

Since V is harmonic, one has:

$$G = -\iint (V + \lambda) \frac{dV}{dn} \, dS,$$

no matter what the constant λ, and, as a consequence, by virtue of the inequality (**14**):

$$G^2 < HJ,$$

where:

$$J = \iint \left(\frac{dV}{dn} \right)^2 dS.$$

Upon giving λ the value that corresponds to the minimum of H, it becomes:

$$\frac{G^2}{\Delta} < \frac{J}{S}, \tag{15}$$

in which S denotes the total extent of the given surface.

If one is given the values of dV/dn on that surface then *the right-hand side of the preceding inequality is known.*

19. – However, Poincaré has established a second inequality between the two quantities G and Δ: he has shown that *for any function V* (harmonic or not) *one may assign an upper bound to the ratio Δ/G that depends only on the form of the surface.*

First suppose that it is convex and that there exists an upper bound ρ to the ratio l / cos(l, n), in which l is the length of a chord of this surface and (l, n) is the angle that this chord makes with the normal to one of its extremities.

In addition, let L be the maximum of the length l. In the quadruple integral Δ one may express dS' by means of the spherical angle:

$$d\varpi = \frac{dS' \cos(l, n)}{l^2},$$

in which one sees dS' from a point of dS. By hypothesis, one has:

$$\Delta < \frac{\rho^2}{2} \int\iiint (V - V') \cos(l, n) dS d\varpi.$$

I will exhibit the successive integrations upon supposing that one first performs them relative to the element dS of S, and then relative to the element $d\varpi$ of the sphere S on which one successively makes the different spherical representations of center dS. In addition, if we replace $(V - V')$ by a linear integral that is taken along l then one has:

$$\Delta < \frac{\rho^2}{2} \iint_\Sigma d\varpi \iint \cos(l, n) dS \left[\int_0^l \frac{dV}{dl} dl \right]^2.$$

However, from the Schwarz inequality, one may write:

$$\left(\int_0^l \frac{dV}{dl} dl \right)^2 < l \int_0^l \left(\frac{dV}{dl} \right)^2 dl < L \int_0^l \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] dl.$$

It therefore results that:

$$\Delta < \frac{\rho^2 L}{2} \iint_\Sigma d\varpi \iint_S \cos(l, n) dS \int_0^l \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] dl\,,$$

or, since cos(l, n) dS is the projection of dS on a plane that is perpendicular to the direction of l:

$$\Delta < \frac{\rho^2 L}{2} \iint_\Sigma d\varpi \iiint \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] d\tau.$$

The triple integral is none other than G (for any imaginable element $d\varpi$).

We therefore have:

(16) $$\frac{\Delta}{G} < \lambda,$$

$$\lambda = \frac{\rho^2 L}{2} \iint_\Sigma d\varpi = 2\pi\rho^2 L.$$

20. – We seek to pass from a domain bounded by a surface that satisfies the conditions of the preceding section to a domain that is bounded by another arbitrary surface. In general, if the connection remains the same, i.e., if we are always dealing with a unique simply connected surface, then we may establish a correspondence between the points (x, y, z) and (x', y', z') of the two domains such that the coordinates of the second are continuous with respect to the first, their first-order derivatives are finite and continuous on the boundary surface, and the modulus of the functional determinant:

$$\frac{D(x', y', z')}{D(x, y, z)}$$

remains constantly greater than a given positive quantity.

I say that under these conditions, the ratio:

$$\frac{\Delta G'}{\Delta' G},$$

(Δ' and G' are quantities that are analogous to Δ, G) has a well-defined positive maximum. This will obviously suffice for one to infer a further inequality from the inequality of the preceding section:

$$\frac{\Delta'}{G'} < \lambda'.$$

For the proof, we set:

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \varphi\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$dx'^2 + dy'^2 + dz'^2 = f(dx, dy, dz).$$

The forms f and φ have the square of the functional determinant for their discriminants. These two forms represent ellipsoids whose axes fall between well-defined limits. One therefore knows a lower bound for the quantity:

$$\frac{\varphi}{\left(\frac{\partial V}{\partial x'}\right)^2 + \left(\frac{\partial V}{\partial y'}\right)^2 + \left(\frac{\partial V}{\partial z'}\right)^2} = \frac{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2}{\left(\frac{\partial V}{\partial x'}\right)^2 + \left(\frac{\partial V}{\partial y'}\right)^2 + \left(\frac{\partial V}{\partial z'}\right)^2},$$

The ratio $\frac{dS'}{dS}$ also has a minimum. Therefore, the same is true for $\frac{G'}{G}$. On the other hand, $\frac{dS'}{dS}$ has a given minimum with respect to the sections of the ellipsoid $f = 1$, and the sphere of radius 1. One therefore has precisely a minimum for $\frac{\Delta}{\Delta'}$.

21. – One may proceed in a different fashion and generalize the method of proof in the preceding section by using curvilinear chords with 4 parameters instead of rectilinear chords, such that there is always one and only one of them that passes between any two points of the surface; the length l and ratio $\frac{l}{\cos(l,n)}$ will have upper bounds L and ρ. Finally, one may distribute them in families that depend on each of the two parameters a, b, and are such, that if s denotes the arc described on an arbitrary chord of the family then the modulus of:

$$\frac{D(x,y,z)}{D(a,b,s)},$$

remains greater than a well-defined number. Under these conditions, the reasoning of no. **19** remains valid.

22. – Now combine the inequality (**16**) with the previously obtained inequality (**15**) (no. **18**); it becomes:

$$G < \frac{\lambda J}{S},$$

$$\Delta < \frac{\lambda^2 J}{S}.$$

These are the inequalities that we had in mind.

One may remark that these inequalities give an upper bound to the integral:

$$I = \iiint \left(\frac{\partial \Phi}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial \Phi}{\partial z}\frac{\partial V}{\partial z} \right) d\tau,$$

(in which one designs a given arbitrary function by Φ), since one has – as always, from the Schwarz inequality:

$$I^2 < G \iiint \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] d\tau$$

$$< \lambda \frac{J}{S} \iiint \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] d\tau .$$

23. – The inequalities that we arrive at with the aid of the Neumann method play the same role relative to the second boundary-value problem as the propositions that we recalled in no. **1** play relative to the Dirichlet problem; however, they are a long way from providing limits for the oscillation of the desired function and the moduli of its derivatives that are as precise as in the latter. For example, in the Neumann method they may constitute only *alternate* methods that are similar to the ones that we used in the solution of the Dirichlet problem. We know that there exists a number that the ratio of the oscillation of V to the maximum modulus of dV / dn remains less than; however, the existence itself of this number is the only thing we know in that regard.

This absence of precise givens on the coefficient that we just spoke of is one of the principal lacunae in our knowledge of the second boundary-value problem.

§ 5. – THE FUNCTIONS OF Fr. NEUMANN AND KLEIN.

24. – The considerations that were discussed in § 4 proved the existence of a solution, but they did not provide any simple expression for it.

We shall confirm that one may write such expressions if one knows how to construct functions that are analogous to the Green functions for the second boundary-value problem, since one knows the essential role they play in the theory of the Dirichlet problem.

Like the Green functions, the functions γ_A^M that we consider will be harmonic in the given domain, except for a point A, which is chosen arbitrarily, and at which they become infinite (in the three-dimensional case) like the inverse of the quantity $r = AM$.

Such functions will not satisfy equation (**1**) of no. **1** when one is concerned with an interior point. This equation is true only if one adds the surface of a small sphere Σ that has A for its center to the given boundary surface S. Now, when the radius of this sphere grows infinitely small the integral $\iint_\Sigma \frac{d\gamma_A}{dn} d\Sigma$ goes to 4π, since γ_A then reduces to its principal term $1/r$. On S, equation (**1**) of no. **4** must therefore be replaced by the following formula:

(**17**) $$\iint_\Sigma \frac{d\gamma_A}{dn} d\Sigma = 4\pi.$$

1. *Franz Neumann's function. – Exterior problem.* Let γ_A^M be a function of the coordinates of the point M that is harmonic in the given normal domain except for the point A, has a null normal derivative on the surface, and becomes infinite at A like $1/r$, i.e., such that $\gamma_A - 1/r$ remains harmonic at A. The determination of such a function is obviously a particular case of the second boundary-value problem. However, it will

suffice to solve this same problem in the general case. If V is a harmonic function that satisfies the conditions of the problem – in other words, such that $dV/dn = F$ on S – it will suffice to consider V and γ_A^M as in the theory of Green functions in order to obtain:

$$V_A = -\frac{1}{4\pi}\iint \gamma_A^M F dS. \tag{E}$$

Interior problem. – By virtue of equation (**17**), one may no longer take:

$$\frac{d\gamma_A^M}{dn} = 0,$$

on the surface; we shall only assume that this derivative is constant. Evidently, one will therefore have:

$$\frac{d\gamma_A^M}{dn} = \frac{4\pi}{S}. \tag{17′}$$

Upon once more applying Green's theorem to the functions V and γ_A^M, this time we get:

$$V_A = -\frac{1}{4\pi}\iint \gamma_A^M F dS - \frac{1}{S}\iint V dS.$$

However, we seek V_A only up to an additive constant, i.e., we seek the difference $V_A - V_{A'}$, in which A' is an arbitrary point of origin. We may therefore neglect the complementary term, which is the same for all points.

25. – *2nd Klein function.* – In the case of the interior problem, the function γ_A^M of Fr. Neumann is itself defined only up to an additive constant. This fact constitutes an inconvenience from several standpoints. This is why, under these conditions, it does not seem possible to establish a symmetry property for the function γ_A^M that is analogous to the well-known relation:

$$g_A^M = g_M^A \tag{18}$$

that applies to the case of the classical Green function.

In order to obviate this inconvenience, Klein ([14]) was led to form a function that possesses not just one, but two, poles of opposite signs. Such a function satisfies equation (**1**) of no. **4**. Nothing prevents its normal derivative on the surface from being everywhere null. The Klein function is therefore defined by this condition and by the following ones:

1. That it be harmonic in the given domain, except for two points A and A_0.

([14]) In Pockels: *Über die partielle Differentialgleichung* $\Delta u + K^2 u = 0$ *und deren Auftreten in der Mathematischen Physik;* Leipzig, 1891.

2. That it be such, that its difference with $1/r$ in the neighborhood of A remains harmonic.

3. That its difference with $-\frac{1}{r} = -\frac{1}{MA_0}$ remains harmonic in a neighborhood of A_0;

4. Finally, that it be annulled at a given point M_0.

The value of such a function at a point M of the domain may be written:

$$\Gamma_{AA_0}^{MM_0}.$$

In reality, this is the increase that any function that satisfies all of the same conditions – except the last one – experiences upon going from M_0 to M.

The function $\Gamma_{AA_0}^{MM_0}$, like the function of Fr. Neumann, permits us to solve the problem that we posed, since one has, from a theorem of Green:

$$\frac{1}{4\pi}\iint \Gamma_{AA_0}^{MM_0} F_M \, dS_M = V_A - V_{A_0}.$$

The arbitrary constant that gets added to the value of V_A is $-V_{A_0}$ here.

On the other hand, since $\Gamma_{A_1A_2}^{M_1M_2}$ represents the increase in a function between the points A_2 and A_1 one may write:

$$\Gamma_{A_1A_2}^{M_1M_3} = \Gamma_{A_1A_2}^{M_1M_2} + \Gamma_{A_1A_2}^{M_2M_3},$$

and, in particular:

$$\Gamma_{A_1A_2}^{M_1M_2} = -\Gamma_{A_1A_2}^{M_2M_1}.$$

One may perform the same operations on the lower indices:

$$\Gamma_{A_1A_3}^{M_1M_2} = \Gamma_{A_1A_2}^{M_1M_2} + \Gamma_{A_2A_3}^{M_1M_2},$$

as one sees immediately upon referring to the definition of the symbol Γ. In particular:

$$\Gamma_{A_1A_2}^{M_1M_2} = -\Gamma_{A_2A_1}^{M_1M_2}.$$

I say that one has, in addition, that:

(19) $$\Gamma_{A_1A_2}^{M_1M_2} = \Gamma_{M_1M_2}^{A_1A_2},$$

i.e., the property that is analogous to the corresponding property of Green functions.

Indeed, if we set, to abbreviate:

$$\Gamma_{(M)} = \Gamma_{A_1A_2}^{M_1M_2}$$

$$\Gamma'_{(M)} = \Gamma_{M_1M_2}^{A_1A_2},$$

then one has, from Green's theorem:

$$\iint_S \left(\Gamma \frac{d\Gamma'}{dn} - \Gamma' \frac{d\Gamma}{dn}\right) dS = 4\pi\left(\Gamma_{A_1A_2}^{M_1M_2} - \Gamma_{M_1M_2}^{A_1A_2}\right).$$

However, the left-hand side is null, so the inequality is therefore proved.

26. – As we shall see, it is not at all necessary to abandon the function of Fr. Neumann in order to preserve the property of symmetry. It suffices to complete its definition.

Indeed, the function γ_A^M is defined only up to an additive constant that does not depend upon the point M but is a function of the point A. We shall confirm that it suffices to conveniently determine the additive quantity in question for the Neumann function to possess the desired property. First of all, one may easily pass from the Neumann function to the Klein function.

Indeed, one has:

$$\Gamma_{A_1A_2}^{M_1M_2} = \gamma_{A_1}^{M_1} - \gamma_{A_2}^{M_1} - \gamma_{A_1}^{M_2} + \gamma_{A_2}^{M_{21}},$$

because the right-hand side possesses the properties that characterize the symbol:

$$\Gamma_{A_1A_2}^{M_1M_2}.$$

It results from this that the Neumann symbol has a property that is completely analogous to the one expresses by equations (**18**), (**19**), provided that one conveniently determines the arbitrary parameter – a function of the point A – that figures in the latter. Indeed, from the preceding relation, equation (**19**) may be written:

$$\gamma_{A_1}^{M_1} - \gamma_{A_2}^{M_1} - \gamma_{A_1}^{M_2} + \gamma_{A_2}^{M_{21}} = \gamma_{M_1}^{A_1} - \gamma_{M_2}^{A_1} - \gamma_{M_1}^{A_2} + \gamma_{M_2}^{A_{21}},$$

from which, upon setting:

(**20**) $$\gamma_A^M - \gamma_M^A = \varphi(A, M),$$

we obtain:

$$\varphi(A_1, M_1) - \varphi(A_1, M_2) - \varphi(A_2, M_1) + \varphi(A_2, M_2) = 0,$$

a functional relationship that the function φ must satisfy, and whose general solution is, as one easily verifies:

$$\varphi(A, M) = \psi(A) - \psi_1(M).$$

By virtue of equation (**20**), it is clear, moreover, that the functions ψ and ψ_1 must be identical.

From this, one may choose the functions γ such that the function φ is always null; it suffices to replace γ_A^M with $\gamma_A'^M = \gamma_A^M - \psi(A)$.

Furthermore, the function $\psi(A)$ may be determined directly in such a manner as to nullify φ. Indeed, I say that in order to do this it suffices to calculate the constant $\psi(A)$ at each point A by the condition:

$$\iint_S \gamma_A'^M dS_M = \iint_S [\gamma_A^M - \psi(A)] dS_M$$
$$= \iint_S \gamma_A^M dS_M - S\psi(A) = K,$$

in which K is a constant that is chosen once and for all, but arbitrarily (for, example, one may take $K = 0$).

Indeed, one will then have:

$$\iint_S \left(\gamma_A'^M \frac{d\gamma_B'^M}{dn} - \gamma_B'^M \frac{d\gamma_A'^M}{dn} \right) dS_M = 0,$$

and, as a consequence, by applying Green's theorem to the functions $\gamma_A'^M$ and $\gamma_B'^M$ one obtains the desired relation:

(**21**) $$\gamma_A'^B = \gamma_B'^A.$$

Conversely, if this relation is true for all pairs of points (A, B) then one has:

(**22**) $$\iint \gamma_A'^M dS_M = K,$$

in which K is a constant that is independent of A.

27. – One knows that the use of the Green function permits us to solve not only the Dirichlet problem, but also the generalized problem that relates to the equation:

(**2**) $$\Delta V = f(x, y, z),$$

in which f is a given function, at every point of the domain envisioned D.

Parallel remarks apply to the present question. Suppose we are given, on the one hand, equation (**2**), and, on the other, the condition on the surface:

(**23**) $$\frac{\overline{dV}}{dn} = F.$$

Condition (**1**) of no. **4** for the interior problem is then replaced by:

(**24**) $$\iiint_D f(x, y, z)\, d\tau + \iint_S F\, dS = 0.$$

By applying Green's theorem to the function V and the Neumann function γ_A^M this becomes:

(25)
$$\begin{cases} 4\pi V_A + \iiint_D \gamma_A^M \, f d\tau + \iint_S \gamma_A^M \, F dS \\ \qquad - C\iint V dS = 0, \end{cases}$$

in which C is zero for the exterior problem and (from equation (**17′**)) the constant $\frac{4\pi}{S}$ for the interior problem. In the two cases, the term $C\iint VdS$ may be ignored, since it is either null or constant.

§ 6. – CASE OF THE SPHERE ([15])

28. – Now that we have established the existence of the solution of the second boundary-value problem and shown how its expression reduces to the search for the function of Fr. Neumann, it remains for us to indicate the cases in which this solution may be obtained effectively, for example, ones in which the Neumann function is known. However, these cases are very few in number.

A general method by which one may propose to express the desired solution consists of representing it by a series of the form:

$$A_0 \Phi_0 + A_1\Phi_1 + \ldots + A_m \Phi_m + \ldots$$

in which the Φ's are well-defined harmonic functions and the *A*'s are arbitrary coefficients. It is clear that one will obtain such a representation if one may put the given values of the normal derivative into the form:

$$A_0 \frac{d\Phi_0}{dn} + A_1 \frac{d\Phi_1}{dn} + \cdots + A_m \frac{d\Phi_m}{dn} + \cdots$$

This is what one often encounters upon taking the Φ's to be the *fundamental functions* that were introduced by Poincaré, Le Roy, Stekloff, etc.

28. – This method immediately leads to the solution in the case of the sphere. One knows that any given function on that surface may be developed into a series of spherical functions, provided that it is continuous and satisfies the Dirichlet conditions on any great circle.

Having said this, first suppose that that we are dealing with the interior problem. We must find the function V, when we know dV/dn, that satisfies the relation:

([15]) Dini, *loc. cit.*

(**1**) $$\iint \frac{dV}{dn} dS = 0 .$$

Suppose that we are given V in the form of a series of spherical polynomials:

$$V = Y_0 + \rho\, Y_1 + \rho^2 Y_2 + \dots$$

From this, one deduces that:

(**26**) $$\frac{dV}{dn} = -(Y_1 + 2RY_2 + \cdots + mR^{m-1}Y_m + \cdots) .$$

It therefore suffices to develop dV/dn in a series of Laplace functions:

$$\Phi_1 + \Phi_2 + \dots + \Phi_m + \dots$$

There must not be a Y_0 term; this is precisely the possibility condition (**1**) for the problem. We will only have to suppose:

$$Y_m = -\frac{\Phi_m}{mR^{m-1}} .$$

If the function dV/dn does not satisfy the Dirichlet conditions, so it may not be developed in a series of spherical functions, then, provided that it is continuous or finite with isolated discontinuities, one may ([16]) develop it into a series whose terms will each be a *sum* of spherical functions (of different degrees, in general), and work with this series we did with the series (**25**) ([17]).

Similarly, for the exterior problem one will have:

$$V = \frac{Y_0}{\rho} + \frac{Y_1}{\rho^2} + \cdots + \frac{Y_m}{\rho^{m+1}} ,$$

and one determines the Y's in a manner that is analogous to the foregoing.

29 (cont.). – **Case of two concentric spheres.** – We apply the same method to the space between two concentric spheres of radii R_1 and R_2 . Here one sets ([18]):

$$V = Y_0 + Y_1\rho + \cdots + Y_m\rho^m + \cdots$$

([16]) Picard, *Traité d'Analyse*, t. I, pp. 248 et. seq.

([17]) Cf. Le Roy, Thesis *Sur l'intégration des équations de la chaleur*, pp. 28-30.

([18]) In the case for which the given values of the normal derivative do not satisfy the Dirichlet conditions one applies the same modification as in the preceding section.

$$+\frac{Y_0'}{\rho}+\frac{Y_1'}{\rho^2}+\cdots+\frac{Y_m'}{\rho^{m+1}}+\cdots$$

One simultaneously determines Y_m and Y_m' by two equations:

$$mR_1^{m-1}Y_m-\frac{(m+1)Y_m'}{R_1^{m+2}}=\Phi_m,$$
$$-mR_2^{m-1}Y_m+\frac{(m+1)Y_m'}{R_2^{m+2}}=\Phi_m',$$

whose determinant is different from zero for $m > 0$. For $m = 0$, the two equations will also be compatible because of the possibility condition (**1**).

30. – In the two problems that we just treated, the solution may be obtained without recourse to a series. We shall see that in either the case of a sphere or that of two concentric spheres one may reduce the Neumann problem to the Dirichlet problem and quadratures.

Case of one sphere. – Let D be the interior (or exterior) domain bounded by the sphere S. Let D' be the domain bounded by the sphere S' that is obtained by adding (or subtracting) dR from the radius R of S.

If V is the desired function then by a simple homothety one may deduce a function V' that is harmonic in the domain D and takes the same values at the various points of S' that V takes at the corresponding points of S. The difference $V-V'$ will be harmonic in D. However, it takes the value $-\frac{dV}{dn}dR$ on S. Therefore, if one sets:

$$V_1=\frac{V-V'}{dR}$$

then one will have calculated V_1 since one knows the solution to the first boundary-value problem.

In other words, and more rigorously: If $V(x, y, z)$ is harmonic then the same is true for $V(\lambda x, \lambda y, \lambda z)$, in which λ is an arbitrary constant. Therefore, the same is also true for $\frac{\partial}{\partial\lambda}V(\lambda x,\lambda y,\lambda z)$, i.e., for $\lambda = 1$, since:

$$V_1=x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\delta\frac{\partial V}{\partial\delta}, \tag{27}$$

in which δ is the distance to the center.

In fact, one directly sees that one has:

$$\Delta\left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}\right)=\left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}+2\right)\Delta V. \tag{28}$$

If V_1 is calculated then one will deduce the V of equation (**27**) by the following rectilinear integral, which is taken over the radius OM:

(**29**) $$V = \int_0^{OM} \frac{V_1}{\delta} d\delta + const. \qquad \text{(interior problem)}$$

(**30**) $$V = -\int_{OM}^{\infty} \frac{V_1}{\delta} d\delta + const. \qquad \text{(exterior problem).}$$

The integral means something in the case of the interior problem because the condition:

$$V_1 = 0$$

at the center of the sphere is, from Gauss's theorem, nothing but the possibility condition for the problem:

(**1**) $$\iint_S \overline{\frac{dV}{dn}} dS = 0.$$

Likewise, the integral means something in the case of the exterior problem because V_1 has been calculated as a function that is regular at infinity.

If V_1 is a function that is regular at the origin and is annulled there then the same will be true for:

$$V = \int_0^{OM} \frac{V_1}{\delta} d\delta,$$

as one sees by replacing V_1 with its development into a Maclaurin series.

Moreover, if V_1 is harmonic then V will be harmonic, as well; now, by virtue of equation (**28**), since ΔV_1 is null, if ΔV is not identically null then it must be a homogeneous function of degree –2, which is absurd, since ΔV is regular at the origin. Furthermore, one verifies the same fact from the expression for V in the form of a definite integral. This latter manner of operation is further applied to expression (**30**), for which the former line of reasoning will fail.

On the sphere of radius R the function V_1 obviously takes the same values as the quantity $-R\dfrac{dV}{dn}$ does in the case of the interior problem and $+R\dfrac{dV}{dn}$ does in the case of the exterior problem. One may thus obtain the expression by solving the Dirichlet problem. From known formulas for the solution of this problem on the sphere, one will have:

$$V_1 = \mp \frac{R}{2\pi} \iint_S \frac{dV}{dn} \frac{d\frac{1}{r}}{dn} dS + \frac{1}{4\pi} \iint_S \frac{1}{r} \overline{\frac{dV}{dn}} dS$$

(the – sign in the first integral refers to the interior problem, the + sign in the other refers to the exterior problem). From this, one deduces – for the interior problem, for example – that:

$$\textbf{(31)}\qquad \begin{cases} V = -\dfrac{R}{2\pi}\displaystyle\iint_S \dfrac{dV}{dn}\,dS\int_0^\rho \dfrac{d\frac{1}{r}}{dn}\,\dfrac{d\delta}{\delta} \\ \quad + \dfrac{1}{4\pi}\displaystyle\iint_S \dfrac{dV}{dn}\,dS\int_0^\rho \dfrac{1}{r}\,\dfrac{d\delta}{\delta}. \end{cases}$$

The simple quadratures are performed without difficulty. Moreover, one is led to the same quadratures upon seeking the Neumann function, as we shall do.

31. – **Neumann function.** – We look for the Neumann function for the interior problem in the case of the sphere. This amounts to a function γ_A^M that is harmonic on the sphere, except for a neighborhood of *A*, and such that $d\gamma / dn$ is constant along the boundary surface.

Set:

$$\textbf{(32)}\qquad \gamma_A^M = \frac{1}{r} + H,$$

in which *H* is everywhere harmonic and its normal derivative must satisfy the equation:

$$\textbf{(33)}\qquad \frac{dH}{dn} = K - \frac{d\frac{1}{r}}{dn}.$$

We may apply the method of solution that we just indicated to this particular case of the preceding problem, and direct the calculations in such a manner as to deduce the Neumann function from the known expression for the Green function.

Let ρ be the distance *OM*, δ, the distance *OA*, and γ, the angle these two lines make with each other. The application of the preceding method first leads us to look for a harmonic function $H_1 = \rho\dfrac{\partial H}{\partial\rho}$ that takes the same values, up to a factor of $-R$, as the left-hand side of equation (**33**), i.e., the quantity:

$$K + \frac{\partial\frac{1}{r}}{\partial\rho}.$$

Now, the function $\dfrac{1}{r}$ is homogeneous of degree –1 with respect to ρ and δ, and one has, as a consequence:

$$\textbf{(34)}\qquad \rho\frac{\partial\left(\frac{1}{r}\right)}{\partial\rho} + \delta\frac{\partial\left(\frac{1}{r}\right)}{\partial\delta} = -\frac{1}{r},$$

from which, for $\rho = R$:

(**35**) $$H_1 = -R\frac{dH}{dn} = -KR + \delta\frac{\partial\left(\frac{1}{r}\right)}{\partial\delta} + \frac{1}{r} .$$

Now let:

$$g_A^M = \frac{1}{r} - h_A^M$$

be the Green function for the interior Dirichlet problem on our sphere. When the point *M* is on its surface, one has:

$$h = \frac{1}{r},$$

and this inequality, which is true for any point *A*, may be differentiated with respect to δ. One thus obtains:

(**36**) $$H_1 = -KR + h + \delta\frac{\partial h}{\partial\delta} .$$

This equation is true on the surface and, as a consequence, in the entire volume, since both sides of the equation represent harmonic functions. As for *h*, one knows that it is obtained by considering the point A', which is the image of *A* (*fig.* 3) situated on the ray *OA* at a distance from the center of $OA' = \delta' = \frac{R^2}{\delta}$. Upon denoting the distance MA' by r', one obtains:

(**37**) $$h = \frac{R}{\delta r'} .$$

This quantity *h* is homogenous and of degree 0 as a function of ρ, γ, and δ' (since $\frac{1}{r'}$ is of degree –1, and $\frac{1}{\delta}$ is of degree +1), in such a way that one has:

(**38**) $$\frac{\delta'\partial h}{\partial\delta'} = -\delta\frac{\partial h}{\partial\delta} = -\rho\frac{\partial h}{\partial\rho} .$$

Since the function H_1 is determined by formula (**36**), all that remains for us to do is to calculate *H* by the equation:

$$H_1 = \rho\frac{\partial H}{\partial\rho} .$$

If one takes the identity (**38**) into account then one sees that one will have:

(**39**) $$H = h + \int_0^\rho \left(\frac{R}{\delta r'} - KR\right)\frac{d\rho}{\rho} .$$

As we shall see, the constant K is chosen in such a way that the integral makes sense, i.e., in such a way as to nullify the coefficient of $\frac{d\rho}{\upsilon}$; we must therefore take $K = \frac{1}{R^2}$.

One will perform the quadrature simply by introducing the angle $OA'M = \psi$, which gives:

$$\rho = \frac{\delta' \sin\psi}{\sin(\gamma+\psi)}, \qquad r' = \frac{\delta' \sin\gamma}{\sin(\gamma+\psi)},$$

and, as a consequence,

$$(\mathbf{40}) \qquad \left\{ \begin{aligned} &\int_0^\rho \left(\frac{R}{\delta r'} - \frac{1}{R} \right) \frac{d\rho}{\rho} \\ &= \int_0^\psi \frac{1}{R} \left[\frac{\sin(\gamma+\psi)}{\sin\gamma} [(\cot\psi - \cot(\gamma+\psi)] d\psi - \frac{d\rho}{\rho} \right] \\ &= \int_0^\psi \frac{1}{R} \left(\frac{d\psi}{\sin\psi} - \frac{d\rho}{\rho} \right) = \frac{1}{R} \log \frac{2R^2 \tan\frac{\psi}{2}}{\rho\delta \sin\gamma}. \end{aligned} \right.$$

Formulas (**32**), (**39**), (**40**) give us:

$$(\mathbf{41}) \qquad \gamma_A^M = \frac{1}{r} + \frac{R}{\delta r'} + \frac{1}{R} \log \frac{2R^2 \tan\frac{\psi}{2}}{\rho\delta \sin\gamma}.$$

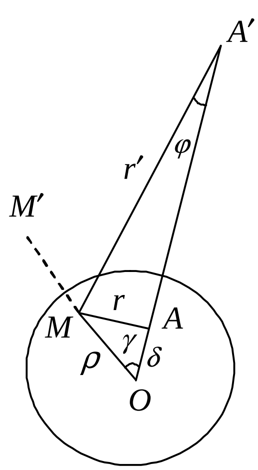

Fig. 3

Conforming to our general conclusions, the quantity thus obtained is symmetric with respect to the two points A, M upon which it depends. Indeed, as is well known, this property comes from the second term $R / \delta r'$ of the preceding formula. It likewise belongs to the angle ψ that figures in the third term, because, if one calls M' the image of the point M then the two triangles $OA'M$, OAM' are, as one knows, similar to each other and to the triangle that has an angle equal to γ between its two sides of length R^2 and $\rho\delta$, respectively.

Upon accounting for the trigonometric relations that the triangle $OA'M$ provides, one may easily write the last term in the form:

$$\frac{1}{R} \log \frac{2\delta'}{r' + \delta' - \rho\cos\gamma} = \frac{1}{R} \log \frac{2 \cdot \overline{OM'}}{\overline{AM'} + \overline{OM'} - \delta\cos\gamma}$$

(upon permuting the points A and M).

When the point M is on the sphere the value of γ_M^A reduces to:

$$\frac{2}{r}-\frac{1}{R}\log\frac{r+R-\delta\cos\gamma}{2R},$$

a value that was obtained (up to a constant) by Fr. Neumann ([19]) by summing a series of spherical functions, and it will suffice to substitute this in formula (E) of no. **24**.

Finally, if one is concerned with the exterior problem, not the interior problem, then the equalities (**35**) and (**36**) persist upon taking $K = 0$. The formula (**39**) must be replaced by:

$$H = h - \int_\rho^\infty \frac{R}{\delta r'}\frac{d\rho}{\rho},$$

and one finally has:

$$\textbf{(41}'\textbf{)} \qquad \gamma_A^M = \frac{1}{r} + \frac{R}{\delta r'} + \frac{1}{R}\log\left(\tan\frac{\psi}{2}\tan\frac{\gamma}{2}\right).$$

32. – We again attempt to apply the same method to the case of *two concentric spheres*. Let φ_1 be the given values of the normal derivative on the interior sphere S_1 (of radius R_1); φ_2 are the corresponding values for the exterior sphere S_2 of radius R_2. Of course, one supposes that possibility condition:

$$\textbf{(42)} \qquad \iint_{S_1} \varphi_1 dS + \iint_{S_2} \varphi_2 dS = 0.$$

Furthermore, consider the auxiliary function:

$$V_1 = x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \rho\frac{\partial V}{\partial \rho}.$$

This function will take the values $R_1\varphi_1$ on the interior sphere and $-R_2\varphi_2$ on the exterior sphere. The solution of the Dirichlet problem will thus be known. If one is then given the values V_0 of V on a concentric sphere S_0 that is intermediate to or coincident with one of the boundary spheres then one will have:

$$V = V_0 + \int_{M_0}^M V_1 \frac{d\rho}{\rho},$$

in which the points M_0 and M are taken on the same ray.

([19]) *Vorles. über die Theorie des Potentials und der Kugelfunctionen*, pp. 275. As for the interior and exterior values of the function γ_A^M, they are due to Bjerknes (*loc. cit.*, 1871) and Beltrami (*loc. cit.*, tome III, pp. 370-371; 1873.)

Obviously one may not choose the values V_0 and V on the sphere S_0 in an arbitrary manner. These values will be determined by the condition that one have:

$$\Delta V = 0,$$

in the given space.

Now, from the identity (**28**) it already results that the condition $\Delta V_1 = 0$ forces ΔV to be a homogenous function of degree –2.

It will therefore be necessary and sufficient to assure that ΔV is null on S_0.

Let ρ, θ, φ be polar coordinates. One has:

(**43**)
$$\Delta V = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial V}{\partial\rho}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}\right]$$

and here:

$$\Delta V = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}(\rho V_1)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}\right] = 0..$$

The first term is therefore known, as a consequence, and one knows the sum of the other two terms.

33. – The sum of these other terms is nothing but the *second-order differential parameter* for the sphere.

In a general manner, let:

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

be the area element of a surface.

One knows that one gives the name of *first-order differential parameter* of an arbitrary function V to the quantity:

$$\Delta_1 V = \frac{G\left(\frac{\partial V}{\partial u}\right)^2 - 2F\frac{\partial V}{\partial u}\frac{\partial V}{\partial v} + G\left(\frac{\partial V}{\partial v}\right)^2}{EG - F^2},$$

and the name *second-order differential parameter* to the quantity:

(**44**)
$$\Delta_2 V = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{G\frac{\partial V}{\partial u} - F\frac{\partial V}{\partial v}}{H}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{E\frac{\partial V}{\partial v} - F\frac{\partial V}{\partial u}}{H}\right),$$

in which $H = \sqrt{EG - F^2}$ satisfies the identity:

$$dS = H\,du\,dv.$$

(See Darboux, *Leçons sur la théorie des surfaces,* Book VII, chap. I.)

If $\Delta_1(V, W)$ is the polar form of $\Delta_1(V)$ then one has a formula that is analogous to the Green formula:

(**45**) $$\iint \Delta_1(V,W)dS + \int W \frac{dV}{dn} dS + \iint W\Delta_2 V dS = 0,$$

in which the double integrals are taken over a certain region on the surface and the simple integral is taken along the bounding contour of this domain.

If the functions V and W are regular on the whole surface S, which is assumed to be closed, then one may suppose that the domain of integration comprises the entire surface; the simple integral then disappears.

From this, one deduces that there is no regular non-constant function that satisfies either the equation $\Delta_2 V = 0$ or the equation:

(**46**) $$\Delta_2 V = \text{const.}$$

on the whole surface.

34. – The second-order parameter $\Delta_2 V$ may be further defined geometrically in the following manner: Make two rectangular geodesics pass through the point M on the surface considered, or, furthermore, two normal rectangular sections, and on each of these lines consider the second derivative $\frac{d^2V}{ds^2}$, in which s is the arc-length of the curve. The sum $\frac{d^2V}{ds_1^2} + \frac{d^2V}{ds_2^2}$ of the values of $\frac{d^2V}{ds^2}$ on the two geodesics or normal sections does not vary when they revolve around the point M while remaining rectangular to each other, and will be equal to $\Delta_2 V$, precisely.

It is easy to deduce a relation from the relation that exists between the parameter $\Delta^2 V$ and the symbol ΔV that relates to space such that the identity (**43**) is only a special case. It suffices to refer it to three rectangular axes such that one is the normal Mn to the surface and the other two Mx_1 and Mx_2 are tangent to our two normal sections, respectively. If R_1 and R_2 are their radii of curvature, taken to be positive in the direction of Mn, then one will have:

$$\frac{\partial^2 V}{\partial x_1^2} = \frac{d^2V}{ds_1^2} - \frac{1}{R_1}\frac{\partial V}{\partial n},$$

$$\frac{\partial^2 V}{\partial x_2^2} = \frac{d^2V}{ds_2^2} - \frac{1}{R_2}\frac{\partial V}{\partial n},$$

and, as a consequence:

$$\Delta V = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial n^2} = \Delta_2 V + \frac{\partial^2 V}{\partial n^2} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\partial V}{\partial n}.$$

35. – The equation that we shall have to integrate is of the form:

(**47**) $$\Delta_2 V = f,$$

in which f is a function that is given at every point of the surface. Moreover, this problem is possible (since V is an everywhere regular function) only if one has:

(**48**) $$\iint_S f dS = 0,$$

as equation (**45**) shows when extended to the whole surface for $W = 1$. If it is possible then there is only one solution, just as we concluded when we discussed the equation $\Delta_2 V = 0$.

In order to integrate equation (**47**) one must define a function that is analogous to the Green function in the plane, i.e., one that has logarithmic singular points. There exists no function that has just one logarithmic singular point and satisfies the equation $\Delta_2 V = 0$. However, one may, with Picard, consider either a function that satisfies this equation and has *two* singular points (like the function of Klein that was considered in no. **25**) or a function that satisfies equation (**46**) and has only one singular point A. In other words, it is regular everywhere except for a point A and infinite at A like $1/r$, in which r is the geodesic distance to A. Let g_A^M be such a function:

$$g_A^M = \frac{1}{r} + H,$$

in which H is a function that is regular on the whole surface. In formula (**45**), take:

$$W = 1,$$
$$V = g_A^M.$$

One then has:

$$\iint \Delta_2 g_A^M dS + \int \frac{dg_A^M}{dn} ds = 0.$$

If one takes the integral over the whole surface, minus a small circle around the point A, then the simple integral reduces to -2π, and one obtains:

$$KS + 2\pi = 0,$$

in which K is the constant value of $\Delta_2 g_A^M$, which is therefore found to be determined. As in the foregoing, one may dispose of the function of A that remains arbitrary in the definition of g_A^M in such a manner that:

(**49**) $$g_A^M = g_M^A.$$

In order to do this, it suffices to impose the condition on g that:

$$\iint_S g_A^M \, dS_M = c,$$

in which c is a constant that is independent of the point A (for example, $c = 0$).

Since g_A^M is known, the solution to equation (**47**) is obtained by applying formula (**45**) (taken over the whole surface, minus a small circular curve that encircles the point A) to the functions V and g_A^M. We also obtain:

(**50**) $$2\pi V_A = -\iint_S f_M \, g_A^M \, dS_M + const.$$

Here, one easily proves that the value for V thus obtained answers the question, since the difficulty that results from the boundary conditions does not exist here. If suffices to remark that, on the one hand, by virtue of the symmetry relation (**49**) the function g_A^M also satisfies equation (**46**) when one considers it to be a function of the point A. On the other hand, since the expression under the $\iint$ sign in formula (**50**) is irregular in the same fashion as a logarithmic potential, the differentiation under the $\iint$ sign is legitimate, so we can form the symbol Δ_2 with the condition that the quantity $-2\pi f_A$ be added to the result. Now, the result of this differentiation under the $\iint$ sign is null by virtue of condition (**48**).

The function g_A^M is easily obtained on the sphere. From formula (**43**), equation (**46**) will be written:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\varphi^2} = const.$$

If one takes the point A for the point $\theta = 0$ then g will obviously be independent of φ. Furthermore, under these conditions, the preceding equation admits (up to an additive constant and a constant factor) only one regular solution for $\theta = 0$, namely:

(**51**) $$g = -\log\sin\frac{\theta}{2}.$$

36. – Now we return to the problem posed. It remains for us to determine a regular function on a sphere S_0 that satisfies the condition:

$$\Delta^2 V_0 = -\frac{\partial}{\partial\rho}(\rho V_1).$$

From formulas (**50**) and (**51**), this function will be:

$$2\pi V_0 = \iint \frac{\partial}{\partial \rho}(\rho V_1) \log \sin \frac{\theta}{2} dS + const.,$$

which serves to determine the solution to the problem. We must nevertheless assure that the possibility condition (**48**) is satisfied, which, in this case, is:

(**52**)
$$\iint_{S_0} \frac{\partial}{\partial \rho}(\rho V_1) dS_0 = 0.$$

However, this condition will result from the condition:

(**42**)
$$\iint_{S_1} \varphi_1 dS_1 + \iint_{S_2} \varphi_2 dS_2 = 0.$$

Indeed, it suffices to show that relation (**52**) will be verified, at least on an intermediate sphere S_0.

Now, since $d\omega$ is the element of the sphere of radius 1 that corresponds to the elements dS_1 or dS_2 one has:

$$\varphi_1 dS_1 = \frac{V_1}{\rho_1} dS_1 = V_1 \rho_1 d\omega,$$
$$\varphi_2 dS_2 = \frac{V_1}{\rho_2} dS_2 = V_1 \rho_2 d\omega.$$

Therefore, the integral $\iint_S V_1 \rho d\omega$, which is taken over a sphere of radius ρ, takes the same value for $\rho = \rho_1$ and $\rho = \rho_2$. Its derivative with respect to ρ is thus annulled, at least for a value ρ_0 of ρ that is somewhere between ρ_1 and ρ_2, in such a way that one may take a sphere of radius ρ_0 for S_0 ([20]).

37. – One may demand to know whether considerations that are analogous to the ones that we just presented for the sphere will permit us to solve the second boundary-value problem for other surfaces. The response is negative, at least for the method that we just developed. Indeed, let α, β, γ be the direction cosines of the normal to a closed surface S, and suppose further that one knows the quantity:

$$\frac{\partial V}{\partial n} = \alpha \frac{\partial V}{\partial x} + \beta \frac{\partial V}{\partial y} + \gamma \frac{\partial V}{\partial z}$$

at every point.

([20]) Of course, under these conditions the integral $\iint V_1 \rho d\omega$ will be null, not only for one value of ρ, but for any value of ρ that is between ρ_1 and ρ_2. Moreover, it is easy to see that this integral is a linear function of r, as a consequence of the equation $\Delta V_1 = 0$, and abstracting from condition (**42**).

In order to imitate the path that we followed in the case of the sphere we must know three functions X, Y, Z that are proportional to α, β, γ on the surface S, and are such, that the equation $\Delta V = 0$ entails that:

(**53**)
$$\Delta\left(X\frac{\partial V}{\partial x} + Y\frac{\partial V}{\partial y} + Z\frac{\partial V}{\partial z}\right) = 0 .$$

We saw in no. **30** that the same is true for $X = x$, $Y = y$, $Z = z$. Furthermore, it seems to be a consequence of this fact that:

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$$

is an infinitesimal transformation of a certain one-parameter group.

Conversely, if X, Y, Z are three functions that satisfy relation (**53**) for any harmonic function then consider the one-parameter group that has $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$ for its infinitesimal transformation. In other words, write the differential equations:

$$\frac{dx'}{X(x',y',z')} = \frac{dy'}{Y(x',y',z')} = \frac{dz'}{X(x',y',z')} = d\lambda ,$$

and let:

(**54**)
$$\begin{cases} x' = \xi(x,y,z,\lambda), \\ y' = \eta(x,y,z,\lambda), \\ z' = \varsigma(x,y,z,\lambda), \end{cases}$$

be the solution that reduces to x, y, z for $\lambda = 0$. The right-hand side of relation (**53**) may be written $\frac{\partial}{\partial \lambda}\Delta V(x',y',z')$, and, under the hypotheses that we accepted, this quantity is null for an arbitrary value of λ since the corresponding transformation (**54**) preserves the equation $\Delta V = 0$.

Now, the conditions for this to be true are:

(**55**)
$$\begin{cases} \left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2 + \left(\frac{\partial z'}{\partial x}\right)^2 = \left(\frac{\partial x'}{\partial y}\right)^2 + \left(\frac{\partial y'}{\partial y}\right)^2 + \left(\frac{\partial z'}{\partial y}\right)^2 \\ \qquad = \left(\frac{\partial x'}{\partial z}\right)^2 + \left(\frac{\partial y'}{\partial z}\right)^2 + \left(\frac{\partial z'}{\partial z}\right)^2 \\ \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial z} + \frac{\partial y'}{\partial y}\frac{\partial y'}{\partial z} + \frac{\partial z'}{\partial y}\frac{\partial z'}{\partial z} = \frac{\partial x'}{\partial z}\frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial z}\frac{\partial y'}{\partial x} + \frac{\partial z'}{\partial z}\frac{\partial z'}{\partial x} \\ \qquad = \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial x}\frac{\partial z'}{\partial y} = 0, \end{cases}$$

$$(\mathbf{56}) \qquad \frac{\partial^2 x'}{\partial x^2}+\frac{\partial^2 x'}{\partial y^2}+\frac{\partial^2 x'}{\partial z^2}=\frac{\partial^2 y'}{\partial x^2}+\frac{\partial^2 y'}{\partial y^2}+\frac{\partial^2 y'}{\partial z^2}=\frac{\partial^2 z'}{\partial x^2}+\frac{\partial^2 z'}{\partial y^2}+\frac{\partial^2 z'}{\partial z^2}=0.$$

Therefore, if the transformation $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$ preserves the equation $\Delta V = 0$ then the derivatives of the left-hand sides of equations (**55**) and (**56**) with respect to λ must be null at the same time as these left-hand sides themselves – as is easily verified by direct computation.

Moreover, conditions (**55**), (**56**) are verified for $\lambda = 0$ since x', y', z' reduce to x, y, z. Therefore, they are verified for any value of λ.

However, equations (**55**) define conformal transformations of space, and equations (**56**), in turn, exclude any transformations beside displacements and similarities.

If one adds the condition that X, Y, Z be the direction cosines of the normal to a surface then it is easy to see that this surface can be nothing but a sphere.

§ 7. – MIXED PROBLEMS

38. – Up till now, we were occupied with the problem in which the normal derivative is given on the entire boundary surface. One must not think that this problem and the Dirichlet problem, in which the values of the functions V themselves are given on the entire boundary, are the only ones that we may be required to solve. Not only does one encounter analogous questions in the theory of heat that involve the values of $\frac{dV}{dn}+hV$ (h being a negative number), but also in hydrodynamics, where one is, in general, led to neither the Dirichlet problem nor the one that we just treated, but to a *mixed* problem in which the values of the desired harmonic function V are given on a subset of the surface and those of its normal derivative on the other subset ([21]).

As in the foregoing, if this problem has a solution then it has only one; the classical reasoning by which one proves this fact for the first two problems applies again without modification. However, this negative result is almost the only one that we possess in that regard.

39. – The study of a limiting case permits us to at least account for the nature of the difficulty. We propose to solve the problem for the subset of space that is situated over the xy-plane, with the value of dV / dn being given at every point of this plane, with the exception of a certain area Σ in which one gives V itself. In addition, one imposes the condition on the function V that it go to zero at infinity like a potential (of course, the given values on the boundary are assumed to be compatible with this condition). The problem is then well-defined.

([21]) Rigorously, this problem may regarded as a particular case of the one that one studies in the theory of heat by considering h to be sometimes null and sometimes infinity.

If the value of V is given over the entire plane then the solution will be a two-sheeted potential of weight $\frac{V}{2\pi}$. On the contrary, if the derivative is given everywhere then V will be a one-sheeted potential of density $\frac{1}{2\pi}\frac{dV}{dn}$.

We take the values of the density of this one-sheeted potential on the interior of the area Σ to be unknown auxiliary variables. They will be determined by the condition that the corresponding potential has known values on this area (namely, the given values minus the ones that the one-sheeted potential takes exterior to Σ).

The problem thus posed is: *to distribute a one-sheeted potential over a given bounding area* Σ *such that it has a given value at each point of* Σ, which may be reduced to the Dirichlet problem in the following manner:

Construct very small lengths λ on either side of the normal to Σ that annihilate the contour. The set of all of them thus forms two leaves that together constitute a closed surface S. The problem that consists of finding a given one-sheeted potential at each point of S is none other than that of finding the electric distribution on S (which is assumed to be placed in a given electric field); it reduces to the Dirichlet problem. The desired density on Σ is obviously twice the limit that the density on S tends to when the lengths λ tend to zero.

When the area Σ is circular or elliptic one may take an infinitely flat ellipsoid for S that has Σ for its principal section.

The known methods of solution for the Dirichlet problem on the ellipsoid then give the solution of the problem.

Unfortunately, the preceding procedure does not apply (at least in the same form) outside of the case of a planar boundary (which is devoid of any real significance). For example, consider a sphere whose surface is divided into two parts; the value of V is given on one of them Σ, whereas dV/dn is given on the other. If Σ is reduced to 0 then the value of the harmonic function V will be given by formula (**31**). Therefore, if one takes the values of dV/dn in the area Σ to be unknown auxiliary variables then they will be determined by the condition that the integral:

$$-\frac{1}{4\pi}\iint \gamma_A^M\,dS = -\frac{1}{4\pi}\iint\left(\frac{1}{r}+\frac{R}{\delta r'}+\frac{1}{R}\log\frac{2R^2\tan\frac{\psi}{2}}{\rho\delta\sin\gamma}\right)dS$$

have given values on Σ. This is a much more difficult problem than the first one. True, it belongs to a category of questions that have been solved by the recent work of Fredholm ([22]); however, this solution has a relatively complicated form.

40. – Meanwhile, there exist several exceptional cases in which the problem is easily reduced to the Dirichlet problem. Some of them are, for example:

([22]) Among others, see C.R. Ac. des Sc., 27 January – 30 June 1902.

1. The case of a prism or a right cylinder (the *z*-axis parallel to the edge), in which V isg given on the lateral surface and dV/dn is given on the base. The harmonic function dV/dn will then be given by the Dirichlet conditions.

2. The case of a portion of a sphere that is bounded by a polyhedron or cone that has its vertex at the center, in which V is given on the polyhedral or conical surface and dV/dn is given on the spherical surface. (A remarkable particular case is that of the hemisphere, for which the polyhedral surface reduces to a plane.) One operates as in no. **30**.

3. The case of a portion of a volume of revolution that is bounded by two planes that intersect the axis, in which dV/dn is given on these planes and V is given on the surface. One takes the harmonic function $x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}$ (in which the *z*-axis is the axis of revolution) to be the new unknown (as is easily insured by the considerations of nos. **30** and **37**).

One likewise calculates a function that is harmonic in the space between two concentric spheres and given by its values on one of the spheres and its normal derivative on the other in the form of a series of spherical functions (compare no. **29** (cont.)).

41. – On the other hand, one generalizes the theory of the Green function that we applied to the Neumann problem to the mixed problem without difficulty. However, this Green function is unknown, in general.

41 (cont.). – As in no. **5**, it is clear that one must not consider the problem that we just spoke of and the one in which one is concerned, no longer with the Laplace equation, but with equation (**2**), with the same boundary conditions, as being essentially different.

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CHAPTER II

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WAVES FROM THE KINEMATICAL VIEWPOINT

§ 1. – CLASSICAL RESULTS ([23])

a) *Results concerning deformations*

43. – The position of a deformable medium, such as a fluid, is defined when one assigns a position to each point, in other words, when the coordinates *x, y, z* of an arbitrary point are given by relations of the form:

$$(\mathbf{1})\qquad \begin{cases} x = F_1(a, b, c), \\ y = F_2(a, b, c), \\ z = F_3(a, b, c), \end{cases}$$

in which *x, y, z* are parameters that are intended to distinguish the various points of the fluid from each other, for example, the coordinates that the respective points of the medium have at a definite position of the medium.

If there is *motion,* i.e., if the position of the medium depends on time *t,* then formulas (**1**) become:

$$(\mathbf{1}')\qquad \begin{cases} x = F_1(a, b, c, t), \\ y = F_2(a, b, c, t), \\ z = F_3(a, b, c, t). \end{cases}$$

The figure formed by the set of points, each of which has the Cartesian coordinates *a, b, c* that correspond to a point of the medium will be called the *initial state* of that medium. This initial state may be, for example, the one in which the medium is found at a definite instant t_0 of its motion. However, it is not necessary that this be the case in order for that state to be physically realizable. Its role is limited to permitting us to mathematically express that *the particle* that occupies the position (x_2, y_2, z_2) at the instant t_2 is *the same* as the one that is at (x_1, y_1, z_1) at the instant t_1. One recognizes that this is the case when the values for *a, b, c* that give $x = x_1$, $y = y_1$, $z = z_1$, for $t = t_1$ are the same as the ones that give $x = x_2$, $y = y_2$, $z = z_2$ for $t = t_2$.

In any other case, nothing will prevent us from changing the initial state; in other words, from expressing *x, y, z* in formulas (**1′**), no longer as functions of *a, b, c,* but as

([23]) See Lord Kelvin and Tait, *Treatise of Natural Philosophy;* Kirchhoff, *Mécanique; Traité de Mécanique rationelle,* from Appell, tome III, ch. XXII and XXIII.

functions of other parameters, a', b', c', provided that the latter are functions of *a, b, c* uniquely, and not of time, with *a, b, c* being likewise calculable as functions of a', b', c'.

43. – In the particular case where one considers the initial state to be the first position that is occupied by the by the medium, formulas (**1**) define a *deformation* that permits us to pass from this first position to the one whose coordinates are *x, y, z.*

One calls a deformation for which the second position coincides with the first position an *identity deformation,* so the functions F_1, F_2, F_3 would be none other than *a, b, c.*

44. – We shall always suppose that when one is given *x, y, z,* and *t* equations (**1′**) admit a unique solution for *a, b, c;* in other words, that two different particles may not occupy the same position at the same instant *t.* This obviously expresses the *impenetrability* of matter.

45. – We likewise suppose that the functions *x, y, z* are continuous, in general. Nevertheless, we remark that as far as continuity with respect to *a, b, c* is concerned, this second hypothesis is much less legitimate than the first. We account for this by remarking that two liquids or two gases are finally brought into opposition by their mixing, in general. In this case, it is clear that molecules that are originally separated by finite intervals – namely, the ones that belong to the two fluids, respectively, and are not originally situated on their contact surface – later come into immediate contact with each other. There is obviously no reason to suppose that the different parts of the same fluid do not diffuse into each other as one supposes for the molecules of the two different fluids. If this is the case then *x, y, z* – which are always continuous with respect to *t* – will be totally discontinuous functions of *a, b, c.*

Other than that case, the hypothesis of continuity seems to sufficiently account for the phenomena that are found in a large number of cases. We shall adopt this hypothesis in what follows, and suppose, moreover, that the functions *x, y, z* are differentiable.

45 (cont.). – From what we said above, it is clear that this restricts us in the choice of the initial state to a certain degree, if it is to be arbitrary. When we replace the initial coordinates *a, b, c* by other ones a', b', c' we must have that they are differentiable functions of the first.

46. – We do not exclude the case in which the functions *x, y, z* are discontinuous with respect to *a, b, c* on isolated surfaces, and we will study it later on.

In particular, whereas, by virtue of our first hypothesis (no. **44**) the two portions of the medium may never interpenetrate each other, the opposite may be true: It may very well be the case that the two regions are originally contiguous and then separate from each other in such a manner as to create a cavity between them.

Unless indicated to the contrary, one nevertheless supposes – as we shall do in all of what follows – *that such cavities are not produced,* and reserve treatment of the case in which they come about.

47. – We shall first recall the principles that relate to deformations.

When equations (**1**) are differentiated, they give:

$$(\mathbf{2}) \qquad \begin{cases} dx = a_1 da + b_1 db + c_1 dc, \\ dy = a_2 da + b_2 db + c_2 dc, \\ dz = a_3 da + b_3 db + c_3 dc, \end{cases}$$

in which a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are the partial derivatives of x, y, z with respect to a, b, c. The determinant:

$$(\mathbf{3}) \qquad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix},$$

which is the functional determinant of x, y, z with respect to a, b, c, must not change sign when one varies a, b, c. If this were not true then it would be easy to see that this sign change between an initial position surface S_0 and the present position S would take the regions of the initial medium that are close to S_0 and on either side of it to regions of the present medium S, but on the same side of it; this is contrary to the hypothesis made in no. **44**. The determinant D must not be zero in equation $(\mathbf{1}')$ for any a, b, c for a certain value of t. Otherwise, one would know that x, y, z are distinct functions of a, b, c, which is contrary to the same hypothesis. This determinant will therefore have an invariant sign; we always suppose it is positive ([24]). This determinant is related to the *density* ρ of the medium at the point considered. Indeed, it is defined by the condition that the mass element dm is equal to $\rho\, dx\, dy\, dz$. On the other hand, since this same element is equal to $\rho_0\, da\, db\, dc$, in which ρ_0 is a function of a, b, c that is independent of t, one has:

$$(\mathbf{3}) \qquad D = \frac{\rho_0}{\rho};$$

we suppose that ρ is never infinite so that D is never zero.

One further refers to the determinant $D = \frac{\rho_0}{\rho}$ by the name of *dilatation* of the state (x,y,z) with respect to the state (a,b,c).

([24]) It is clear that this implies a new restriction on the choice of initial state.

48. – If the medium considered occupies only a bounded portion of space then it is impossible (under the theoretical hypotheses that we adopted) for a point that is situated on the boundary surface of the initial state to become an interior point or vice versa. This is because a small arc of a tangent curve that varies continuously about the point in question will necessarily change into an analogous arc that necessarily will or will not be completely contained in the medium, according to whether this point is or is not interior, respectively.

49. – If one is confined to studying what happens in the neighborhood of a definite point of the medium while neglecting higher-order infinitesimals, and one transports the origin of the coordinates at this point in both the initial and present space then one may replace *da, db, dc, dx, dy, dz* by *a, b, c, x, y, z* in formulas (**2**) and write:

$$
(\mathbf{2}') \qquad \begin{cases} x = a_1 a + b_1 b + c_1 c, \\ y = a_2 a + b_2 b + c_2 c, \\ z = a_3 a + b_3 b + c_3 c. \end{cases}
$$

Thus, the deformation of the medium around the point O considered essentially coincides with the one that is defined by the linear substitution $(\mathbf{2}')$.

Geometrically speaking, the transformation that is defined by equations $(\mathbf{2}')$ (which we sometimes call an *affinity*) is a homographic transformation that preserves the plane at infinity. It may be considered to be characterized by the property that it changes two arbitrary parallel lines into two parallel lines and alters two arbitrary segments taken from these two lines in the same fashion.

50. – In general, as one knows, equations $(\mathbf{2}')$ may be put into the form:

$$
(\mathbf{5}) \qquad \begin{vmatrix} a_1 - s & b_1 & c_1 \\ a_2 & b_2 - s & c_2 \\ a_3 & b_3 & c_3 - s \end{vmatrix} = 0;
$$

however, when the preceding equation has multiple roots it may not always be possible to reduce the substitution $(\mathbf{2}')$ to the form (**4**). Furthermore, equation (**5**) may have imaginary roots.

On the contrary, another form that is given to the substitution $(\mathbf{2}')$ that has great importance in the mechanics of fluids is always possible in real form; this is how introduces the notion of a *pure deformation.*

One says that the substitution $(\mathbf{2}')$ represents a *pure deformation* when it may be put into the form (**4**) with the planes $A = 0$, $B = 0$, $C = 0$ forming a tri-rectangular trihedron. The necessary and sufficient condition for this to be the case is that the quantity:

$$(\mathbf{6}) \qquad x\,da + y\,db + z\,dc$$

be an exact differential; in other words, that the determinant (**3**) be symmetric. The first part of this proposition relates to the well-known theory of surfaces of 2nd degree. The second part results from the fact that the expression (**6**) is invariant under any change of rectangular coordinates that is performed simultaneously on x, y, z and on a, b, c and, on the other hand, the fact that if one takes the planes $A = 0$, $B = 0$, $C = 0$ to be the new coordinate planes (which is always possible in the case of a pure deformation) then equations (**4**) become:

$$(\mathbf{4}') \qquad x = s_1 a, \qquad y = s_2 b, \qquad z = s_3 c\,.$$

In this latter form, one sees that the pure deformation amounts to a system of three dilatations performed in rectangular directions.

51. – Any deformation in the form $(\mathbf{2}')$ may be replaced by a rotation that is preceded by or followed by a pure deformation. To that effect, it is sufficient to consider the *ellipsoid of dilatation* or the *ellipsoid of deformation.* For an arbitrary deformation that is defined by equations (**1**), one therefore refers to the ellipsoid $\varphi = 1$ by either term, where φ is the quadratic form that is defined by the identity:

$$\begin{aligned}\varphi(da, db, dc) &\equiv (1 + 2\varepsilon_1)\, da^2 + (1 + 2\varepsilon_2)\, db^2 + (1 + 2\varepsilon_3)\, dc^2 \\ &\qquad + 2\gamma_1\, db\, dc + 2\gamma_2\, dc\, da + 2\gamma_3\, da\, db \\ &\equiv dx^2 + dy^2 + dz^2.\end{aligned}$$

The coefficients $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$ that figure in the preceding formula, i.e., the quantities that are given by the formulas:

$$(\mathbf{7}) \qquad \begin{cases} 1+2\varepsilon_1 = \left(\dfrac{\partial x}{\partial a}\right)^2 + \left(\dfrac{\partial y}{\partial a}\right)^2 + \left(\dfrac{\partial z}{\partial a}\right)^2, \quad 1+2\varepsilon_2 = \left(\dfrac{\partial x}{\partial b}\right)^2 + \left(\dfrac{\partial y}{\partial b}\right)^2 + \left(\dfrac{\partial z}{\partial b}\right)^2, \\ 1+2\varepsilon_3 = \left(\dfrac{\partial x}{\partial c}\right)^2 + \left(\dfrac{\partial y}{\partial c}\right)^2 + \left(\dfrac{\partial z}{\partial c}\right)^2, \\ \gamma_1 = \dfrac{\partial x}{\partial b}\dfrac{\partial x}{\partial c} + \dfrac{\partial y}{\partial b}\dfrac{\partial y}{\partial c} + \dfrac{\partial z}{\partial b}\dfrac{\partial z}{\partial c}, \quad \gamma_2 = \dfrac{\partial x}{\partial c}\dfrac{\partial x}{\partial a} + \dfrac{\partial y}{\partial c}\dfrac{\partial y}{\partial a} + \dfrac{\partial z}{\partial c}\dfrac{\partial z}{\partial a}, \\ \gamma_3 = \dfrac{\partial x}{\partial a}\dfrac{\partial x}{\partial b} + \dfrac{\partial y}{\partial a}\dfrac{\partial y}{\partial b} + \dfrac{\partial z}{\partial a}\dfrac{\partial z}{\partial b}, \end{cases}$$

are called the *components of deformation.* The dilatation D is a function of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$; this is because the discriminant of φ is equal to D^2.

When the deformation is of the form $(\mathbf{2}')$ the ellipsoid of deformation:

$$(\mathbf{7}') \qquad \varphi(a, b, c) = 1$$

is the locus of points that are found on the sphere:

$$x^2 + y^2 + z^2 = 1$$

once this deformation has been performed.

Three diametrically conjugate planes of the quadric $(\mathbf{7}')$ become three rectangular planes under the deformation $(\mathbf{2}')$. There thus exists one and only one tri-rectangular trihedron T to which there corresponds a tri-rectangular trihedron T'; it is formed by the principal planes of the quadric in question. A rotation of either space makes the trihedrons T and T' coincide, and after that one will require only a pure deformation to make the two media correspond completely.

If the substitution $(\mathbf{2}')$ is a pure deformation then the quadric $f = 1$ that is defined by the identity:

$$(\mathbf{6}') \qquad \tfrac{1}{2} df = x\,da + y\,db + z\,dc$$

has the same principal planes as the ellipsoid $(\mathbf{7}')$ with the axes of one having the squares of the axes of the other for length. This is what one immediately sees upon regarding the pure deformation in the form $(\mathbf{4}')$.

52. – All of these results that relate to deformations are well known, but we must insist, for the moment, on a particular case that one may associate with the general case by considering circular cross-sections of the ellipsoid $(\mathbf{7}')$, as Lord Kelvin and Tait did.

Let P be such a plane of circular section and let P' be its homologue. A circle that is traced in the plane P has a circle in the plane P' for its transform. In other words, all of the curves in the plane P are dilated by the same ratio, in such a way that any figure of the plane P is similar to its own homologue in the plane P'. In order to transform the initial medium into another one that is similar to the final medium (in other words, that it may be obtained by a homothety and a displacement) it is therefore sufficient to perform a deformation of the form $(\mathbf{2}')$ in which all of the points of the plane P remain unaltered. These are the deformations that we shall study in particular.

If, to simplify, we take the plane P to be the xy-plane then formulas $(\mathbf{2}')$ may obviously be written:

$$(\mathbf{8}) \qquad \begin{cases} x = a + \lambda c \\ y = b + \mu c \\ z = c(1 + \nu), \end{cases}$$

since one must have $x = a$, $y = b$, $z = c$ for $c = 0$.

These formulas show that the displacements of all of the points have the same direction and are proportional to the distance from these points to the plane P. In order to completely understand the deformation in question it suffices to give the segment (λ, μ, ν). This segment, which represents the displacement of a point that is situated at a unit distance from the plane P, may therefore be called the *characteristic segment* of the deformation.

The triangle M_0m_0M (*fig.* 4) that is formed by an arbitrary point M_0, its projection on P, and its new position after the deformation is always similar to the analogous triangle that is formed with the aid of a definite point that is situated a unit of distance from the plane P and has the characteristic segment for one of its sides.

In order for one to be dealing with a pure deformation, it is necessary and sufficient that $\lambda = \mu = 0$, i.e., that the characteristic segment is normal to the plane P. The deformation then reduces to a dilatation that is normal to that plane.

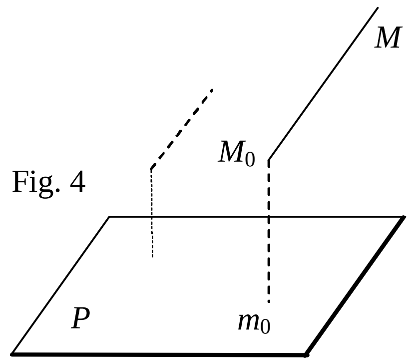

Fig. 4

Suppose that this is not the case. We may always nonetheless suppose that $\mu = 0$ by taking a plane that is parallel to the characteristic segment to be the *xz*-plane. Take this plane to be the plane of the figure.

Let M be the final position of the point that was originally situated at M_0 in the plane of the figure. The perpendicular altitude in this plane to the medium M_0M cuts the plane P at a point O (*fig.* 5) such that $OM = OM_0$. At a second point O' that likewise belongs to the trace of the plane of the figure on the plane P we construct a line $O'M_0'$ that is equal and parallel to OM_0. The parallelogram $OM_0O'M_0'$ will be transformed into another parallelogram $OMO'M'$; it will be deformed in the manner of an articulated parallelogram.

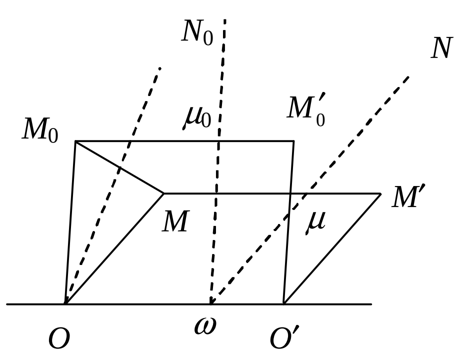

Fig. 5

It is easy to deduce the displacement of an arbitrary point N_0 in the plane of the figure from this deformation. Indeed, if we construct a parallel to OM_0 at this point that cuts OO' at ω and M_0M_0' at μ_0 then this point μ_0 may obviously be considered as the result of the motion of the superior base M_0M_0' of the articulated parallelogram; this is how we will know its new position μ. If we then take a segment ωN that is equal to ωN_0 on the line $\omega\mu$ then one will have the new position of the point N_0.

The motion of a point that is not situated in the plane of the figure will then be defined by the motion of its projection on this plane. As for the rest, in order to obtain the displacements of all of the points of space at the same time it obviously suffices to replace our articulated parallelogram by an articulated right parallelepiped that has this parallelogram for its base.

53. – If one takes OO' to be equal to OM_0 then the parallelogram $OM_0O'M_0'$ is a rhombus, and it remains one after deformation. The two diagonals of this rhombus and the perpendicular to the plane of the figure thus constitute the axes of the trihedron T, which is tri-rectangular in the initial state as well as in the final state. If one wants to decompose the deformation into a pure deformation and a rotation then one chooses the perpendicular to the plane of the figure for the rotational axis and the angle through which the diagonals to the rhombus turn – namely, $\frac{\overline{M_0OM}}{2}$ – to be its angle.

54. – The ratio of the densities – in other words, the ratio by which the volume of the articulated parallelepiped is altered – is obviously nothing but the ratio by which the surface of the parallelogram that serves for its base is altered, or again the height of this parallelogram. One will thus have:

$$\frac{\rho_0}{\rho} = 1 + \nu,$$

in which ν is the normal component of the characteristic segment.

If this component is null then it is clear that any figure that is situated in a plane P_1 that is parallel to P is subjected to a simple translation in its plane that is parallel to the fixed direction (λ, μ, 0) and proportional to the distance between the two planes PP_1. Such a deformation goes by the name of a *slide.*

Upon decomposing the characteristic segment into two pieces, one of which is parallel to the plane P and the other of which is perpendicular to this plane, one will decompose the deformation with which we occupy ourselves into a slide and a pure dilatation.

55. – If the plane P, instead of being the xy-plane, has the direction cosines α, β, γ then formulas (**8**) will obviously be replaced by:

$$\text{(9)} \qquad \begin{cases} x = a + \lambda(\alpha a + \beta b + \gamma c), \\ y = b + \mu(\alpha a + \beta b + \gamma c), \\ z = c + \nu(\alpha a + \beta b + \gamma c). \end{cases}$$

The characteristic segment will be the one that has l, m, n for its projections, and the density ratio, which is related, as we have seen, to the normal component of this segment, will be:

$$\text{(10)} \qquad \frac{\rho_0}{\rho} = 1 + \lambda\alpha + \mu\beta + \nu\gamma.$$

This ratio will likewise be the one by which the distance of an arbitrary point to the plane P is altered.

56. – The preceding permits us to represent a deformation (no longer homographic, but arbitrary) in a neighborhood of a surface S that leaves all of the points of this surface unaltered. Indeed, in a neighborhood of an arbitrary one of these points O this may be included in its tangent plane, in such a way that the displacement of an arbitrary point M_0 of the neighboring space to O is obtained by multiplying a definite segment (λ, μ, ν) by the normal distance to a point M_0 of the surface. This being the case, the segment (λ, μ, ν) varies with the position of a point O on S. The partial derivatives of x, y, z with respect to a, b, c at the point O will be:

$$\text{(11)} \qquad \begin{cases} \dfrac{\partial x}{\partial a} = 1 + \lambda\alpha, & \dfrac{\partial x}{\partial b} = \lambda\beta, & \dfrac{\partial x}{\partial a} = \lambda\gamma, \\ \dfrac{\partial y}{\partial a} = \mu\alpha, & \dfrac{\partial y}{\partial b} = 1 + \mu\beta, & \dfrac{\partial y}{\partial c} = \mu\gamma, \\ \dfrac{\partial z}{\partial a} = \nu\alpha, & \dfrac{\partial z}{\partial b} = \nu\beta, & \dfrac{\partial z}{\partial c} = 1 + \nu\gamma, \end{cases}$$

in which α, β, γ are the direction cosines of the normal to S at O. The dilatation at this point, which is equal to the ratio by which the shortest distance to the point M_0 to S is changed, will be given by formula (**10**), which represents the determinant of the matrix (**11**), moreover.

57. – *Higher-order deformations.* – If, having accounted for first-order infinitesimals in an arbitrary deformation, as we have done, one would like to introduce infinitesimals of higher order then in the general case one will be led a study that is extremely complicated and actually seems useless. We will need to do this study only in an extremely important particular case that is completely analogous to the one that we must treat.

We consider a deformation that not only leaves all of the points of a certain surface S unaltered, but also coincides with the identity deformation at each of its points up to infinitesimals of the n^{th} order, i.e., it is such that all of the partial derivatives of x, y, z with respect to a, b, c up to order $n - 1$ inclusive are null on S, with the exception of the derivatives $\dfrac{\partial x}{\partial a}, \dfrac{\partial y}{\partial b}, \dfrac{\partial z}{\partial c}$, which will be equal to 1. We seek relations that will couple the derivatives of order n under these conditions.

The method that shall serve for us will, moreover, replace the one that we used in the preceding section for the case of $n = 1$.

First, let $n = 2$, to fix ideas, and let:

$$f(a, b, c) = 0$$

denote the equation for S, and let f_a, f_b, f_c denote the partial derivatives $\dfrac{\partial f}{\partial a}, \dfrac{\partial f}{\partial b}, \dfrac{\partial f}{\partial c}$.

By hypothesis, the relation:

$$\frac{\partial x}{\partial a} = 1$$

is true for all of the points of S. Therefore, we may differentiate it on that surface. In other words, the relation:

$$\frac{\partial^2 x}{\partial a^2} da + \frac{\partial^2 x}{\partial a \partial b} db + \frac{\partial^2 x}{\partial a \partial c} dc = 0$$

is true for all values of da, db, dc that satisfy the relation:

(12) $$f_a\, da + f_b\, db + f_c\, dc = 0,$$

which gives:

$$\frac{\dfrac{\partial^2 x}{\partial a^2}}{f_a} = \frac{\dfrac{\partial^2 x}{\partial a \partial b}}{f_b} = \frac{\dfrac{\partial^2 x}{\partial a \partial c}}{f_c}.$$

Similar, the equations $\dfrac{\partial x}{\partial b} = 0, \dfrac{\partial x}{\partial c} = 0,$ when differentiated on S, give:

$$\frac{\dfrac{\partial^2 x}{\partial a \partial b}}{f_a} = \frac{\dfrac{\partial^2 x}{\partial b^2}}{f_b} = \frac{\dfrac{\partial^2 x}{\partial b \partial c}}{f_c},$$

$$\frac{\dfrac{\partial^2 x}{\partial a \partial c}}{f_a} = \frac{\dfrac{\partial^2 x}{\partial b \partial c}}{f_b} = \frac{\dfrac{\partial^2 x}{\partial c^2}}{f_c}.$$

Together, these relations imply:

(13) $$\frac{\dfrac{\partial^2 x}{\partial a^2}}{f_a^2} = \frac{\dfrac{\partial^2 x}{\partial a \partial b}}{f_a\, f_b} = \frac{\dfrac{\partial^2 x}{\partial a \partial c}}{f_a\, f_c} = \frac{\dfrac{\partial^2 x}{\partial b^2}}{f_b^2} = \frac{\dfrac{\partial^2 x}{\partial b \partial c}}{f_b\, f_b} = \frac{\dfrac{\partial^2 x}{\partial a \partial c}}{f_c^2},$$

or, upon designating the common value of the preceding ratios by λ:

(14) $$\begin{cases} \dfrac{\partial^2 x}{\partial a^2} = \lambda f_a^2, & \dfrac{\partial^2 x}{\partial b^2} = \lambda f_b^2, & \dfrac{\partial^2 x}{\partial c^2} = \lambda f_c^2, \\ \dfrac{\partial^2 x}{\partial b \partial c} = \lambda f_b\, f_c, & \dfrac{\partial^2 x}{\partial c \partial a} = \lambda f_c\, f_a, & \dfrac{\partial^2 x}{\partial a \partial b} = \lambda f_a\, f_b, \end{cases}$$

relations that one may express as follows: the symbolic equation:

(14′) $$\left(\frac{\partial}{\partial a} da + \frac{\partial}{\partial b} db + \frac{\partial}{\partial c} dc \right)^2 x = \lambda (f_a da + f_b db + f_c dc)^2$$

is an identity with respect to the differentials *da*, *db*, *dc*.

If we remark that the difference $x - a$ is null at the point of *S* we are considering, as well as its first derivatives, then we see that one may write, up to third-order infinitesimals:

$$x = a + \frac{\lambda f^2}{2}. \tag{15}$$

Similarly, upon introducing new numbers μ, ν one may write:

$$\begin{cases} y = b + \dfrac{\mu f^2}{2}, \\ z = c + \dfrac{\nu f^2}{2}. \end{cases} \tag{15$'$}$$

58. – The quantities f_a, f_b, f_c are proportional to the direction cosines α, β, γ of the normal to *S*. They are *equal* to these direction cosines, respectively, if the equation for *S* has been taken in a convenient form. Suppose that this is true. f will then represent the normal distance δ to the point (a, b, c) on *S*, up to second-order infinitesimals. We then see that the deformation under consideration is known, up to third-order infinitesimals, when it is given at each point of the segment (λ, μ, ν). The displacement of an arbitrary point M_0 near *S* may be obtained by constructing the normal $M_0m = \delta$ at that point and multiplying the corresponding segment (λ, μ, ν) at the point *m* by $\delta^2/2$. A small segment of the normal to *S* becomes a segment of a parabola after deformation.

59. – Things happen in a completely analogous fashion for *n* greater than 2. By hypothesis (for $p + q + r = n - 1$), one will have:

$$\frac{\partial^{n-1} x}{\partial a^p \partial b^q \partial c^r} = 0,$$

which, when differentiated on *S*, will give:

$$\frac{\partial^n x}{\partial a^{p+1} \partial b^q \partial c^r} da + \frac{\partial^n x}{\partial a^p \partial b^{q+1} \partial c^r} db + \frac{\partial^n x}{\partial a^p \partial b^q \partial c^{r+1}} dc = 0,$$

by means of relation (**12**), and, as a consequence:

$$\frac{\partial^n x}{\partial a^{p+1} \partial b^q \partial c^r} : f_a = \frac{\partial^n x}{\partial a^p \partial b^{q+1} \partial c^r} : f_b = \frac{\partial^n x}{\partial a^p \partial b^q \partial c^{r+1}} : f_c.$$

In other words, the ratio:

$$\textbf{(16)} \qquad \frac{\partial^n x}{\partial a^p \partial b^q \partial c^r} : (f_a^p f_b^q f_c^r), \qquad (p + q + r = n)$$

is independent of the choice of indices p, q, r, provided that their sum is equal to n. One denotes this ratio by λ, and, similarly, there will exist two other numbers μ, ν such that one has:

$$\textbf{(16}'\textbf{)} \qquad \begin{cases} \dfrac{\partial^n y}{\partial a^p \partial b^q \partial c^r} = \mu f_a^p f_b^q f_c^r, \\ \dfrac{\partial^n z}{\partial a^p \partial b^q \partial c^r} = \nu f_a^p f_b^q f_c^r, \end{cases}$$

which then gives:

$$\left(\frac{\partial}{\partial a} da + \frac{\partial}{\partial b} db + \frac{\partial}{\partial c} dc\right)^n x = \lambda (f_a da + f_b db + f_c dc)^n,$$

$$\left(\frac{\partial}{\partial a} da + \frac{\partial}{\partial b} db + \frac{\partial}{\partial c} dc\right)^n y = \mu (f_a da + f_b db + f_c dc)^n,$$

$$\left(\frac{\partial}{\partial a} da + \frac{\partial}{\partial b} db + \frac{\partial}{\partial c} dc\right)^n z = \nu (f_a da + f_b db + f_c dc)^n.$$

Suppose further that the equation for S is taken in such a manner that f represents the normal distance to the point (a, b, c) at S, and f_a, f_b, f_c represent the direction cosines of the normal to S. We see that the displacement of a neighboring point of S is of the form $\left(\frac{\lambda f^n}{n!}, \frac{\mu f^n}{n!}, \frac{\nu f^n}{n!}\right)$. Small segments of the normal to S are transformed into segments of parabolas of degree n.

60. – Density, which depends on first-order derivatives, remains unaltered to higher order under the deformations that we shall consider; only its derivatives of order greater than or equal to $n - 1$ are modified. It is easy to see what these modifications will be for order $n - 1$.

Indeed, start with formulas (**3**), (**3′**). Here, all of the elements of the determinant D are equal to zero, except for the ones on the principal diagonal that have the value 1. Form the derivative:

$$\frac{\partial^{n-1}}{\partial a^p \partial b^q \partial c^r} \frac{\rho_0}{\rho} = \frac{\partial^{n-1} D}{\partial a^p \partial b^q \partial c^r}.$$

Insofar as we do not make the weights of the differentiation depend on one and the same element, the term that we obtain will be null, since only derivatives of order less than n appear in it, and they are null by hypothesis (except for order *one*). On the other hand, if we perform the operation $\frac{\partial^{n-1}}{\partial a^p \partial b^q \partial c^r}$ on a non-principal element then we must multiply the result that we obtain by the minor that relates to that element, which is null. There thus remain only three terms that are obtained by differentiating the elements of the principal diagonal $n-1$ times, namely (since the corresponding minors of these elements are equal to 1):

$$\frac{\partial^{n-1}}{\partial a^p \partial b^q \partial c^r}\left(\frac{\rho_0}{\rho}\right) = \frac{\partial^n x}{\partial a^{p+1} \partial b^q \partial c^r} + \frac{\partial^n y}{\partial a^p \partial b^{q+1} \partial c^r} + \frac{\partial^n z}{\partial a^p \partial b^q \partial c^{r+1}},$$

and consequently, by virtue of formulas $(\mathbf{16}), (\mathbf{16'})$:

$$\frac{\partial^{n-1}}{\partial a^p \partial b^q \partial c^r}\left(\frac{\rho_0}{\rho}\right) = f_a^p f_b^q f_c^r (\lambda f_a + \mu f_b + \nu f_c); \tag{17}$$

for example, for $n = 2$ one will have:

$$\begin{cases} \dfrac{\partial}{\partial x}\dfrac{\rho_0}{\rho} = f_a(\lambda f_a + \mu f_b + \nu f_c), \\ \dfrac{\partial}{\partial y}\dfrac{\rho_0}{\rho} = f_b(\lambda f_a + \mu f_b + \nu f_c), \\ \dfrac{\partial}{\partial z}\dfrac{\rho_0}{\rho} = f_c(\lambda f_a + \mu f_b + \nu f_c). \end{cases} \tag{17'}$$

One may replace $D = \rho_0/\rho$ in the preceding formulas (which will be useful somewhat later on) by the quantity log D, whose derivative with respect to D is equal to 1 on S (which is also true for formula (**17**), whose terms on the left-hand side contain derivatives of higher order in the logarithm and are thus null by the hypotheses we made).

a) *Results relating to velocity*

61. – Having occupied ourselves with the deformation that was represented by formulas (**1**) in the foregoing, we now pass to the study of motion, properly speaking, i.e., we make t vary in formulas (**1′**).

Under these conditions, the system of independent variables that we have employed up till now – namely, the initial coordinates *a*, *b*, *c*, and time *t* – is not the only one that we have to consider. One may also have to express the various quantities on which one operates as a function of the *present* coordinates *x*, *y*, *z*, and *t*. When it will cause no confusion, we denote the derivatives that are taken in first system by δ and the partial derivatives for which *x*, *y*, *z*, *t* are considered to be independent variables by the symbol ∂. Therefore, the components of velocity will be $u = \frac{\delta x}{\delta t}$, $v = \frac{\delta y}{\delta t}$, $w = \frac{\delta z}{\delta t}$; those of acceleration will be $\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2}$.

It is necessary to write the relations that exist between the partial derivatives of the same quantity in the two systems. If one first considers the derivatives with respect to *a, b, c* or *x, y, z* then one must consider *t* to be constant, and one has:

$$\begin{aligned}
\frac{\delta}{\delta a} &= \frac{\delta x}{\delta a}\frac{\partial}{\partial x} + \frac{\delta y}{\delta a}\frac{\partial}{\partial y} + \frac{\delta z}{\delta a}\frac{\partial}{\partial z}, \\
\frac{\delta}{\delta b} &= \frac{\delta x}{\delta b}\frac{\partial}{\partial x} + \frac{\delta y}{\delta b}\frac{\partial}{\partial y} + \frac{\delta z}{\delta b}\frac{\partial}{\partial z}, \\
\frac{\delta}{\delta c} &= \frac{\delta x}{\delta c}\frac{\partial}{\partial x} + \frac{\delta y}{\delta c}\frac{\partial}{\partial y} + \frac{\delta z}{\delta c}\frac{\partial}{\partial z}.
\end{aligned}$$

As far as the derivatives $\frac{\delta}{\delta t}$ and $\frac{\partial}{\partial t}$ are concerned, *x, y, z* are functions of *t*, since *a, b, c* are given, and their derivatives with respect to *t* are the components of *u, v, w* of the velocity, and one has:

(**18**)
$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

61 (cont.). – We must employ a system of variables that is intermediate between the preceding two. In the latter system, in order to study what happens at an arbitrary definite instant t_0 the state of the medium at that instant has to be the initial state. It is a function of the initial coordinates thus defined and of time *t* that gives us the coordinates of the different points at the instants neighboring t_0. It is clear that the derivatives with respect to the initial coordinates for this manner of operation will be $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$; however, the derivative with respect to time will not be $\frac{\partial}{\partial t}$, but the derivative $\frac{\delta}{\delta x}$ that figures in the left-hand side of formula (**18**).

62. – If the state of the medium at the instant t_0 is taken to be the initial state then we compare it to the state at the instant $t_0 + \delta t$. At this new state, the coordinates of the different points will be given by the formulas:

$$\begin{aligned}
x' &= x + u\,\delta t, \\
y' &= y + v\,\delta t, \\
z' &= z + w\,\delta t.
\end{aligned}$$

Upon differentiating these equations with respect to *x, y, z* we obtain formulas that correspond to (**2′**), which will be:

(19)
$$\begin{cases} dx' = dx + \delta t du = dx\left(1+\dfrac{\partial u}{\partial x}\delta t\right) + \dfrac{\partial u}{\partial y}\delta t dy + \dfrac{\partial u}{\partial z}\delta t dz, \\ dy' = \dfrac{\partial v}{\partial x}\delta t dx + dy\left(1+\dfrac{\partial v}{\partial y}\delta t\right) + \dfrac{\partial v}{\partial z}\delta t dz, \\ dz' = \dfrac{\partial w}{\partial x}\delta t dx + \dfrac{\partial w}{\partial y}\delta t dy + dz\left(1+\dfrac{\partial w}{\partial z}\delta t\right). \end{cases}$$

Here, the determinant D will be:

$$\begin{vmatrix} 1+\dfrac{\partial u}{\partial x}\delta t & \dfrac{\partial u}{\partial y}\delta t & \dfrac{\partial u}{\partial z}\delta t \\ \dfrac{\partial v}{\partial x}\delta t & 1+\dfrac{\partial v}{\partial y}\delta t & \dfrac{\partial v}{\partial z}\delta t \\ \dfrac{\partial w}{\partial x}\delta t & \dfrac{\partial w}{\partial y}\delta t & 1+\dfrac{\partial w}{\partial z}\delta t \end{vmatrix} = 1 + \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right)\delta t,$$

(upon neglecting powers of δt greater than the first). Since this determinant is equal to $\dfrac{\rho}{\rho+\delta\rho} = 1 - \dfrac{\delta\rho}{\rho+\delta\rho}$, one will have:

(20)
$$\frac{\delta\rho}{\rho\delta t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which gives us $\dfrac{\delta\rho}{\delta t}$, and by means of this expression, relation (**18**) gives $\dfrac{\partial\rho}{\partial t}$ as:

(21)
$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0.$$

On the other hand, we now occupy ourselves with the task of decomposing the preceding linear substitution, namely, the substitution whose coefficients are:

(19)
$$\begin{pmatrix} 1+\dfrac{\partial u}{\partial x}\delta t & \dfrac{\partial u}{\partial y}\delta t & \dfrac{\partial u}{\partial z}\delta t \\ \dfrac{\partial v}{\partial x}\delta t & 1+\dfrac{\partial v}{\partial y}\delta t & \dfrac{\partial v}{\partial z}\delta t \\ \dfrac{\partial w}{\partial x}\delta t & \dfrac{\partial w}{\partial y}\delta t & 1+\dfrac{\partial w}{\partial z}\delta t \end{pmatrix},$$

into a pure deformation and a rotation. We do this by the intermediary of the two substitutions:

$$\textbf{(22)}\qquad \begin{pmatrix} 1+\dfrac{\partial u}{\partial x}\delta t, & \frac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)\delta t, & \frac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right)\delta t \\ \frac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\delta t, & 1+\dfrac{\partial v}{\partial y}\delta t, & \frac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right)\delta t \\ \frac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right)\delta t, & \frac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right)\delta t, & 1+\dfrac{\partial w}{\partial z}\delta t \end{pmatrix},$$

and:

$$\textbf{(23)}\qquad \begin{pmatrix} 1, & \frac{1}{2}\left(\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}\right)\delta t, & \frac{1}{2}\left(\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\right)\delta t \\ \frac{1}{2}\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)\delta t, & 1, & \frac{1}{2}\left(\dfrac{\partial v}{\partial z}-\dfrac{\partial w}{\partial y}\right)\delta t \\ \frac{1}{2}\left(\dfrac{\partial w}{\partial x}-\dfrac{\partial u}{\partial z}\right)\delta t, & \frac{1}{2}\left(\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\right)\delta t, & 1 \end{pmatrix},$$

whose product gives (**19′**), when one neglects terms in δt^2.

The first represents a pure deformation, since the matrix (**22**) is symmetric; the second represents the effect of the rotation whose components are:

$$\textbf{(24)}\qquad \begin{cases} p = \frac{1}{2}\left(\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\right), \\ q = \frac{1}{2}\left(\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\right), \\ r = \frac{1}{2}\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right), \end{cases}$$

during the time interval δt.

The rotation whose components are the quantities p, q, r that we just wrote is called the *instantaneous molecular rotation* or *vorticity*.

63. – One may justify the name of instantaneous molecular rotation with the remark that Stokes – and later on, Helmholtz – once made, that if one mentally isolates a small spherical portion of the moving fluid around the point considered at the instant considered, and one briefly supposes it to be solidified then the instantaneous rotation experienced by the solid so obtained will have the components p, q, r, precisely. Beltrami ([25]) has shown that this conclusion remains true as long as the principal axes of inertia of the solidified portion coincide with those of the pure deformation (**22**), i.e., with those of the quadric:

([25]) *Principii dell' Idrodynamica razionale*, Mem. de l'Ac. de Bologne, 3rd series, tome I, pp. 458-459, 1871.

$$(25)\qquad \begin{cases} x^2+y^2+z^2+\varphi = x^2\left(1+\dfrac{\partial u}{\partial x}\right)+y^2\left(1+\dfrac{\partial v}{\partial y}\right)+z^2\left(1+\dfrac{\partial w}{\partial z}\right) \\ + yz\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right)+zw\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right)+xy\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)=(1+\alpha)x^2 \\ \quad (1+\alpha')y^2+(1+\alpha'')z^2+2\beta yz+2\beta' zx+2\beta'' xy = 1, \end{cases}$$

whose coefficients are those of the substitution (**22**), up to the quantity $x^2 + y^2 + z^2$.

In a general manner, let:

$$(26)\qquad \Phi = Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy = 1$$

be the equation of the ellipsoid of inertia of the molecule, which one supposes to be isolated at the instant t_0 and briefly solidified, as we shall explain, with the origin taken to be the center of gravity O of that molecule. In order to find its movement around O after solidification, it will suffice, as one knows, to find the total moments of the quantities of movement relative to three rectangular axes issuing from O; in other words, write:

$$\begin{aligned} \frac{1}{2}\frac{\partial\Phi}{\partial p_1} &= Ap_1 + B''q_1 + B'r_1 = \sum m\left(y_1\frac{\delta z_1}{\delta t} - z_1\frac{\delta y_1}{\delta t}\right), \\ \frac{1}{2}\frac{\partial\Phi}{\partial q_1} &= B''p_1 + A'q_1 + Br_1 = \sum m\left(z_1\frac{\delta x_1}{\delta t} - x_1\frac{\delta z_1}{\delta t}\right), \\ \frac{1}{2}\frac{\partial\Phi}{\partial r_1} &= B'p_1 + Bq_1 + A''r_1 = \sum m\left(x_1\frac{\delta y_1}{\delta t} - y_1\frac{\delta x_1}{\delta t}\right), \end{aligned}$$

in which p_1, q_1, r_1 denote the components of the rotation after solidification, assuming that x_1, y_1, z_1 denote the coordinates taken with respect to a system of axes that are parallel to our fixed axes, but whose origin always coincides with the point O. Since $\frac{\delta x_1}{\partial t}, \frac{\delta y_1}{\partial t}, \frac{\delta z_1}{\partial t}$ are noting but the variations that are felt by u, v, w when one passes from the point O to an infinitely close point, one may write, up to higher-order infinitesimals:

$$\begin{aligned} \frac{\delta x_1}{\delta t} &= \frac{\partial u}{\partial x}x_1 + \frac{\partial u}{\partial y}y_1 + \frac{\partial u}{\partial z}z_1 \\ &= \frac{\partial u}{\partial x}x_1 + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)y_1 + \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)y_1 + qz_1 - ry_1 \\ &= \frac{1}{2}\frac{\partial\varphi}{\partial x_1} + qz_1 - ry_1 \end{aligned}$$

$$\frac{\delta y_1}{\delta t} = \frac{\partial v}{\partial x} x_1 + \frac{\partial v}{\partial y} y_1 + \frac{\partial v}{\partial z} z_1 = \frac{1}{2}\frac{\partial \varphi}{\partial y_1} + r x_1 - p z_1$$

$$\frac{\delta z_1}{\delta t} = \frac{\partial w}{\partial x} x_1 + \frac{\partial w}{\partial y} y_1 + \frac{\partial w}{\partial z} z_1 = \frac{1}{2}\frac{\partial \varphi}{\partial z_1} + p y_1 - q x_1$$

in which p, q, r are the components of the vorticity, as we have always written. If we substitute into the preceding equations, we get:

$$\begin{aligned} A p_1 + B'' q_1 + B' r &= \sum \frac{m}{2}\left(y_1 \frac{\partial \varphi}{\partial z_1} - z_1 \frac{\partial \varphi}{\partial y_1}\right) + \sum m[p(y^2 + z^2) - qxy - rxz] \\ &= Ap + B''q + B'r - \sum m[y_1(\beta' x_1 + \beta y_1 + \alpha'' z_1) \\ &\qquad - z_1(\beta'' x_1 + \alpha' y_1 + \beta z_1)] \\ &\qquad + \beta'' B' \end{aligned}$$

$$\begin{aligned} B'' p_1 + A' q_1 + B r_1 &= B''p + A'q + Br + \sum m[z_1(\alpha x_1 + \beta'' y_1 + \beta' z_1) \\ &\qquad - x_1(\beta' x_1 + \beta y_1 + \alpha'' z_1)] \\ &= B''p + A'q + Br - \alpha B' - \beta'' B + \beta'(A - A'') + \beta B'' \\ &\qquad + \alpha'' B' \end{aligned}$$

$$\begin{aligned} B' p_1 + B q_1 + A'' r_1 &= B'p + Bq + A''r + \sum m[x_1(\beta'' x_1 + \alpha' y_1 + \beta z_1) \\ &\qquad - y_1(\alpha x_1 + \beta'' y_1 + \beta' z_1)] \\ &= B'p + Bq + A''r + \beta''(A' - A) - \alpha' B'' - \alpha' B'' - \beta B' \\ &\qquad + \alpha B'' + \beta' B. \end{aligned}$$

One therefore sees that p_1, q_1, r_1 coincide with p, q, r when the ellipsoid of inertia reduces to a sphere, and also when its axes coincide in direction with those of the quadric (**25**) (because nothing prevents us from supposing that one has taken the coordinate axes parallel to the axes in question, which implies that one will have $B = B' = B'' = \beta = \beta' = \beta'' = 0$).

On the other hand, we must point out the form of the complementary segment:

$$\textbf{(27)} \qquad \begin{cases} \beta'' B' - \beta' B'' + \beta(A'' - A') - B(\alpha'' - \alpha'), \\ \beta B'' - \beta'' B + \beta'(A - A'') - B'(\alpha - \alpha''), \\ \beta' B - \beta B' + \beta''(A' - A) - B''(\alpha' - \alpha), \end{cases}$$

that figures in the expression for the resulting moment of the quantities of motion. This segment remains unaltered up to sign when one exchanges the quadratic forms that figure in equations (**25**), (**26**).

64. – A simple geometric interpretation, except on segment (27), at least for the direction, may be obtained in the following fashion:

Let $\lambda x + \mu y + \nu z = 0$ be an arbitrary plane through the origin. The locus of the directions for which the planes of the conjugate diameters, with respect to the quadrics (**25**) and (**26**), respectively, cut this plane is, as one knows, a cone of second order that has the equation:

$$
(\mathbf{28}) \qquad \begin{vmatrix} \lambda & \mu & \nu \\ Ax + B''y + B'z & B''x + A'y + Bz & B'x + By + A''z \\ \alpha x + \beta''y + \beta'z & \beta''x + \alpha'y + \beta z & \beta'x + \beta y + \alpha''z \end{vmatrix} = 0.
$$

Furthermore, this cone passes through the three sides of the trihedron *T* that are mutually conjugate to the two quadrics in question (a triad that is always real since one of these quadrics is an ellipsoid). Conversely, the equation of any cone that passes through the sides of the trihedron *T* may be put into the preceding form.

We suppose that the cone (**28**) is capable of having a tri-rectangular trihedron inscribed in it. We find:

$$
\begin{gathered}
\lambda(\beta'B'' - \beta''B' + \beta(A' - A'') - B(\alpha' - \alpha'')) + \mu(\beta''B - \beta B'' \\
+ \beta'(A'' - A) - B'(\alpha'' - \alpha)) + \nu(\beta B' - B\beta' + \beta''(A - A') - B''(\alpha - \alpha')) = 0;
\end{gathered}
$$

in other words: the plane $\lambda x + \mu y + \nu z = 0$ must pass through the direction (**27**).

However, among all of the second-order cones that pass through the sides of the trihedron *T* one immediately perceives three of them that admit tri-rectangular trihedra. These are the ones that are formed by an arbitrary face of the trihedron *T* and the plane perpendicular to that face through the opposite side. Moreover, as one knows, the three planes so constructed intersect along the same line Δ. Since the condition that is imposed on a cone of second degree in order to make it admit a tri-rectangular trihedron is linear with respect to the coefficients the cones (**28**) that admit tri-rectangular trihedra will be the one that pass through the sides of the trihedron *T* and the line Δ.

What is more, it is clear that if one takes the conjugate diameter planes of Δ with respect to the two quadrics (**25**) and (**26**) then their intersection will provide the direction (**27**).

65. – One arrives at an interpretation of the segment (**27**) itself upon considering not only the quadric (**26**), but also the quadric (**25**) (or, what amounts to the same thing, the quadric $\varphi = 1$ that one deduces upon dividing $x^2 + y^2 + z^2$ by the left-hand side) to be the ellipsoid of inertia. Consequently, upon setting not only:

$$
\begin{aligned}
A &= \sum m(y^2 + z^2), & A' &= \sum m(z^2 + x^2), & A'' &= \sum m(x^2 + y^2), \\
B &= -\sum myz, & B' &= -\sum mzx, & B'' &= -\sum mxy,
\end{aligned}
$$

(in which, for the sake of simplicity, we have suppressed the useless initial factors); however:

$$\alpha = \sum m'(y'^2 + z'^2), \quad \alpha' = \sum m'(z'^2 + x'^2), \quad \alpha'' = \sum m'(x'^2 + y'^2),$$
$$\beta = -\sum m'y'z', \qquad \beta' = -\sum m'z'x', \qquad \beta'' = -\sum m'x'y',$$

and one finds the values:

$$\begin{aligned}
&\sum mm'[x'y'zx - z'x'xy - y'z'(y^2 - z^2) + yz(y'^2 - z'^2)] \\
&\qquad = \sum mm'(xx' + yy' + zz')(y'z - z'y) \\
&\sum mm'[y'z'xy - x'y'yz - z'x'(z^2 - x^2) + zx(z'^2 - x'^2)] \\
&\qquad = \sum mm'(xx' + yy' + zz')(z'x - x'z) \\
&\sum mm'[z'x'yz - y'z'zx - x'y'(x^2 - y^2) + xy(x'^2 - y'^2)] \\
&\qquad = \sum mm'(xx' + yy' + zz')(x'y - y'x)
\end{aligned}$$

for the quantities (**27**).

One knows that the expression $xx' + yy' + zz'$ does not change under an arbitrary transformation of rectangular coordinates, and also that the same is true for the geometric significance of the segment $(y'z - z'y, z'x - x'z, x'y - y'x)$. We have therefore illustrated this property for the segment (**27**).

66. – If the instantaneous molecular rotation is everywhere null then the expression $udx + vdy + wdz$ is an exact differential.

There then exists a function, which is called the *velocity potential*, whose partial derivatives are the components u, v, w. If the medium considered completely occupies space then this function – which is defined up to a constant – is unique in the whole medium. The same is true when it occupies only a portion of space if this portion is *simply linearly connected*, i.e., when any closed curve that is traced in the medium is reducible to a point by a continuous deformation that is performed without leaving the medium.

In the contrary case (for example, if the volume that is being occupied has the form of a torus) then the velocity potential may have *periods*, i.e., it is augmented by a constant when the point (x, y, z) describes a closed path that is not reducible to a point (in the case of the torus, when this point amounts to its original position after having been turned around the axis by 2π).

67. – When the instantaneous molecular rotation (p, q, r) is no longer null the differential equations:

(**29**) $$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

define a double infinitude of curves that are called *vortex lines.*

It may happen that the vortex lines close on themselves; however, in general, their character (like that of all lines defined by differential equations) is considerably more complicated. Each of them returns as close as one wants to its point of departure an infinitude of times, but without ever passing through that point precisely. Similar observations apply to the *vortex tubes*, or the surfaces that are formed by the vortex lines that issue from the points of an arbitrary closed curve. The misunderstanding of these circumstances has sometimes led people to state erroneous conclusions.

By contrast, the analogous reasoning, when applied to the *current lines,* i.e., the ones that are defined by the equations:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w},$$

in the case where there exists a velocity potential F, is exact (at least if the medium is simply linearly connected and, consequently, the function F is unique). Indeed, F is increasing along the current lines (since its differential $dF = udx + vdy + wdz$ is proportional to $u^2 + v^2 + w^2$). They may neither close on themselves nor exhibit the complicated character in question; they necessarily terminate on the boundary surface (or at infinity, if the medium is not bounded).

68. – If the instantaneous molecular rotation is non-null then one may further write:

$$u = \frac{\partial F}{\partial x}, \qquad v = \frac{\partial F}{\partial y}, \qquad w = \frac{\partial F}{\partial z}.$$

Clebsch has proposed to put the components of velocity into the form:

$$\text{(30)} \qquad \begin{cases} u = \dfrac{\partial F}{\partial x} + \psi \dfrac{\partial \chi}{\partial x}, \\ v = \dfrac{\partial F}{\partial y} + \psi \dfrac{\partial \chi}{\partial y}, \\ w = \dfrac{\partial F}{\partial z} + \psi \dfrac{\partial \chi}{\partial z}, \end{cases}$$

in which ψ and χ are two other functions that are left to be determined. He based the possibility of such a reduction on the following reasoning:

If one differentiates the second of equations (**30**) with respect to z, and the third one with respect to y, and then divides them then one obtains:

$$\text{(31)} \qquad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = -2p = \frac{\partial \psi}{\partial z}\frac{\partial \chi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \chi}{\partial z}.$$

This equation and the two analogous ones:

$$(\mathbf{31'}) \qquad \begin{cases} -2q = \dfrac{\partial\psi}{\partial x}\dfrac{\partial\chi}{\partial z} - \dfrac{\partial\psi}{\partial z}\dfrac{\partial\chi}{\partial x}, \\ -2r = \dfrac{\partial\psi}{\partial y}\dfrac{\partial\chi}{\partial x} - \dfrac{\partial\psi}{\partial x}\dfrac{\partial\chi}{\partial y}, \end{cases}$$

show that ψ and χ satisfy the partial differential equation:

$$p\frac{\partial\psi}{\partial x} + q\frac{\partial\psi}{\partial y} + r\frac{\partial\psi}{\partial z} = 0;$$

in other words, they are integrals of the system of differential equations (**29**) that defines the vortex lines.

However, the functions *p, q, r* satisfy the relation:

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} = 0.$$

From this, it results that the system (**29**) admits unity as a *multiplier* ([26]). The theory of the multiplier informs us that under these conditions one may find two integrals ψ and χ for this system that verify relations (**31**), (**31′**), from which one may derive (**30**) without difficulty.

Indeed, this demonstration proves precisely that one may give the form (**30**) to the components *u, v, w in a sufficiently small region R of the medium.* However, things are very different, in general, if one envisions the medium globally. Indeed, the existence of the integrals ψ and χ is established in a region such as *R*, and there is such a region in the neighborhood of every point of our medium. However, if – as is true except in exceptional cases – the vortex lines exhibit the complicated behavior referred to above all of the time then it is obviously impossible for ψ and χ to be well-defined in the entire volume considered.

By contrast, the form:

$$u = \frac{\partial F}{\partial x} + \frac{\partial\psi}{\partial z} - \frac{\partial\chi}{\partial y},$$
$$v = \frac{\partial F}{\partial y} + \frac{\partial\chi}{\partial x} - \frac{\partial\varphi}{\partial z},$$
$$w = \frac{\partial F}{\partial z} + \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x},$$

which has likewise been proposed for the components of the velocity, may always be obtained. In order to show this, it obviously suffices to establish that by means of the

([26]) See, for example, Jordan, *Cours d'Analyse,* vol. III, ch. 1.

condition $\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}=0$ one may always find three functions φ, ψ and χ that satisfy the equations:

$$A=\frac{\partial\psi}{\partial z}-\frac{\partial\chi}{\partial y},$$
$$B=\frac{\partial\chi}{\partial x}-\frac{\partial\varphi}{\partial z},$$
$$C=\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}.$$

However, the proof of this proposition ([27]) avoids the difficulty that we have always pointed out, and (by means of some simple precautions) this is also true if the region is multiply connected.

§ 2. – STUDY OF DISCONTINUITIES – IDENTITY CONDITIONS

69. – In the foregoing, we supposed that the coordinates *x*, *y*, *z* and their derivatives of various order were continuous. Meanwhile, this hypothesis is far from being the only one that is convenient to envision, and the study of motions in which one of the derivatives in question experiences abrupt variations is indispensable in a multitude of physical theories. The propagation of discontinuities in that space has been determined by Riemann for the case of rectilinear movement in a gas in a celebrated memoir that we shall discuss later on. Later on, in 1877, Christoffel ([28]) repeated the results of Riemann for the extension to motions in three dimensions, but he confined himself to waves of a very exceptional nature – viz., shock waves (first order waves) – whose existence had been discovered by Riemann, and, moreover, since the study of these waves presented special difficulties, he only considered a limiting case, the one for which the discontinuities are infinitely small. It was Hugoniot ([29]) who, in 1887, without any knowledge of the work of Riemann and Christoffel, moreover, showed the importance of the discontinuities that we shall discuss and made a general study; he illuminated a fundamental notion, that of *compatibility,* on which we shall later insist, and whose necessity seems to have been apparent to Christoffel, although it was indicated by Riemann in the case of rectilinear movement.

([27]) See, for example, Picard, *Traité d'Analyse,* tome I.

([28]) *Annali di Matematica,* tome VIII; 1877.

([29]) *Journal de l'Ecole Polytechnique,* tome XXXIII, 1887; *Journal de Math.,* tome III, series IV, 1887.

70. – The discontinuities that we shall study will not exactly be the most general that we are required to consider. For example, we shall not envision singularities such as the ones that we alluded to in sec. **45**.

On the contrary, we suppose that the discontinuities in question affect only an isolated surface at an arbitrary instant. The equation of one such, when referred to the initial state, will be:

(**32**) $$f(a, b, c,) = 0,$$

i.e., the preceding relation expresses the condition that must replace the initial coordinates *a, b, c* of a particle in order for it to posses a discontinuity at the instant *t*. Likewise, one may further envision the equation:

(**33**) $$\varphi(x, y, z) = 0,$$

of the same surface with respect to the present coordinates *x*, *y*, *z*. At each instant *t* this equation expresses the locus of present positions of the particle that are affected by the discontinuity at that instant, and one deduces the first equation from this one by eliminating (*a*, *b*, *c*) with the aid of relations (**1**). We let S_0 denote the surface as represented in Cartesian coordinates by equation (**32**). We further let *S* denote the one that is represented by equation (**33**), and which is transformed into the former by the deformation (**1**).

The surface S_0 divides the spatial locus of points (*a*, *b*, *c*) (or the surface *S* divides the spatial locus of points (*x*, *y*, *z*)) into two regions 1 and 2. In each of them (at least up until one encounters a new discontinuity surface) we assume that the coordinates *x*, *y*, *z* and their derivatives exist and are continuous.

71. – We shall complete this hypothesis with another one: Not only the derivatives that we shall discuss will be continuous in the interior of each of the regions 1 and 2 but we further assume that each of them tend toward a definite limit when the point (*a*, *b*, *c*) tends towards a limiting position that is situated on *S*, while always remaining inside the same region.

In other words, let Φ be an arbitrary function of *x*, *y*, *z*, *a*, *b*, *c*, *t*, and the partial derivatives of all orders of *x*, *y*, *z* with respect to *a*, *b*, *c*, *t*. This quantity, once it is expressed with the aid of the independent variables *a*, *b*, *c* for a definite value of *t*, will give a function Φ_1 that is defined at every interior point of region 1 and is continuous at those points. *Likewise, this function will have a definite value* Φ_0^1 *at an arbitrary point* (a_0, b_0, c_0) *of S*. Furthermore, it will be continuous, in the sense that Φ_1 tends toward Φ_0^1 when the point (*a*, *b*, *c*) tends toward (a_0, b_0, c_0) *without ceasing to belong to region* 1 *at any moment.*

Similarly, Φ, when considered in region 2, will be a function Φ_2 of *a*, *b*, *c* that will be defined and continuous in that entire region. That function will take a definite value Φ_0^2 at the point (a_0, b_0, c_0) of *S*; at this point, it will be continuous for displacements that are interior to region 2.

However, the two values Φ_0^1 and Φ_0^2 that correspond to the same point (a_0, b_0, c_0, t) might not be equal to each other, and this is what constitutes the discontinuity. Therefore, the value of Φ will be subject to an *abrupt variation* upon passing S. The value of that variation $\Phi_0^2 - \Phi_0^1$ will be denoted by the notation $[\Phi]$.

72. – Because of the preceding hypotheses, we must prove an analytical lemma that is necessary for all of what follows.

Suppose that a, b, c vary along a curve that is completely situated in region 1. Then, since we make the same hypotheses on the partial derivatives of Φ as we do for Φ itself, and, as a consequence, these derivatives will exist and be continuous in region 1, we will obtain the differential of Φ by applying the composite function theorem.

Will the same be true when the point (a, b, c) is situated on S and displaced on that surface? This is entirely obvious. Indeed, Φ_1 has no place on S, in terms of partial derivatives, properly speaking, since it is defined only on one side of S and not in all of a neighborhood of the point considered; one is therefore not within the purview of the usual condition for the application of the theorem in question.

Nevertheless, the conclusion remains exact. We shall confirm this by placing ourselves in the two-dimensional case, for the sake of simplicity.

We then have a function Φ that is defined on only one side of a curve $\mathcal{S}$ (*fig.* 6). In its region of existence, it will have partial derivatives that tend towards definite limits (which we designate by $\frac{\partial\Phi}{\partial x_0}$ and $\frac{\partial\Phi}{\partial y_0}$) when the point ($x$, y) tends towards a point M (x_0, y_0) of $\mathcal{S}$.

This amounts to saying that F will have a derivative that is given by the relation:

$$\frac{d\Phi}{ds} = \frac{\partial\Phi}{\partial x_0}\frac{dx_0}{ds} + \frac{\partial\Phi}{\partial y_0}\frac{dy_0}{ds},$$

along the arc $\mathcal{S}$.

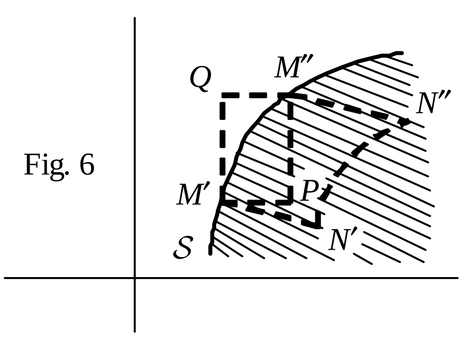

Fig. 6

One may respond to the question above by employing the classical proof in a convenient manner. If M' and M'' are two points on $\mathcal{S}$ that are infinitely close to each other then it consists of introducing either the point P that has the same abscissa as M' and the same ordinate as M'' or the point Q that has the same abscissa as M'' and the same ordinate as M'.

However, of the two broken lines $M'PM''$ and $M'QM''$, there is, in general, one of them (*fig.* 6) that is situated completely within the region of existence for Φ, and which permits us to apply this reasoning as a consequence.

One may further arrive at this result, as Painlevé [30] indicated, by taking small segments from the different points of the arc $M'M''$ that are equal and parallel to ones that are situated in the region of existence for Φ. The locus of extremities of these segments is an arc $N'N''$, on which one may write (since the derivatives of Φ exist this time):

$$\Phi_{N''} - \Phi_{N'} = \int_{N'N''} \left(\frac{\partial \Phi}{\partial x} \frac{dx}{ds} + \frac{\partial \Phi}{\partial y} \frac{dy}{ds} \right) ds.$$

When the length of the segment tends toward 0 the quantities $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial y}$ tend – uniformly, moreover – toward $\frac{\partial \Phi}{\partial x_0}$ and $\frac{\partial \Phi}{\partial y_0}$. One will thus have, in the limit:

$$\Phi_{M''} - \Phi_{M'} = \int_{M'M''} \left(\frac{\partial \Phi}{\partial x_0} \frac{dx_0}{ds} + \frac{\partial \Phi}{\partial y_0} \frac{dy_0}{ds} \right) ds,$$

which is obviously equivalent to the desired result.

Each of the two preceding lines of reasoning may obviously be extended unchanged to a greater number of dimensions in such a way that our lemma is proved.

73. – Having said this, suppose that the function Φ is not subject to any abrupt variation on S_0, but that its first derivatives are, on the contrary, discontinuous; in other words, that one has:

$$[\Phi] = 0, \qquad \left[\frac{\partial \Phi}{\partial a}\right], \left[\frac{\partial \Phi}{\partial b}\right], \left[\frac{\partial \Phi}{\partial c}\right] \neq 0.$$

The preceding lemma permits us to see that the changes in the values $\left[\frac{\partial \Phi}{\partial a}\right], \left[\frac{\partial \Phi}{\partial b}\right], \left[\frac{\partial \Phi}{\partial c}\right]$ that are experienced by these partial derivatives may not be arbitrary.

Indeed, describe an arbitrary path that is situated on the surface S_0 at the point (a, b, c). The function Φ_1 is defined at each point of this path; one may, moreover, differentiate it on S_0 by applying our lemma, and write:

$$d\Phi_1 = \frac{\delta \Phi_1}{\delta a} da + \frac{\delta \Phi_1}{\delta b} db + \frac{\delta \Phi_1}{\delta c} dc.$$

The same considerations apply to the function Φ_2; one obtains:

[30] Sur les lignes singuliéres des functions analytiques, *Annales scientifiques de l'Ecole Normale supérieure*, 1887, 1st part, ch. II, no. 2.

$$d\Phi_2 = \frac{\delta\Phi_2}{\delta a}da + \frac{\delta\Phi_2}{\delta b}db + \frac{\delta\Phi_2}{\delta c}dc.$$

Divide the corresponding sides of these equations; the left-hand sides cancel since Φ is assumed to be continuous when passing S_0. We get:

$$\left(\frac{\delta\Phi_1}{\delta a} - \frac{\delta\Phi_2}{\delta a}\right)da + \left(\frac{\delta\Phi_1}{\delta b} - \frac{\delta\Phi_2}{\delta b}\right)db + \left(\frac{\delta\Phi_1}{\delta c} - \frac{\delta\Phi_2}{\delta c}\right)dc$$
$$= \left[\frac{\delta\Phi}{\delta a}\right]da + \left[\frac{\delta\Phi}{\delta b}\right]db + \left[\frac{\delta\Phi}{\delta c}\right]dc = 0.$$

However, the differentials, *da, db, dc* are clearly arbitrary, except for the condition that they satisfy the differential equations for S_0:

$$f_a da + f_b db + f_c dc = 0$$

(in which we have denoted the partial derivatives of f by f_a, f_b, f_c, as before).

Therefore, one must have:

$$\left[\frac{\delta\Phi}{\delta a}\right] : f_a = \left[\frac{\delta\Phi}{\delta b}\right] : f_b = \left[\frac{\delta\Phi}{\delta c}\right] : f_c. \tag{34}$$

74. – Now suppose that not only Φ, but also its first derivatives remain continuous. What may we say about the abrupt variations of the second derivatives?

We may apply the preceding mode of reasoning to the function $\delta\Phi/\delta a$; this gives:

$$\left[\frac{\delta^2\Phi}{\delta a^2}\right] : f_a = \left[\frac{\delta^2\Phi}{\delta a\,\delta b}\right] : f_b = \left[\frac{\delta^2\Phi}{\delta a\,\delta c}\right] : f_c,$$

and similarly for the functions $\frac{\delta\Phi}{\delta b}$, $\frac{\delta\Phi}{\delta c}$, which gives:

$$\left[\frac{\delta^2\Phi}{\delta a\,\delta b}\right] : f_a = \left[\frac{\delta^2\Phi}{\delta b^2}\right] : f_b = \left[\frac{\delta^2\Phi}{\delta b\,\delta c}\right] : f_c,$$
$$\left[\frac{\delta^2\Phi}{\delta a\,\delta c}\right] : f_a = \left[\frac{\delta^2\Phi}{\delta b\,\delta c}\right] : f_b = \left[\frac{\delta^2\Phi}{\delta c^2}\right] : f_c.$$

As before, these equalities show that one has:

$$
\textbf{(35)} \qquad \begin{cases}
\left[\dfrac{\delta^2\Phi}{\delta a^2}\right] = \lambda f_a^2, & \left[\dfrac{\delta^2\Phi}{\delta b^2}\right] = \lambda f_b^2, \quad \left[\dfrac{\delta^2\Phi}{\delta c^2}\right] = \lambda f_c^2, \\
\left[\dfrac{\delta^2\Phi}{\delta b\,\delta c}\right] = \lambda f_b\, f_c, & \left[\dfrac{\delta^2\Phi}{\delta c\,\delta a}\right] = \lambda f_c\, f_a, \quad \left[\dfrac{\delta^2\Phi}{\delta a\,\delta b}\right] = \lambda f_a\, f_b,
\end{cases}
$$

when λ is a suitably chosen number.

In a general manner, if the function Φ is continuous, along with its derivatives up to order $n-1$, then one will have the following series of proportions between the partial derivatives of order *n:*

$$
\textbf{(35}'\textbf{)} \qquad \left[\frac{\delta^n\Phi}{\delta a^n}\right] : f_a^n = \cdots = \left[\frac{\delta^n\Phi}{\delta a^p\,\delta b^q\,\delta c^r}\right] : f_a^p\, f_b^q\, f_c^r = \cdots = \left[\frac{\delta^n\Phi}{\delta c^n}\right] : f_c^n .
$$

Upon denoting the common value of these rations by λ, one will have for any *da, db, dc:*

$$
\left(\left[\frac{\delta}{\delta a}\right]da + \left[\frac{\delta}{\delta b}\right]db + \left[\frac{\delta}{\delta c}\right]dc\right)^n \Phi = \lambda(f_a da + f_b db + f_c dc)^n .
$$

75. – The discontinuities that we shall study might be of different orders. To begin with, it may happen that the coordinates *x, y, z* themselves are discontinuous, not only as functions of time (we never assume that a molecule passes instantaneously from one position to another), but also as functions of *a, b, c*. Such discontinuities will be called *of order zero* or *absolute.*

In the contrary case, the discontinuity does not depend on the coordinates themselves, but on their derivatives; these are classified by their orders.

We call the total order of differentiation with respect to *a, b, c, t,* namely, $p + q + r = n$, the *order* of a derivative $\dfrac{\delta^n x}{\delta a^p\,\delta b^q\,\delta c^r\,\delta t^s}$.

Often, there will be good reason to establish categories within the derivatives of the same order, according to the number *s* of differentiations that are performed with respect to *t.* This latter number will be called the *index* of the derivative considered.

For example, there are 30 second-order derivatives of *x, y, z,* 18 of which have index zero, namely:

$$
\frac{\delta^2 x}{\delta a^2}, \frac{\delta^2 x}{\delta a\,\delta b}, \ldots, \frac{\delta^2 x}{\delta c^2}; \frac{\delta^2 y}{\delta a^2}, \ldots, \frac{\delta^2 z}{\delta c^2},
$$

9 of which have index one, namely:

$$
\frac{\delta^2 x}{\delta a\,\delta t}, \frac{\delta^2 x}{\delta b\,\delta t}, \frac{\delta^2 x}{\delta c\,\delta t}; \frac{\delta^2 y}{\delta a\,\delta t}, \ldots, \frac{\delta^2 z}{\delta c\,\delta t},
$$

and 3 of which have index two:

$$\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2},$$

i.e., the components of acceleration.

The *order of a discontinuity* will be the least of the orders of the derivatives that are affected.

76. – We must point out that one may be forced to consider discontinuities of order infinity.

Indeed, if a function is regularly analytic around the point (a_0, b_0, c_0) then it is determined by the Taylor formula when one gives values to all of its derivatives at that point, which is not true if the function is arbitrary, either because it ceases to be analytic or because it ceases to be regular.

Therefore, suppose that the motion is analytic in all of region 1. It may happen that one has continuity of the derivatives of all orders upon crossing S_0, and that, nevertheless, the motion in region 2 cannot be analytically continued into the first region, either because it is not itself analytic, or because it presents suitable singularities on S_0 (for example, like those that the function e^{1/x^2} presents at $x = 0$).

The study of discontinuities of this nature presents particular difficulties. We shall not go into them in what follows.

77. – Leaving aside the absolute discontinuities, we shall first occupy ourselves with discontinuities of the first order.

There are then three derivatives of index one and nine derivatives of index zero. The first are the components of velocity. At the moment, we have no observation to make in regard to them.

On the contrary, from the lemma that we proved it always results that the abrupt variations of the derivatives of index zero cannot be arbitrary. Indeed, since, by hypothesis, there is no discontinuity of order zero and x is continuous one must have:

$$\frac{\left[\frac{\delta x}{\delta a}\right]}{f_a} = \frac{\left[\frac{\delta x}{\delta b}\right]}{f_b} = \frac{\left[\frac{\delta x}{\delta c}\right]}{f_c},$$

or, upon denoting the common value of these ratios by λ:

$$\textbf{(36)} \qquad \left[\frac{\delta x}{\delta a}\right] = \lambda f_a, \quad \left[\frac{\delta x}{\delta b}\right] = \lambda f_b, \quad \left[\frac{\delta x}{\delta c}\right] = \lambda f_c.$$

Similarly, upon introducing two other numbers μ and ν one will have:

$$(\mathbf{36'})\qquad \begin{cases} \left[\dfrac{\delta y}{\delta a}\right] = \mu f_a, & \left[\dfrac{\delta y}{\delta b}\right] = \mu f_b, & \left[\dfrac{\delta y}{\delta c}\right] = \mu f_c, \\ \left[\dfrac{\delta z}{\delta a}\right] = \nu f_a, & \left[\dfrac{\delta z}{\delta b}\right] = \nu f_a, & \left[\dfrac{\delta z}{\delta c}\right] = \nu f_a. \end{cases}$$

We consider λ, μ, ν to be the projections of a vector (the locus of points traced out by starting with a point (x, y, z) in space). *This vector suffices to define the variations of the nine derivatives of index zero.*

In order to obtain the abrupt variations of all of the first-order derivatives one adds the vector that represents the abrupt variation of the velocity – which we call, by analogy, (λ_1, μ_1, ν_1) – to this latter vector.

78. – The considerations that we developed in no. **56** permit us to give a simple geometrical interpretation to the preceding result. To that effect, imagine – which is clearly possible – a fictitious state of the medium that coincides with the present state envisioned in region 1, but in such a way that the coordinate derivatives are continuous upon passing the surface S_0.

To abbreviate, we call the state thus defined in all of space "the state of region 1;" as one sees, it is the state of region 1 prolonged into region 2.

Let x', y', z' be the coordinates of an arbitrary particle of region 2 in this new state.

The quantities:

$$\left[\frac{\delta x}{\delta a}\right], \left[\frac{\delta x}{\delta b}\right], \left[\frac{\delta x}{\delta c}\right]$$

are obviously nothing but the values of the expressions:

$$\frac{\delta(x-x')}{\delta a}, \frac{\delta(x-x')}{\delta b}, \frac{\delta(x-x')}{\delta c}$$

(considered with respect to region 2) at an arbitrary point of S_0.

However, since x', y', z' coincide with x, y, z at any point of S_0 the deformation that permits us to pass from the point (x', y', z') to the point (x y, z) falls into the category that we studied in no. **56**. In other words, the displacement that is experienced by a point M that is infinitely close to a definite point of S_0 during this deformation has constant direction and is proportional to the distance from M to S_0.

Moreover, this results from the preceding formulas. Indeed, if we start with a point of S_0 and give increments da, db, dc to a, b, c then since, on the other hand, f is null on S_0, one will naturally have:

$$df = f = f_a da + f_b db + f_c dc,$$

in such a way that one may write, up to higher-order infinitesimals:

$$\textbf{(37)} \qquad \begin{cases} x - x' = \lambda f, \\ y - y' = \mu f, \\ z - z' = \nu f. \end{cases}$$

One therefore sees quite well that the displacement of our particle upon passing from the state of region 1 to that of region 2 is represented by the segment (λ, μ, ν) multiplied by the number f, which is itself proportional to the distance from the point (a, b, c) to the surface $f = 0$.

79. – It results immediately from this remark that the segment we just introduced, which we represented by the numbers λ, μ, ν, is independent of the direction of the axes with respect to which the (x, y, z) is defined. In other words, if one refers to other rectangular axes of space at this point then the new values of λ, μ, ν will be the projections of the same segment on the new axes. This segment represents the displacement of the point considered (upon passing from (x', y', z') to (x, y, z)) divided by the value of f, which is calculated on the initial state and is independent of the present coordinates.

79 (cont.). – However, the choice of axes in space is not the only arbitrary element that exists in our mode of representation.

In the first place, the surface of discontinuity, when referred to the initial state, was represented by an equation $f(a, b, c) = 0$. It is clear that there are therefore an infinitude of ways of representing the same surface. Nothing will prevent us from multiplying f by an arbitrary constant number, or, more generally, by an arbitrary function that is non-null on S_0.

In the second place, we may choose the initial state that we refer the molecules to in an entirely arbitrary fashion. We therefore must demand to know what sort of influence the choice of initial state will have on (λ, μ, ν).

We first occupy ourselves with this question. Suppose that one changes the initial state (a, b, c) into another one (a', b', c'), but without changing the function f (in other words, we are content to replace a, b, c with their values as a function of (a', b', c') in this function). Therefore, *the segment* (λ, μ, ν) *will not change.* It results immediately from the interpretation that we just indicated that this segment is the quotient of the displacement of an arbitrary point when one passes from the state in region 1 to the state in region 2 with the value of f at that point.

80. – Now suppose, on the contrary, that without changing a, b, c, one multiplies the function f by a (continuous and differentiable) factor that is non-null at the points considered. With these conditions, f_a, f_b, f_c will obviously be multiplied by the same number ([31]) since they are proportional to the direction cosines of the normal to S_0. Therefore, λ, μ, ν *must be divided by that number* in formulas (**36**), (**36′**).

We thus see that it is necessary to specify the form in which one writes the equation of the surface S_0 in order to define the components λ, μ, ν in a precise fashion.

The convention that is natural make in this regard consists of assuming, for example, that α, β, γ are *equal* to the direction cosines of the normal to S_0, respectively, and taking the normal distance from the point (a, b, c) to that surface to be f. We adopt this convention in what follows.

It is, moreover, easy to write the components λ, μ, ν thus defined when the equation of S_0 is given in an arbitrary form. The direction cosines α, β, γ will then have the values:

$$\frac{f_a}{\sqrt{f_a^2+f_b^2+f_c^2}}, \qquad \frac{f_b}{\sqrt{f_a^2+f_b^2+f_c^2}}, \qquad \frac{f_c}{\sqrt{f_a^2+f_b^2+f_c^2}}.$$

It will then suffice to replace formulas (**36**), (**36′**) with:

$$\textbf{(38)} \quad \begin{cases} \left[\dfrac{\delta x}{\delta a}\right] = \lambda\dfrac{f_a}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta x}{\delta b}\right] = \lambda\dfrac{f_b}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta x}{\delta c}\right] = \lambda\dfrac{f_c}{\sqrt{f_a^2+f_b^2+f_c^2}}, \\ \left[\dfrac{\delta y}{\delta a}\right] = \mu\dfrac{f_a}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta y}{\delta b}\right] = \mu\dfrac{f_b}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta y}{\delta c}\right] = \mu\dfrac{f_c}{\sqrt{f_a^2+f_b^2+f_c^2}}, \\ \left[\dfrac{\delta z}{\delta a}\right] = \nu\dfrac{f_a}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta z}{\delta b}\right] = \nu\dfrac{f_b}{\sqrt{f_a^2+f_b^2+f_c^2}}, & \left[\dfrac{\delta z}{\delta c}\right] = \nu\dfrac{f_c}{\sqrt{f_a^2+f_b^2+f_c^2}}. \end{cases}$$

81. – Meanwhile, it remains for us to choose the sign of the radical $\sqrt{f_a^2+f_b^2+f_c^2}$. We suppose that the direction cosines α, β, γ are those of the normal to S_0 that is directed into region 2. The sign of the radical must therefore be that of f in that region.

On the contrary, if f has been chosen in such a manner as to permit the application of formulas (**36**), (**36′**) then, to that effect, it must be equal (at least up to higher-order infinitesimals) to the normal distance from the point (a, b, c) to S_0, where this distance must be considered as positive in region 2 and negative in region 1.

82. – The preceding convention resolves the difficulty relating to the form of the function f. However, it spoils the validity of the remark that is always made that the choice of initial state seems to have no influence on the result. Indeed, for two different initial states (a, b, c) and (a', b', c') the quantity $f_a^2+f_b^2+f_c^2$ that figures in the

([31]) This number is, moreover, the value that the factor in question takes at the point considered.

denominators of formulas (**38**) has different values ([32]). As a consequence, depending upon whether one has adopted one or the other of them, the segment (λ, μ, ν) must be multiplied by a different factor.

It is, moreover, easy to see the significance of the ratio of these two factors. Indeed, each of them represents the quantity by which f must be multiplied in order for it to represent the normal distance from a point to the surface of discontinuity in the corresponding initial state. Their ratio is therefore the normal dilatation to that surface when passing from one of these states to the other.

83. – Therefore, we may now speak of the segment (λ, μ, ν) only upon indicating which initial state it was formed from.

For certain questions (for example, in elasticity) the initial state is indicated by the nature of the problem itself. The same is not true in hydrodynamics. We then agree to take the actual state of region 1 at the instant considered to be the initial state. It is true that this state is defined only in a part of the medium. However, one may prolong it into region 2, as we shall do in a moment, since the partial derivatives of the coordinates x, y, z with respect to the coordinates, a, b, c (the coordinates of the arbitrary initial state that we originally chose) remain continuous. That fictitious state may be taken to be the new initial state without having the condition that was stated in no. **45** (cont.) cease to apply.

84. – If one inverts the roles of the regions 1 and 2 then it is immediately clear that one must change the signs of the quantities:

$$\left[\frac{\delta x}{\delta a}\right] = \left(\frac{\delta x}{\delta a}\right)_2 - \left(\frac{\delta x}{\delta a}\right)_1, \left[\frac{\delta x}{\delta b}\right], \left[\frac{\delta x}{\delta c}\right].$$

The same will be true of:

$$\left[\frac{\delta y}{\delta a}\right], \dots, \left[\frac{\delta z}{\delta c}\right].$$

However, on the other hand, the denominator $\sqrt{f_a^2 + f_b^2 + f_c^2}$ will experience a double change. On the one hand, there will be a change of sign. On the other hand, from what we just said, there will be a multiplication by a factor that is equal to the dilatation normal to S that is associated with passing from state 1 to state 2, i.e., unity plus the normal component $\lambda\alpha + \mu\beta + \nu\chi$ of our segment.

([32]) $\varphi(da', db', dc')$ represents the linear element $da^2 + db^2 + dc^2$ expressed with the aid of the variables da', db', dc'; Φ is the form added to φ; D is the functional determinant $\frac{D(a,b,c)}{D(a',b',c')}$, and the quantity $\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2$ is equal to $\frac{1}{D^2}\Phi\left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c},\right)$.

By definition, depending on whether or not the direction cosines α, β, γ will be changed into $-\alpha, -\beta, -\gamma,$ the components that we must concern ourselves with will change into:

$$
\textbf{(39)} \qquad \begin{cases} \lambda' = \dfrac{\lambda}{1+\lambda\alpha+\mu\beta+\nu\gamma}, \qquad \mu' = \dfrac{\mu}{1+\lambda\alpha+\mu\beta+\nu\gamma}, \\ \nu' = \dfrac{\nu}{1+\lambda\alpha+\mu\beta+\nu\gamma}. \end{cases}
$$

85. – Before passing to the general case, we again treat the case of second-order discontinuities, by reason of its importance. First imagine the derivatives of index 0. Since the function x is continuous, as well as its derivatives of first order, the proposition of no. **74** shows us the existence of a number λ such that one has:

$$
\textbf{(40)} \qquad \begin{cases} \left[\dfrac{\delta^2 x}{\delta a^2}\right] = \lambda f_a^2, \qquad \left[\dfrac{\delta^2 x}{\delta b^2}\right] = \lambda f_b^2, \qquad \left[\dfrac{\delta^2 x}{\delta c^2}\right] = \lambda f_c^2, \\ \left[\dfrac{\delta^2 x}{\delta b\,\delta c}\right] = \lambda f_b\,f_c, \quad \left[\dfrac{\delta^2 x}{\delta c\,\delta a}\right] = \lambda f_c\,f_a, \quad \left[\dfrac{\delta^2 x}{\delta a\,\delta b}\right] = \lambda f_a\,f_b. \end{cases}
$$

Similarly, if μ and ν denote two conveniently chosen numbers, one may write:

$$
\textbf{(40}'\textbf{)} \qquad \begin{cases} \left[\dfrac{\delta^2 y}{\delta a^2}\right] = \mu f_a^2, \quad \left[\dfrac{\delta^2 y}{\delta b^2}\right] = \mu f_b^2, \quad \left[\dfrac{\delta^2 y}{\delta c^2}\right] = \mu f_c^2, \quad \left[\dfrac{\delta^2 y}{\delta b\,\delta c}\right] = \mu f_b\,f_c, \\ \qquad \left[\dfrac{\delta^2 y}{\delta c\,\delta a}\right] = \mu f_c\,f_a, \quad \left[\dfrac{\delta^2 y}{\delta a\,\delta b}\right] = \mu f_a\,f_b, \\ \left[\dfrac{\delta^2 z}{\delta a^2}\right] = \nu f_a^2, \quad \left[\dfrac{\delta^2 z}{\delta b^2}\right] = \nu f_b^2, \quad \left[\dfrac{\delta^2 z}{\delta c^2}\right] = \nu f_c^2, \quad \left[\dfrac{\delta^2 z}{\delta b\,\delta c}\right] = \nu f_b\,f_c, \\ \qquad \left[\dfrac{\delta^2 z}{\delta c\,\delta a}\right] = \nu f_c\,f_a, \quad \left[\dfrac{\delta^2 z}{\delta a\,\delta b}\right] = \nu f_a\,f_b. \end{cases}
$$

The numbers λ, μ, ν will again be considered the as the components of a segment.

It is clear that this result may be interpreted as in the preceding. If we prolong the state in region 1 into region 2, in such a manner that the second derivatives remain continuous, then in order to pass from state 1 thus prolonged into region 2, a transformation that belongs to the category we studied in no. **57** must act on this latter region, in such a way that if x', y', z' are the coordinates of the prolonged state 1, and x, y, z are those of state 2 then one will have formulas (**14**) for $x'-x, y'-y, z'-z,$ and analogous ones from no. **57**, which will be completely equivalent to the preceding ones.

These formulas show that one has:

$$\text{(41)} \qquad x - x' = \lambda \frac{f^2}{2}, \qquad y - y' = \mu \frac{f^2}{2}, \qquad z - z' = \nu \frac{f^2}{2}.$$

The may furthermore be written:

$$\left(\left[\frac{\delta}{\delta a} \right] da + \left[\frac{\delta}{\delta b} \right] db + \left[\frac{\delta}{\delta c} \right] dc \right)^2 (x, y, z)$$
$$= (\lambda, \mu, \nu)(f_a\, da + f_b\, db + f_c\, dc)^2.$$

86. – We pass to the derivatives of index 1. They will be of the form:

$$\frac{\delta^2 x}{\delta a\, \delta t}, \dots, \frac{\delta^2 z}{\delta c\, \delta t}.$$

We apply the lemma of no. **72** to the quantity $\delta x / \delta t$; we will have:

$$\text{(42)} \qquad \left[\frac{\delta^2 x}{\delta a\, \delta t} \right] = \lambda_1 f_a, \qquad \left[\frac{\delta^2 x}{\delta b\, \delta t} \right] = \lambda_1 f_b, \qquad \left[\frac{\delta^2 x}{\delta c\, \delta t} \right] = \lambda_1 f_c,$$

and similarly:

$$\text{(42}'\text{)} \qquad \begin{cases} \left[\dfrac{\delta^2 y}{\delta a\, \delta t} \right] = \mu_1 f_a, & \left[\dfrac{\delta^2 y}{\delta b\, \delta t} \right] = \mu_1 f_b, & \left[\dfrac{\delta^2 y}{\delta c\, \delta t} \right] = \mu_1 f_c, \\ \left[\dfrac{\delta^2 z}{\delta a\, \delta t} \right] = \nu_1 f_a, & \left[\dfrac{\delta^2 z}{\delta b\, \delta t} \right] = \nu_1 f_b, & \left[\dfrac{\delta^2 z}{\delta c\, \delta t} \right] = \nu_1 f_c, \end{cases}$$

in which $(\lambda_1, \mu_1, \nu_1)$ are a new segment.

Finally, the discontinuities λ_2, μ_2, ν_2 experienced by the three derivatives of index two may be considered to be the components of a third segment, which is the abrupt variation of the acceleration.

The geometric interpretation of this latter segment is therefore self-evident. As for that of the segment $(\lambda_1, \mu_1, \nu_1)$, one may obtain it by attributing fictitious velocities to the different points that are equal to the present values in region 1, but in such a way that the partial derivatives of their components are continuous. Of course, this will change the new positions that are acquired by the points of region 2 during a time δt, and this change may be interpreted as a deformation that belongs to the category that was studied in no. **56**. Since this deformation is proportional to δt, it will have a characteristic segment of the form $(\lambda_1 \delta t, \mu_1 \delta t, \nu_1 \delta t)$, in which λ_1, μ_1, ν_1 are the quantities that figure in formulas (**42**), (**42′**).

As before, it results from this that the segments (λ, μ, ν) and $(\lambda_1, \mu_1, \nu_1)$ do not change when one changes the initial state without changing the function f (in the previously-explained sense), but only changes its form by altering the components of each by the same ratio ([33]). It is then convenient to take f equal to the normal distance to the point a, b, c of S_0, or, what amounts to the same thing, to replace the preceding formulas by:

$$\left[\frac{\delta^2 x}{\delta a^2}\right] = \lambda\alpha^2, \ldots, \left[\frac{\delta^2 x}{\delta b\,\delta c}\right] = \lambda\beta\gamma, \ldots; \left[\frac{\delta^2 y}{\delta a^2}\right] = \mu\alpha^2, \ldots, \left[\frac{\delta^2 z}{\delta a\,\delta b}\right] = \nu\alpha\beta,$$

in which α, β, γ again denote the direction cosines of the normal to S_0 that is directed into region 2. Likewise, as in the case of first-order discontinuities, one must specify what the choice of initial state is. When there is nothing in the nature of the problem to suggest a choice, one takes the present state at the instant considered.

Here, contrary to what takes place in the first-order case, it makes no difference if the state is chosen from region 1 or region 2. Indeed, the deformation that permits us to pass from one of these states to the other one coincides with the identity transformation up to second-order infinitesimals in a neighborhood of the points of S_0 . Therefore, there is no normal dilatation at these points, and consequently no change in the quantity $f_a^2 + f_b^2 + f_c^2$.

If one inverts the roles of the two regions then the coefficients f_a, f_b, f_c will undergo simple sign changes. As a consequence, the same will be true for λ, μ, ν, whereas λ_1, μ_1, ν_1 remain unaltered.

88. – The results that relate to the case of arbitrary n now appear by themselves. There will exist $n+1$ segments, of which, the first one (λ, μ, ν) will make the variations of the derivatives of index 0 known, by means of the formulas:

$$\textbf{(43)} \qquad \begin{cases} \left[\dfrac{\delta^n x}{\delta a^p\,\delta b^q\,\delta c^r}\right] = \lambda\alpha^p\beta^q\gamma^r, \quad \left[\dfrac{\delta^n y}{\delta a^p\,\delta b^q\,\delta c^r}\right] = \mu\alpha^p\beta^q\gamma^r, \\ \qquad\qquad \left[\dfrac{\delta^n z}{\delta a^p\,\delta b^q\,\delta c^r}\right] = \nu\alpha^p\beta^q\gamma^r, \end{cases} \qquad (p+q+r=n),$$

or

$$\left(\left[\frac{\delta}{\delta a}\right]da + \left[\frac{\delta}{\delta b}\right]db + \left[\frac{\delta}{\delta c}\right]dc\right)^n (x, y, z) = (\lambda, \mu, \nu)(\alpha da + \beta db + \gamma dc)^n;$$

the second segment (λ_1, μ_1, ν_1) exhibits the variations of the derivatives of index 1 by the formulas:

([33]) This ratio is equal to the ratio of the two values of f, as it relates to the segment (λ_1, μ_1, ν_1); however, for the segment (λ, μ, ν), it has a value equal to the square of the first one.

$$\textbf{(43}'\textbf{)}\qquad \begin{cases} \left[\dfrac{\delta^n x}{\delta a^p \delta b^q \delta c^r \delta t}\right] = \lambda_1 \alpha^p \beta^q \gamma^r, \\ \left[\dfrac{\delta^n y}{\delta a^p \delta b^q \delta c^r \delta t}\right] = \mu_1 \alpha^p \beta^q \gamma^r, \qquad (p+q+r=n-1) \\ \left[\dfrac{\delta^n z}{\delta a^p \delta b^q \delta c^r \delta t}\right] = \nu_1 \alpha^p \beta^q \gamma^r, \end{cases}$$

The h+1th segment (λ_h, μ_h, ν_h) will give the variations of the derivatives of index h:

$$\textbf{(43}''\textbf{)}\qquad \begin{cases} \left[\dfrac{\delta^n x}{\delta a^p \delta b^q \delta c^r \delta t^h}\right] = \lambda_h \alpha^p \beta^q \gamma^r, \\ \left[\dfrac{\delta^n y}{\delta a^p \delta b^q \delta c^r \delta t^h}\right] = \mu_h \alpha^p \beta^q \gamma^r, \qquad (p+q+r=n-h), \\ \left[\dfrac{\delta^n z}{\delta a^p \delta b^q \delta c^r \delta t^h}\right] = \nu_h \alpha^p \beta^q \gamma^r, \end{cases}$$

and so on, up the n+1th segment (λ_n, μ_n, ν_n), which will be nothing but:

$$\left[\frac{\delta^n x}{\delta t^n}\right], \qquad \left[\frac{\delta^n y}{\delta t^n}\right], \qquad \left[\frac{\delta^n z}{\delta t^n}\right].$$

The geometric interpretation of the segments (λ, μ, ν) and (λ_1, μ_1, ν_1) will be the same as always. That of the segment (λ_2, μ_2, ν_2) is obtained by correcting the accelerations of the points of region 2 in such a way as to make the derivatives of order $n-2$ continuous when passing S_0, while leaving the positions and velocities unaltered. A deformation results from this during the infinitely small time δt, which is proportional to δt^2, and whose characteristic segment will be ($\lambda_2\, \delta t^2$, $\mu_2\, \delta t^2$, $\nu_2\, \delta t^2$).

The same situation will apply to the remaining segments upon introducing accelerations of higher order.

From this (or, further, from the obvious fact that the derivatives of x, y, z transform like the variables themselves under a change of absolute coordinates), one deduces that the preceding segments do not depend on the choice of coordinates in (x, y, z) space. They are independent of the choice of function f, since the preceding formulas contain only the direction cosines of the normal to S_0.

If no particular state is suggested by the problem then one chooses the present state at the instant considered to be in either of the two regions (which is indifferent to the fact that n is greater than unity).

The inversion of regions 1 and 2 will change the sense of the segments whose index has the same parity as n and leave the others unaltered.

§ 3. – STUDY OF DISCONTINUITIES (cont.)
KINEMATICAL COMPATIBILITY CONDITIONS

89. – We now must demand to know whether the relations that we obtained up till now are the only ones that the elements of our discontinuities are subject to.

Suppose that we are given the numbers λ, μ, ν; λ_1, μ_1, ν_1;…; λ_n, μ_n, ν_n arbitrarily at each point a_0, b_0, c_0 of S_0. On the other hand, let X, Y, Z; X_1, Y_1, Z_1; …;X_n, Y_n, Z_n be functions of a, b, c that are everywhere continuous, as well as their derivatives of all orders. At the instant considered t_0, we give the different points of the medium:

Positions (X, Y, Z) in region 1 and positions:

$$\left(X+\frac{\lambda f^n}{n!}, Y+\frac{\mu f^n}{n!}, Z+\frac{\nu f^n}{n!}\right),$$

in region 2;

Velocities (X_1, Y_1, Z_1), in region 1 and velocities:

$$\left(X_1+\frac{\lambda_1 f^{n-1}}{(n-1)!}, Y_1+\frac{\mu_1 f^{n-1}}{(n-1)!}, Z_1+\frac{\nu_1 f^{n-1}}{(n-1)!}\right),$$

in region 2;

Accelerations (X_2, Y_2, Z_2) in region 1 and accelerations:

$$\left(X_2+\frac{\lambda_2 f^{n-2}}{(n-2)!}, Y_2+\frac{\mu_2 f^{n-2}}{(n-2)!}, Z_2+\frac{\nu_2 f^{n-2}}{(n-2)!}\right),$$

in region 2, etc.

Finally, there will be accelerations of order n that are equal to (X_n, Y_n, Z_n) in region 1 and ($X_n + \lambda_n$, $Y_n + \mu_n$, $Z_n + \nu_n$) in region 2.

In the preceding expressions it is convenient to give λ, μ, ν, …, λ_n, μ_n, ν_n the values that they have at the point (a_0, b_0, c_0), which is the foot of the normal that is based at the point (a, b, c) on S_0.

One thus obtains a discontinuity of order n at the instant t_0 for which the segments defined above have the arbitrarily chosen values (λ, μ, ν), …, (λ_n, μ_n, ν_n) at each point of S_0.

90 – It remains for us to see whether this system of velocities and accelerations actually corresponds to a *motion* that satisfies that condition of impenetrability that was stated in no. **44**, as well as the supplementary hypothesis that was made in no. **46**.

However, if we consider two media that occupy two contiguous regions of space, 1 and 2 (with bounding surface S), at the instant t, and if *we suppose these regions to be mutually independent,* and we give velocities and accelerations to the various points in each medium at that instant that are continuous of various orders, but vary (for the points of the contact surface) when one passes from one to the other then these media will, in general, cease to be contiguous at the instants later than t. They will separate, or, on the contrary, merge, with certain points of region 1 entering into region 2, and conversely. In order for things to be otherwise, it is obviously necessary for certain conditions to hold. These conditions, which we will recall later on, are the following ones: *For each point of S the normal components of successive velocities and accelerations must be the same in both regions.*

It therefore seems that this must be verified in the context of the actual problem.

91. – We shall see that things do not always happen exactly this way. However, to that effect, we have two fundamental cases to distinguish:

1. The discontinuity constantly affects the same molecules; in other words, the equation of the surface of discontinuity does not contain t. We then say that the discontinuity is *stationary.*

2. The equation of the surface of discontinuity depends on time. As a consequence, we must write:

$$f(a, b, c, t) = 0, \tag{44}$$

and it is soluble for t when a, b, c are given (at least in a certain region). The discontinuity then affects different molecules depending on the instant considered; we say that it *propagates.* We further give such a discontinuity the name of *wave.*

The preceding stated conditions are essentially necessary in the case of stationary discontinuities.

This is no longer the case for propagating discontinuities.

92. – Nevertheless, if we place ourselves, as we have every right to do, *in a region and an interval of time during which the surface of discontinuities remain unique* then the segments (λ, μ, ν), $(\lambda_1, \mu_1, \nu_1)$, …, $(\lambda_n, \mu_n, \nu_n)$ might not be arbitrary.

Suppose that the instant t_0 at which we take the surface of discontinuity S_0 subdivides such a time interval. With these conditions, we must express that this surface is unique, *not only for the value t_0 of t, but also for the other values, before and after.*

When this is the case we say, with Hugoniot, that the two motions that occur in regions 1 and 2 at the instant t_0 are *compatible.*

The conditions for the media to be compatible vary with the dynamical problem that one must solve. However, we shall confirm that there is something that all of these problems have in common: These conditions are necessary in order for compatibility to be kinematically possible.

93. Case of stationary discontinuities. – First of all, consider a stationary discontinuity of order n, and let $\frac{\delta^n x}{\delta a^p \delta b^q \delta c^r \delta t^h}$ be a derivative of order n that is discontinuous. Suppose that the index h is different from zero.

At a point (a_0, b_0, c_0) of the surface of discontinuity, the h^{th} derivative with respect to time of $\left[\dfrac{\delta^{n-h} x}{\delta a^p \delta b^q \delta c^r}\right]$ is different from zero, *and this is true for a continuous sequence of values of t*, since, by hypothesis, the discontinuity does not cease to depend on the molecule (a_0, b_0, c_0).

Therefore, $\left[\dfrac{\delta^{n-h} x}{\delta a^p \delta b^q \delta c^r}\right]$ may not be null, except for particular values of t. If we abstract from this, we see that a derivative of order $n - h$ is discontinuous, and that, as a consequence, the discontinuity is of order less than n, which is contrary to hypothesis.

Thus, h must ultimately be null.

Therefore, *for a stationary discontinuity the first derivatives that are discontinuous are of index zero.*

94. – In particular, suppose that a derivative of the form $\dfrac{\delta^n x}{\delta t^n}, \dfrac{\delta^n y}{\delta t^n}, \dfrac{\delta^n z}{\delta t^n}$ is discontinuous. Then, from the preceding reasoning, the same will be true for at least one of the coordinates x, y, or z. It will therefore have an *absolute discontinuity.* The two portions, 1 and 2, of the medium behave like two different bodies and slide over each other while constantly remaining in contact.

These are precisely the conditions that we imposed upon ourselves in no. **90**. Therefore, as we said in that section, if the velocity, acceleration, etc., are discontinuous then their discontinuities may not be arbitrary. It is easy to find the conditions that they must satisfy (while always assuming, as we did in no. **46**, that the two partial media, 1 and 2, remain in contact and do not indeed separate).

Indeed, let S be the surface of discontinuity *as considered for the present state,* and let:

$$\varphi(x, y, z) = 0, \tag{45}$$

be the equation of S. A particle, either in region 1 or region 2, that belongs to that surface at an arbitrary moment will not cease to do so (no. **48**). One will thus have:

$$\frac{\partial \varphi}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial \varphi}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial \varphi}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial \varphi}{\partial t} = 0.$$

Let (x_1, y_1, z_1), (x_1, y_1, z_1) be particles that are situated at the same point (x, y, z) of S at the instant t, one of which belongs to portion 1, and the other belongs to portion 2 of the medium. One may replace x, y, z by x_1, y_1, z_1, as well as by x_2, y_2, z_2, in the preceding equation. Upon dividing the respective sides of the two relations so obtained, we get:

$$\frac{\partial \varphi}{\partial x}\left[\frac{\delta x}{\delta t}\right] + \frac{\partial \varphi}{\partial y}\left[\frac{\delta y}{\delta t}\right] + \frac{\partial \varphi}{\partial z}\left[\frac{\delta z}{\delta t}\right] = 0. \tag{46}$$

Therefore, *the abrupt variation of the velocity is a segment that is tangent to S.*

For some particular instant, it may happen that $\left[\frac{\delta x}{\delta t}\right], \left[\frac{\delta y}{\delta t}\right], \left[\frac{\delta z}{\delta t}\right]$ are null. Then *the variation of the acceleration* (if it is non-zero) *is tangent to S* along its course. One sees this upon differentiating equation (**45**) twice, which gives:

$$(\mathbf{47}) \qquad \begin{cases} \dfrac{\partial^2\varphi}{\partial x^2}\left(\dfrac{\delta x}{\delta t}\right)^2 + \cdots + 2\dfrac{\partial^2\varphi}{\partial x\partial y}\dfrac{\delta x}{\delta t}\dfrac{\delta y}{\delta t} + 2\left(\dfrac{\partial^2\varphi}{\partial x\partial t}\dfrac{\delta x}{\delta t} + \dfrac{\partial^2\varphi}{\partial y\partial t}\dfrac{\delta y}{\delta t} + \dfrac{\partial^2\varphi}{\partial z\partial t}\dfrac{\delta z}{\delta t}\right) \\ \qquad + \dfrac{\partial^2\varphi}{\partial t^2} + \dfrac{\partial\varphi}{\partial x}\dfrac{\partial^2 x}{\partial t^2} + \dfrac{\partial\varphi}{\partial y}\dfrac{\partial^2 y}{\partial t^2} + \dfrac{\partial\varphi}{\partial z}\dfrac{\partial^2 z}{\partial t^2} = 0, \end{cases}$$

in which everything is continuous, except for the terms:

$$\frac{\partial\varphi}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial\varphi}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial\varphi}{\partial z}\frac{\partial^2 z}{\partial t^2}.$$

Therefore, the set of them does not vary over the discontinuity.

If the acceleration itself is discontinuous then a conclusion that is analogous to the preceding one will be true for the third-order acceleration, and so on.

We further remark that *none of the derivatives of x, y, z is discontinuous when it is considered as a function of time,* with *a, b, c* fixed; the point (*a, b, c*) must belong to either region 1 or region 2, and everything is continuous in each of them.

95. Case of waves. – Contrary to the latter case, the discontinuities that propagate never give rise to absolute discontinuities. Indeed, two molecules that are infinitely close at a given instant may cease to be that way only by passing through the discontinuity. However, since this passage can happen only during an infinitely small time interval their positions will be altered only by an infinitesimal amount during this time.

Contrary to the case of stationary discontinuities, the derivatives that will be discontinuous will be the derivatives with respect to time. This is because an arbitrary specific molecule will pass from one region to the other at the moment when it contacts the wave.

Finally, if there is compatibility then it will not be true that the derivatives of index 0 will be discontinuous to the exclusion of the other derivatives of the same order as in the case of stationary discontinuities. Indeed, on the contrary, we shall see that all of them vary in time.

96. – We thus occupy ourselves with expressing the concept that there is compatibility, in the sense of no. **92**. The motion will be continuous in each of the two regions, 1 and 2, and they will be separated by a surface whose position varies with time, but which is unique at each instant, and whose equation we express by:

$$(\mathbf{44}) \qquad f(a, b, c, t) = 0.$$

To that effect, it will be convenient for us to use the language of four-dimensional geometry and consider *a, b, c, t* to be the coordinates of a point in a four-dimensional space E_4.

In this conception, ordinary space [which relates to the point (*a, b, c*)], when considered at the instant *t*, must be regarded as the section of the space E_4 by the multiplicity t = const.

Equation (**44**) will represent a triply extended multiplicity $\mathcal{S}_0$, which is the section for t = const. The multiplicity $\mathcal{S}_0$ divides E_4 into two regions, 1 and 2, which are generated by the previously considered regions 1 and 2 of ordinary space, respectively, when time varies.

The preceding hypothesis that we made (nos. **70-71**) consists of demanding that the quantities that we operate on and their derivatives are continuous outside of $\mathcal{S}_0$ and on $\mathcal{S}_0$ itself, but may be discontinuous upon crossing $\mathcal{S}_0$.

97. – This being the case, we apply the method of no. **73**, no longer to the surface S_0, but the multiplicity $\mathcal{S}_0$. Let Φ be a function on this multiplicity that is continuous, but whose derivatives are discontinuous (always under the conditions indicated in nos. **70-71**). As in the preceding, one verifies that one has:

$$\left[\frac{\delta\Phi}{\delta a}\right]da+\left[\frac{\delta\Phi}{\delta b}\right]db+\left[\frac{\delta\Phi}{\delta c}\right]dc+\left[\frac{\delta\Phi}{\delta t}\right]dt=0 \tag{48}$$

by means of the differential equation of the multiplicity $\mathcal{S}_0$, which may be written this time as:

$$f_a\,da + f_b\,db + f_c\,dc + f_t\,dt = 0, \tag{49}$$

in which f_t denotes the derivative $\frac{\delta f}{\delta t}$. Equations (**34**) may thus be completed, and one may write:

$$\left[\frac{\delta\Phi}{\delta a}\right]: f_a=\left[\frac{\delta\Phi}{\delta b}\right]: f_b=\left[\frac{\delta\Phi}{\delta c}\right]: f_c=\left[\frac{\delta\Phi}{\delta t}\right]: f_t. \tag{50}$$

Now, if one likewise supposes that the derivatives of Φ up to order n are continuous then one will have, instead of the proportions (**35**):

$$\left[\frac{\delta^n\Phi}{\delta a^n}\right]: f_a^n=\cdots=\left[\frac{\delta^n\Phi}{\delta a^p\,\delta b^q\,\delta c^r\,\delta t^h}\right]: f_a^p\,f_b^q\,f_c^r\,f_t^h=\cdots=\left[\frac{\delta^n\Phi}{\delta t^n}\right]: f_t^n, \tag{51}$$

which may be summarized, as before, by the identity (with respect to *da, db, dc, dt*):

$$\left(\left[\frac{\delta}{\delta a}\right]da+\left[\frac{\delta}{\delta b}\right]db+\left[\frac{\delta}{\delta c}\right]dc+\left[\frac{\delta}{\delta t}\right]dt\right)^n \Phi$$
$$= \lambda(f_a\,da + f_b\,db + f_c\,dc + f_t\,dt)^n,$$

in which λ is the common value of the ratios (**51**).

98. – Suppose, as we agreed to do in no. **80**, that the derivatives f_a, f_b, f_c are *equal* to the direction cosines α, β, γ, respectively, of the normal to the to the surface S_0 that is represented by equation (**44**) at the instant t. f represents (up to second-order infinitesimals) the normal distance of a point to that surface, *when measured on the adopted initial state* and regarded as positive in region 2.

Therefore, let $f_0(a, b, c)$ be the left-hand side of equation S_0, i.e., the function f when one does not vary t; let S_0' be the surface that is analogous to S_0 and corresponds to $t + dt$. The normal distance from a point of S_0' to S_0 (which is taken to be positive in region 2 and negative in region 1) will obviously be equal to the value of f_0, or, what amounts to the same thing, of df_0. Let:

$$dn = df_0 = f_a\,da + f_b\,db + f_c\,dc = -f_t\,dt.$$

The quantity dn/dt is the *velocity of propagation* of the wave, as measured in the initial state considered. We denote it by the letter θ in such a way that one has:

$$\theta = -f_t. \tag{52}$$

It is positive or negative according to whether the propagation is from region 1 into region 2 or vice versa.

If f is the left-hand side of the equation for S_0, when taken in an arbitrary form, then in order to obtain the direction cosines α, β, γ one must divide the coefficients of (**49**) by $\sqrt{f_a^2 + f_b^2 + f_c^2}$. One will then have:

$$\theta = \frac{-f_t}{\sqrt{f_a^2 + f_b^2 + f_c^2}}, \tag{52′}$$

with the radical always being taken to have the sign of f in region 2.

99. – Thanks to the presence of the radical, the velocity θ depends on the choice of initial state, and, when one changes this, it is obviously altered by the same ratio as the normal distances to the surface of discontinuity.

As in the foregoing, we usually choose the present state at the instant considered to be the initial state by specifying that it amounts to the state of region 1 in the first-order case.

100. – Along with the velocity of propagation, one may need to introduce the *velocity of displacement* of the wave, i.e., the velocity that the surface of discontinuity moves with *when considered in the present space.*

Let:

$$\varphi(x, y, z, t) = 0 \tag{45}$$

be the equation of that surface (φ is the value taken by f when one replaces a, b, c by their values as deduced from equations (**1′**)). Let S be its position at the instant t and let S' be its position at the instant $t + dt$. The velocity of displacement, which we designate by T, will be the quotient of the normal distance to the surfaces S, S' by dt (with the same sign convention as always).

The argument that was presented above obviously gives:

$$T = \frac{-\dfrac{\partial\varphi}{\partial t}}{\sqrt{\left(\dfrac{\partial\varphi}{\partial x}\right)^2 + \left(\dfrac{\partial\varphi}{\partial y}\right)^2 + \left(\dfrac{\partial\varphi}{\partial z}\right)^2}} \tag{53}$$

for the value of that velocity.

The velocity T is distinct from the velocity of propagation, even when it is computed in an initial state that is identical with the present state at the instant t; indeed, S_0 coincides with S in this case. However, S_0' does not coincide with S'. S_0' is related to the positions that are occupied at the instant t by the particles that form the surface S' at the instant $t + dt$.

It is, moreover, easy to find the relation that exists between the two velocities. Indeed, suppose, to simplify, that f is chosen in such a manner that f_a, f_b, f_c are equal to the direction cosines α, β, γ of the normal to S_0 (i.e., to S).

These direction cosines will also be equal to $\dfrac{\partial\varphi}{\partial x}, \dfrac{\partial\varphi}{\partial y}, \dfrac{\partial\varphi}{\partial z}$ at the instant t, since x, y, z are nothing but a, b, c at that time. One will have $\theta = -f_t$ and $T = -\dfrac{\partial\varphi}{\partial t}$.

However, φ is nothing but f when it is expressed with the aid of the variables x, y, z, t, and, as a consequence, $\dfrac{\partial\varphi}{\partial t}$ is nothing but the derivative $\dfrac{\partial f}{\partial t}$, which is related to $\dfrac{\delta f}{\delta t} = f_t$ by relation (**18**). One therefore has:

$$T = -\left(f_t - u\frac{\partial u}{\partial x} - v\frac{\partial\varphi}{\partial y} - w\frac{\partial\varphi}{\partial z}\right) = \theta + v_n, \tag{54}$$

in which v_n denotes the component of the velocity (u, v, w) that is normal to S at the point and instant considered.

Moreover, one will arrive at the same result by applying the theorem on the composition of motions. Indeed, the motion of S in space may be considered to be the result of:

1. Its relative motion with respect to the medium, which, when considered only during the time interval dt, brings S into agreement with the position of S_0', and is nothing but the propagation of the wave.

2. Its ensuing motion, which is also the motion of the medium.

The velocity T is therefore the sum of the normal velocities of these two motions, which are precisely θ and v_n.

100 (cont.). – Formulas (**52′**), (**53**) are, moreover, susceptible to a geometric interpretation that will be seem clearer to us if we first consider motions in two dimensions.

Envision a motion that takes place in a plane, in such a way that it has two initial coordinates a, b, and two present coordinates x, y. The discontinuities that we consider will then define curves whose position at the instant of time t_0 will be represented in the initial state by S_0 (*fig.* 7).

We may take time t to be the third coordinate, with the t axis taken to be vertical and the horizontal coordinates taken to be a, b. The multiplicity $\mathcal{S}_0$, which represents the progress of a wave, according to a convention that is analogous to the one that was made in no. **96**, will be a surface here (*fig.* 7), whose section S_0' by an arbitrary horizontal plane $t = t'$ will give the new position of the wave at the corresponding instant. It will no longer be necessary to represent this curve in the same plane as the one in which S_0 figures, which will obviously be in the horizontal projection s_0' on that plane (*fig.* 7).

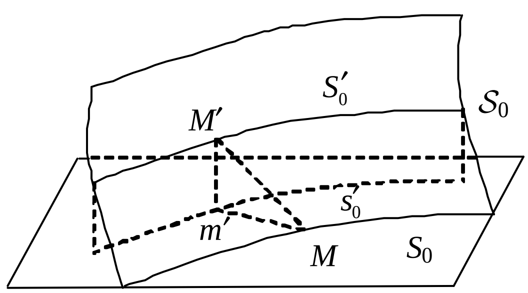

Fig. 7

If t' is infinitely close to t_0 then the normal displacement dn of the wave will be described by the normal distance Mm' (*fig.* 7) between the curves S_0, s_0'. Moreover, since dt is nothing but the normal distance from the point m' to the point M' of S_0', which is its projection, one sees that the *velocity of propagation* dn/dt is nothing but the limit of $\dfrac{m'M}{m'M'}$, i.e., the *inverse of the slope of the surface* $\mathcal{S}_0$ *with respect to the horizontal plane* (or the cotangent of the angle that this surface forms with that plane).

However, the formulas of analytical geometry give an expression for the quantity thus obtained that is completely analogous to (**52′**).

If one takes the actual coordinates x, y to be the horizontal coordinates, i.e., if one constructs not only $\mathcal{S}_0$, but $\mathcal{S}$, then the inverse of the slope of the latter will give the velocity of displacement in a form that is analogous to (**53**).

It is clear that these considerations may be similarly extended to motions in space with no more difficulty than the introduction of four-dimensional geometry, and one thus recovers the formulas (**52′**), (**53**) that we wrote for them.

In what follows, we will need to employ figures that relate to planar motions (such as *fig.* 7) in order to represent the reasoning that we shall make on motions in space (in such a way that when we speak of a *surface* S_0 in the text we will trace out a *curve* S_0 in the figure, and so on).

101. – As before, we first treat discontinuities of the first and second order.

For the first order case, x is continuous, but not its derivatives; we therefore write (no. **97**):

$$\left[\frac{\delta x}{\delta a}\right] : \alpha = \left[\frac{\delta x}{\delta b}\right] : \beta = \left[\frac{\delta x}{\delta c}\right] : \gamma = \left[\frac{\delta x}{\delta t}\right] : -\theta.$$

If the common value of the first three ratios is λ (in the system of notations of no. **77**), whereas the variation of $\delta x / \delta t$ is denoted by λ_1, then one obtains:

(**55**) $$\lambda_1 = -\theta\lambda .$$

Similarly:

(**55′**) $$\begin{cases} \mu_1 = \left[\dfrac{\delta y}{\delta t}\right] = -\theta\mu, \\ \nu_1 = \left[\dfrac{\delta z}{\delta t}\right] = -\theta\nu. \end{cases}$$

Therefore, *the two segments* (λ, μ, ν), (λ_1, μ_1, ν_1) *have the same direction. Their ratio is equal, up to sign, to the velocity of propagation* θ.

102. – This relation persists for any adopted initial state, but whereas λ_1, μ_1, ν_1 will be independent of that initial state, on the contrary, that choice will influence λ, μ, ν, and θ. If one chooses the present state of region 1, or that of region 2, then the quantity θ will have two values θ_1 and θ_2, which are generally different, and whose ratio is equal to the normal dilatation $1 + \lambda\alpha + \mu\beta + \nu\gamma$.

By reason of that circumstance, there is often an advantage to introducing the velocity of displacement T for a first-order discontinuity that is independent of the choice of initial state. This velocity is related to θ_1 and θ_2 by the formula (**54**), in which v_n – which is, in general, affected by the discontinuity – has two different values v_{1n} and v_{2n}. One has:

(**56**) $$T = \theta_1 - v_{1n} = \theta_2 - v_{2n} .$$

103. – Now suppose we have a second-order discontinuity. Our lemma, when applied to the quantity x, which is continuous, as well as its first-order derivatives, gives us:

$$\left[\frac{\delta^2 x}{\delta a^2}\right] : \alpha^2 = \cdots = \left[\frac{\delta^2 x}{\delta a\,\delta t}\right] : -\alpha\theta = \cdots = \left[\frac{\delta^2 x}{\delta t^2}\right] : \theta^2 .$$

However, the ratio $\left[\frac{\delta^2 x}{\delta a^2}\right] : \alpha^2$, as well as all of the analogous ratios that relate to the derivatives of index zero, is equal to λ (no. **85**). Similarly, we have set:

$$\left[\frac{\delta^2 x}{\delta a\,\delta t}\right] : \alpha = \left[\frac{\delta^2 x}{\delta b\,\delta t}\right] : \beta = \left[\frac{\delta^2 x}{\delta c\,\delta t}\right] : \gamma = \lambda_1 ,$$

and:

$$\left[\frac{\delta^2 x}{\delta t^2}\right] = \lambda_2 .$$

Therefore:

(**57**)
$$\lambda = \frac{\lambda_1}{-\theta} = \frac{\lambda_2}{\theta^2} .$$

Since one has, analogously:

(**57′**)
$$\begin{cases} \mu = \dfrac{\mu_1}{-\theta} = \dfrac{\mu_2}{\theta^2} , \\ \nu = \dfrac{\nu_1}{-\theta} = \dfrac{\nu_2}{\theta^2} , \end{cases}$$

the three segments (λ, μ, ν), (λ_1, μ_1, ν_1), (λ_2, μ_2, ν_2) *have the same direction and define a geometric progression, with the ratio of that progression being* $-\theta$.

The preceding proportions may then be written:

$$\left(\left[\frac{\delta}{\delta a}\right]da + \left[\frac{\delta}{\delta b}\right]db + \left[\frac{\delta}{\delta c}\right]dc + \left[\frac{\delta}{\delta t}\right]dt\right)^2 x$$
$$= \lambda(\alpha\, da + \beta\, db + \gamma\, dc - \theta\, dt)^2,$$

$$\left(\left[\frac{\delta}{\delta a}\right]da + \left[\frac{\delta}{\delta b}\right]db + \left[\frac{\delta}{\delta c}\right]dc + \left[\frac{\delta}{\delta t}\right]dt\right)^2 y$$
$$= \mu(\alpha\, da + \beta\, db + \gamma\, dc - \theta\, dt)^2,$$

$$\left(\left[\frac{\delta}{\delta a}\right]da + \left[\frac{\delta}{\delta b}\right]db + \left[\frac{\delta}{\delta c}\right]dc + \left[\frac{\delta}{\delta t}\right]dt\right)^2 z$$
$$= \nu(\alpha\, da + \beta\, db + \gamma\, dc - \theta\, dt)^2.$$

These considerations may be generalized. For arbitrary *n, the* $n + 1$ *segments* (λ, μ, ν), $(\lambda_1, \mu_1, \nu_1)$, …, $(\lambda_n, \mu_n, \nu_n)$ *have the same direction and form a geometric progression whose ratio is equal, up to sign, to the velocity of propagation;* one has:

$$\textbf{(58)} \qquad \begin{cases} \lambda = \dfrac{\lambda_1}{-\theta} = \cdots = \dfrac{\lambda_h}{(-\theta)^h} = \cdots = \dfrac{\lambda_n}{(-\theta)^n}, \\ \mu = \dfrac{\mu_1}{-\theta} = \cdots = \dfrac{\mu_h}{(-\theta)^h} = \cdots = \dfrac{\mu_n}{(-\theta)^n}, \\ \nu = \dfrac{\nu_1}{-\theta} = \cdots = \dfrac{\nu_h}{(-\theta)^h} = \cdots = \dfrac{\nu_n}{(-\theta)^n}. \end{cases}$$

These relations are true for any choice of initial state, as well as their influence on the values of the quantities that they figure in. Of course, for orders higher than unity it is useless to specify, when one chooses the initial state to be the present state, whether it is state 1 or state 2. If one inverts the order of the two regions then θ undergoes a simple change of sign.

104. – The case of stationary discontinuities obviously corresponds to $\theta = 0$. However, if one nullifies θ in formulas (**58**) then one then gets: $\lambda_h = \mu_h = \nu_h = 0$ (for $h \geq 1$). One thus comes back to the result obtained in no. **93** precisely: *for a stationary discontinuity of order n, the only derivatives of order n that are discontinuous have index zero.*

105. – On first sight, the case in which relations (**58**) are verified seems to be much more particular than the one where the segments $(\lambda_h, \mu_h, \nu_h)$ are arbitrary. Meanwhile, it results from the foregoing that compatibility must be regarded as the rule and the contrary case, the exception. Indeed, if there is no compatibility then the surface of discontinuity might not be unique, and, at the very least, it doubles into two leaves that are separated before the instant *t* and are something different afterwards. Without discontinuity, the discontinuity that exists at the instant *t* must therefore be regarded as being, in reality, the superposition of two or more others whose surfaces momentarily coincide ([34]). One supposes only that the absence of compatibility is true an infinitude of times in a finite interval of time, because then the surfaces of discontinuity, which double an infinitude of times, will not be isolated.

106. – From now on, there is therefore reason to suppose, unless indicated to the contrary, that the $n + 1$ segments have the relations (**58**) between them. In particular, it results from this that *a discontinuity is completely defined by a segment at each point of*

([34]) Such a case is, moreover, the exceptional case; when two discontinuities propagate independently of each other, they encounter each other only along a line, in general, without the wave surfaces ever coinciding at any moment. (See chap. VII).

the wave surface: the segment (λ, μ, ν) that corresponds to the derivatives of index zero, *and a number:* the velocity of propagation.

107. – This segment and this number may be arbitrary, moreover. In other words, this time, equations (**58**) give precisely all of the relations that exist between the variations of the derivatives of order n in order for there to be a discontinuity of that order when one knows nothing about the dynamical nature of the motion.

Indeed, suppose that we are given equation (**44**) for $\mathcal{S}_0$ arbitrarily. We may obviously choose it in such a manner that S_0 has a given position at a definite instant and θ has a given value at each point. Suppose that we are also given the segment (λ, μ, ν), arbitrarily at every point of S_0, and similarly $\mathcal{S}_0$. Finally, let X, Y, Z be functions of a, b, c, t that are continuous, along with their derivatives of all orders. The equations:

$$x = X, \quad y = Y, \quad z = Z, \qquad \text{in region 1,}$$

$$\text{(59)} \qquad \left.\begin{aligned} x &= X + \lambda \frac{[f(a,b,c,t)]^n}{n!} \\ y &= Y + \mu \frac{[f(a,b,c,t)]^n}{n!} \\ z &= Z + \nu \frac{[f(a,b,c,t)]^n}{n!} \end{aligned}\right\} \qquad \text{in region 2,}$$

define a motion (and no longer just a system of velocities and accelerations, as in no. **89**) that presents a discontinuity of order n whose elements are precisely the ones that we were given.

Moreover, we see precisely why the motion thus defined does not cease to verify the general hypotheses of nos. **44-46**, even though it does not satisfy the conditions of no. **90**. The reason for this is the same as the one that explains why the discontinuities that propagate do not give rise to absolute discontinuities, as we saw in no. **95**. For an arbitrary particle, the continuity of velocities or accelerations ceases at the moment when it encounters the wave. However, it ceases only during an infinitesimal time and is then re-established, in such a way that the continuity of motion is not disturbed.

108. – In the case where there is no compatibility, the discontinuities of order n are divided, in general, into discontinuous partial derivatives that are likewise of order n (the segments (λ_h, μ_h, ν_h) that relate to them have a geometric sum for each value of h that is equal to the analogous segment that corresponds to the original discontinuity).

However, we remark that other hypotheses are possible. For example, a discontinuity of order n may subdivide into discontinuities of order *greater than n*. A discontinuity that depends on the velocities may be replaced by two others that depend only on the accelerations.

In order to account for this, consider, to simplify, a motion that is performed along the axis of the abscissa with each point of the initial state being defined by just one coordinate a, and its actual position by just one coordinate x. The variation of x as a function of a and t may then be represented by a surface.

Suppose that this surface is composed of two semi-planes that form a dihedral; we then have a first-order discontinuity.

Construct two lines SA and SB through a point S in the intersection of the dihedral (which corresponds to a value a_0 of a and a value t_0 of t), which are contained in the two faces (*fig.* 8), and relate these two lines by a tangent conical nappe with the lines as generators in the two faces of the dihedral. We may then suppress the portion of it that is contained between the two lines and the portion of the intersection (*fig.* 8) that corresponds to $t > t_0$ in order to replace them with the conical nappe, and we therefore have a motion that presents a discontinuity of order 1 for $t = t_0$ and two discontinuities of order 2 for $t > t_0$.

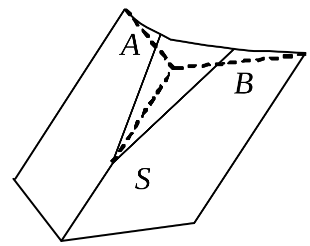

Fig. 8

109. – Returning to the case of compatibility, we propose to calculate the variations experienced by the principal elements that we considered in the first part of this chapter.

Density. – In order to study density, we suppose that one has taken the present state to be the initial state.

First, consider a first-order wave. We know that the deformation of state 2 with respect to state 1 belongs to the category that we studied in no. **56**, and we have learned to evaluate the normal dilatation, which is equal to the ratio of the densities ρ_1/ρ_2. This dilatation is obtained by adding unity to the normal component $\lambda\alpha + \mu\beta + \nu\gamma$ of the discontinuity ([35]), which is assumed to be the state 1 of the medium; therefore, one has:

(60) $$\frac{\rho_1}{\rho_2} = 1 + \lambda\alpha + \mu\beta + \nu\gamma.$$

This expression may be transformed in various manners by considering the indicated relations in nos. **101**-**102**. First of all, upon multiplying the normal component of the discontinuity by $-\theta$ we obtain the normal component of the variation of the velocity. Therefore:

(61) $$\frac{\rho_1}{\rho_2} = 1 - \frac{[v_n]}{\theta}.$$

In this formula, v_n denotes the normal component of velocity and must receive the value θ_1. One will clearly obtain the same value for ρ_1 / ρ_2 upon changing the sign of $[v_n]$

([35]) That is, the segment which, when combined with the number θ, defines the discontinuity, as we just saw.

and replacing θ_1 with θ_2, and ρ_1 / ρ_2 with ρ_2 / ρ_1; one verifies this without difficulty with the aid of the preceding relations (no. **84**).

On the other hand, we have seen that the normal dilatation is equal to the ratio of the velocities θ_1 and θ_2. If we derive this from the double equality (**56**) then we obtain:

(**62**)
$$\frac{\rho_1}{\rho_2} = \frac{\theta_2}{\theta_1} = \frac{T - v_{2n}}{T - v_{1n}},$$

or, if one prefers:

(**61′**)
$$\frac{\rho_1}{\rho_2} - 1 = \frac{-[v_n]}{T - v_{1n}} = \frac{-[v_n]}{\theta_1}.$$

110. – One proves the same fact directly by considering the portion of space that is comprised of the interior of a small cylinder whose two bases C, C' are situated in the two positions S, S', respectively (*fig.* 9), that are occupied by the wave surface at the instant t and $t + dt$, and whose volume is $CT\,dt$.

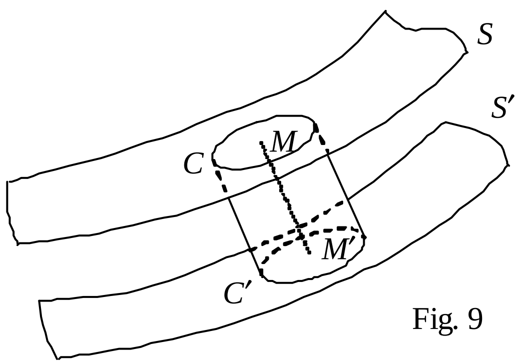

Fig. 9

This portion of space takes state 1 to state 2 during the time dt.

However, the volume that is generated by the face C in state 1 is given by the expression $Cv_{1n}\,dt$, but the volume that is terminated by the face C' in state 2 is equal to $Cv_{2n}\,dt$. As for the volume generated or terminated by the lateral faces, it is negligible if we suppose, as we have the right to do, that the ratio dt / C is infinitesimal.

One comes back to relation (**62**) upon writing that the volume of the remaining subset varies with a ratio that is inverse to that of the densities.

111. – For the discontinuities of higher order, the abrupt variation does not depend on the density itself, but only on its derivatives of order $n - 1$ (if the wave considered is of order n). Since the quantity ρ_0/ρ is continuous, as well as its derivatives up to order $n - 2$, inclusive, there will exist a number κ such that one has:

(**63**)
$$\left[\frac{\delta^{n-1} \log \dfrac{\rho_0}{\rho}}{\delta a^p \delta b^q \delta c^r \delta t^h}\right] = \kappa \alpha^p \beta^q \gamma^r (-\theta)^h.$$

In order to evaluate κ, we may confine ourselves to considering derivatives of index zero. If we take the state of region 1 to be the initial state then the quantity $\log(\rho_0/\rho)$ will be identically null in that region. On the other hand, since the deformation of region 2 with respect to region 1 has the properties that were studied in no. **59**, the value of $\dfrac{\delta^{n-1}\log\dfrac{\rho_0}{\rho}}{\delta a^p\,\delta b^r\,\delta c^r}$ in region 2 will be given by formula (**17**). One will therefore have:

$$\kappa = \lambda\alpha + \mu\beta + \nu\gamma. \tag{64}$$

Since ρ_0 is, in general, continuous, as well as all of its derivatives, we therefore have the variation of the derivatives of $\log(1/\rho)$

111 (cont.). – One may, moreover, likewise start with the derivatives of non-null index by using formula (**20**). For $n = 2$, this gives:

$$\frac{\delta}{\delta t}\left(\log\frac{1}{\rho}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

The left-hand side may be written: $\dfrac{\delta u}{\delta a} + \dfrac{\delta v}{\delta b} + \dfrac{\delta w}{\delta c} = \dfrac{\delta^2 x}{\delta a\,\delta t} + \dfrac{\delta^2 y}{\delta b\,\delta t} + \dfrac{\delta^2 z}{\delta c\,\delta t}$ if a, b, c coincide with x, y, z ([36]). One therefore arrives at formula (**64**) precisely. Moreover, one generalizes to the case of arbitrary n by differentiating formula (20) a sufficient number of times.

If one would like to find the change of density, with the discontinuity being referred to an arbitrary initial state, then one must, of course, transform the formulas that we just found with the aid of the previously established principles.

112. – **Components of deformation.** – More generally, we seek the influence of a discontinuity on the components of deformation (**7**) (no. **51**). This time, we shall make no hypothesis on the choice of initial state.

Again, let x', y', z' be the corresponding coordinates of the state of region 1 when prolonged into region 2. First take $n = 1$; one will have, up to second-order infinitesimals:

$$x = x' + \lambda f, \qquad y = y' + \mu f, \qquad z = z' + \nu f,$$

and, as a consequence:

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= (dx' + \lambda df)^2 + (dy' + \mu df)^2 + (dz' + \nu df)^2 \\ &= dx'^2 + dy'^2 + dz'^2 + 2df\,(\lambda dx' + \mu dy' + \nu dz') \\ &\qquad + (\lambda^2 + \mu^2 + \nu^2)\,df^2 \\ &= dx'^2 + dy'^2 + dz'^2 \end{aligned}$$

([36]) This transformation is applicable in the two regions whenever the initial state is in region 1, thanks to the fact that it introduces only *one* differentiation with respect to a, b, c.

$$+ 2(f_a da + f_b db + f_c dc)(\lambda dx' + \mu dy' + \nu dz')$$
$$+ (\lambda^2 + \mu^2 + \nu^2)(f_a da + f_b db + f_c dc)^2.$$

The variations that are felt by the components of the deformation are thus half of the coefficients of the polynomial:

(65)
$$\begin{cases} 2(f_a da + f_b db + f_c dc)(\lambda dx' + \mu dy' + \nu dz') \\ \quad + (\lambda^2 + \mu^2 + \nu^2)(f_a da + f_b db + f_c dc)^2. \end{cases}$$

113. – These results take a much simpler form for $n > 1$. In that case, it is no longer the components of deformation, but their derivatives of order $n - 1$, that are discontinuous. As always, we take the most important case, that of $n = 2$, in which:

$$x = x' + \frac{\lambda f^2}{2}$$
$$y = y' + \frac{\mu f^2}{2}$$
$$z = z' + \frac{\nu f^2}{2}$$

from which:

(66)
$$\begin{cases} dx = dx' + \lambda f df \\ dy = dy' + \mu f df \\ dz = dz' + \nu f df \end{cases}$$

upon neglecting the terms that contain f^2 as a factor. The simplification that results from this is the fact that upon squaring and adding one may neglect the squares of λdf, μdf, νdf. One then obtains:

$$dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2 + 2\, f df\, (\lambda dx' + \mu dy' + \nu dz'),$$

in such a way that the variations of ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 are the coefficients of the quadratic polynomial $f df\, (\lambda dx' + \mu dy' + \nu dz')$.

As in no. **47**, denote the first derivatives of x, y, z with respect to a, b, c by a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3. These derivatives coincide with those of x', y', z' at every point of the wave surface, since the discontinuity is of second order, and one has:

(67)
$$\lambda dx' + \mu dy' + \nu dz' = L da + M db + N dc,$$

with:

(67′)
$$L = \lambda a_1 + \mu a_2 + \nu a_3, \quad M = \lambda b_1 + \mu b_2 + \nu b_3, \quad N = \lambda c_1 + \mu c_2 + \nu c_3 .$$

Since df is equal to $f_a\, da + f_b\, db + f_a\, dc$, the quantities by which ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 vary in passing from state 1 to state 2 are (as always, up to terms of order f^2):

$$\textbf{(68)}\qquad \begin{cases} fLf_{a,} & fLf_{b,} & fLf_{c,} \\ f(Mf_c + Nf_b), & f(Nf_a + Lf_c), & f(Lf_b + Mf_a). \end{cases}$$

Finally, since f is null on S_0 and has f_a, f_b, f_c for its partial derivatives, one obtains:

$$\textbf{(69)}\qquad \begin{cases} \left[\dfrac{\delta\varepsilon_1}{\delta a}\right] : f_a = \left[\dfrac{\delta\varepsilon_1}{\delta b}\right] : f_b = \left[\dfrac{\delta\varepsilon_1}{\delta c}\right] : f_c = Lf_a \\ \left[\dfrac{\delta\varepsilon_2}{\delta a}\right] : f_a = \left[\dfrac{\delta\varepsilon_2}{\delta b}\right] : f_b = \left[\dfrac{\delta\varepsilon_2}{\delta c}\right] : f_c = Mf_b \\ \left[\dfrac{\delta\varepsilon_3}{\delta a}\right] : f_a = \left[\dfrac{\delta\varepsilon_3}{\delta b}\right] : f_b = \left[\dfrac{\delta\varepsilon_3}{\delta c}\right] : f_c = Nf_c \\ \left[\dfrac{\delta\gamma_1}{\delta a}\right] : f_a = \left[\dfrac{\delta\gamma_1}{\delta b}\right] : f_b = \left[\dfrac{\delta\gamma_1}{\delta c}\right] : f_c = Mf_c + Nf_b \\ \left[\dfrac{\delta\gamma_2}{\delta a}\right] : f_a = \left[\dfrac{\delta\gamma_2}{\delta b}\right] : f_b = \left[\dfrac{\delta\gamma_2}{\delta c}\right] : f_c = Nf_c + Lf_b \\ \left[\dfrac{\delta\gamma_3}{\delta a}\right] : f_a = \left[\dfrac{\delta\gamma_3}{\delta b}\right] : f_b = \left[\dfrac{\delta\gamma_3}{\delta c}\right] : f_c = Lf_c + Mf_b, \end{cases}$$

which is, of course, a result that one verifies without difficulty with the aid of formulas **(40)**, **(40′)**.

If the discontinuity is of order n then we must replace f by $\dfrac{f^{n-1}}{(n-1)!}$ in formulas (**66**), and, as a consequence, in formulas (**68**). Formulas (**69**) will then be replaced by:

$$\left[\frac{\delta^{n-1}\varepsilon_1}{\delta a^p \delta a^q \delta a^r}\right] : f_a^p f_b^q f_c^r = Lf_a,$$

$$\ldots$$

$$\left[\frac{\delta^{n-1}\gamma_1}{\delta a^p \delta a^q \delta a^r}\right] : f_a^p f_b^q f_c^r = Mf_c + Nf_b,$$

$$\ldots$$

We have written only the formulas that relate to the derivatives with respect to a, b, c. However, we may obviously deduce the variations of all the derivatives of order $n - 1$ from them, since the lemma of no. **97** is applicable to ε_1, ε_2, ε_3, γ_1, γ_2, γ_3.

From the preceding formulas, one may deduce the variation of the density (in the first-order case) or its derivatives, since the dilatation is a function of the components of deformation.

113 (cont.) – Similarly, when the discontinuity is of first order the result is entirely similar to the one that we just described if that discontinuity is very small in such a manner that one may neglect the squares of λ, μ, ν. The polynomial (**65**) then reduces to its linear part with respect to these three quantities, and we get (on account of (**67**)):

(**69**)
$$\begin{cases} [\varepsilon_1] = Lf_a, & [\varepsilon_1] = Lf_a, & [\varepsilon_1] = Lf_a, \\ [\gamma_1] = Mf_c + Nf_b, & [\gamma_2] = Nf_a + Lf_c, & [\gamma_3] = Lf_b + Mf_a. \end{cases}$$

114. Molecular rotation. – Suppose we have a second-order discontinuity. Take the present state to be the initial state, while always supposing that f_a, f_b, f_c are equal to the direction cosines α, β, γ of the normal to the wave. The components:

$$p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$
$$q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$
$$r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

of the molecular rotation may be written here as:

$$\frac{1}{2}\left(\frac{\delta^2 z}{\delta b\,\delta t} - \frac{\delta^2 y}{\delta c\,\delta t}\right), \qquad \frac{1}{2}\left(\frac{\delta^2 x}{\delta c\,\delta t} - \frac{\delta^2 z}{\delta a\,\delta t}\right), \qquad \frac{1}{2}\left(\frac{\delta^2 y}{\delta a\,\delta t} - \frac{\delta^2 x}{\delta b\,\delta t}\right).$$

From our formulas, its variation will therefore be:

(**70**)
$$[p] = \frac{\theta}{2}(\mu\gamma - \nu\beta), \qquad [q] = \frac{\theta}{2}(\nu\alpha - \lambda\gamma), \qquad [r] = \frac{\theta}{2}(\lambda\beta - \mu\alpha).$$

The geometric interpretation of these expressions is well known: With the point considered as origin, if one constructs the segment (λ, μ, ν) that characterizes the discontinuity, and constructs, on the other hand, a segment that is normal to the wave and has the magnitude $\theta/2$ then *the variation of the molecular rotation is equal to the moment of one of these segments with respect to the extremity of the other.*

Or, if one prefers, *the variation of the molecular rotation is obtained by turning the projection of the segment* (λ, μ, ν) *into the plane that is tangent to the wave through a right angle and multiplying it by* $\theta/2$.

One sees that *the variation of the molecular rotation is always a segment that is tangent to the surface of discontinuity.*

It is clear that one can write down formulas that are analogous to (**70**) for the variations of the derivatives of p, q, r in the discontinuities of order greater than 2.

115. – The *direction of a discontinuity* is the direction of the segment (λ, μ, ν) that characterizes it, a direction that we also know to be the direction of the segments $(\lambda_1, \mu_1, \nu_1), \ldots, (\lambda_n, \mu_n, \nu_n)$.

If this direction is normal to the wave surface S, *when considered in the present state,* then the discontinuity will be called *longitudinal;* if it is tangent then the discontinuity will be called *transversal.*

It results from the formulas above that *a longitudinal discontinuity has no influence on the molecular rotation* and *a transversal discontinuity has no influence on the density.*

116. Sign of a discontinuity. – Suppose that we are given a discontinuity of arbitrary order that relates to the instant t, and imagine that upon starting at that instant, each molecule continues to move with the same velocity and the same initial acceleration – in other words, that one makes these velocities and accelerations continuous with respect to time.

As we remarked in no. **90**, the fictitious motion thus obtained no longer verifies the hypotheses that were stated in nos. **44-46**, in general. Either the two partial media, 1 and 2, penetrate each other, or, on the contrary, they separate from each other and cease to be contiguous.

In the first case, the discontinuity considered will be called *positive* or *compressive;* in the second case, it will be called *negative* or *dilative.*

From this definition, the sign of a discontinuity is not modified when one inverts the roles of the two regions that it separates. It changes if one reverses the motion, i.e., if one substitutes the anterior instants for the posterior instants, and vice versa (which amounts to changing the sign of t in the equations of motion).

From what we said in no. **94**, it is obvious that the sign in question is related, in a sense, to the difference that exists between the two values of the normal component of the velocity or of one of the successive accelerations. To obtain the exact form of that relation, it suffices to repeat the present argument in that context.

Let S_1 be the boundary surface of the medium 1 (which coincides with the surface of discontinuity S at the given instant t_0) in our fictitious motion, and let (x_1, y_1, z_1) be any of its points, with:

$$\varphi(x_1, y_1, z_1) = 0, \tag{71}$$

for its equation.

Agree, to fix ideas, that medium 1 is situated on the $\varphi < 0$ side of that surface.

Furthermore, let S_2 be the boundary surface of medium 2, and let (x_2, y_2, z_2) be one of its points that coincides with (x_1, y_1, z_1) at the instant t_0, and this point is likewise assumed to be animated by our fictitious motion. There will be penetration of the two media if one has:

$$\varphi(x_2, y_2, z_2, t) = 0, \tag{72}$$

for $t = t_0 + \varepsilon$.

In order for this to be true, it suffices that the derivative of the left-hand side is negative for $t = t_0$, if it is not null. Since the left-hand side of equation (**71**) is assumed to be identically null, this gives (compare equation (**46**)):

$$\frac{\partial\varphi}{\partial x}\left[\frac{\delta x}{\delta t}\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\delta y}{\delta t}\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\delta z}{\delta t}\right]<0. \tag{73}$$

However, since φ is assumed to be negative on the side of region 1, $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ have the signs of the direction cosines of the normal to S that is directed into region 2. Therefore, inequality (**73**) expresses that *when passing from region* 1 *into region* 2 *the normal component of the velocity increases by a segment that is directed toward region* 1.

On the contrary, if the change of the normal component of the velocity is directed towards region 2 when one enters into the latter region then the quantity $\varphi(x_2, y_2, z_2, t)$ will be positive at the instant $t_0 + \varepsilon$ and, as consequence, the point (x_2, y_2, z_2) will be external to the new position that is occupied by the medium 1 at this instant. In a word, the discontinuity will be dilative.

Now, if the normal component of the velocity remains continuous then the inequality (**73**) is replaced by an equality that must express the fact that the second derivative of $\varphi(x_2, y_2, z_2, t)$ is negative. To that effect, as in no. **94**, one will have only to differentiate equation (**71**) and inequality (72) twice. In the case for which the velocity itself is continuous (and no longer only its normal component), one therefore obtains for a compressive discontinuity (compare equation (**47**)):

$$\frac{\partial\varphi}{\partial x}\left[\frac{\delta^2 x}{\delta t^2}\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\delta^2 y}{\delta t^2}\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\delta^2 z}{\delta t^2}\right]<0,$$

and, for a dilative discontinuity:

$$\frac{\partial\varphi}{\partial x}\left[\frac{\delta^2 x}{\delta t^2}\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\delta^2 y}{\delta t^2}\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\delta^2 z}{\delta t^2}\right]>0.$$

These expressions express the same conditions as always, except that velocity is replaced by acceleration.

If the latter is continuous throughout the motion then it will suffice to replace it with the third-order acceleration, and so on.

117. – The preceding conditions persist even when there is no compatibility (a fact that will be useful in the sequel).

Of course, it makes sense to speak of stationary discontinuities, which, from the foregoing, are neither compressive nor dilative.

If there is compatibility then one may give a somewhat different form to the result.

To fix ideas, take the first-order case; the difference of the normal velocities is equal to $-\theta$ multiplied by the normal component of the segment (λ, μ, ν); however, the latter is equal to $\rho_1/\rho_2 - 1$.

Therefore, the discontinuity is dilative or compressive according to whether propagation is toward the region of increasing density or the region of decreasing density.

For a second-order discontinuity it is no longer the velocity but the acceleration that is discontinuous. Similarly, if the discontinuity is of order n then it will depend only upon the acceleration of order n, and its sign will consequently depend on the sign of the normal component of the segment $(\lambda_n, \mu_n, \nu_n)$.

On the other hand, consider the successive derivatives of the density with respect to time. The first n – 2 of them are continuous; for the $(n-1)^{\text{st}}$ one, one has (no. **111**):

$$\left[\frac{\delta^{n-1}\log\frac{1}{\rho}}{\delta t^{n-1}}\right] = (\lambda\alpha + \mu\beta + \nu\gamma)(-\theta)^{n-1},$$

which, from formulas (**58**), may be written:

$$\left[\frac{\delta^{n-1}\log\frac{1}{\rho}}{\delta t^{n-1}}\right] = -\frac{1}{\theta}(\lambda_n\alpha + \mu_n\beta + \nu_n\gamma).$$

The discontinuity will be compressive if the parenthesis on the right-hand side is negative, i.e., *if the propagation is towards the region in which the* $(n-1)^{\text{st}}$ *derivative of the dilatation with respect to time is largest.*

118. – One may append certain simple remarks to the preceding considerations that relate to the splitting of a discontinuity.

For example, consider a compressive first-order discontinuity. Suppose that there is no compatibility, but the splitting relates to just two waves, and that, moreover, these two waves propagate in opposite senses. It is clear that at least one of the two is compressive. In order for this to be the case, the propagation must be towards the region of lower density, and, as a consequence, *the intermediate state that gives birth to the two states will be more condensed than at least one of the two states.*

On the contrary, if the given discontinuity is dilative then the intermediate state will be less condensed than at least one of them.

§ 4. – STUDY OF DISCONTINUITIES (cont.) HIGHER-ORDER COMPATIBILITY CONDITIONS

119. – In the foregoing, we studied a discontinuity of order n from the viewpoint of its effects on the derivatives of order n. What relations does this discontinuity imply between the variations of the derivatives of order than n (upon supposing that they, like the latter, take definite values on each side of the wave)?

These relations are notably more complicated than the latter ones. We form them only in the simplest case.

Consider a first-order discontinuity, and introduce the second derivatives. We have found that the conditions regarding the variable x are:

(36) $$\left[\frac{\delta x}{\delta a}\right] = \lambda f_a, \quad \left[\frac{\delta x}{\delta b}\right] = \lambda f_b, \quad \left[\frac{\delta x}{\delta c}\right] = \lambda f_c,$$

(74) $$\left[\frac{\delta x}{\delta t}\right] = \lambda f_t,$$

the first three of which provide identity conditions and the last of which provides a kinematical compatibility condition.

We suppose, to fix ideas, that f represents (rigorously, this time) the distance from the point a, b, c to S_0. With these conditions, we may remark that $\frac{\delta^2 f}{\delta a\,\delta t}, \frac{\delta^2 f}{\delta b\,\delta t}, \frac{\delta^2 f}{\delta c\,\delta t}$ – viz., the derivatives of the direction cosines of the normal to the surface f = const. with respect to time – will be known when f_t is given on that surface. Of course, the same will be true for the second derivatives of f with respect a, b, c. All of these quantities will therefore be known when one is given S_0 and the left-hand sides of equations (**36**), (**74**).

We compose the following differential forms by means of them:

(75) $$\begin{cases} f_1(da, db, dc) = f_a da + f_b db + f_c dc, \\ f_2(da, db, dc) = \left(\frac{\delta}{\delta a} da + \frac{\delta}{\delta b} db + \frac{\delta}{\delta c} dc\right)^2 f = \frac{\delta^2 f}{\delta a^2} da^2 + \cdots, \\ f_3(da, db, dc) = \frac{\delta^2 f}{\delta a\,\delta t} da + \frac{\delta^2 f}{\delta b\,\delta t} db + \frac{\delta^2 f}{\delta c\,\delta t} dc, \end{cases}$$

and we likewise consider the analogous forms that contain the variations of the derivatives of x:

(76) $$\begin{cases} \xi_2(da, db, dc) = \left(\left[\frac{\delta}{\delta a}\right] da + \left[\frac{\delta}{\delta b}\right] db + \left[\frac{\delta}{\delta c}\right] dc\right)^2 x \\ \xi_3(da, db, dc) = \left[\frac{\delta^2 x}{\delta a\,\delta t}\right] da + \left[\frac{\delta^2 x}{\delta b\,\delta t}\right] db + \left[\frac{\delta^2 x}{\delta c\,\delta t}\right] dc. \end{cases}$$

First, form the identity conditions. Differentiate the equation $[x]$ = const. twice on the surface S_0. We see that one must have:

$$\xi_2 + \left[\frac{\delta x}{\delta a}\right] d^2 a + \left[\frac{\delta x}{\delta b}\right] d^2 b + \left[\frac{\delta x}{\delta c}\right] d^2 c = 0,$$

by means of the relations:

(77) $$f_1 = f_a\, da + f_b\, db + f_c\, dc = 0$$

and:

(78) $$f^2 + f_a\, d^2a + f_b\, d^2b + f_c\, d^2c = 0.$$

By virtue of relations (**36**), this amounts to saying that equation (**77**) implies $\xi_2 - \lambda f_2 = 0$. In order for this to be true, it is necessary that there exist a linear polynomial $Ada + Bdb + Cdc$ such that one has, for any *da, db, dc:*

(79) $$\xi_2 = \lambda f_2 + 2f_1(Ada + Bdb + Cdc) = 0.$$

Likewise, differentiate the first equation (**36**) on S_0. The differential of $\dfrac{\delta f}{\delta a}$ is:

$$\frac{\delta^2 f}{\delta a^2}da + \frac{\delta^2 f}{\delta a\,\delta b}db + \frac{\delta^2 f}{\delta a\,\delta c}dc = \frac{1}{2}\frac{\partial f_2}{\partial(da)},$$

and, similarly, the differential of $\left[\dfrac{\delta x}{\delta a}\right]$ is $\dfrac{1}{2}\dfrac{\partial \xi_2}{\partial(da)}$. One therefore has:

$$\frac{1}{2}\frac{\partial \xi_2}{\partial(da)} = f_a\,d\lambda + \frac{\lambda}{2}\frac{\partial f_2}{\partial(da)}.$$

However, upon differentiating identity (**79**) with respect to *da* one obtains (since $f_1 = 0$):

$$\frac{1}{2}\frac{\partial \xi_2}{\partial(da)} = \frac{\lambda}{2}\frac{\partial f_2}{\partial(da)} + f_a(Ada + Bdb + Cdc),$$

and comparing these two formulas gives:

(80) $$d\lambda = Ada + Bdb + Cdc \qquad \text{(on } S_0\text{)}.$$

Formulas (**79**) and (**80**), as well as analogous formulas that relate to *y*, *z* (which introduce six more auxiliary coefficients that are analogous to *A*, *B*, *C*), are the identity conditions.

120. – We now pass on to the compatibility conditions. To that effect, we must complete the preceding calculation, on the one hand, by differentiating on $\mathcal{S}_0$ and, on the other hand, on S_0, by using (**74**). If we first differentiate the latter on S_0 then we get:

$$\xi_1' = f_t\,d\lambda + \lambda f_1' = \lambda f_1' + f_t(Ada + Bdb + Cdc).$$

Since this must be true for all values of *a, b, c* that verify condition (**77**), one has, upon introducing the auxiliary quantity λ':

(**81**)
$$\begin{cases} \left[\dfrac{\delta^2 x}{\delta a\,\delta t}\right] = \lambda\dfrac{\delta^2 f}{\delta a\,\delta t} + Af_t + \lambda' f_a, \\ \left[\dfrac{\delta^2 x}{\delta b\,\delta t}\right] = \lambda\dfrac{\delta^2 f}{\delta b\,\delta t} + Bf_t + \lambda' f_b, \\ \left[\dfrac{\delta^2 x}{\delta c\,\delta t}\right] = \lambda\dfrac{\delta^2 f}{\delta c\,\delta t} + Cf_t + \lambda' f_c. \end{cases}$$

If we now make a, b, c, t vary on $\mathcal{S}_0$ then we must have, no longer (**77**), but now:

(**77′**)
$$f_1 + f_t\,dt = 0.$$

If the first equation (**36**) is differentiated under these new conditions then it will give (from (**81**)):

$$\frac{1}{2}\frac{\partial \xi_2}{\partial(da)} + \left(\lambda\frac{\delta^2 f}{\delta a\,\delta t} + Af_t + \lambda' f_a\right)dt = f_a\,d\lambda + \lambda\left(\frac{1}{2}\frac{\partial f_2}{\partial(da)} + \frac{\delta^2 f}{\delta a\,\delta t}\right).$$

If we replace ξ_1 with its expression (**79**) and take (**77′**) into account then we have, after making all reductions:

(**80′**)
$$A\,da + B\,db + C\,dc + \lambda' dt = d\lambda \qquad \text{(on } \mathcal{S}_0\text{)}.$$

If we suppose that the abrupt variations of the second derivatives of x are known then everything in the left-hand side of (**80′**) is known (thanks to equations (**79**) and (**81**)), and this gives us the infinitesimal variation of λ when time varies.

Finally, recall (**74**) in order to differentiate it on $\mathcal{S}_0$, which gives (on account of (**80′**)):

$$\xi_1' + \left[\frac{\delta^2 x}{\delta t^2}\right]dt = f_t\,(A\,da + B\,db + C\,dc + \lambda' dt) + \lambda\left(f_1' + \frac{\delta^2 f}{\delta t^2}\,dt\right).$$

We replace ξ_1' by its value from (**81**) and take (**77′**) into account; only the terms in dt remain. This gives:

(**82**)
$$\left[\frac{\delta^2 x}{\delta t^2}\right] = 2\lambda' f_t + \lambda\frac{\delta^2 f}{\delta t^2}.$$

If we always suppose that the second derivatives of x are known then this formula gives us $\dfrac{\delta^2 f}{\delta t^2}$, i.e., the motion of the wave surface up to third-order infinitesimal elements. However, one will likewise obtain $\dfrac{\delta^2 f}{\delta t^2}$ by means of analogous equations that correspond to y and z. Upon equating the expressions thus obtained, one gets new compatibility conditions.

If one sets:

(83)
$$\begin{cases} X_2 = \xi_2 + 2\xi_1' dt + \left[\dfrac{\delta^2 x}{\delta t^2}\right] dt^2 \\ \qquad = \left(\left[\dfrac{\delta}{\delta a}\right] da + \left[\dfrac{\delta}{\delta b}\right] db + \left[\dfrac{\delta}{\delta c}\right] dc + \left[\dfrac{\delta}{\delta t}\right] dt\right)^2 x, \\ F_2 = f_2 + 2 f_1' dt + \dfrac{\delta^2 f}{\delta t^2} dt^2 = \left(\dfrac{\delta}{\delta a} da + \dfrac{\delta}{\delta b} db + \dfrac{\delta}{\delta c} dc + \dfrac{\delta}{\delta t} dt\right)^2 f, \\ F_1 = f_1 + f_t dt, \end{cases}$$

then the preceding results may be summarized in the identity:

(79′)
$$X_2 = \lambda F_2 + 2F_1 (A da + B db + C dc + \lambda' dt).$$

121. Third-order conditions. Now add the following forms to (**75**), (**76**):

$$f_3 = \left(\frac{\delta}{\delta a} da + \frac{\delta}{\delta b} db + \frac{\delta}{\delta c} dc\right)^3 f,$$

$$f_2' = \left(\frac{\delta}{\delta a} da + \frac{\delta}{\delta b} db + \frac{\delta}{\delta c} dc\right)^2 \frac{\delta f}{\delta t} = \frac{\delta^3 f}{\delta a^2 \delta t} da^2 + \cdots,$$

$$\xi_3 = \left(\left[\frac{\delta}{\delta a}\right] da + \left[\frac{\delta}{\delta b}\right] db + \left[\frac{\delta}{\delta c}\right] dc\right)^3 x = \left[\frac{\delta^3 x}{\delta a^3}\right] da^3 + \cdots$$

$$\xi_2' = \left(\left[\frac{\delta}{\delta a}\right] da + \left[\frac{\delta}{\delta b}\right] db + \left[\frac{\delta}{\delta c}\right] dc\right)^2 \frac{\delta x}{\delta t} = \left[\frac{\delta^2 x}{\delta a^2 \delta t}\right] da^2 + \cdots$$

Differentiate the relation $[x] = 0$ three times on S_0; we obtain:

$$\xi_3 + \frac{3}{2}\left(\frac{\partial \xi_2}{\partial(da)} d^2 a + \frac{\partial \xi_2}{\partial(db)} d^2 b + \frac{\partial \xi_2}{\partial(dc)} d^2 c\right)$$
$$+ \left[\frac{\delta x}{\delta a}\right] d^3 a + \left[\frac{\delta x}{\delta b}\right] d^3 b + \left[\frac{\delta x}{\delta c}\right] d^3 c = 0,$$

by means of equations (**77**), (**78**) and:

(84)
$$\begin{cases} f_3 + \dfrac{3}{2}\left(\dfrac{\partial f_2}{\partial(da)}d^3a + \dfrac{\partial f_2}{\partial(db)}d^3b + \dfrac{\partial f_2}{\partial(dc)}d^3c\right) \\ \qquad + f_a d^3a + f_b d^3b + f_c d^3c = 0. \end{cases}$$

The third differentials are eliminated immediately by condition (**36**). If we replace ξ_2 by its value from (**79**) then we get the relation:

$$\xi_3 = \lambda f_3 + 3f_2(Ada + Bdb + Cdc),$$

which must persist because of (**77**). In other words, we may write:

(85)
$$\xi_3 = \lambda f_3 + 3f_2(Ada + Bdb + Cdc) + 3f_1\psi_2,$$

in which $\psi_2(da, db, dc)$ is a quadratic form in da, db, dc.

In order to understand the significance of ψ_2, consider equation (**79**), which is an identity with respect to da, db, dc. Change da, db, dc into $d'a, d'b, d'c$, and then differentiate them on S_0 (without varying $d'a, d'b, d'c$); this gives:

$$\begin{aligned} &\frac{1}{3}\left(\frac{\partial\xi_3}{\partial(d'a)}da + \frac{\partial\xi_3}{\partial(d'b)}db + \frac{\partial\xi_3}{\partial(d'c)}dc\right) \\ &\qquad = f_2(d'a, d'b, d'c)(Ada + Bdb + Cdc) \\ &\qquad + \frac{\lambda}{3}\left(\frac{\partial f_3}{\partial(d'a)}da + \frac{\partial f_3}{\partial(d'b)}db + \frac{\partial f_3}{\partial(d'c)}dc\right) \\ &\qquad + 2f_1(d'a, d'b, d'c)(dAd'a + dBd'b + dCd'c) \\ &\qquad + (Ad'a + Bd'b + Cd'c)\left(\frac{\partial f_2}{\partial(da)}d'a + \frac{\partial f_2}{\partial(db)}d'b + \frac{\partial f_2}{\partial(dc)}d'c\right). \end{aligned}$$

If we replace ξ_3 by its value (**85**) then the polynomial $f_1(d'a, d'b, d'c)$ becomes a factor, and the equation reduces to:

$$dAd'a + dBd'b + dCd'c) = \frac{1}{2}\left(\frac{\partial\psi_2}{\partial(da)}d'a + \frac{\partial\psi_2}{\partial(db)}d'b + \frac{\partial\psi_2}{\partial(dc)}d'c\right).$$

Since $d'a, d'b, d'c$ are arbitrary, one has:

(86)
$$dA = \frac{1}{2}\frac{\partial\psi_2}{\partial(da)}, \qquad dB = \frac{1}{2}\frac{\partial\psi_2}{\partial(db)}, \qquad dC = \frac{1}{2}\frac{\partial\psi_2}{\partial(dc)}, \qquad (\text{on } S_0),$$

equations from which one may deduce the value of $d^2\lambda$ (by virtue of (**80**)):

(87) $$d^2\lambda = A\, d^2a + B\, d^2b + C\, d^2c + \psi_2.$$

Once we have the identity conditions, we obtain the compatibility conditions by differentiating all of the conditions (**79**) and (**82**) on $\mathcal{S}_0$.

We carry out the calculations for one piece upon introducing the expressions (**83**), and:

$$X_3 = \xi_3 + 3\xi_2' dt + 3\left(\left[\frac{\delta^3 x}{\delta a \delta t^2}\right] da + \cdots\right) dt^2 + \left[\frac{\delta^3 x}{\delta t^3}\right] dt^3$$

$$= \left(\left[\frac{\delta}{\delta a}\right] da + \left[\frac{\delta}{\delta b}\right] db + \left[\frac{\delta}{\delta c}\right] dc + \left[\frac{\delta}{\delta t}\right] dt\right)^3 x$$

$$F_3 = f_3 + 3 f_2' dt + 3\left(\frac{\delta^3 f}{\delta a \delta t^2} da + \frac{\delta^3 f}{\delta b \delta t^2} db + \frac{\delta^3 f}{\delta c \delta t^2} dc\right) dt^2 + \frac{\delta^3 f}{\delta t^3} dt^3$$

$$= \left(\frac{\delta}{\delta a} da + \frac{\delta}{\delta b} db + \frac{\delta}{\delta c} dc + \frac{\delta}{\delta t} dt\right)^3 f.$$

We may then rewrite the preceding formulas by replacing the ξ's with X's, f with F, and introducing terms in dt, d^2t, d^3t everywhere they exist in terms of da, db, dc; d^2a, d^2b, d^2c; d^3a, d^3b, d^3c. In this fashion, the identity (**85**) will be replaced by:

(85′) $$X_3 = \lambda F_3 + 3F_2(A da + B db + C dc + \lambda' dt) + 3F_1 \Psi_2,$$

in which Ψ_2 is a quadratic form in da, db, dc, dt.

Comparing the preceding formula with (**85**) shows that the part of Ψ_2 that is independent of dt is nothing but ψ_2; one may set:

(88) $$\Psi_2 = \psi_2 + 2dt(A' da + B' db + C' dc) + \lambda'' dt^2.$$

Upon continuing to follow the method that we have always presented, one verifies that the complete differentials of A, B, C, λ' on S_0 are:

(86′) $$\begin{cases} dA = \dfrac{1}{2}\dfrac{\partial \Psi_2}{\partial(da)} = \dfrac{1}{2}\dfrac{\partial \psi_2}{\partial(da)} + A' dt, \\ dB = \dfrac{1}{2}\dfrac{\partial \Psi_2}{\partial(db)} = \dfrac{1}{2}\dfrac{\partial \psi_2}{\partial(db)} + B' dt, \\ dC = \dfrac{1}{2}\dfrac{\partial \Psi_2}{\partial(dc)} = \dfrac{1}{2}\dfrac{\partial \psi_2}{\partial(dc)} + C' dt, \\ d\lambda' = \dfrac{1}{2}\dfrac{\partial \Psi_2}{\partial(dt)} = \dfrac{1}{2}\dfrac{\partial \psi_2}{\partial(dt)} + A' da + B' db + C' dc + \lambda'' dt, \end{cases}$$

in such a way that (compare (**87**)):

(**87′**) $$\Psi_2 + A d^2 a + B d^2 b + C d^2 c + \lambda' d^2 t = d^2 \lambda \qquad (\text{on } 6_0).$$

Finally, if one equates the coefficients of the different powers of dt in (**85′**) then one has, in addition to the identity (**85**), the relations:

(**89**) $$\begin{cases} \xi_2' = \lambda f_2' + \lambda' f_2 + 2 f_1'(A da + B db + C dc) + f_1 \psi_2 \\ \qquad + 2 f_1 (A' da + B' db + C' dc), \\ \left[\dfrac{\delta^3 x}{\delta a \, \delta t^2}\right] = \lambda \dfrac{\delta^3 f}{\delta a \, \delta t^2} + A \dfrac{\delta^2 f}{\delta t^2} + 2\lambda' \dfrac{\delta^2 f}{\delta a \, \delta t} + 2A' f_t + \lambda'' f_a, \\ \left[\dfrac{\delta^3 x}{\delta t^3}\right] = 3\lambda' \dfrac{\delta^2 f}{\delta t^2} + 3\lambda'' f_t + \lambda \dfrac{\delta^3 f}{\delta t^3}. \end{cases}$$

If we are given the position of the discontinuity surface *at the instant t* then these various formulas and their analogues relating to *y*, *z* permit us to calculate the parameters, A', B', C', λ'', and their analogues as functions of the abrupt variations of the first, second, and third derivatives of the coordinates. Similarly, one eliminates these quantities between them, in such a manner as to obtain the compatibility conditions. Finally, the last of formulas (**89**) gives us $\dfrac{\delta^3 f}{\delta t^3}$, i.e., the third-order acceleration of the wave surface, and one has two other compatibility conditions upon equating the value thus obtained to the one that one deduces from the analogous equations that relate to *y* and *z*.

122. If these conditions are not satisfied then the *first-order* discontinuity surface might remain unique. However, if one adds at least one wave of higher order then it separates from the first in the neighboring instants to the one considered.

123. Case of one wave of second order. – We further propose to find the third-order conditions for a second-order discontinuity.

We may arrive at them by calculations that are analogous to the ones that we just carried out. However, we may also deduce this result from the preceding one, because one obtains a second-order discontinuity upon setting $\lambda = 0$ in formulas (**79**) and (**79′**) (the analogous quantities μ and ν are likewise null).

The second-order conditions must then coincide with those of nos. **85**, **86**, **103**; indeed, this is easy to confirm. From relation (**80′**), it suffices to remark that $A da + B db + C dc + \lambda' dt$ must be annulled because of condition (**77′**). One may therefore write:

(**90**) $$A = \frac{\lambda}{2} f_a, \qquad B = \frac{\lambda}{2} f_b, \qquad C = \frac{\lambda}{2} f_c, \qquad \lambda' = \frac{\lambda}{2} f_t,$$

(in which, λ is, of course, no longer the quantity that we have always denoted by that symbol, which is null here). Relation (**79′**) then becomes:

$$X_2 = \lambda F_1^2,$$

which is precisely the result that we found in no. **103**.

Now consider the expression Ψ_2 that figures in the third-order conditions. From (**87′**), this expression must be such that:

$$\Psi_2 + A\,d^2a + B\,d^2b + C\,d^2c + \lambda' d^2t = d^2\lambda,$$

are nullified because of the relations $F_1 = 0$, and:

$$F_2 + f_a\,d^2a + f_b\,d^2b + f_c\,d^2c + f_t\,d^2t = 0.$$

From the values (**90**) of A, B, C, λ', this gives:

(**91**) $$\Psi_2 = \frac{\lambda}{2} F_2 + F_1(A_1 da + B_1 db + C_1 dc + \lambda_1' dt),$$

in which A_1, B_1, C_1, λ_1' denote new parameters.

All three relations (**86′**) are then reduced (for example, since $A = \frac{\lambda}{2} f_a$ and $df_a = \frac{1}{2}\frac{\partial F_2}{\partial(da)}$) to:

$$d\lambda = A_1\,da + B_1\,db + C_1\,dc + \lambda_1' dt,$$

and relations (**85**), (**85′**) become:

$$\begin{aligned}
X_3 &= 3\lambda F_1 F_2 + 3F_1^2(A_1 da + B_1 db + C_1 dc + \lambda_1' dt),\\
\xi_3 &= 3\lambda f_1 f_2 + 3 f_1^2(A_1 da + B_1 db + C_1 dc),\\
\xi_3' &= \lambda f_1 f_2 + 2 f_1[\lambda f_1' + f_t(A_1 da + B_1 db + C_1 dc)] + f_1^2 \lambda_1',\\
\left[\frac{\delta^3 x}{\delta a\,\delta t^2}\right] &= A_1 f_1^2 + 2\lambda f_t \frac{\delta^2 f}{\delta a\,\delta t} + f_a\left(\lambda\frac{\delta^2 f}{\delta t^2} + 2\lambda_1' f_t\right),\\
\left[\frac{\delta^3 x}{\delta t^3}\right] &= 3 f_t\left(\lambda\frac{\delta^2 f}{\delta t^2} + \lambda_1' f_t\right).
\end{aligned}$$

Naturally, these equations no longer give $\frac{\delta^3 f}{\delta t^3}$, but only $\frac{\delta^2 f}{\delta t^2}$, which appears in the last two derivatives, and, as a consequence, may be deduced from six different equations (counting the one that correspond to y, z).

Moreover, the distinction between the identity conditions and the compatibility conditions is different from the one we had before. Indeed, condition (**74**) is given, and, as a consequence, furnishes the identity conditions, instead of making it part of the compatibility conditions.

CHAPTER III

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THE FORMULATION OF THE HYDRODYNAMICAL PROBLEM
AS AN EQUATION

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§ 1. THE INTERNAL EQUATIONS AND THE SUPPLEMENTARY CONDITION

124. – One knows that the equations of hydrodynamics are deduced from those of hydrostatics by d'Alembert's principle. As is well known, this deduction, upon which we will not insist here, leads to the following equations:

$$
(\mathbf{1}) \qquad \begin{cases} X = \dfrac{\delta^2 x}{\delta t^2} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ Y = \dfrac{\delta^2 y}{\delta t^2} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ Z = \dfrac{\delta^2 z}{\delta t^2} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial z}, \end{cases}
$$

in which the symbols ∂, δ have the same significance as in chapter II. The quantities $u = \dfrac{\delta x}{\delta t}$, $v = \dfrac{\delta y}{\delta t}$, $w = \dfrac{\delta z}{\delta t}$ denote the components of velocity, in such a way that $\dfrac{\delta^2 x}{\delta t^2}$, for example, is equal to $\dfrac{\delta u}{\delta t}$; equation (**18**) of chapter II further permits us to write:

$$
(\mathbf{2}) \qquad \begin{cases} \dfrac{\delta^2 x}{\delta t^2} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\delta^2 y}{\delta t^2} = \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = Y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \dfrac{\delta^2 z}{\delta t^2} = \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = Z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z}, \end{cases}
$$

Here, the quantity ρ – which is the density, as expressed by equations (**3**) and (**3′**) of no. **47** – is:

(3)
$$\frac{\rho_0}{\rho} = \begin{vmatrix} \frac{\delta x}{\delta a} & \frac{\delta x}{\delta b} & \frac{\delta x}{\delta c} \\ \frac{\delta y}{\delta a} & \frac{\delta y}{\delta b} & \frac{\delta y}{\delta c} \\ \frac{\delta z}{\delta a} & \frac{\delta z}{\delta b} & \frac{\delta z}{\delta c} \end{vmatrix}.$$

Equations (**1**) and (**3**) (through which one may express $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$, with the aid of the partial derivatives of *p*, *x*, *y*, *z* with respect to *a*, *b*, *c*), when combined with one supplementary condition that remains to be defined, are the ones that define the unknown quantities *x*, *y*, *z*, *p*, ρ as functions of *a*, *b*, *c*, *t*. However, one knows that it is most often advantageous to substitute the Eulerian formulation of this equation, in which *a*, *b*, *c* do not figure, for this mode of formulation, which is that of Lagrange. In the Eulerian formulation, the independent variables are *x*, *y*, *z*, *t*, the unknown functions are *p*, ρ, and the components of the velocity are *u*, *v*, *w*.

In this manner of operation, equations (**1**) are replaced by equations (**2**), and as for equation (**3**), which defines ρ, it will be replaced by the equation:

(4)
$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

that was derived in no. **62**. The latter equation, which is called the *equation of continuity*, is nothing but the one that expresses the conservation of mass.

125. – Equations (**1**) and (**3**), which are of the Lagrangian form, and equations (**2**) and (**4**), which are of the Eulerian form, are insufficient in number to determine the unknown functions since there are five of them, and it is necessary to add a fifth equation to them that is called the *supplementary condition.* The latter condition is the one by which the physical nature of the fluid intervenes, and has not been assumed in the formulation of the equations that we have written up till now.

For liquids (which are assumed to be perfectly incompressible) this equation is:

$$\rho = \text{constant.}$$

As far as compressible fluids are concerned, the formation of the supplementary condition involves greater difficulty. One knows that one then considers the fluid as characterized – from the physical viewpoint – by a relation of the form:

(5)
$$F(p, \rho, T) = 0,$$

between the density ρ of a portion of the fluid, the pressure *p*, and temperature *T*. For example, for the perfect gas this relation is:

$$\frac{p}{\rho T} = \text{constant.} \tag{5′}$$

On the other hand, this relation provides the supplementary condition that we seek if one knows the law by which the temperature varies. When one supposes that it is constant – as had been done until the time of Laplace – the supplementary condition is given by the law of Mariotte:

$$\frac{p}{\rho} = \text{constant.} \tag{6}$$

If, on the contrary – and research into the velocity of sound has proved that this hypothesis is much closer to reality than the first – one admits that the gas has null conductivity, in such a way that the release or absorption of heat by the contraction or dilation of the different parts serves only to warm or cool the molecules themselves that are situated in these parts (*adiabatic* contraction or dilatation) then one confirms that relation (**6**) must be replaced by the following one:

$$\frac{p}{\rho^m} = \text{constant,} \tag{7}$$

in which m is a constant coefficient (the ratio of the two specific heats of the gas).

126. – One must note an essential difference that separates equation (**7**) from equations (**5′**) and (**6**). The constant that figures in the right-hand side of (**5′**) is an absolute constant that is known *a priori* for a gas of a given nature. Up to a constant factor, it is the one that one often calls the *density* of the gas, i.e., the ratio of the weight of an arbitrary volume of fluid to the weight of same volume of air under the same conditions of temperature and pressure. The same is true for the quantity that equation (**6**) refers to if one is given the temperature. On the contrary, the constant that is introduced in equation (**7**) is an integration constant that depends on the original state that the fluid starts from and varies adiabatically. If this state is unique for any mass then the same is true for the constant in question. However, it is not in the least bit necessary for this to be true; in the contrary case, the right-hand side of equation (**7**), which is independent of *t, is a function of a, b, c.*

127. – However, either of formulas (**6**) and (**7**) presupposes relation (**5**). Now, its existence itself does not come about without raising several objections in hydrodynamics. True, it is indubitable (at least under the conditions that rational hydrodynamics places on it) any time there is equilibrium, and relation (**5**) has been established for perfect gases, for example, by a series of experiments in which one compares diverse equilibrium states of the same gas. However, no analogous experiment has been carried out to verify this relation for a gas in more or less rapid motion.

Therefore, it has not been established that equation (**5**) preserves its form in this latter case. Similarly, Bjerknes ([37]) has studied the hypotheses under which that relation will be modified by terms that correspond to motion, i.e., ones containing velocity or acceleration.

On the contrary, a line of reasoning that has the objective of establishing that relation (**5**) persists in every case has been presented by Duhem ([38]). It consists of reducing this fact to a hypothesis that relates to the form of the quantity that is called *the thermodynamic potential.* This represents a magnitude $\mathcal{F}$ that permits the application of the principle of virtual velocities when one accounts for the changes of temperature and pressure, just as the potential function for force permits us to write this same principle in a simple manner when one is in the domain of classical mechanics.

This thermodynamic potential is composed of the exterior force potential plus a complementary part that we call the *internal thermodynamic potential.*

Duhem assumes that the latter has an expression of the form:

$$\iiint \rho\Phi\, dx dy dz,$$

in which Φ depends on density and temperature. As a consequence, this potential is a function of the position of the medium, but not the velocity or acceleration of its points. One obtains the equilibrium conditions of a fluid by writing that the total variation of the thermodynamic potential is null or positive for any infinitesimal modification that is compatible with the constraints.

If one defines the modification in question by successively taking infinitesimal modifications that do not change the density at any point and do not interrupt the continuity then one proves the existence of a function p that is *continuous at each point,* and whose introduction verifies the classical equations of hydrostatics.

Upon introducing a modification that creates a cavity one obtains the condition $p > 0$.

Finally, upon considering a modification that varies the density one arrives at the relation:

$$p = \rho^2 \frac{\partial \Phi}{\partial \rho},$$

and this relation is of the form (**5**), precisely.

128. – If one now writes equations of motion instead of equations of equilibrium then – by virtue of the general laws of thermodynamics – the principle that one applies is that of Hamilton (in which, the thermodynamics potential replaces the force potential).

([37]) *Acta Mathematica,* tome IV, pp. 121-170.

([38]) *Cours de Physique mathématique, Hydrodynamique, Élasticité, Acoustique,* tome I; Paris, Hermann, 1891.

However, the application of this principle leads to the same result as d'Alembert's principle, namely, equations (**1**), and formula (**5′**) persists in the case of motion as in the case of equilibrium.

129. – By virtue of the foregoing, does assuming equation (**5**) force equation (**7**) to follow as a consequence in the case of adiabatic compression or relaxation?

An objection that is similar to the preceding one is posed with regard to this. Indeed, the reasoning that permits us to pass from one of these relations to the other rests on the study of the specific heat of the gas, namely, the formula:

$$dQ = C\frac{\partial T}{\partial v}dv + c\frac{\partial T}{\partial p}dp, \tag{8}$$

which represents the quantity of heat released by an infinitesimal modification as a function of the variation of the volume dv and the variation of the pressure dp. Now, the values C and c of the specific heat have been established, like the coefficients of dilatation, by experiments that are essentially static, i.e., one in which the motion of the gaseous mass to be studied has very little rapidity, or by experiments in which the velocities of these movements are poorly known (the experiments of Clements and Desormes). Does the same formula remain valid in the general case of hydrodynamics?

Thermodynamics permits us to respond to this question. To that effect, it suffices to start with the fundamental equation that expresses the principle of equivalence:

$$d\mathcal{T} - \tfrac{1}{2}d\Sigma mV^2 = EdQ + dU\,, \tag{9}$$

in which $d\mathcal{T}$ represents the work done by the external forces that are applied to the system, $\Sigma m\mathrm{V}^2$ is the vis viva, E is the mechanical equivalent of a calorie, dQ is the quantity of heat released, and U is the internal energy, i.e., a certain function of the internal state of the system.

The external forces that are applied to a gaseous mass will be of two types:

1. forces applied to the mass elements, such as gravity, electricity, etc.
2. external pressures that are applied to the surface.

The elementary work that is done by the latter will be:

$$dt\iint p[u\cos(n,x) + v\cos(n,y) + w\cos(n,z)]dS, \tag{10}$$

in which dS is the element of the boundary surface, n is the normal to that element, and u, v, w are the components of velocity, as before. The left-hand side of equation (**9**) is therefore written, letting $d\mathcal{T}_0$ denote the work done by gravity and other forces of the first category:

$$d\mathcal{T}_0 + dt\iint p[u\cos(n,x) + v\cos(n,y) + w\cos(n,z)]dS - \tfrac{1}{2}d\Sigma mV^2 = EdQ + dU. \tag{11}$$

Under the conditions that are imposed by the experimental measurement of the specific heat of the gas, the work $d\mathcal{T}_0$ is negligible. The same is true for the vis viva since the velocities are immeasurable. Since the double integral (**10**) represents the quantity $p d\mathcal{V}$, for constant p, in which $d\mathcal{V}$ is the variation of the volume, the preceding formula may be written:

(**12**) $$dQ = \frac{1}{E}(p\,d\mathcal{V} - dU).$$

It is clear that this formula is what gives us the heat that is released under an arbitrary change of volume or pressure, and, as a consequence, its right-hand side coincides with the quantity (**8**).

Now we place ourselves in the general case and start with the equations of motion (**1**). Multiply these equations by udt, vdt, wdt, and add them. Multiply again by $\rho\, dx\, dy\, dz$, and integrate over the volume occupied by our fluid. On the left-hand side, we obtain the work $d\mathcal{V}_0$.

As for the quantity:

$$dt\iiint \rho\left(u\frac{\delta^2 x}{\delta t^2} + v\frac{\delta^2 y}{\delta t^2} + w\frac{\delta^2 z}{\delta t^2}\right)dxdydz,$$

it is (by virtue of the relations $\frac{\delta^2 x}{\delta t^2} = \frac{\delta u}{\delta t}$, $\frac{\delta^2 y}{\delta t^2} = \frac{\delta v}{\delta t}$, $\frac{\delta^2 z}{\delta t^2} = \frac{\delta w}{\delta t}$) the differential of one-half the vis viva. The last term obtained in the right-hand side may be written:

$$-dt\left\{\iint p[u\cos(n,x) + v\cos(n,y) + w\cos(n,z)]dS + \iiint p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz\right\},$$

by virtue of Green's theorem. Finally, formula (**11**) then becomes:

$$EdQ + dU = dt\iiint p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz.$$

As one knows, the quantity:

$$dt\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz$$

represents the change of volume $d\mathcal{V}$ felt during the instant dt by the mass $\rho\, dx\, dy\, dz$. Therefore, the quantity of heat released by this element will be:

$$dQ = \frac{1}{E}(pdV - dU),$$

in which dU is the internal energy of that element.

As in the case of motion, we therefore arrive at formula (**12**). Moreover, this is true independently of the velocity that is imprinted on the fluid, as have stated.

130. – Nevertheless, one must remark that our reasoning excludes the possibility of abrupt variations in the velocity; in other words, percussions that are exerted in the interior of the mass. Indeed, this assumption is necessary in order to write the relation:

$$d\,\tfrac{1}{2}\Sigma mV^2 = dt\iiint \rho\left(u\frac{\delta^2 x}{\delta t^2} + v\frac{\delta^2 y}{\delta t^2} + w\frac{\delta^2 z}{\delta t^2}\right)dxdydz,$$

which expresses the variation of one-half the vis viva.

The relation in question is obviously analogous to the theorem of vis viva, and one knows that in the theory of percussions one is led to replace the theorem of vis viva with a relation of a different form: Carnot's theorem.

Moreover, it is clear that all of the preceding considerations relate to the case in which such percussions do not exactly exist. In particular, it is no longer true that pressure remains continuous when they are produced, as we shall have occasion to remark later on.

131. – When the fluid considered is neither an incompressible liquid nor a perfect gas, relations (**5**) will nevertheless continue to be true if one adopts the hypothesis of Duhem; however, it will have a different form from (**5′**). We will succeed in determining the internal equations of motion if we are given, in addition, the manner by which the temperature varies *a priori*, as must always be the case. In the case for which it remains constant, relation (**5**) obviously takes the form:

(**13**) $$F(\rho, p) = 0.$$

From a line of reasoning that is completely similar to the preceding one, the same will be true for the case of adiabatic relaxation or compression if the velocity remains continuous.

We have nothing to say in general about the analytic form of the relations thus obtained. However, they all satisfy a common condition of inequality: *pressure is an increasing function of density*. In other words, upon adding the pressure felt by the fluid, one diminishes its volume. This condition expresses the *stability* of the internal equilibrium of the medium ([39]).

([39]) Duhem, *loc. cit.*, pp. 80-83. – See also, ch. VI, no. 272.

§ 2. – INTRODUCTION OF THE BOUNDARY CONDITIONS

132. – The motion of an arbitrary fluid is determined, on the one hand, by internal equations such the ones that we wrote in the foregoing, and, on the other hand, by the initial conditions, and finally by the boundary conditions.

The initial conditions consist of being given the positions of the different particles and their velocities at an instant t_0 from which one begins to study the motion.

Boundary conditions will be of two types: On all or part of the surface, the fluid will be in contact with solid walls whose motion we assume to be given. We thus have to write that a part of this surface (which is, as we studied in no. **48**, constantly formed from the same molecules) coincides with the wall at each instant.

If there exists a free surface then we suppose that the value of pressure is given on that surface, i.e., the quantity p that figures in the equations of motion.

133. – In rational mechanics, when one writes the differential equations of motion for a system that is subject to arbitrary given constraints these equations permit us to calculate, in the first place, the accelerations of the different points at an arbitrary instant when one is given the positions of these points and their velocities at this instant, with the two conditions:

1. the given position of the system satisfies these constraints,
2. the given velocities of the different points are the ones that these points must receive under a motion that is compatible with these constraints at the instant in question.

We thus occupy ourselves with the problem of solving this question in the present case, in other words, of calculating the accelerations of the different points at the instant t_0 if we know:

1. the forces that act upon the fluid,
2. the positions of the points and their velocities,
3. the motion of the wall and the pressure on the free surface as well as its derivatives with respect to time at that instant.

The question presents itself in a very different fashion, depending on whether one is concerned with a liquid or a gas.

(1.) Case of liquids. – We employ the system of independent variables that we indicated in no. **61** (cont.), and, other than time t, we make the initial coordinates coincide with the present coordinates at the instant considered.

Since the fluid is assumed to be incompressible, relation (**13**) reduces to:

$$\rho = \text{constant},$$

and the value of p cannot be known from this. As for relation (**4**), it reduces to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{14}$$

The other equations of motion, namely equations (**1**), give us the sums:

$$\frac{\delta u}{\delta t}+\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\delta v}{\delta t}+\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{\delta w}{\delta t}+\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

If one differentiates the first of these equations with respect to *x*, the second with respect to *y*, the third with respect to *z*, and adds them term-by-term, then this will produce a result of the form:

(15)
$$\frac{1}{\rho}\Delta p+\frac{\partial}{\partial x}\left(\frac{\delta u}{\delta t}\right)+\frac{\partial}{\partial y}\left(\frac{\delta v}{\delta t}\right)+\frac{\partial}{\partial z}\left(\frac{\delta w}{\delta t}\right)=F,$$

in which F is a known function of x, y, z in the interior of the liquid volume.

However, equation (**14**) applies at any instant (x, y, z are the present coordinates at that instant). One must therefore take the derivative $\frac{\partial}{\partial t}$ of it and write:

(16)
$$\frac{\partial^2 u}{\partial x \partial t}+\frac{\partial^2 v}{\partial y \partial t}+\frac{\partial^2 w}{\partial z \partial t}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right)=0.$$

We replace $\frac{\partial u}{\partial t},\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}$ by their values as functions of $\frac{\delta u}{\delta t},\frac{\delta v}{\delta t},\frac{\delta w}{\delta t}$, as given by relations (**2**); we therefore obtain the value of $\frac{\partial}{\partial x}\left(\frac{\delta u}{\delta t}\right)+\frac{\partial}{\partial y}\left(\frac{\delta v}{\delta t}\right)+\frac{\partial}{\partial z}\left(\frac{\delta w}{\delta t}\right)$:

(16)
$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{\delta u}{\delta t}\right)+\frac{\partial}{\partial y}\left(\frac{\delta v}{\delta t}\right)+\frac{\partial}{\partial z}\left(\frac{\delta w}{\delta t}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) \\ \quad +\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right), \end{cases}$$

and if we substitute this in (**15**) then we get:

(17)
$$\Delta p = F_2,$$

since F_2 is likewise known at each point.

134. – On the other hand, the coordinates of the molecule that are in contact with the wall must not cease to verify the equation of the surface for the latter. This equation:

$$f(x, y, z, t) = 0$$

may or may not depend on time, but we nonetheless suppose that it is known at any instant. If we differentiate it twice with respect to t then we get:

$$\textbf{(18)} \qquad \frac{\partial f}{\partial x}\frac{\delta^2 x}{\delta t^2}+\frac{\partial f}{\partial y}\frac{\delta^2 y}{\delta t^2}+\frac{\partial f}{\partial z}\frac{\delta^2 z}{\delta t^2}+\left(\frac{\partial}{\partial x}\frac{\delta x}{\delta t}+\frac{\partial}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial}{\partial z}\frac{\delta z}{\delta t}+\frac{\partial}{\partial t}\right)^2 f=0.$$

Everything in this relation is known except for $\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2}$. Since the quantities $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are proportional to the direction cosines α, β, γ of the normal n to the wall, one thus obtains the normal component j_n of the acceleration at each point of it.

Finally, if one replaces $\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2}$ by their values obtained from equations (**1**) then one sees that *the preceding equation gives us the value of the normal derivative* $\frac{dp}{dn}$ *at every point of the wall.*

If the contact with the wall takes place all along the boundary surface of the fluid then the search for the quantity p as a function of x, y, z is, as a consequence, reduced to the problem that was the object of chapter I. As we saw, the problem assumes a condition of possibility, namely:

$$\iint \frac{dp}{dn}\,dS = \iiint F_2\,dx\,dy\,dz.$$

It is easy to interpret this condition. Indeed, it is obtained by integrating equation (**16′**) – or, what amounts to the same thing, equation (**16**), which is equivalent by relations (**2**) – over the entire volume that is occupied by the fluid. Now, the latter equation, when differentiated with respect to time as in (**14**), expresses the fact that the second derivative with respect to t of the elementary volume that is occupied by an infinitesimal portion of the fluid is null. After integration, it will express the fact that the total volume bounded by the given wall has a null second derivative; the necessity of this condition is obvious *a priori* ([40]).

If this condition is assumed to be satisfied then the considerations that were developed in chapter I show the possibility of the problem for all of the types of containers for which the Neumann method is applicable. We know, moreover, that the solution is unique, up to an arbitrary constant that may be added to the value of p and that has no influence on the desired accelerations.

These accelerations are therefore determined.

135. One similarly obtains the accelerations of higher order (provided that the expressions for the forces X, Y, Z are given at any instant). It suffices to differentiate equations (**1**), (**16′**), (**18**) with respect to time, and to determine the successive derivatives

([40]) It is easy to see that for one or the other of these forms the condition of possibility gives us the integral $\iint j_n\,dS$, when taken over the surface of the wall, as a function of position and velocity.

of pressure with the aid of the equations thus differentiated, as we have determined them from the original equations.

136. – Nevertheless, the preceding reasoning seems to suppose that the desired accelerations are distributed in a continuous manner. One may demand this (and the following chapters will show that a degree of doubt is justified) if one does not arrive at a different conclusion when one abandons this.

It is easy to confirm that such is not the case. Indeed, if we suppose that the velocities are continuous with respect to time then the pressure p must be continuous. On the other hand, the same will be true for the normal component of acceleration. Indeed, suppose that a discontinuity is produced along a certain surface. Either, this discontinuity will be stationary, and, in this case, since the velocities are assumed to be continuous, the fact in question will result from no. **94**, or it will propagate ([41]), and then there will be compatibility at other instants than the instant considered itself, at least for infinitely close instants. If this is true, and the density does not vary then the normal component of the discontinuity will be null, and, as a consequence, the normal component of acceleration will be continuous.

Now, in the preceding expression that we found for acceleration the only unknown elements are the derivatives of pressure. Therefore, if the normal component of acceleration is continuous (and the givens of the problem are, too) then dp/dn is continuous.

On the other hand, let the function p_1 that verifies equation (**17**) and the boundary conditions that we have always used to determine p be continuous, as well as its derivatives. The difference $p - p_1$ is a function that is harmonic except for its discontinuities, and which is continuous on passing to them, as well as its normal derivative. From a remark that was made in no. **1**, it is therefore harmonic everywhere in the volume that is occupied by the liquid, and since it is, in addition, determined by the givens on the null boundary, one sees that p is identically equal to p_1.

Of course, as we said before, the foregoing supposes that there is no discontinuity in the givens in question, namely, in the components of velocity and their derivatives with respect to x, y, z. The contrary hypothesis will be examined later on (ch. V).

137. – Now suppose that the liquid has a free surface. One knows only dp/dn on it; however, we suppose that one is given the value of p at each point.

This time, we are thus reduced to the mixed problem that was posed in nos. **38-41**(cont.): p is given on the free surface and dp/dn is given on the wall.

It is therefore certain that the solution to this problem is unique; however, it remains for us to show that one actually exists.

The remarks that we just made in the preceding sections continue to apply, moreover.

([41]) In reality, *no* discontinuity may propagate in the case of liquids.

138. – By contrast, the preceding conditions lose their validity if the pressure, when determined in the manner we just described, becomes negative. The corresponding solution becomes inadmissible, moreover.

Since the condition $p > 0$ is destined (no. **127**) to assure equilibrium (and, as a consequence in the case of motion, equilibrium after introducing forces of inertia) under a virtual modification in which cavities are created, such a modification will be produced under the present hypothesis.

We shall not undertake a discussion of the general case for this type of situation. It will be quite difficult if one wants to account for all of the possible circumstances, despite the fact that in all of the practical cases that are presented the solution is simple, in general.

139. (2.) Case of gases. – Now, if we pass on to the case of compressible fluids then relation (**13**) must be solved with respect to *p*. Since the value of density at each point is one of the givens of the problem, as one knows the positions of the diverse molecules at the given instant as a function of their initial positions, one sees that, contrary to what took place in the preceding case, *p is known directly at all of the points.*

Moreover, the equations of motion give us all of the accelerations.

However, at the points that are situated on the bounding surface these accelerations must verify condition (**18**). It is therefore necessary that the value (given at every point) of dp/dn is precisely the entity that verifies that relation, with the components of acceleration being calculated as we just explained.

Now, there is no sort of argument that would make this is true. What is more, when this concordance is presented in the instants that precede the one in question, it might disappear in the instants that follow by a change of the normal acceleration to the wall.

There is thus a contradiction.

140. – This contradiction is also found in the calculation of the higher-order accelerations. It is clear that when we differentiate equations (**1**) with respect to time, as was indicated for the case of liquids, one will know the accelerations in question at every point of the fluid, and there will seem to be, *a priori*, no reason for the surface of acceleration to coincide with that of the wall.

We shall, moreover, return to this again later on when we reason by analogy with the situation in classical mechanics. In the problems where this is posed, at the moment when one may calculate the accelerations as a function of the position and velocity of the system at any instant, it results that the motion of that system is determined entirely by being given its position and velocity at a given instant. Now, we just saw that one may calculate the accelerations of all the molecules as functions of the positions and velocities here, without taking into account the motion of the wall. Therefore, the ultimate motion must be likewise determined independently of the motion of the wall.

More generally, the motion of an arbitrary portion of the fluid will be determined without having to take into account the neighboring regions; this is obviously absurd.

In the following chapters we shall learn how to eliminate this apparent contradiction.

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CHAPTER IV

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RECTILINEAR MOTION OF GASES

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§ 1. – CASE OF CONSTANT PROPAGATION VELOCITY

141. – In order to elucidate the difficulty that defined the subject of the preceding chapter, we shall first consider a particularly simple case for which the equations of the problem may be integrated: It is the case of *rectilinear motion* or *motion by fronts.*

The study of this movement was the subject of the important memoir of Riemann ([42]) to which we alluded in no. **69**. Likewise, Hugoniot ([43]) dedicated two of his memoirs that were published in 1887 to this study, in which a large part of Riemann's results were recovered without his knowledge of that fact.

One supposes that the receptacle in which the gas is constrained has the form of a right cylinder whose lateral surface is fixed, and whose bases are formed by movable pistons. Moreover, we assume that the density is constant in any section parallel to the bases at an arbitrary instant, as well as the velocity, which is parallel to the generatrices. Under these conditions, the state of an arbitrary point of the medium and its velocity depends only on the abscissa of this point, measured parallel to the generatrices. The problem then comes down that of to expressing the present abscissa x as well as the density ρ and pressure p as a function of the original abscissa a and time t. Here, instead of ρ, it will be more convenient for us to introduce the inverse quantity $\frac{\rho_0}{\rho} = \omega$, or *dilatation,* which will have the expression:

$$\omega = \frac{\rho_0}{\rho} = \frac{\delta x}{\delta a}. \tag{1}$$

If the density ρ_0 of the initial state is constant then ω is everywhere inversely proportional to ρ. Unless stated to the contrary, we shall always suppose that the initial state is chosen in such a fashion that this is true.

From the conclusions that we reached in the preceding chapter, the pressure will be a function of ω. If one adopts the Mariott law, i.e., if one supposes that the temperature is constant, then one will have:

([42]) *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungweite* (Mémoires de l'Ac. des Sc. de Göttingue, tome VIII; 1860). The French translation, due to Stouff, occupies pages 177-203 of the edition of the Oeuvres de Riemann that was translated by L. Laugel (Paris, Gauthier-Villars, 1898).

([43]) *Journal de l'Ecole Polytechnique,* tome XXXIII; 1887.

(**2**) $$p = K\rho = \frac{k}{\omega} \qquad (k = K\rho_0).$$

On the contrary, if one chooses the adiabatic law of Poisson, as one has the right to do, then one will write:

(**2′**) $$p = K\rho^m = k\omega^{-\mu} \qquad (k = K\rho_0^m),$$

in which, as we saw (no. **126**), k is a function of the initial abscissa a, but it reduces to a constant at an arbitrary instant if the fluid has been kept at a pressure and temperature that is uniform over all of its mass.

We remark that formula (**2**) may be considered to be a particular case of (**2′**); the former is deduced from the latter by setting $m = 1$.

For the moment, we shall not specify the form of the relation that exists between pressure and density, and we write it here in the general form of (no. **131**), which is:

(**3**) $$p = \varphi(\omega),$$

in which the function $\varphi(\omega)$ may depend on a.

142. – Nevertheless, we must recall that the function φ may be completely arbitrary *a priori*. Indeed, we know (no. **131**) that upon increasing the pressure, the density must increase, and the specific volume must decrease. Therefore, one must have, in any case:

(**4**) $$\frac{d\varphi}{d\omega} < 0.$$

143. – We use the Lagrange variables, which are a and t here. This time, it will be, moreover, useful to employ the notation δ to denote the derivatives that are taken under this hypothesis, if this will not create confusion.

Equations (**1**) of the preceding chapter (no. **124**) reduce to the first of them; in it, the quantity dp/dx must replaced by:

$$\frac{\dfrac{\partial p}{\partial a}}{\dfrac{\partial x}{\partial a}} = \frac{1}{\omega}\frac{\partial p}{\partial a}.$$

The equation then becomes:

(**5**) $$\frac{1}{\rho_0}\frac{\partial p}{\partial a} = X - \frac{\partial^2 x}{\partial t^2}.$$

In the general case, p may be a function of $\omega = \partial x/\partial a$ and a. For example, if the law of constraint is that of Poisson then the quantity k that figures in formula (**2′**) will be a function of a. Equation (**5**) will then be written:

$$\text{(6)} \qquad \frac{1}{\rho_0}\left[\frac{dk}{da}\left(\frac{\partial x}{\partial a}\right)^{-m} - mk\left(\frac{\partial x}{\partial a}\right)^{m-1}\frac{\partial^2 x}{\partial a^2}\right] = X - \frac{\partial^2 x}{\partial t^2}.$$

Nevertheless, we shall principally consider the case for which relation (**3**) is the same for all values of *a*, and for which, at the same time, there are no external forces acting, in such a way that X is null in equation (**5**). If we then set:

$$\text{(7)} \qquad -\frac{1}{\rho_0}\varphi'(\omega) = \psi(\omega)$$

then this equation becomes:

$$\text{(8)} \qquad \frac{\partial^2 x}{\partial t^2} = \psi(\omega)\frac{\partial^2 x}{\partial a^2} = \psi\left(\frac{\partial x}{\partial a}\right)\frac{\partial^2 x}{\partial a^2},$$

a second-order partial differential equation that determines the unknown x as a function of the independent variables a, t.

144. – To begin with, we shall further simplify the question by replacing the function $\psi(\omega)$ (which is positive, from inequality (**4**)) with a constant θ^2. This is what one is led to do when one studies the *small motions* of the fluid. Indeed, one supposes that the distances from the molecules to their initial positions are infinitesimal, and that their velocities $\psi(\omega)$ differ very little from the value $\psi(1)$ that they take in the initial state.

Furthermore, the hypothesis $\psi(\omega) = \theta^2 = \text{const.}$ applies to motions of finite amplitude if one takes the expression:

$$\text{(3}'\text{)} \qquad \psi(\omega) = C - \rho_0\theta^2\omega,$$

for the function φ, with C being a constant.

However, it is clear that such an expression for the pressure will not be admissible, at least for a gas, since it must become negative for a sufficiently large ω.

In theory, a law of this type might be entirely appropriate for a slightly compressible fluid. Indeed, in such a fluid, p varies between very sizable limits for very small variations of ω and is annulled for a certain finite value of this quantity. Of course, in reality, the phenomenon will be limited to a certain positive non-null value of p by the vaporization of the liquid.

145. – By means of the preceding simplification, equation (**8**) may be written:

$$\text{(8}'\text{)} \qquad \frac{\partial^2 x}{\partial t^2} = \theta^2\frac{\partial^2 x}{\partial a^2}.$$

This is the equation for a vibrating string. Its general integral is well known; it may be written:

(**9**) $$x = \tfrac{1}{2}[f_1(a+\theta t) + f_2(a-\theta t)].$$

The whole issue then reduces to that of choosing the functions f_1 and f_2 in such a manner that they satisfy:

1. the initial conditions,
2. the boundary conditions.

Take one of the extremities of the pipe to be the origin of the coordinates, and let l be its length; a will thus vary between 0 and l.

We suppose that the values of x and $\frac{\partial x}{\partial t}$ are given for $t = 0$, namely, x_0 and $\left(\frac{\partial x}{\partial t}\right)_0$; one will then have, for $0 < a < l$:

(**10**) $$\begin{cases} f_1(a) + f_2(a) = 2x_0 \\ \theta[f_1'(a) - f_2'(a)] = 2\left(\dfrac{\partial x}{\partial t}\right)_0. \end{cases}$$

The second of these to equations may be integrated, and gives:

(**11**) $$\theta\,[f_1(a) - f_2(a)] = F(a) - F(0) + \text{constant},$$

in which $F(a)$ is the primitive of $2\left(\frac{\partial x}{\partial t}\right)_0$. As for the constant, one may assume that it is null because the value (**9**) of x does not change if one adds a constant to the function f_1. However, if one subtracts the same constant from the function f_2 then this operation adds an arbitrary constant to the left-hand side of equation (**10**).

From equations (**10**), (**11**), we know the functions f_1 and f_2 for all values of the argument between 0 and l.

146. – We now include the boundary conditions. We assume that the positions of the pistons that close the pipe at its two extremities are known at every instant. We thus have the values of x that correspond to $a = 0$ and $a = l$ for each positive value of t.

It is easy ([44]) to show directly that these conditions succeed in determining the unknown functions. Here, it will be convenient for us to employ a geometric representation.

We consider a, t, and x to be coordinates in space, in which the t-plane is taken to be the horizontal plane of projection. The desired solution will then be represented by a portion of that surface. Since a is, by definition, between 0 and l, and t is positive, this portion of the surface will necessarily be bounded by three planes: one of them, T, is the x-plane, the second one, O, is the tx-plane, and the third one, L, is the $a = l$ plane.

([44]) For example, see JORDAN, *Cours d'Analyse*, tome III.

In figure 10, we use the auxiliary planes T, O, L as the plane of projection, in addition to the horizontal plane.

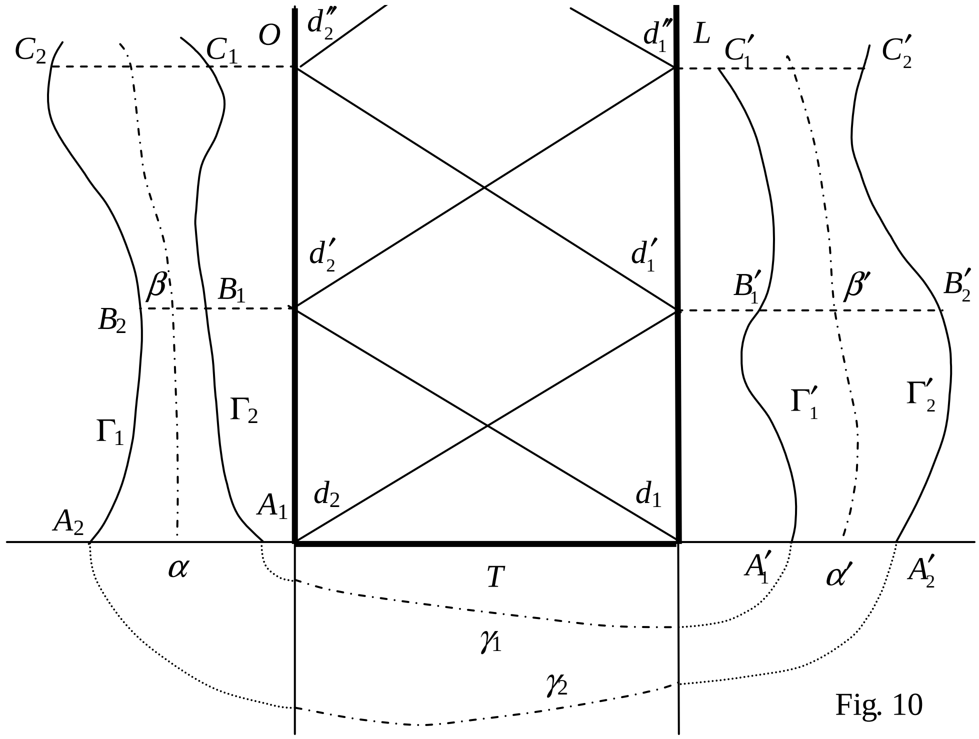

Fig. 10

One may set:

$$x = \frac{x_1 + x_2}{2},$$

in which x_1 represents the function $f_1(a + \theta t)$ and x_2 represents the function $f_2(a - \theta t)$.

The equations:

$$x_1 = f_1(a + \theta t),$$
$$x_2 = f_2(a - \theta t),$$

represent two cylinders K_1 and K_2, one of which will have its generators parallel to the line d_1 and represented by the equations:

(12) $$x = 0, \qquad a + \theta t = l,$$

and the other will have its generators parallel to the line d_2 and represented by the equations:

(12′) $$x = 0, \qquad a - \theta t = 0;$$

the desired value of x will be the mean of the ordinates of these two cylinders.

From equations (**10**) and (**11**) we know the curves γ_1, γ_2 (*fig.* 10), along which our two cylinders cut the plane T.

We therefore know a portion of the cylinder K_1 that is projected horizontally along the triangle formed by the *a*-axis, the *t*-axis, and the line d_1. Similarly, we know a portion of the cylinder K_2 that is projected along the triangle bounded by the *t*-axis, the trace of L, and the line d_2. If we call the sections of our cylinder by the *O*-plane, Γ_1 and Γ_2, and the sections of our cylinder by the *L*-plane, Γ_1' and Γ_2', then we immediately will have a portion A_1B_1 of the curve Γ_1 and a portion $A_2'B_2'$ of the curve Γ_2'.

147. – Having said this, the effect of the boundary conditions is to make known the sections $\alpha\beta, \alpha'\beta'$ (*fig.* 10) of the desired surface by the planes *O* and *L*. Once we have obtained the arc A_1B_1 we may deduce the corresponding arc A_2B_2, of Γ_2 by the same construction that gives us the arc $A_1'B_1'$ of Γ_1' that corresponds to $A_2'B_2'$.

We thus have two new nappes for our cylinders. From the first one, we now know everything that is projected between the traces of the planes *O*, *T*, *L*, and the line d_1', which is represented by the equation $a + \theta t = 2l$; from the second one, we know the part that is projected between the same traces and the line d_2', which has the equation $\theta t - a = l$.

The two new nappes determine a new arc B_1C_1 of Γ_1, and a new arc $B_2'C_2'$ of Γ_2', from which one deduces an arc $B_1'C_1'$ of Γ_1' and an arc $B_2'C_2'$, etc., by means of the curves $\alpha\beta, \alpha'\beta'$.

The solution of the problem therefore presents no difficulty.

148. – It remains for us to demand to know what form the contradiction that we encountered in the preceding chapter will take. This contradiction is apparent in equation (**8′**). Indeed, at the initial instant the quantity $\frac{\partial^2 x}{\partial t^2}$ depends only on the motion of the wall. Now, it must be equal to the quantity $\theta^2 \frac{\partial^2 x}{\partial a^2}$, which depends only on the initial state, and these two givens are independent of each other since the fluid molecules are only required to be in contact with the wall at the instant $t = 0$ as far as their initial *positions* are concerned.

Indeed, we have seen that when we know only the initial state, an abstraction made from the motion of the piston lets us know a part of the cylinder K_1, and, as a consequence, the value of x_1 for all values of *a* and *t* (positive) that are sufficiently close to 0. On the contrary, the cylinder K_2 is not completely known in a neighborhood of the ordinate $a = 0$, $t = 0$ under these conditions, since the known portion is bounded by the generator projected along d_2. The curvature of that known portion of the cylinder K_2 obviously depends upon the value of $\frac{\partial^2 x}{\partial a^2}$.

The construction of the nappe along the cylinder K_2 itself depends on the arc A_2B_2, and, as a consequence, on the motion of the piston; the curvature of this nappe thus depends on the initial acceleration of this piston.

From the foregoing, it is obvious and, moreover, quite easy to confirm directly, that the condition for the curvature of the two neighboring regions on the cylinder K_2 to be the same is precisely the equality of the two quantities $\frac{\partial^2 x}{\partial t^2}$, and $\theta^2 \frac{\partial^2 x}{\partial a^2}$ when this is of issue.

149. – When this equality does not hold, since the curvature of the cylinder K_2 is discontinuous all along the projection of the generator on d_2 the same is true for the derivatives $\frac{\partial^2 x}{\partial a^2}, \frac{\partial^2 x}{\partial a \partial t}, \frac{\partial^2 x}{\partial t^2}$.

As a consequence, if we consider a positive, but small, value of t, which amounts to cutting the figure by a plane that is parallel to the ax-plane, then we see that the derivative $\frac{\partial^2 x}{\partial a^2}$ of the dilatation, which is continuous in general (at least for the points in a neighborhood of the extremity 0 of the cylinder), will exhibit a discontinuity at the point whose projection is on the line d_2. As t increases, the particles between which this discontinuity is produced move away from the extremity.

Similarly, if we consider a well-defined molecule near the extremity, which amounts to cutting the figure by a plane perpendicular to the a-axis, then we confirm that the acceleration of this molecule exhibits a discontinuity at an instant that is very close to the initial instant if the molecule in question is very close to the extremity, and the value t that corresponds to the discontinuity increases as one considers more remote points of the piston.

In a word, we recognize a *second-order wave*, such as the ones that we studied in chapter II. This wave propagates in the positive sense with a velocity (referred to the initial state that we chose) that is nothing but θ, since the equation of the line d_2 is $a = \theta t$.

Thanks to the presence of this discontinuity, the contradiction discussed in the preceding chapter disappears. When $\theta t - a$ is very small and negative, the two quantities $\frac{\partial^2 x}{\partial t^2}$ and $\theta^2 \frac{\partial^2 x}{\partial a^2}$ have the same value, which is deduced from the initial state at the extremity $a = 0$. If $\theta t - a$ is very small and positive then they have the same value again, which is the initial acceleration of the piston.

150. – If an analogous phenomenon takes place at the opposite extremity of the cylinder then it will give rise to a discontinuity that obviously affects the cylinder K_1, and no longer the cylinder K_2. It is produced at all of the points of the projection of the generator on d_1. It therefore propagates with a velocity of θ again, but in the negative sense this time.

151. – In particular, this will be true (except in the exceptional cases) when the wave begins at the extremity $a = 0$ at the initial instant with a propagation that is represented by the line d_2 and reaches the extremity $a = l$. Indeed, since the cylinder K_2 has its curvature discontinuous along the generator that corresponds to this wave, the same will be true for the curve Γ_2'. If the curvature of the line $\alpha'\beta'\gamma'$, … does not present a variation of appropriate magnitude before this point then the line Γ_1 and, as a consequence, the cylinder K_1, will have discontinuous curves.

In a word, the original wave that begins at the extremity $a = 0$ and propagates with the velocity θ is *reflected* by the piston $a = l$; i.e., when the wave encounters the piston, it generates an analogous wave that propagates with the velocity $-\theta$.

152. – If the two quantities $\frac{\partial^2 x}{\partial t^2}$ and $\theta^2 \frac{\partial^2 x}{\partial a^2}$ are equal to each other for the extremity 0 at the origin of time then the curvature of the cylinder K_2 will be continuous. However, the singularity that we studied in the preceding chapter is produced for the third-order derivatives of x. Equation (**8′**) gives, effectively:

$$\theta^2 \frac{\partial^3 x}{\partial a^2 \partial t} = \frac{\partial^3 x}{\partial t^3}, \tag{13}$$

an equality whose left-hand side is provided for us by the initial state of the gas, and whose right-hand side is provided by the motion of the piston. When this equality is true precisely it will produce a third-order discontinuity that will successively affect the different points of figure 10 that are projected on d_2, and, as a consequence, also propagate with a velocity of θ.

As before, such third-order discontinuities may be of two types that propagate with the same velocity θ but in different senses; one begins at the extremity $a = 0$ of the pipe and the other at the extremity $a = l$.

Meanwhile, if equation (**13**) is verified then it may, in turn, give rise to a fourth-order discontinuity, and so on.

153. – Similarly, one clearly sees here how a discontinuity of infinite order may be produced. This is what happens when the first nappe of the cylinder K_2 (which is furnished by the initial state) is analytic, so the second nappe is precisely the analytic continuation of the first, but with a contact of order infinity.

154. – Finally, one may take the viewpoint that was adopted in no. **140** upon successively considering two motions M_1 and M_2, that coincide at the initial instant, but for which the motions of the piston $a = 0$ will be different after that instant. The cylinder K_2 will then be modified by means of the projection of the generator along d_2, a line that defines the law of propagation for this modification.

Similarly, when the motion of the piston $a = l$ is unaltered the cylinder K_2 will change (as a result of the change of the curve Γ_2') after the moment when this propagation reaches the extremity of the tube; once again, there will be *reflection.*

155. – We return to the case of a discontinuity that is created at the initial instant and at the extremity $a = 0$.

At a point that is projected on d_2, but in a region where the curvature of the cylinder K_1 is continuous along with its derivatives, there will be *compatibility:* The discontinuity will remain unique not only at the instant that corresponds to this point, but at the preceding and following instants.

The same will be true for a point that is projected on d_1 only if – at the very least – a discontinuity is produced for $a = l$ at the time origin.

On the contrary, consider the point of intersection of the lines d_1, d_2. This point corresponds to a value t_1 of t for which there exists a unique discontinuity. Only this discontinuity affects the two cylinders K_1 and K_2 at the same time. *There is no point of compatibility:* Any value of t that is different from t_1 will correspond to a perpendicular to the axis of t that cuts d_1 and d_2 at two distinct points. Here one sees quite well that a discontinuity without compatibility is nothing but the superposition of two discontinuities that intersect at an isolated instant, conforming to the general considerations of no. **105**.

In the present problem, the compatibility condition presents itself in a very simple particular form: It is obvious that only one of the functions f_1 and f_2 has discontinuous derivatives. For example, if θ is positive then we must have that the second derivative of the function f_1 does not experience any variation. Now, one has:

$$\left[\frac{\partial^2 x}{\partial a^2}\right] = [f_1''] + [f_2''] \tag{14}$$

$$\left[\frac{\partial^2 x}{\partial a \partial t}\right] = \theta\{[f_1''] - [f_2'']\}. \tag{14′}$$

Upon eliminating $[f_2'']$, which is different from zero, in general, we obtain:

$$2[f_1''] = 0 = \left[\frac{\partial^2 x}{\partial a^2}\right] + \frac{1}{\theta}\left[\frac{\partial^2 x}{\partial a \partial t}\right]. \tag{15}$$

One will obtain a third condition that is analogous to (**14**), (**14′**) upon envisioning the acceleration. However, in the present problem, this is not a distinct quantity; it must be considered as determined by equation (**8**) with the aid of the other second-order derivatives. The condition that one obtains by introducing them is obviously none other than (**14**).

Similarly, if the discontinuity is of arbitrary order n then one will have to consider only the derivatives of index *zero* or *one* since all of the others are calculated as a function of the first by means of the partial differential equation. The corresponding compatibility condition will thus be:

$$\left[\frac{\partial^n x}{\partial a^n}\right] + \frac{1}{\theta}\left[\frac{\partial^n x}{\partial a^{n-1} \partial t}\right] = 0. \tag{16}$$

156. – However, one does not have just one condition of this type that corresponds to the same order of discontinuity. No matter what this order n_0 is, one must also write all of the conditions (**16**), which are infinite in number and correspond to different values of n that are greater than n_0, and which will be the *higher-order compatibility conditions,* whose kinematical part we obtained in nos. **119-123**.

For example, for a second-order discontinuity that propagates in the positive sense, one must have not only conditions (**15**), but also conditions that express that the function f_1 has continuous third, fourth, etc., derivatives.

If these conditions are not verified at an arbitrary instant then the second-order discontinuity that progresses in the positive sense will double into a discontinuity of third, fourth, etc., order, which will separate from the first at instants that are near t_1 and propagate in the negative sense.

As one sees, these compatibility conditions of different orders are independent from each other here.

157. – The consideration of the two functions f_1 and f_2 will permit us express the compatibility in the present problem, and similarly for a discontinuity of infinite order (no. **76**).

Indeed, suppose that x and $\partial x/\partial t$ are analytic functions of a for $a < a_1$ at the initial instant in a part of the tube, and that for $a > a_1$ these same functions cease to be the analytical continuations of the first ones, and meanwhile for $a = a_1$ their derivatives of all orders with respect to a are continuous. The condition for there to be compatibility with propagation in the positive sense in this discontinuity of infinite order is that the function f_1 be analytic and regular. Now, this function may be calculated with the aid of the preceding givens by the intermediary of equations (**10**) and (**11**).

If, at the same time as the motion M_1 one considers another one M_2 that coincides with M_1 at the initial instant in a region R of the tube and is distinct from it in another region R' that is contiguous with the first, then the two motions M_1 and M_2 will be identical at an arbitrary instant t in a certain region R_t and distinct in another region R'_t. The point of separation of these two regions generally moves towards the region R with velocity θ as they displace. One may further say that there is compatibility if, on the contrary, the displacement of this point is towards the region R'. In order for this to be the case, one of the functions f_1 and f_2, when calculated in the way we just described, must be the same for M_1 and M_2.

158. – The study of the propagation of discontinuities such as we just encountered agrees with the theory of characteristics of second-order partial differential equations, so we shall summarize its principles and refer to the well-known treatises of Darboux and Goursat ([45]) for the details. Moreover, we shall recover this theory in a more general form in the following chapters (chap. VII).

Consider the *Monge-Ampére* equation:

([45]) Darboux, *Leçons sur la théorie des surfaces,* tome III, pp. 263 et seq. – Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre,* tome I.

(**17**) $$A(rt - s^2) + Br + 2Cs + B't + D = 0,$$

in which r, s, and t denote the second-order partial derivatives of an unknown function z of x and y. Meanwhile, A, B, B', C, D are given functions of x, y, z, as well as the first-order partial derivatives p and q. If x, y, z are considered to be the coordinates of a point of space then any function z of x and y that satisfies this equation will represent an integral surface.

Such a surface is, in general, determined by the *Cauchy conditions*, which consist of giving the values of z and its first derivatives p, q at all of the points of a curve γ in the xy-plane. This obviously must be such that the relation:

(**18**) $$dz = pdx + qdy,$$

is verified for a displacement that is effected along γ.

Geometrically, this amounts to being given a skew curve Γ (which projects on γ) that passes through the desired surface as well the tangent plane to that surface at each point of the curve.

In order to solve the *Cauchy problem* – i.e., to determine the solution from these givens – one first seeks the values of r, s, and t at each point of γ; these quantities obviously must satisfy the conditions:

(**19**) $$\begin{cases} dp = rdx + sdy, \\ dq = sdx + tdy, \end{cases}$$

(the differentials always correspond to a displacement performed along γ), and equation (**17**), for that matter. The latter is of second degree, at least when $A \neq 0$. Meanwhile, if one obtains the values of two of the quantities r, s, and t as functions of the third one from equations (**19**) then the left-hand side of (**17**) will be of first degree with respect to them.

(If one considers r, s, t, instead of x, y, z, to be the Cartesian coordinates then this amounts to saying that the line that is represented by equations (**19**) is parallel to a generatrix of the asymptotic cone of the quadric that corresponds to (**17**)).

Upon taking s to be the unknown, one finds:

(**20**) $$\begin{cases} s[A(dpdx + dqdy) + Bdy^2 - 2Cdxdy + B'dx^2] \\ \qquad = Adpdq + Bdpdy + B'dqdx + Ddxdy. \end{cases}$$

In the equations that we must study the coefficient A is, moreover, null, and the linear character of equations (**17**), (**19**) appears at first glance.

A completely similar calculation gives us the third-order derivatives $\frac{\partial^3 z}{\partial x^3}$, $\frac{\partial^3 z}{\partial x^2 \partial y}$, $\frac{\partial^3 z}{\partial x \partial y^2}$, and, in general, the derivatives of all orders of the desired function at each point of Γ.

Hence, if one knows that this function is holomorphic then it is perfectly determined, since one has all of the coefficients in its development.

Conversely, when the givens are analytic and regular then one proves, with the aid of the theorem of Kowalewsky ([46]), that the solution thus determined exists; it results from the preceding that this solution is unique.

159. – However, things are different if solving the first-degree equation (**20**) is impossible or indeterminate, which follows from:

$$(\mathbf{21}) \qquad A(dp\,dx + dq\,dy) + B dy^2 - 2C dx\,dy + B'\,dx^2 = 0.$$

If this condition is verified then the problem that consists of looking for r, s, t is, in general, impossible. Putting aside this hypothesis, to which we shall return later on, we assume that equations (**17**) and (**19**) are compatible. The condition for this to be true is that one have the following:

$$(\mathbf{22}) \qquad A dp\,dq + B dp\,dy + B'\,dq\,dx + D dx\,dy = 0,$$

in addition to equation (**21**).

If A is null then equation (**21**) reduces to:

$$(\mathbf{21'}) \qquad B dy^2 - 2C dx\,dy + B'\,dx^2 = 0.$$

For each system of values of x, y, z, p, q this defines two values λ_1 and λ_2 of the angular coefficient $\lambda = dy/dx$ for the tangent to γ.

If one has, for example, $dy/dx = \lambda_1$ then relation (**22**) becomes:

$$(\mathbf{22'}) \qquad \lambda_1(B dp + D dx) + B'\,dq = 0.$$

When conditions (**21**) and (**22**) are verified, our equations no longer determine $r, s,$ and t, and it seems that one of these three quantities must be chosen in a completely arbitrary fashion at each point of γ.

However, this is not exactly what happens; indeed, if one considers the following derivatives $\frac{\partial^3 z}{\partial x^3}$, $\frac{\partial^3 z}{\partial x^2 \partial y}$, $\frac{\partial^3 z}{\partial x \partial y^2}$ then one sees that the equations that they must imply likewise have the left-hand side of (**21**) for their determinant (which is obvious for $A = 0$, since the coefficients of these equations are then the same as those of equations (**17**), (**19**)). There will therefore be a condition of possibility, which consists of a first-order linear differential equation that $r, s,$ and t must satisfy. The choice of the latter therefore involves only one arbitrary *constant.* As for the third-order derivatives, if they verify the differential equation that we just described then *they must be, in turn, indeterminate,* or

([46]) Compare ch. VII, no. 281.

rather, as the consideration of the fourth derivatives shows, they must depend on a new arbitrary constant.

Therefore, each order of derivative introduces a new constant.

There is good reason to believe that the problem thus posed admits an infinitude of solutions this time; one can prove ([47]) that this is indeed the case.

160. – Now suppose that one starts with a given integral surface Σ. On this surface, the differential equation (**21**) will define two families (a curve of each family passes through an arbitrary point of the surface), and on each of them one will have condition (**22**) moreover, since the contrary would be in contradiction with the existence of the surface Σ itself.

The curves thus defined are called the *characteristics* that are situated on the surface.

161. – We now demand that there may exist another integral surface that is tangent to the first one all along a curve Γ. From the foregoing, *the necessary and sufficient condition for this to be the case is that* Γ *be a characteristic,* at least if we suppose that the two surfaces are analytic.

Our reasoning does not rigorously exclude the existence of two (non-analytic) integral surfaces that have a contact of infinite order ([48]) along a curve Γ that is or is not characteristic.

162. – From the fundamental property of characteristics, it obviously results that these curves will be preserved under any change of variables.

More generally, *the characteristics are preserved under any contact transformation* ([49]).

For example, consider the Legendre transformation that makes the variables $x, y, z,$ and the partial derivatives p, q correspond to analogous quantities x_1, y_1, z_1, p_1, q_1 that are defined by the formulas:

$$x_1 = p, \quad y_1 = q, \quad z_1 = px + qy - z,$$
$$p_1 = x, \quad q_1 = y.$$

Under this transformation, the new values of the second derivatives are:

$$r_1 = \frac{t}{rt - s^2}, \quad s_1 = -\frac{s}{rt - s^2}, \qquad t_1 = \frac{r}{rt - s^2}.$$

([47]) See below, ch. VII, no. 319.

([48]) See note I at the end of this work.

([49]) For the definition and the fundamental properties of contact transformations, see Goursat, *Leçons sur l'intégration aux dérivées partielles du premier ordre*, ch. XI, Paris, Hermann.

When this transformation is applied to equation (**17**) we obtain an analogous equation in which A, B, B', C, D are changed into D, B', B, $-C$, A.

Furthermore, two arbitrary tangent surfaces are changed into two tangent surfaces. Therefore, the characteristics of the new equation correspond to those of the original one.

What we just said about the Legendre transformation may also be repeated for any contact transformation; they change an arbitrary equation of the form (**17**) into an equation of the same form and the characteristics into characteristics.

163. – However, one must recall that an integral of one of the equations does not always produce a corresponding integral of the other, properly speaking, since the transformation under consideration may make a curve correspond to a surface (or even to a unique point). This is why the Legendre transformation (which, as one knows, is equivalent to a transformation by reciprocal poles) changes any developable surface into a curve. Lie ([50]) has indicated a general definition of the integral of an equation such as (**17**), from which such a curve may be considered as an integral of this equation, the same as a surface. Without going into the considerations of that definition, one may say that a curve is a *degenerate integral* of equation (**17**) if the developable surface into which it is changed by the Legendre transformation is an integral of the transformed equation.

164. – Finally, if A is null then the coefficients B, B', C are functions of just x, y, and equation $(\mathbf{21'})$ may be considered to be an ordinary differential equation in these two quantities. If the discriminant $BB' - C^2$ is different from 0 then this equation will have two series of distinct integral curves that we may denote by X = const., Y = const. Upon taking X and Y for new independent variables one makes r and t disappear from the equation.

165. – The application of these results to the theory of the propagation of motion is immediate.

Indeed, if two motions of our fluid mass are second-order discontinuous then according to the geometric representation that has served us up till now there are two corresponding mutually tangent surfaces all along a line, since the first derivatives do not change value at any point of discontinuity. From the preceding results, such a line is necessarily a characteristic.

If the discontinuity is of higher than second order then this conclusion will not be modified. Indeed, we have seen that if the first and second derivatives of the desired integral are given along the curve Γ then the third derivatives will have perfectly determined values (in such a way that there may not be a third-order discontinuity produced) if the curve Γ is not a characteristic, whereas, in the contrary case they may not change.

([50]) See Goursat, *Leçons sur l'intégration aux dérivées partielles du premier ordre*, tome I, pages 49-51.

Therefore, the compatible motions will obviously be the ones whose representative surfaces agree along a line Γ.

The coefficients for equation $(\mathbf{8'})$ are 1, 0, and θ^2, respectively. The characteristic will therefore correspond to $\frac{da}{dt} = \pm\theta$. Since the quantity $\pm\theta$ obviously represents the velocity of propagation, the two families of characteristics correspond to the two senses in which this propagation may be effected.

§ 2. – GENERAL CASE

166. – We now study the propagation of these discontinuities in terms of the equation of motion that we originally obtained, and not equation $(\mathbf{8'})$, which we have substituted arbitrarily.

This study presents no difficulty, moreover, when one either employs the considerations that were developed in chapter II or when one makes direct recourse to the theory of characteristics.

To that effect, we begin with the equation of motion in its most general form (**5**), as we obtained it in no. **143**, namely:

$$(\mathbf{5}) \qquad \frac{1}{\rho_0}\frac{\partial p}{\partial a} = \frac{1}{\rho_0}\left(\frac{\partial \varphi}{\partial \omega}\frac{\partial^2 x}{\partial a^2} + \frac{\partial \varphi}{\partial a}\right) = X - \frac{\partial^2 x}{\partial t^2} \qquad (p = \varphi(\omega, a))$$

Suppose that the motion presents a second-order discontinuity for a definite value of a and a definite instant t. Furthermore, suppose that there is compatibility, and let θ be the velocity of propagation. If we let the indices 1 and 2 denote things that refer to the two regions that are separated by discontinuity, respectively, then one must have:

$$(\mathbf{23}) \qquad \begin{cases} \left(\dfrac{\partial^2 x}{\partial a \partial t}\right)_2 - \left(\dfrac{\partial^2 x}{\partial a \partial t}\right)_1 = -\theta\left[\left(\dfrac{\partial^2 x}{\partial a^2}\right)_2 - \left(\dfrac{\partial^2 x}{\partial a^2}\right)_1\right] \\ \left(\dfrac{\partial^2 x}{\partial t^2}\right)_2 - \left(\dfrac{\partial^2 x}{\partial t^2}\right)_1 = \theta^2\left[\left(\dfrac{\partial^2 x}{\partial a^2}\right)_2 - \left(\dfrac{\partial^2 x}{\partial a^2}\right)_1\right]. \end{cases}$$

However, the quantities $\left(\frac{\partial^2 x}{\partial a^2}\right)_1$, $\left(\frac{\partial^2 x}{\partial t^2}\right)_1$; $\left(\frac{\partial^2 x}{\partial a^2}\right)_2$, $\left(\frac{\partial^2 x}{\partial t^2}\right)_2$ must separately satisfy equation (**5**), in which the first derivatives have the same values in either case. Upon subtracting the relations so obtained side-by-side, we get:

$$\left[\frac{\partial^2 x}{\partial t^2}\right] = -\frac{1}{\rho_0}\frac{\partial \varphi}{\partial \omega}\left[\frac{\partial^2 x}{\partial a^2}\right],$$

or:

(**24**) $$\theta^2 = -\frac{1}{\rho_0}\frac{\partial \varphi}{\partial \omega},$$

or

(**24′**) $$\theta^2 = \psi(\omega),$$

in which $\psi(\omega)$ is the quantity that is defined by formula (**7**).

We thus obtain the value of the velocity of propagation θ. The theory of characteristics will lead us to the same result, with θ being nothing but the angular coefficient da/dt of the tangent to the characteristic, which is furnished by equation (**21′**) .

The quantity θ is the *velocity of sound* in the gas for the pressure and temperature considered. Indeed, it is the velocity with which an arbitrary motion (such as a sound vibration) is propagated under these conditions.

167. – In addition, relations (**23**) tell us about compatibility. If one considers the motion of a fluid as being determined by the positions and velocities of the molecules at the instant t then $\frac{\partial^2 x}{\partial a^2}$ will be an unknown that one must obtain from equation (**5**).

On the contrary, $\frac{\partial^2 x}{\partial t^2}$ and $\frac{\partial^2 x}{\partial a \partial t}$ are the givens of the question, which are discontinuous for the value of a considered. Between these givens, one must have the first relation of (**23**) for compatibility, in which θ denotes the square root of expression (**24**), with a sign that depends on the sense of propagation ([51]).

In the contrary case, the discontinuity divides into two ([52]), such that one of them propagates with the velocity $+\sqrt{\frac{1}{\rho_0}\frac{\partial \varphi}{\partial \omega}}$, and the other with the velocity $-\sqrt{\frac{1}{\rho_0}\frac{\partial \varphi}{\partial \omega}}$.

If the discontinuity is of order higher than the second then we know from the theory of characteristics that the velocity of propagation does not change in value. In order to obtain the same result by starting from kinematical considerations it obviously will suffice to remark that the p^{th} derived variations (where p is the order of the discontinuity) will then form a geometric progression of ratio θ and, on the other hand, that one must substitute these variations in any one of the equations that are obtained by differentiating (**5**) p – 2 times.

Likewise, it is clear that the expression for the velocity of propagation does not change if, among the accelerating forces, one appears that is a function of velocity, or if, in general, the force X depends arbitrarily on not only x, a, and t, but on the first derivatives of x with respect to a and t.

([51]) Compare ch. V, no. **243**.

([52]) We shall return to this doubling in the general case of space in ch. V, **251**.

168. – From the preceding, it is impossible to treat the dynamics of gases without taking into account the discontinuities that propagate in them. Their absence assumes an exceptional agreement between the initial givens and the motion of the wall.

This circumstance does not present itself in the dynamics of liquids, in which one always studies only motions that are continuous, or, at least, ones with stationary discontinuities ([53]).

This constitutes a special difficulty in the study of gases. Indeed, one sees that before setting out to define the equations of motion, it is necessary to determine the domain in which these equations are valid. This domain is bounded by waves whose propagation must be studied.

Furthermore, there are very few general results on the motions of gases. Almost all of them relate to the rectilinear motion that was treated by Riemann and Hugoniot in the cited memoirs, with which we shall now occupy ourselves.

169. – We confine ourselves to the case in which the gas is elementary (or, at least, may have been at an arbitrary instant), with a uniform pressure and temperature, in such a way that the equation of motion reduces to:

$$\frac{\partial^2 x}{\partial t^2} = \psi(\omega)\frac{\partial^2 x}{\partial a^2}, \tag{8}$$

in which, as we have seen, $\psi(\omega)$ represents the function $-1/\rho_0/\varphi'(\omega)$, and ω is the partial derivative $\partial x/\partial a$. As for the other partial derivative $\partial x/\partial t$, it is nothing but the velocity u.

The velocity of sound θ is equal to $\pm\sqrt{\psi(\omega)}$, and this is what equation (**21′**) expresses. On the other hand, write equation (**22′**): it gives us (here, p is replaced by u, and q by ω):

$$du = \theta\, d\omega = \pm\sqrt{\psi(\omega)}\, d\omega.$$

This equation is integrable. Upon setting:

$$\sqrt{\psi(\omega)} = \chi'(\omega), \tag{25}$$

in which one intends that the radical on the left-hand side should be given the + sign, one gets:

$$u \pm \xi(\omega) = \text{constant}. \tag{26}$$

170. – Therefore, each family of characteristics admits an integrable combination. This remark led Riemann to take the quantities:

([53]) See ch. V, nos. **244-246**.

$$\text{(27)} \qquad \begin{cases} u + \chi(\omega) = \xi, \\ u - \chi(\omega) = \eta, \end{cases}$$

as independent variables, which gives:

$$\text{(28)} \qquad u = \frac{\xi + \eta}{2},$$

$$\text{(28}'\text{)} \qquad \chi(\omega) = \frac{\xi - \eta}{2}.$$

In order to perform this change of variables we begin by performing a Legendre transformation, i.e., by taking u and ω to be the independent variables, and the combination:

$$\text{(30)} \qquad z = \omega a + ut - x,$$

to be the unknown function, in which the derivatives with respect to u and ω are nothing but t and a. The new values of the second derivatives are calculated by means of the formulas of no. **162**, and equation (**8**) becomes:

$$\text{(31)} \qquad \frac{\partial^2 z}{\partial \omega^2} = \psi(\omega) \frac{\partial^2 z}{\partial u^2}.$$

Now it is easy to pass to the variables ξ, η. We get:

$$4 \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{\chi''(\omega)}{\chi'^2(\omega)} \left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \right),$$

or, after deriving ω from equation (**28′**):

$$\text{(32)} \qquad \frac{\partial^2 z}{\partial \xi \partial \eta} - f(\xi - \eta) \left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \right) = 0,$$

in which f is a function that is defined by the relation:

$$\text{(33)} \qquad f[2\chi(\omega)] = \frac{1}{4} \frac{\chi''(\omega)}{\chi'^2(\omega)}.$$

The equation is thus referred to its characteristics; it has the Laplace form:

$$\text{(34)} \qquad \frac{\partial^2 z}{\partial \xi \partial \eta} + a \frac{\partial z}{\partial \xi} + b \frac{\partial z}{\partial \eta} + cz = 0.$$

171. – It is precisely this example that led Riemann ([54]) to imagine his method of integration, which was, as we know, extended to the general equation (**34**) by Darboux.

We recall that this method is, in many ways, analogous to the one that we discussed in chapter I. It rests on an identity that is completely similar to that of Green, namely:

$$\iint [\varsigma \mathcal{F}(z) - z\mathcal{G}(\varsigma)]\, d\xi\, d\eta = \int M\, d\eta - N\, d\xi, \tag{35}$$

in which:

z and ζ denote two arbitrary regular functions,

$\mathcal{F}(z)$ is the left-hand side of equation (**34**), and

$\mathcal{G}(z)$ is the left-hand side of equation:

$$\mathcal{G}(\zeta) = \frac{\partial^2 \varsigma}{\partial\xi\partial\eta} - a\frac{\partial\varsigma}{\partial\xi} - b\frac{\partial\varsigma}{\partial\eta} + \left(c - \frac{\partial a}{\partial\xi} - \frac{\partial b}{\partial\xi}\right)\varsigma = 0, \tag{36}$$

which is called the *adjoint* of the one that was proposed.

M and N are the expressions:

$$\begin{cases} M = az\varsigma + \frac{1}{2}\left(\varsigma\dfrac{\partial z}{\partial\eta} - z\dfrac{\partial\eta}{\partial\eta}\right), \\ N = bz\varsigma + \frac{1}{2}\left(\varsigma\dfrac{\partial z}{\partial\xi} - z\dfrac{\partial\varsigma}{\partial\xi}\right); \end{cases} \tag{37}$$

the double integral is taken over an arbitrary area in the plane of ξ, η, and the simple integral in the right-hand side is taken over the boundary contour of that area.

Once this identity has been posed, the solution of the Cauchy problem for equation (**34**), in which the curve γ is an arbitrary arc in the $\xi\eta$-plane, subject to the condition that it be cut by an arbitrary parallel to the ξ-axis at only one point, also be cut by an arbitrary parallel to the η-axis at only one point – in other words, the calculation of an integral z of equation (**34**) that is given by its values and first derivatives at each point of γ, at an arbitrary point $A(\xi', \eta')$, (*fig.* 11) – comes down to that of constructing a function $g(\xi, \eta; \xi', \eta')$ that one may regard as corresponding to the Green function. This function g is defined by the three-fold condition:

([54]) The unknown that was considered by Riemann was not z, but a quantity w that he defined to be an integral of an exact total differential by the formula:

$$dw = -\frac{z}{2}(d\xi + d\eta) + \omega\chi'(\omega)\left(\frac{\partial z}{\partial\xi}d\xi - \frac{\partial z}{\partial\eta}d\eta\right),$$

which is equivalent to formula (**3**) (sec. II) of Riemann's memoir (page 183 of the French translation); the relations that allow us to pass from Riemann's notion to our own are:

$$\sqrt{\varphi'(\rho)} = \omega\chi'(\omega), \quad f(\rho) = -\chi(\omega), \quad r = \frac{\eta}{2}, \quad s = -\frac{\xi}{2}.$$

1. It must satisfy the equation $\mathcal{G} = 0$, when one considers it as a function of ξ, η (ξ' and η' are constant).

2. It must reduce to $\exp \int_{\xi'}^{\xi} b(\xi, \eta)\, d\xi$ for $\eta = \eta'$, and to $\exp \int_{\eta'}^{\eta} a(\xi, \eta)\, d\eta$ for $\xi = \xi'$.

This presents a reciprocity property that is analogous to that of the Green function, and is proved in a completely similar manner ([55]); the expression $g(\xi, \eta; \xi', \eta')$ does not change when one permutes the points ξ, η, ξ', η' along with the differential polynomials $\mathcal{G}$, $\mathcal{F}$ (a fact that is immediately obvious for condition 2, but not condition 1).

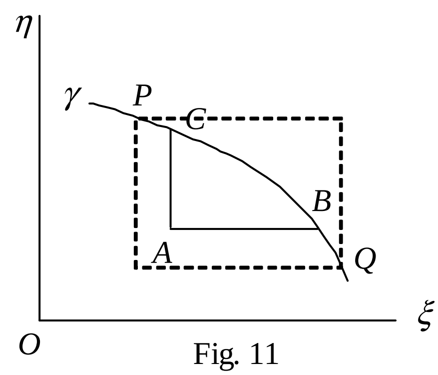

Fig. 11

Once the function $\mathcal{G}$ is constructed, one substitutes it for ζ in identity (**35**), by means of which the left-hand side will disappear, in which z is the desired unknown function. As for the area integration, one bounds it, on the one hand, by the curve g, and the other, by the parallels AB, AC (*fig.* 11), which are parallel to the ξ and η axes, respectively, at the point A.

By virtue of the properties that we assumed for the function g the integral $\int M\,d\eta - N\,d\xi$ reduces to $\frac{1}{2}[(zg)_B - z_A]$ on AB and $\frac{1}{2}[(zg)_C - z_A]$ on AC, and one has:

(**38**) $$z_A = \tfrac{1}{2}[(zg)_B + (zg)_A] - \int_B^C M\,d\eta - N\,d\xi\,,$$

a formula that resolves the question, since everything in the right-hand side is expressible directly with the aid of the givens.

Conversely, *if the curve γ satisfies the geometric condition that was indicated above,* and, of course, if the given values of z and its derivatives on γ satisfy the relation:

(**39**) $$dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta\,,$$

then the preceding formula defines a function that satisfies the conditions of the problem precisely.

If, instead of integrating $\mathcal{F} = 0$, one must integrate the equation:

$$\mathcal{F} = f,$$

([55]) DARBOUX, *Loc. cit.,* no. **359**.

in which f is a given function of ξ, η, then if we use the same method again formula (**38**) is completed simply by taking the double integral $\iint fgd\xi d\eta$ over our curvilinear triangle; this results immediately from the general formula (**35**).

172. – Finally, the same method applies just as well to the case in which the curve γ is replaced by a system of two characteristics, and *only* the values of z are given on each of these two lines. Formula (**38**) (or rather, the equation in the right-hand side) is then replaced with:

$$z_A = (gz)_0 - \int_C^0 g\left(\frac{\partial z}{\partial \xi} + bz\right) d\xi - \int_B^0 g\left(\frac{\partial z}{\partial \eta} + az\right) d\eta. \tag{40}$$

173. – One does not simply describe a general method for the computation of the Riemann function $g(\xi,\eta;\xi',\eta')$. However, one is assured of the existence of this function such that the coefficients of the equation are analytic and regular ([56]), or, more generally, that they are continuous and differentiable ([57]).

Under these conditions, it obviously results from the foregoing that that if the curve γ is cut by a parallel to the ξ-axis at only one point and a parallel to the η-axis at only one point (as we have assumed), then the Cauchy problem admits one and only one solution.

More precisely, if the function z and its derivatives are given along an *arc PQ*, and it satisfies the preceding condition then this function is defined over the entire rectangle that has *PQ* for its diagonal and sides that are parallel to the axes.

174. – This allows us to fill a lacuna whose existence we pointed out a little earlier (no. **161**) in the context of the equations with which we are presently occupied. Indeed, suppose that equation (**17**) has the Laplace form. We may then confirm that two integral surfaces – analytic or not – may not have a contact of the same infinite order along an entire line without this line being a characteristic, since one may then consider, in particular, an arc of the line in question along which ξ and η are each constantly increasing in such a way that the given of z and its first derivatives along this arc completely determine this function in the neighborhood.

This conclusion extends to equation (**8**). Indeed, consider the Cauchy problem for that equation, i.e., assume that one is given a series of values for a, t, w, u, x that depend on one parameter. This problem may be immediately reduced to the analogous problem that relates to equation (**31**), since one knows z and its derivatives $\partial z/\partial \omega = a$ and $\partial z/\partial u = t$ for a series of values of ω and u.

The necessary and sufficient condition for this problem to cease to be well-defined is therefore that the series of values thus considered be characteristic.

([56]) DARBOUX, *Loc. cit.*, tome II, pages 91-94.

([57]) DARBOUX, *Loc. cit.*, tome IV, pages 355-359 (Note from Picard).

Physically speaking, if one imagines two successive motions M and M' of our fluid mass that coincide for $t \le t_0$, $a \le a_0$ then these motions will again coincide for a value of t that is greater than t_0 until it reaches the value of a that is attained by the wave that starts at a_0 and propagates with negative velocity $-\sqrt{\psi(\omega)}$; in other words, up to the value:

$$a = a_0 - \int_{t_0}^{t} \sqrt{\psi(\omega)}\,dt\,.$$

Up till now, this result has been true in full rigor only by assuming that the motions in question are analytic for either wave.

175. – In the case of a perfect gas, we have found:

$$\text{(2')} \qquad \varphi(\omega) = k\omega^{-m},$$

and, as a result:

$$\text{(41)} \qquad \psi(\omega) = k'\omega^{-m-1}, \qquad k' = \frac{mk}{\rho_0}.$$

$\chi'(\omega)$ is therefore equal to $\sqrt{k'}\,\omega^{-\frac{m+1}{2}}$.

Since this quantity represents the velocity of sound, the constant $\sqrt{k'}$ is nothing but *the velocity of sound in the initial state* λ; one has:

$$\text{(42)} \qquad \lambda = \sqrt{k'} = \sqrt{\frac{mk}{\rho_0}}\,.$$

If m different from unity, as is the case in Poisson's law of springs [Trans. Note: *détente*], then we get (neglecting an additive constant):

$$\text{(43)} \qquad \begin{cases} \psi(\omega) = -\dfrac{2\sqrt{k'}}{m-1}\,\omega^{-\frac{m-1}{2}} = -\dfrac{2\lambda}{m-1}\,\omega^{-\frac{m-1}{2}} \\[2ex] \dfrac{\chi''(\omega)}{\chi'^2(\omega)} = -\dfrac{m+1}{2\sqrt{k'}}\,\omega^{-\frac{m-1}{2}} = -\dfrac{m+1}{2\lambda}\,\omega^{-\frac{m-1}{2}} \end{cases}$$

and, as a consequence, the function f that was defined above (formula **33**) has the value:

$$\text{(44)} \qquad f(\xi - \eta) = \frac{\beta}{\xi - \eta},$$

in which:

$$\text{(45)} \qquad \beta = \frac{1}{2}\,\frac{m+1}{m-1}\,.$$

One is then led to *Euler's equation:*

(46)
$$\frac{\partial^2 z}{\partial\xi\partial\eta} - \frac{\beta}{\xi-\eta}\left(\frac{\partial z}{\partial\xi} - \frac{\partial z}{\partial\eta}\right) = 0\,.$$

When β is an integer the general integral of this equation is expressed in finite terms; it is ([58]):

(47)
$$z = \frac{\partial^{2\beta-2}}{\partial\xi^{\beta-1}\partial\eta^{\beta-1}}\left(\frac{X-Y}{\xi-\eta}\right),$$

in which X and Y are arbitrary functions of ξ, η, respectively.

One finds that this case is approximately the same as in Poisson's law. The general value that is assumed for the coefficient m (the ratio of the two specific values) is 1.41; the hypothesis m = 1.40 gives β = 3.

176. – However, the Riemann method allows us to solve the Cauchy problem for any β. Indeed, one may form the quantity g. One finds that ([59]):

(48)
$$g(\xi,\eta;\xi',\eta') = (\eta'-\xi)^{-\beta}(\eta-\xi')^{-\beta}(\eta-\xi)^{2\beta}F(\beta,\beta,1,\sigma)\,,$$

in which σ denotes the quotient:

$$\sigma = \frac{(\xi-\xi')(\eta-\eta')}{(\xi-\eta')(\eta-\xi')}$$

and F denotes the hypergeometric series:

(49)
$$\begin{cases} F(\beta,\beta,1,\sigma) = 1 + \dfrac{\beta^2}{1^2}\sigma + \left[\dfrac{\beta(\beta+1)}{1\cdot 2}\right]^2\sigma^2 + \cdots \\ \qquad + \left[\dfrac{\beta(\beta+1)\cdots(\beta+n-1)}{n!}\right]^2\sigma^n + \cdots \end{cases}$$

177. – We just excluded the case in which m = 1 – i.e., the case of Mariotte's law – in which $\varphi(\omega)$ is given by formula **(2)**. In this case, one has:

(50)
$$\chi'(\omega) = \frac{\sqrt{k'}}{\omega}\,,$$

(50′)
$$\chi(\omega) = \sqrt{k'}\log\omega\,,$$

([58]) DARBOUX, *loc. cit.*, no. **353** (tome II, pp. 65).
([59]) *Ibid.*, no. **360**.

and the quantity $\frac{\chi''(\omega)}{\chi'^2(\omega)}$ simply gives the constant $-4l = -\frac{1}{\sqrt{k'}}$.

Equation (**32**) is therefore:

(**51**) $$\frac{\partial^2 z}{\partial\xi\partial\eta} + l\left(\frac{\partial z}{\partial\xi} - \frac{\partial z}{\partial\eta}\right) = 0,$$

which may be further transformed into:

(**52**) $$\frac{\partial^2 z}{\partial\xi\partial\eta} + l^2 z_1 = 0$$

by a change of variables $z = e^{l(\xi-\eta)} z_1$.

Finally, one may likewise reduce l to unity by taking $l\xi$ and $l\eta$ for new independent variables. The equation thus obtained, or rather, an equation that is easily reducible to it, is known by the name of the *telegraph equation.* The Riemann function $g(\xi, \eta; \xi', \eta')$ is likewise known for the telegraph equation, and as a result, for equation (**51**); one has:

(**53**) $$g(\xi, \eta; \xi', \eta') = e^{l[\eta-\eta'-(\xi-\xi')]} J\sqrt{l(\xi-\xi')(\eta-\eta')},$$

in which J is the Bessel function:

$$J(X) = 1 - \frac{X^2}{1^2} + \frac{X^4}{(2!)^2} + \cdots + (-1)^n \frac{X^{2n}}{(n!)^2} + \cdots$$

178. – Since Mariotte's law is a limiting case of the one that is represented by formula (**2′**), the results that we just obtained must be deduced from the ones that were the object of nos. **175-176**. At first, it seems as if this is not the case and that, for example, equation (**51**) may not be derived from (**46**).

Indeed, in order to reach this conclusion it is necessary to account for the additive constant h that will have to be added to the right-hand side of (**43**). As one knows, it is by making this constant ($h = \frac{2\sqrt{k'}}{m-1}$) increase indefinitely that one passes from formula (**43**) to formula (**50′**) for an infinitesimal $m - 1$.

We thus augment $2\chi(\omega)$, or its equal $\xi - \eta$, by the constant h, and simultaneously replace β with lh. Equation (**46**) then becomes:

$$\frac{\partial^2 z}{\partial\xi\partial\eta} + \frac{lh}{\xi-\eta-h}\left(\frac{\partial z}{\partial\xi} - \frac{\partial z}{\partial\eta}\right) = 0,$$

and if one now makes h increase indefinitely then one arrives at equation (**51**).

The same calculation may obviously be performed on the Riemann function that is given by formula (**48**), and which may be written:

$$g(\xi,\eta;\xi',\eta') = \left(\frac{\eta-\xi}{\eta'-\xi}\right)^{\beta}\left(\frac{\eta-\xi}{\eta-\xi'}\right)^{\beta} F(\beta,\beta,1,\sigma)\,.$$

If we replace $\eta - \xi$, $\eta' - \xi$, with $\eta - \xi + h$, $\eta' - \xi + h$, and β with lh then the first factor becomes:

$$\left(\frac{\eta-\xi+h}{\eta'-\xi+h}\right)^{lh} = \left(1+\frac{\eta-\eta'}{h+\eta'-\xi}\right)^{lh},$$

and tends to $e^{l(\eta-\eta')}$ for infinite h. Similarly, the second factor will have the limit $e^{l(\xi-\xi')}$.

As for the hypergeometric series $F(\beta, \beta, 1, \sigma)$, its limit is precisely the Bessel function that appears in formula (**53**); indeed, the general term of the series F is:

$$\sigma^n \frac{[\beta(\beta+1)\cdots(\beta+n-1)]^2}{n!^2}$$
$$= \frac{[(\xi-\xi')(\eta-\eta')]^n}{[(\xi-\eta'-h)(\eta-\xi'+h)]^n}\frac{[lh(lh+1)\cdots(lh+n-1)]^2}{n!^2},$$

which is precisely $\dfrac{[-l^2(\xi-\xi')(\eta-\eta')]^2}{(n!)^2}$ for $h = \infty$.

179. – Is the Cauchy problem, when it is solved by the Riemann method, as we have just seen, the mathematical translation of a physical problem that is posed to us?

In order to respond to that question, first consider the case of an indefinite pipe, and suppose that one is given the positions of the molecules and their velocities at every point at an initial instant. Under these conditions, x and its first two derivatives ω and u will be known for $t = 0$, no matter what the value a. We are therefore led to the Cauchy problem that relates to equation (**8**).

Now, we have seen that this problem amounts to the analogous problem for equation (**32**). Nevertheless, an objection may be presented in this spirit. We have remarked that whether or not the Cauchy problem can be posed is subordinate to the issue of whether ξ and η are always increasing or always decreasing on the curve γ. However, there is no reason for this to be true when ξ and η are deduced from the given distribution of molecules, and, for example, the velocity ξ may have a maximum when a varies as t remains null. Still, this does not imply the impossibility of the problem that was originally posed. Indeed, z might have several different values for the same system of values for u and ω if several systems of values for the *given* independent variables a, t correspond to this pair of values for u, ω. Not only might this be true, but u and γ may be constant, while z takes all possible values; this is what happens in the simplest case of a fluid at rest ($x = a; u = 0, \omega = 1$). In a word – and this fact, which we shall encounter again in the following sections, is obvious from what we saw in no. **163** – a singularity of z, when considered as a function of u, ω, does not necessarily give a singularity of x, when considered as a function of a, t, under a Legendre transformation.

Unfortunately, the inverse may obviously produce them. Once one has obtained a value for z as a function of u and ω, if one is to consider these values as acceptable then one must calculate the derivatives $\partial z/\partial\omega = a$, $\partial z/\partial u = t$ and assure that:

1. These quantities may be taken as independent variables.
2. They take precisely all of the systems of values that are possible when $t \geq 0$.

The examination of the conditions under which this is true undoubtedly presents some difficulties.

180. – Now consider the case of a cylinder that is bounded by pistons. x and its derivatives will then be known only for $0 \leq a \leq t$, and one will therefore have only an *arc* of a curve in the $\xi\eta$-plane along which z and its derivatives with respect to u and ω are known. These givens permit us to calculate z in a rectangle of the $\xi\eta$-plane (*fig.* 11). In the *at*-plane, this rectangle will obviously correspond to the sequence of portions of the tube that are not, moreover, attained by the waves that issue from the extremities.

Outside of the region in the *at*-plane that we just obtained one must take into account the conditions that are implied by the motion of the piston. Now, it is easy to see that they may not be transformed as in the foregoing. Indeed, they give us the value of x for each value of t, and, as a consequence, that of u, with a being equal to 0 or l. However, *the value of* ω *is not given to us.* It is therefore impossible to trace the corresponding curve in the $\xi\eta$-plane, *a priori*.

Therefore, under these conditions, the Legendre transformation and the Riemann method might not lead to the determination of the motion. There is good reason to attempt this determination by a direct study, and it is easy to see what sort of analytical problem one is thus led to.

Indeed, the desired motion must be compatible with the original motion such that it propagates by following a wave whose progress is known. We therefore know the value of x along a line in the *at*-plane, namely, the one that represents this wave, and which is a characteristic ([60]). On the other hand, x is likewise given (by the motion of the piston) along another line that is secant to the first, namely, the line $a = 0$ (or $a = l$). It is by this double condition that one may determine a solution to equation (**8**).

The problem thus posed is considerably more difficult that the Cauchy problem, even in the case of a linear equation. One knows ([61]) that when the givens are analytic one may establish the existence of a holomorphic solution, but one may not put this solution into a form that is sufficiently simple and useful.

([60]) There must be concordance (along that line) in not only the values of x, but also in the values of the derivatives of u and ω. However, as we will see later on (note on page ?) this concordance of the derivatives will result from the latter, provided that it is true initially, i.e., provided that the initial velocity of the piston is equal to that of the molecules that it bounds.

([61]) PICARD, in DARBOUX, *Loc. cit.*, tome IV, pp. 361-362; GOURSAT, *Leçons sur l'integration des équations aux dérivées partielles du second ordre,* tome II, pp. 303. – See below, chap. VII.

On the contrary, Hugoniot showed that this result may be attained in an important case, that of a gas that is originally at rest.

181. Motions that are compatible with rest. – Suppose that the gas is at rest for t = 0, and has a uniform pressure and temperature in a portion that we take to be an initial state. We associate a motion to one of the two pistons – the one that corresponds to $a = 0$ – that is arbitrary, but never has an abrupt change in velocity.

A motion will originate at the point of contact with the piston, a motion that propagates in the positive sense with the velocity of sound $\lambda = \chi'(1)$. This motion and the original state of the gas – namely, the rest state – will be *compatible.*

This ceases to be true starting from the moment when the motion thus created encounters the analogous motion that is produced by the piston that is situated at the extremity l if this piston is moving. At this moment, a third motion arises that is compatible with the first two, but not with the rest state. Hugoniot's theorem, as we have presented it, is no longer applicable to this third motion.

Nevertheless, we can write its equation once we know the first two motions.

Indeed, the integral surface that it represents agrees with each of the first two motions along two characteristics of the different systems. In particular, there is one characteristic ξ = const. for the first of the two motions that we spoke of above (since the propagation of the third motion is in the negative sense), and a characteristic η = const. for the second. One knows the series of values for a, t, x, u, ω on each of these characteristics, and, as a consequence, the value of z and its first derivatives. Indeed, all of these quantities are assumed to be known for the first two motions and are not altered by the discontinuity since it is of second order or higher. If z is known on the two characteristics ([62]) then one comes back to the problem of no. **172**.

No matter whether the second piston is at rest or not, one will always arrive at a moment when a new motion originates; it is the moment when the wave that starts from the extremity $a = 0$ reaches the opposite extremity.

At this moment, as we saw in no. **151**, there will be reflection, and the discontinuity will be retarded. The new motion that is produced is no longer compatible with the rest state. The study of this motion is not carried out according to the method that we just indicated. It depends on the considerations that we developed in no. **180**, but, up till now, we do not possess the methods that would permit us to calculate them explicitly.

([62]) It seems that the problem is impossible because of too large a number of conditions, since one of them gives the agreement between z *and its derivatives,* whereas the one given z will suffice to determine the solution. However, if the values of a function z that satisfies the Laplace equation (**34**) are given on one characteristic ξ = const. then the values of $\partial z / \partial \xi$ will satisfy the relation (a particular case of (**22**)):

$$\frac{d}{d\eta}\left(\frac{\partial z}{\partial \xi}\right) + a\frac{\partial z}{\partial \xi} = -\left(b\frac{\partial z}{\partial \eta} + cz\right),$$

on that line, which determines the quantity $\partial z/\partial \xi$ by which those values are given at a point. Now, this relation is verified by the first given motion. Therefore, if one determines the third motion by formula (**40**) then the values of $\partial z/\partial \xi$ will coincide precisely. Indeed, this coincidence is true at one point, namely, the point that the two characteristics have in common, and the integral surfaces that correspond to the three original motions are then mutually tangent.

182. – In the previously studied case, in which the velocity of propagation θ is constant, the equilibrium state is the one in which the functions f_1 and f_2 that we introduced in no. **145** have the expressions $f_1(a + \theta t) = a + \theta t$; $f_2(a - \theta t) = a - \theta t$.

The motions that are compatible with the rest state are characterized by the fact that one of the two functions reduces to the same variable that it depends on. The corresponding representative surface is obviously a cylinder.

We propose to determine these same motions in the case for which the function φ is arbitrary, and find an integral surface Σ that agrees with the plane $x = a$ along a characteristic Γ that corresponds to:

$$\frac{da}{dt} = +\sqrt{\psi(\omega)} = +\chi'(1).$$

Choose the characteristic Γ' of the system that is opposite to Γ at an arbitrary point of that surface. The quantity $u + \chi(\omega)$ is constant on Γ'. Now, the line Γ' necessarily encounters Γ, and the quantity $u + \chi(\omega)$ is everywhere equal to $\chi(1)$ on Γ.

Thus, the desired surface Σ satisfies the first-order partial differential equation:

$$u + \xi(\omega) = \chi(1). \tag{54}$$

Therefore, *during a motion that is compatible with the rest state, velocity and density are functions of each other. These two quantities increase simultaneously* (when velocity is given its algebraic value) since $\chi'(\omega)$ is positive, and ρ increases when ω diminishes.

183. – One easily proves ([63]) that if a surface is such that there exists a relation between the angular coefficients of its tangent plane that is independent of the coordinates of the point of contact then this surface is developable.

This is therefore the case for the desired surface since u and ω are derivatives of x, which is regarded as a function of a and t. If we choose the various planes at the origin of the coordinates whose directions satisfy equation (**54**), planes whose general equation is:

$$x = \omega a + [\chi(1) - \chi(\omega)]t, \tag{55}$$

then these planes envelop a certain cone C, whose equation is obtained by eliminating ω between the preceding equation and its derivative:

$$a - t\chi'(\omega) = 0. \tag{55′}$$

The generatrices of the developable surface Σ' are parallel to the generatrices of C. Since these generatrices are tangent to the edge of regression one sees that its spherical indicatrix is known.

([63]) JORDAN, *Cours d'Analyse*, 2nd ed., tome I, pp. 476; GOURSAT, *Cours d'Analyse*, tome I, pp. 524.

184. – If we take our ordinates to be, no longer the values of x, but the velocities or dilatations (with the horizontal coordinates still being a and t) then we will have not just a developable surface with which to represent the motion, but a ruled surface with horizontal generatrices, since the velocity and dilatation are constant on each wave.

185. – The preceding conclusions persist for any motion that is compatible with the former as long as the characteristic on which they agree has the same system as Γ.

Therefore, they persist if arbitrary discontinuities (of order at least two) are produced by the motion of the piston at $a = 0$. As we said before, they will be modified only if they propagate in the inverse sense after the moment when one encounters a wave.

One also arrives at analogous results if the molecules are not at rest in their original state but are animated with a uniform motion that is given by the equation:

$$x = \alpha a + \beta t, \tag{56}$$

in which α and β are constant (a motion that obviously satisfies equation (**8**)).

More generally, the line of reasoning that we just used may be applied to any equation of the form (**17**), in which one of the families of characteristics admits an integrable combination dF, when one seeks the integral surface that agrees with a surface that satisfies the equation F = const. *along a characteristic of the other system.*

186. – If one tries to apply the Legendre transformation that we used before to the developable surfaces Σ that we just obtained then one does not produce a surface. Indeed, the reciprocal pole of a developable surface is not a surface, but a line, each point of which corresponds to an infinitude of points of the developable surface, namely, all of the points that are on the same generatrix. Equation (**54**) shows that in the $\xi\eta$-plane this line corresponds to the line $\xi = \chi(1)$. In this, we have an obvious example of the degenerate integrals to which we previously alluded (no. **163**).

187. – The developable surfaces Σ are the only developable surfaces that satisfy equation (**8**). Indeed, if one assumes that one has:

$$\frac{\partial x}{\partial t} = f\left(\frac{\partial x}{\partial a}\right) = f(\omega)$$

then by successively differentiating with respect to t and a one will deduce that:

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial a \partial t} f'(\omega), \qquad \frac{\partial^2 x}{\partial a \partial t} = \frac{\partial^2 x}{\partial a^2} f'(\omega),$$

and equation (**8**) will be verified only if $\psi(\omega) = f'^2(\omega)$ (except when $\partial x/\partial a$ and $\partial x/\partial t$ are constant).

188. – It now remains for us to completely determine the motion whose general properties we just found by assuming that the motion of the piston is given.

To that effect, it suffices to recall the method of representation that we used in nos. **146**, et seq. (*fig.* 10). The motion of the piston lets us known the $\alpha\beta$ section (*fig.* 10) of the desired surface by the plane that we called *O* above. The developable surface will admit a tangent plane at each point of this section that must:

1. Contain the tangent to that curve,
2. Satisfy equation (**54**), or, if one prefers, be parallel to a tangent plane of the cone *C*.

This tangent plane will thus be known. As for the generatrix of contact, it will be parallel to the corresponding generatrix of the cone *C*. The locus of these generatrices will be the desired surface.

As one sees, each of these generatrices represents the propagation of the given motion of the piston at the instant that corresponds to its point of origin.

189. – Analytically speaking, let:

(**57**) $$x_0 = f(t_0),$$

be the equation that gives the abscissa x_0 of the piston as a function of time t_0. One will have the value $u_0 = f'(t_0)$ for the velocity of this same piston. Meanwhile, the relation (**54**) gives the value ω_0 of ω in the immediate neighborhood of this piston. For example, if the law of relaxation (*détente*) is Poisson's law, then, from formula (**43**), ω_0 will have the expression:

(**58**) $$\omega_0 = \left(1 + \frac{(m-1)u}{2\lambda}\right)^{-\frac{2}{m-1}},$$

in which λ is still the velocity of sound in the initial state.

The motion that is communicated to the piston at the instant *t* propagates with the velocity $\chi'(\omega_0)$. At the time $t > t_0$ one arrives at the point whose initial abscissa is:

(**59**) $$a = \chi'(\omega_0)(t - t_0) = (t_0 - t)\frac{du_0}{d\omega_0},$$

and to which the velocity u_0 is communicated at this instant.

Here, we assume that this point has been successively attained by the waves that originate at different instants before t_0; therefore, its actual abscissa will obviously be:

$$x = a + \int_{t_0'=0}^{t_0'=t_0} u_0(t_0')\,dt,$$

in which t designates a function of t_0' under the $\int$ sign that is defined by equation (**59**), namely:

$$t = t_0 - a\frac{d\omega_0}{du_0}.$$

If we replace t by this expression, and, as a consequence, dt by:

$$dt_0 - ad\left(\frac{d\omega_0}{du_0}\right),$$

then we get:

$$\begin{aligned} x &= a + \int_0^{t_0} u_0(t_0')dt_0' - a\int_0 u_0 d\left(\frac{d\omega_0}{du_0}\right) \\ &= a + \int_0^{t_0} u_0 dt_0 - au_0\frac{d\omega}{du_0} + a(\omega_0 - 1) \end{aligned}$$

or (since $\int_0^{t_0} u_0 dt_0 = x_0$, and taking (**59**) into account):

(**60**) $$x = x_0 - (t - t_0)\left(\omega_0\frac{du_0}{d\omega_0} - u_0\right) = x_0 + (t - t_0)(\omega_0\chi'(\omega_0) + u_0),$$

in which x_0, ω_0, u_0 are the functions of t_0 whose method of calculation we just indicated. Eliminating t_0 between equations (**59**) and (**60**) gives the desired result.

190. – One may further imagine that instead of giving the motion of the piston at every value of time, one is given the external pressure that this piston is subjected to. It is clear that the preceding calculations will not be essentially modified. Instead of u_0, one must first calculate ω_0 by solving equation (**2′**). One will then have $u_0 = \chi(1) - \chi(\omega_0)$, and then x_0 by a quadrature; after that, all that remains is to write formulas (**59**), (**60**).

191. – Thanks to the intervention of the discontinuity waves, if the gas is animated with a given motion at a given instant then we may confirm the existence of a motion that satisfies either the internal equations or the boundary conditions *at the instants that follow immediately*. Do we have the right to conclude that the discontinuities that we studied up till now – and they alone – assure us of the existence of the motion *for any later value of time?* Such a conclusion will be vacuously legitimate.

In order to account for this, it suffices to put oneself in the simple case of a constant velocity of propagation. We assume that the gas is at rest at $t = 0$, and put the piston whose abscissa is 0 into motion in the positive sense at this instant, i.e., in such a manner as to compress the fluid. Under our hypotheses, the motion that is thus created at an arbitrary instant t will extend to the points whose initial abscissas fall between zero and θt while the remainder of the mass remains at rest.

Now, by conveniently accelerating the motion of the piston, one may obviously make its abscissa greater than θt at a certain instant.

There is obviously a contradiction, since the neighboring molecules to the piston must coincide with certain molecules that are still at rest at this instant. If we would like to preserve our fundamental hypotheses of impenetrability and continuity (nos. **44-45**) then we shall be obliged to get involved with phenomena that are distinct from the ones that we described in the foregoing.

We must remark that this hypothesis of a wall that is displaced with a velocity that is greater than that of the wave is not purely theoretical; it presents itself in the most important application that one has dreamed of making in Gas Dynamics up till now, namely, the study of the motion of projectiles. Indeed, one knows that their velocity is greater than that of sound.

192. – Nevertheless, before we study the singularities that must therefore be produced when one gives the piston a compressive motion, we must mention one that is produced in the contrary case of a decompressive motion.

In order to calculate the value of ω, we solved equation (**54**) with respect to this quantity.

Now we must demand to know whether this solution is possible. The derivative of the left-hand side with respect to ω is always different from zero (it is equal to $\chi'(\omega) = \sqrt{\psi(\omega)}$). u may therefore take all possible negative values if it tends toward $-\infty$ for $\omega = +\infty$; i.e., the integral:

$$\chi(1) - \chi(\infty) = -\int_1^\infty \chi'(\omega)\,d\omega = -\int_1^\infty \sqrt{\psi(\omega)}\,d\omega\,,$$

is infinite.

This is true for Mariotte's law, for which the function $\chi(\omega)$ is logarithmic. However, this is not the case for Poisson's law. In this case, formula (**43**) gives:

$$\chi(1) - \chi(\infty) = -\frac{2\lambda}{m-1}\,.$$

When the piston reaches the point that it takes on a negative velocity and has a ratio with the corresponding velocity of sound in the initial state that equals $2/(m - 1)$ (in absolute value) *then the fluid will cease to follow the piston*; a vacuum is produced between them just as if one were concerned with a liquid. The only difference is that the latter gas layers will be infinitely dilated (since ω will become infinite ([64])), whereas in order for the aforementioned liquid to be to be compressible the separation will have to happen after a certain finite value of ω.

([64]) Of course, if this indefinite expansion is to be attained then we must assume the completely theoretical hypothesis that the gas maintains its properties down to absolute zero.

On the contrary, if the piston does not attain the negative value $-2/(m-1)$ then its motion will produce a well-defined state at each instant in the neighboring layers that propagates, as we have said, at least during a certain time interval, until it produces singularities such as remain for us to discuss.

§ 3. – THE RIEMANN-HUGONIOT PHENOMENON

193. – If, as we assumed a moment ago, the motion obeys equation (**8′**) then it is easy to see that the singularity whose existence seemed necessary to us in no. **191** will appear (and for the first time) at the moment when an infinitely large compression is produced, with ω becoming null.

Indeed, if the fluid is assumed to be originally at rest then a motion that propagates in the positive sense (starting with the instant, $t = 0$) will have the equation:

$$x = a + f(\theta t - a) \qquad (a < \theta t), \tag{61}$$

with $f(0) = f'(0) = 0$. The function f will then be given by the relation:

$$f(\theta t) = x_0,$$

in which x_0 again represents the space in which the piston moves. If this quantity x_0 is a differentiable function of time then x will be a differentiable function of a and t. Moreover, in any case, exactly one value of x corresponds to each system of values for a and t. In order for the converse to be true – i.e., exactly one value of a corresponds to a system of values of x and t – it is necessary and sufficient that the derivative $\omega = \partial x/\partial a$ does not change sign. If this condition ceases to be satisfied then this will start at the moment when ω is annihilated.

As the values of ω propagate from the extremity $a = 0$ this phenomenon is first produced at the point of contact with the piston.

194. – We shall see, as did Riemann and Hugoniot, that things are different when the function $\psi(\omega)$ no longer reduces to a constant, and, in particular, in the case of Poisson's law.

Indeed, in this case ω can be null only when the velocity becomes infinite. On the other hand, instead of the cylinder that was defined by equation (**61**), we will have a developable surface whose edge of regression will be situated at a finite distance (at least when then velocity of the piston is not constant).

From this, x, when considered as a function of a (with t being regarded as constant), may present two types of singularities:

1. Those for which $\partial x/\partial a$ is null, which, as a consequence, are analogous to the ones that we just spoke of.

2. Those that correspond to the edge of regression of the surface.

One may verify this directly by the study of the derivative $\partial x/\partial a$.

Indeed, if one differentiates formulas (**59**), (**60**) for constant t then one finds:

$$
(\mathbf{62}) \qquad \begin{cases} da = \left[(t-t_0)\chi''(\omega_0)\dfrac{d\omega_0}{dt_0} - \chi'(\omega_0) \right] dt_0 \\ \quad = -\left[\dfrac{(t-t_0)\chi''(\omega_0)}{\chi'(\omega_0)}\dfrac{du_0}{dt_0} + \chi'(\omega_0) \right] dt_0 \\ dx = \omega_0 \left[(t-t_0)\chi''(\omega_0)\dfrac{d\omega_0}{dt_0} - \chi'(\omega_0) \right] dt_0 \\ \quad = -\omega_0 \left[\dfrac{(t-t_0)\chi''(\omega_0)}{\chi'(\omega_0)}\dfrac{du_0}{dt_0} + \chi'(\omega_0) \right] dt_0 . \end{cases}
$$

If we eliminate dt_0 then we will obtain the value of $\partial x/\partial a$ from the quotient of dx/dt_0 and da/dt_0 , at least when the latter of these quantities is not annihilated. As a consequence, except for the cases of $\omega = 0$ and $\omega = \infty$, which, as have seen, are only produced at the point of contact of the piston, there will be no other singularities for which da/dt_0 is annihilated, and, as a consequence, dx/dt_0 as well. As one knows, this characterizes a point of regression for the curve described by the point (a, x) when t_0 varies. The first derivatives of x with respect to a and t therefore remain constant, but the second derivatives become infinite.

Here, from formula (**62**) this point of regression is given by the equation:

$$
(\mathbf{63}) \qquad (t-t_0)\chi''(\omega_0)\frac{d\omega_0}{dt_0} - \chi'(\omega_0) = 0.
$$

195. – The physical interpretation of this circumstance is simple, moreover. It suffices for us to recall that each generatrix of our developable surface represents the propagation of a well-defined motion with the velocity $\chi'(\omega)$, and is characterized by a well-defined system of values of u and ω. A point of regression of our surface corresponds to the intersection of extremely close generatrices, and, as a consequence, the intersection of two consecutive waves, where the second one catches up with the first one.

196. – If, instead of the representative surface of the displacements, one considers the one for which the dilatations appear as a function of a and t, or the one for which the velocities are of concern, then on either of these two surfaces the two consecutive waves that we just discussed will correspond to two generatrices that have the same horizontal projections as the developable surface. Moreover, these new generatrices no longer intersect in space, and the point of intersection of their horizontal projections will be simply the base of their common perpendicular, in such a way that the edge of regression of our developable surface corresponds to the line of "striction" of the surface of dilatations or that of velocity, since these surfaces a vertical tangent plane at each point of that line.

197. – The surfaces thus constructed permit us to analyze the phenomena that we are occupied with in a simple manner by considering their sections with the t = const. planes. Indeed, each point μ (*fig.* 12, 13) of such a section belongs to a certain generatrix that corresponds to a well-defined wave, and we know that these various waves are displaced with unequal velocities depending on whether they are compressed to a greater or lesser degree. As a consequence, during a given time interval $t' - t$ they will describe unequal paths, and the curve of the section will be deformed, instead of the point μ. If the forward waves are the ones that propagate the fastest then the horizontal distances will be increased, and the curve will be stretched in the horizontal sense (*fig.* 12). However, in the opposite case (*fig.* 13) the curve tends to straighten and nothing prevents this from happening up to a certain instant T when one of its tangents goes vertical. If one follows the same deformation at the instants after t'' then one verifies that the inclination of that tangent to the vertical changes sign, and the different parts of the curve separate from each other, just as if it were a wave that broke.

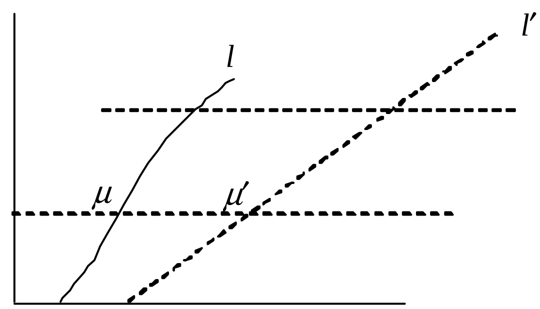

Fig. 12

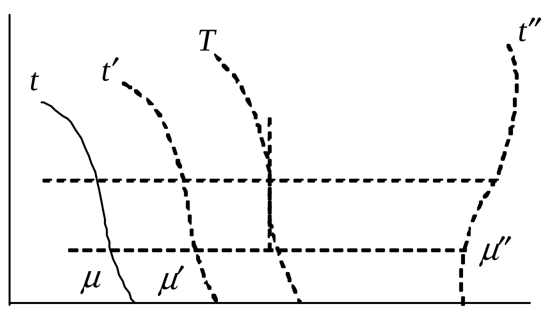

Fig. 13

198. - In the case of Poisson's law – or Mariotte's law – the waves that are compressed the most are the ones that that propagate the fastest; in other words, the velocity $\chi'(\omega)$ is a decreasing function of ω. We therefore make the hypothesis that this condition is verified ([65]). Then, since $1/\omega_0$ is increasing with u_0, it must be the case that this latter quantity is increasing in time – in other words, that the piston has a *positive* acceleration – in order for the wave to overtake another one that originated before it did.

Therefore, if one gives the piston a motion with a negative acceleration du_0 / dt_0 (in other words, directed in the same sense as the decompression) then the waves thus generated will not cross each other. Indeed, formula (**63**), in which one has $\chi''(\omega) < 0$,

([65]) One may remark that the opposite hypothesis, $\chi''(\omega) > 0$, might not be verified, or at least might not be constant, in which case, after a certain moment p will be *less than* an expression of the form $(\mathbf{3}')$, although we have shown the impossibility of this in no. **144**.

shows that the point of contact of the generatrices of our developable surface with the edge of regression corresponds to a negative value of $t - t_0$, and, as a result, of a as well. The representative surface of the motion does not provide any singularity, and will give precisely the equation of a physically possible motion (at least if the phenomenon that was pointed out in no. **192** is not involved).

199. – On the contrary, suppose that the acceleration du_0 / dt_0 is positive at an arbitrary instant. Then, the wave that originates at this instant overtakes the wave that immediately precedes it at an instant t that is given by equation (**63**), namely:

$$t = t_0 - \frac{dt_0}{du_0}\frac{\chi'(\omega_0)^2}{\chi''(\omega_0)}.$$

As one sees, for the same value of velocity, the time that is necessary in order for consecutive waves to meet is just as considerable as the acceleration of the piston is small. In the previously examined case in which one is concerned with infinitesimal motions this encounter will be extended indefinitely.

If the acceleration is due to gravity, and if the gas is originally at the temperature 0° and normal atmospheric pressure then the quantity $\chi'(\omega)$ will be initially (i.e., for $\omega_0 = 1$) equal to the velocity of sound, which is in the neighborhood of 330 meters per second for air. $\dfrac{\chi''(\omega_0)}{\chi'(\omega_0)}$ will have the value $-\dfrac{m+1}{2}\dfrac{1}{\omega_0} = -\dfrac{m+1}{2}$. Upon taking $\dfrac{dt_0}{du_0} = \dfrac{1}{g}$ one finds that the first wave will be overtaken by the ones that follow after about 28 seconds, during which time it will have traveled a little more than 9 kilometers.

On the contrary, in the case for which the gas is compressed by an explosion, as in the experiments of Vielle that we shall describe shortly, the waves will overtake in an interval of several centimeters.

In any case, what is certain is that this time the singularity will not have to be produced at the point of contact with the piston, as was the case under the hypotheses that were treated in no. **193**. The value of $t - t_0$ is always different from 0; it is in the midst of the gas itself that the waves meet.

200. – We have seen that under the present hypotheses the compression might become indefinite. By contrast, it is easy to see that one may make it attain a value as high as one wants before the phenomenon for which this is true will occur.

Indeed, we look for the condition under which this phenomenon does not happen before the time T. The time t that is given by formula (**63**) must be less than T, namely:

$$t_0 + \frac{dt_0}{d\omega_0}\frac{\chi'(\omega_0)}{\chi''(\omega_0)} < T\ .$$

However, this inequality states that the product $A = (T - t_0)\,\chi'(\omega_0)$ is decreasing ([66]). One may always make the compression increase sufficiently slowly that this will be true. This condition is likewise compatible with the condition that ω_0 must have an arbitrarily small given value at the instant T. Indeed, $\chi'(\omega_0)$ is equal to minus the derivative with respect to t of the product that we just discussed at $t = T$, a derivative to which one may assign an arbitrary (negative) value without A ceasing to be decreasing.

This value itself may null, in such a way that one might arrive at an indefinite compression, if one likewise indefinitely increases the velocity of the piston according to a convenient law.

201. – We just assumed that the product A always decreases. On the contrary, what will happen if one governs the motion of the piston in such a way that this product preserves a constant value?

Under these conditions, all of the waves will overlap at the time T. In other words, all of the generatrices of our developable surface will meet at the same point.

This developable surface is therefore reduced to a cone, a cone that is obviously equal to the cone C that was considered in no. **183**. The trace of such a cone on the $a = 0$ plane thus furnishes the convenient motion that must be given to the piston in order for A to be constant.

Equation (**59**) shows that the constant value of A is nothing but the abscissa of the vertex of the cone, i.e., the common point of intersection for the waves.

202. – Furthermore, one may continue this motion only up to the time $t_0 = T$, since the density, and, in turn, the velocity become indefinite at that point.

Suppose that one has continued it up to the instant t_1 for which u_0 will have a certain value u_1, and ω_0 will have a certain value ω_1, and that one then diminishes the acceleration in such a way that the wave that was created at the instant t_1 will no longer be overtaken by the ones that follow before the time T (for example, one renders the motion uniform for $t > t_1$). Then, for an a that is infinitely close to $(t - t_1)\,\chi'(\omega_1)$, but less than that quantity, the velocity will be essentially equal to u_1 and the dilatation equal to ω_1. In particular, for $t = T$ these values of u and ω can be obtained at the point $a = A - \varepsilon$, where ε is an infinitely small positive number.

However, for $t = T$, $a = A + \varepsilon$ one has $u = 0$, $\omega = 1$.

Hence, for $t = T$, $a + A$ the velocity and density will not change abruptly; *one will be in the presence of a first-order discontinuity,* and no longer a second-order one.

203. – Starting with the intersection of consecutive waves, equations (**59**) and (**60**) cease to give a physically acceptable motion.

This is already exhibited by the velocity and dilatation surfaces since the section of one of these surfaces by the plane $t = T$ has a vertical tangent (*fig.* 13), and the sign of the

([66]) This is, moreover, obvious *a priori* from equation (**59**) since it states that at the instant T an arbitrary wave is behind the ones that originated immediately before it.

angular coefficient of the tangent changes at certain points for $t = T + \varepsilon$, in such a way that this curve is cut at several points by a conveniently chosen ordinate. One will then be led to several values for the velocity for the same particle at the same instant.

204. – In order to recognize this fact with the aid of the representative developable surface of motion, we recall that a developable surface is divided into two nappes by its edge of regression, and that an arbitrary generatrix passes from one nappe to the other at the moment when it touches this edge of regression.

In the developable surface that is presently under consideration, if T is the instant when the generatrix G_0 that corresponds to the initial wave touches the edge of regression (if we assume, to fix ideas, that it reaches this edge for a positive t, and, on the other hand, assume that it is the first wave to do so) then the entire part that corresponds to $t < T$ belongs to the first nappe. The section by the plane $t = T - \varepsilon$ will be a certain curve that agrees with the straight line that is the section of the $x = a$ plane (*fig.* 14) at a point of G_0.

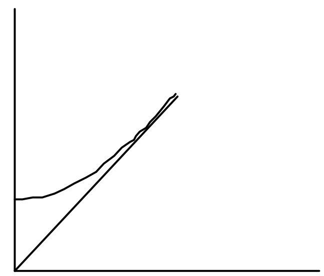

Fig. 14

From what we just said, for $t = T + \varepsilon$ the generatrix G_0 will pass through the second nappe of the surface. Therefore, the section of the latter by the plane $t = T + \varepsilon$ will agree with the line $x = a$ only after crossing the edge of regression.

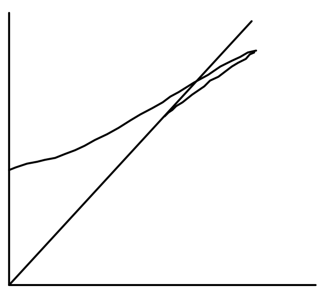

Fig. 14 (cont.)

For such a value of t, x will be represented as a function of a by a line that no longer has the form that is represented in (*fig.* 14), but one which is represented in figure 14 (cont.), and which, together with the line $x = a$, determines a small curvilinear triangle. This is physically absurd, since all of the ordinates that traverse this triangle will give three values of x for one value of a.

205. – A new problem is therefore posed to us: the search for the singularity that is created at the instant T. In the particular case that we recently considered (no. **202**), this singularity is a first-order discontinuity. We are thus led to demand that this not be true in the general case.

To that effect, one must first study the conditions for the propagation of such a discontinuity.

This study, like that of the second-order discontinuities, is carried out with the aid of the equation of motion (**8**); however, the reasoning that led to that equation assumed that the velocity was continuous. On first glance, it likewise seems that the general principles of dynamics imply such a discontinuity, and likewise the existence of acceleration, since this is what makes the force known. We shall nevertheless see that when conveniently applied these principles permit us to account for the phenomenon that remains for us to treat.

If the fluid is always referred to an initial homogenous state then let u_1, u_2 be two values of u on one side and the other of the discontinuity; similarly, let ω_1, ω_2 be those of the dilatation, p_1, p_2, those of the pressure, and θ, the velocity of propagation (relative to our initial state). We first have the kinematical condition:

$$u_1 - u_2 + \theta(\omega_1 - \omega_2) = 0. \tag{64}$$

In order to now write the dynamical relation that exists between the forces that act and the motion, consider two consecutive positions AA', BB' that are occupied at the instants t and $t + dt$ by the discontinuity slice, and whose separation distance is, as a consequence, measured by $\theta\, dt$ on the initial state. We shall apply the fundamental equation of dynamics to the small fluid volume AA', BB' whose mass is $\rho_0 S\theta\, dt$, in which S denotes the cross-sectional area of the tube, and (upon assuming that θ is positive, to fix ideas) assuming that we pass from the state (u_1, p_1, ω_1) to the state (u_2, p_2, ω_2) by writing that the variation of its quantity of motion during the time dt is equal to the total impulse during the same time of the forces that act on it. They are (where they are defined), on the one hand, the forces that are applied to the mass elements, whose impulse will be of order dt^2 (since the forces themselves and the duration of the action will both contain a factor of dt), and, on the other hand, the pressures on the two surfaces AA', BB', whose impulses will be $p_1 S\, dt$ and $p_2 S\, dt$, respectively.

Since the velocity of the portion of the fluid that we envision passes from u_1 to u_2 during the time dt, one obtains (upon dividing by $S\, dt$):

$$p_1 - p_2 = \theta(u_1 - u_2). \tag{65}$$

As one sees, the fact that a finite force (the pressure difference across the discontinuity) produces not only acceleration but an abrupt change of velocity is explained in a completely natural fashion. It amounts to saying that, thanks to the propagation of the discontinuity, the force in question is not applied to a mass of well-defined magnitude, as in the usual situation, but to a mass that is infinitesimal during the time in question.

206. – Before he wrote the two equations (**64**), (**65**), Riemann obtained a third one by expressing the notion that the change of density in an edge traversed by the discontinuity happens without release or absorption of heat and is governed by Poisson's law:

$$p_1\omega_1^m = p_2\omega_2^m, \tag{66}$$

an equality that is, moreover, verified if the gas, starting from a homogenous perfect state, reaches its present state by transformations that satisfy all of Poisson's laws.

As a consequence, if two contiguous regions of the fluid have a first-order discontinuity then if there is to be compatibility, one must satisfy equation (**66**) and the equation:

(**67**) $$(u_1 - u_2)^2 = \frac{1}{\rho_0}(p_1 - p_2)(\omega_2 - \omega_1)$$

that is obtained by eliminating θ from (**64**) and (**65**).

207. – If these conditions are satisfied then the discontinuity will propagate with a velocity θ that is a common solution of equations (**64**) and (**65**). One may express this velocity as a function of pressure and density in such a way that one obtains an expression that is analogous to (**24**). Eliminating $u_1 - u_2$ gives:

(**68**) $$\theta = \sqrt{\frac{p_1 - p_2}{\rho_0(\omega_2 - \omega_1)}}\,.$$

One sees that, contrary to what we found for the expression (**24**), here the velocity depends on two pressures and two densities. If one considers p as the ordinate of a point whose abscissa is ω then the pairs of values (ω_1, p_1) and (ω_2, p_2) correspond to two points that, from (**66**), are situated on the same curve of the form:

(**2′**) $$p\,\omega^m = k.$$

The quantity under the radical in formula (**68**) is, up to a factor of $-1/\rho_0$, the angular coefficient of the line that joins these two points.

Likewise, the analogous quantity that appears in formula (**24**) corresponds to the angular coefficient of the tangent to the curve (**2′**). When the discontinuity is infinitesimal (p_1 is very close to p_2) the velocity θ is, as one naturally suspects, essentially equal to the velocity that corresponds to a second-order discontinuity. However, this is not the case when p_2 is noticeably different from p_1, and, in particular, for a well-defined system of values of p_1 and ω_1, θ may take values that are as large as one wants for a sufficiently large p_2. Thus, the progress of a wave is no longer determined by a characteristic, but by a line of arbitrary direction.

208. – The influence that is thus exerted by a first-order discontinuity on the velocity of propagation appears clearly in the experiments of Vielle ([67]).

These experiments consisted of provoking, either by the detonation of a small quantity of explosive or the rupture (by means of strong air pressure) of either glass ampules or a collodion diaphragm, a sufficiently energetic disturbance whose progress is recorded in a perfectly closed cylinder.

If the wave thus produced is of second order then it results from the preceding considerations that its velocity of propagation will be rigorously independent of the nature of the propagated motion and equal to the velocity of sound in the original medium (about 330 meters per second).

([67]) C.R. *Ac. Sc.* 1898-1899; *Mémorial des Poudres et Salpétres,* tome 10, pp. 177-260; 1900.

Now, by raising the pressure enough Vielle has obtained velocities of propagation that were greater than 1200 meters per second.

One sees that this one fact suffices to exhibit the existence of a first-order discontinuity and to show that it modifies the velocity of propagation.

On the other hand, if one imposes the law of the variation of the pressure at a point with the aid of an appropriate apparatus then one confirms that at a certain distance away from the explosion the pressure immediately attains its maximum value, whereas, at least for certain experiments, the same fact is not true in the immediate neighborhood of the point of departure. The traces thus obtained then show that the discontinuity is initially of second order and changes character in the course of propagation. This is the same phenomenon that we considered in the foregoing.

209. Hugoniot's objection. – The conclusions that we just obtained were established under the hypothesis that Poisson's law is applicable. Hugoniot showed that this hypothesis is no longer legitimate in the case of abrupt condensations or dilatations.

Indeed, recall what we said in no. **129** (ch. III). There, we established that the expression for the quantity of heat released during condensation is the same for any rest state or fluid motion. However, the reasoning that we employed assumed in an essential way that the velocity was continuous. It rests on a combination of the equations of motion that is analogous to the one that led to the theorem of *vis viva* in the dynamics of solid bodies, and which changes form when the velocity varies abruptly.

To see what the true condition for adiabaticity will be, we recall the equation that expresses the conservation of energy, and we regard the assumption that the velocities are continuous or instantaneous to be completely general. We apply this equation as above to a small fluid volume that is defined by the positions AA', BB' of the plane of discontinuity at two successive instants t, $t + dt$.

The work done by the forces that act on the mass elements is negligible, as before. The work done by the pressures will be $(p_1u_1 - p_2u_2)\, dt\, S$. We thus specify that this quantity is the one that varies the sum of the semi-vis viva and the internal energy during the time interval dt. The first term is easy to evaluate since the fluid mass, which is equal to $S\rho_0\theta\, dt$, has passed from the velocity p_2 to the velocity p_1.

As for the internal energy of a perfect gas, its expression is known. If we remark that:

1. It depends only on temperature or, what amounts to the same thing, on the product of volume and pressure,

2. If the gas is subjected to a slow adiabatic release of pressure, then the variation of energy is uniquely measured by the work done by external pressure, i.e., by $p d\mathcal{V}$, in which $\mathcal{V}$ is the volume, then one finds that this energy has the value:

$$U = \frac{p\mathcal{V}}{m-1} = \theta S\, dt \cdot \frac{p\omega}{m-1}.$$

The desired equation is therefore:

$$p_1 u_1 - p_2 u_2 = \frac{\theta}{m-1}(p_1\omega_1 - p_2\omega_2) + \rho_0 \frac{u_1^2 - u_2^2}{2}.$$

It is nevertheless necessary to give this a slightly different form, because on first glance, it seems to contain the two velocities u_1 and u_2, and not just their difference, which is the only thing that must appear if the result is to be independent of a common translational motion of the system. This is what one obtains by multiplying equation (**65**) by $(u_1 + u_2)/2$ and subtracting the preceding equation. One obtains:

(**69**)
$$\frac{(p_1 + p_2)(u_1 - u_2)}{2} = \frac{\theta}{m-1}(p_1\omega_1 - p_2\omega_2).$$

The relation between the two pressures and the two densities is obtained by eliminating $u_1 - u_2$ between (**64**) and (**69**), namely:

(**70**)
$$\frac{(p_1 + p_2)(\omega_2 - \omega_1)}{2} = \frac{1}{m-1}(p_1\omega_1 - p_2\omega_2).$$

210. – This is the relation that Hugoniot has substituted for (**66**) in order to express that the jump in condensation or dilation happens without the release or absorption of heat. One actually gives relation (**66**) the name of the *dynamic adiabatic law,* which is called the *static adiabatic law* when the changes are slow.

When p_2 is very close to p_1 and ω_2 is close to ω_1, both give:

$$\frac{\Delta p}{p} + m\frac{\Delta\omega}{\omega} = 0.$$

In the contrary case, it is easy to see in what sense these two relations differ. That of Poisson gives:

$$\frac{p_2}{p_1} = \left(\frac{\omega_1}{\omega_2}\right)^m,$$

whereas the value that is deduced from formula (**70**) is:

(**70′**)
$$\frac{p_2}{p_1} = \frac{(m+1)\frac{\omega_1}{\omega_2} - (m-1)}{m+1-(m+1)\frac{\omega_1}{\omega_2}}.$$

Let $r = \dfrac{\omega_1}{\omega_2}$. The two functions r^m and $\dfrac{(m+1)r-(m-1)}{m+1-(m-1)r}$ have the logarithmic derivatives $\dfrac{m}{r}$ and:

$$\frac{m+1}{(m+1)r-(m-1)} + \frac{m-1}{m+1-(m-1)r}$$
$$= \frac{4m}{2r(m^2+1)-(1+r^2)(m^2-1)},$$

respectively, for r close to 1, the second of these two fractions being larger than the first ([68]). Therefore, if one regards p_1 and ω_1 as known and considers ω_2 to be an abscissa and p_2 to be an ordinate then equations (**66**) and (**70**) represent two osculating curves at their common point, with the second one increasing faster than the first. In other words, for the same variation of the density, the pressure experiences a larger change from Hugoniot's law than it does from Poisson's.

There is more: in Hugoniot's way of seeing things, the ratio of the pressures is null or infinite, or else the same is true of the densities, namely, of the value:

$$\frac{\omega_1}{\omega_2} = \frac{m+1}{m-1}.$$

The right-hand side of this equality is, as we have seen, almost equal to six for the given value of m. Therefore, at a discontinuity where the density varies from simple to six times as great the pressure necessarily becomes null or infinity.

As for the velocity of propagation, it is clear that if one gives ω_2 (along with p_1 and ω_1) then its value from (**70**) will be greater than its value from (**66**), and the converse will be true if the given is p.

211. – After passing the first-order discontinuity, the product $p\omega^m$ will become constant as a function of time. However, it is clear that, in general, this product will have a different value for each molecule, in such a way that the partial differential equation of motion will no longer have the form (**8**), but, in fact, the form (**6**) (with $X = 0$), and that this is true even if the gas is perfectly homogenous before passing the discontinuity. k will be a function of a whose expression is obtained by calculating the discontinuity at the moment when it reaches the molecule with abscissa a.

The form of that function therefore depends on all of the past circumstances of the motion, and, as a consequence, if one accounts for the objection of Hugoniot then one sees that there exists no equation of the form (**8**) or (**6**) that is verified by all of the motions of a given gas. As Hugoniot remarked, in order to obtain such an equation one

([68]) If we given them the common denominator of $4m$ then the difference of the denominators is:

$$2r(m^2+1) - (1+r^2)(m^2-1) - 4r = -(1-r^2)(m^2-1).$$

must consider k to be an *unknown* function of a in equation (**6**) and eliminate it by differentiating with respect to t. As is easily seen, this gives two fourth-order partial differential equations ([69]).

212. – The experiments of Vielle seem to confirm the views of Hugoniot that we just discussed. For the single case in which one may assign both the pressure difference and the velocity of propagation, the former is around 3 atmospheres and the latter, from 601 to 609 meters per second, a value that is slightly different from the 600 meters per second that corresponds to the law of Hugoniot. Poisson's law gives us a velocity that is somewhat less (around 14 meters per second less).

The divergence between these two hypotheses becomes more acute when one passes to more intense discontinuities, such as those that are produced by the motion of artillery projectiles. They are launched with velocities around 800 to 1200 meters per second. They are preceded by an aerial wave that – at least, its front part – is reasonably plane and propagates with the same velocity as the projectile. However, when the motion of the air is obviously quite different, like the cases that we studied in the preceding chapter ([70]), there exists a remarkable concordance between the resistance experienced by the projectile and the corresponding pressure differences for the observed value of velocity. One therefore finds, for example, a measured resistance of 15 kilograms per square centimeter for a velocity of 1200 meters per second, which corresponds to $p_2 - p_1 = 15.64$ kg. in Hugoniot's theory. On the contrary, Poisson's law demands an overpressure of 17.24 kg. ([71])

213. – Now consider an arbitrary first-order discontinuity, for which the state of the gas is characterized by the quantities p_1, ω_1, u_1 on the left side of this discontinuity, and the quantities, p_2, ω_2, u_2 on the right.

In general, there is no compatibility; a new motion will thus originate and there will be good reason to look for the corresponding values of p, ω, u. We shall do this by first assuming the Riemann hypothesis (in which Poisson's law remains exact), and then that of Hugoniot.

We suppose, moreover, that the state considered in the first case defines a state that is perfectly homogenous beforehand, and that, as a consequence, one has equation (**2′**) for all of the states envisioned.

By once more considering ω and p to be coordinates, this equation represents a curve on which one finds the two points (ω_1, p_1) and (ω_2, p_2), and on which one likewise finds

([69]) In his memoir, Hugoniot obtained a single third-order equation as the result of this elimination. There is an error in that result that is due to the fact that previously the author effectively assumed a certain change of the initial state that assumed that k was known.

([70]) It is clear that, on the one hand, there is a lateral flow, and, on the other hand, that air resistance does not account for the difference between the pressure at the head of the projectile and ordinary atmospheric pressure, but for the difference between pressure at the head and pressure at the tail, which is *smaller than* atmospheric pressure. Compare below, no. **222**.

([71]) VIELLE, *Mém. Poud. Salp., loc. cit.*, pp. 255.

the point (ω, p) that corresponds to the unknown state that will be established in the intermediary cut. Moreover, if u denotes the velocity in this cut then one must have:

$$\textbf{(71)} \qquad \begin{cases} u - u_1 = \sqrt{\dfrac{(p - p_1)(\omega_1 - \omega)}{\rho_0}} \\ u - u_2 = \sqrt{\dfrac{(p - p_2)(\omega_2 - \omega)}{\rho_0}} \end{cases}$$

and, as a consequence, upon eliminating u, we have:

$$\textbf{(72)} \qquad a = u_1 - u_2 = \sqrt{P_2} - \sqrt{P_1}\,,$$

in which P_1 and P_2 designate the quantities that appear under the radicals in the two formulas of (**71**), respectively.

If we put the preceding equation into integer form then it may be written in one of the equivalent forms:

$$\textbf{(73)} \qquad \begin{cases} 4a^2 P_1 = (P_1 - P_2 + a^2)^2 \\ 4a^2 P_2 = (P_1 - P_2 - a^2)^2 \end{cases}$$

in which a denotes $u_1 - u_2$. In the system of coordinates that we adopted, this represents a conic inscribed in the rectangle $A_1A_2B_1B_2$ (*fig.* 15) that has the two points A_1, A_2 for opposite vertices and whose sides are parallel to the axes. The chord, C_1D_1 that joins the points of contact with the sides A_1B_1, A_1B_2 has the equation $P_1 - P_2 + a^2 = 0$, and the analogous chord C_2D_2 (*fig.* 15) has the equation $P_1 - P_2 - a^2 = 0$, whereas $P_1 - P_2 = 0$ represents the diagonal B_1B_2.

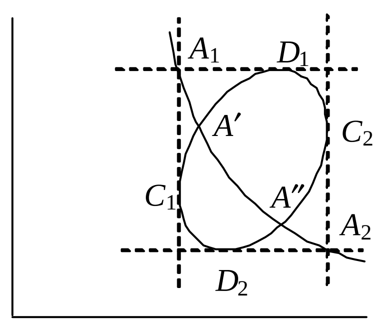

Fig. 15

It is easy to see that when a^2 is taken between 0 and $\dfrac{1}{\rho_0}(p_1 - p_2)(\omega_1 - \omega_2)$, the conic (**73**) is an ellipse and that when a^2 exceeds this limit, $\dfrac{1}{\rho_0}(p_1 - p_2)(\omega_1 - \omega_2)$, it is a hyperbola whose two branches are defined by the opposing sides to the vertex of the angles A_1 and A_2 of the rectangle.

In both cases, this conic cuts our curve $(\mathbf{2'})$ at two points A', A'', one of which is situated on the arc C_1D_1, and the other, on the arc C_2D_2.

However, in order for the solution that corresponds to one of the points A' or A'' to be acceptable, it must satisfy a condition of inequality that we have not written up till now. Since the intermediate cut must be contiguous with the motion (p_1, ω_1, u_1) *on the left* and with the motion (p_2, ω_2, u_2) *on the right,* one obviously must have:

$$\textbf{(74)} \qquad \theta_1 < \theta_2.$$

The quantities θ_1 and θ_2 will be therefore be given by applying formula (**64**) as a function of $u - u_1$ and $u - u_2$, i.e., of $\sqrt{P_1}$ and $\sqrt{P_2}$, in which the radicals have the same signs as in equation (**72**).

One then easily sees that of the two points A' and A'' there is always one and only one of them that satisfies the inequality (**74**), and, as a consequence, that provides the solution to the problem we posed, a solution in which the intermediate motion propagates in a sense that is contrary to the interior of the original two motions. From the foregoing, it is clear that the pressure and density of the new state thus created will or will not be defined between the original pressures and densities, depending on whether the given difference of velocities is less than or greater than the geometric mean of $p_1 - p_2$ and $\frac{1}{\rho_0}(\omega_1 - \omega_2) = \frac{1}{\rho_2} - \frac{1}{\rho_1}$.

214. – However, one may present this same discussion in a form that is simpler in some regards by giving a name to the right-hand sides of equations (**71**).

If p_1, ω_1 are always understood to represent the coordinates of the point A_1, and similarly, p_2, ω_2 always represent the coordinates of a second point A_2 then we agree to give the name of *hyperbolic distance* between the points A_1, A_2 to the expression $\sqrt{\frac{1}{\rho_0}(p_1 - p_2)(\omega_2 - \omega_1)}$ (with the radical being given the + sign), and we denote it by the notation $\overline{A_1A_2}$.

Of course, this hyperbolic distance will be real, moreover, only if the quantities ω_1, ω_2 have an order whose magnitude is inverse to that of p_1 and p_2. However, this will always be the case if the two points considered belong to the curve $(\mathbf{2'})$.

Having said that, if we are given two states of a fluid, between which there exists a first-order discontinuity, then these two states are represented by the two points A_1, A_2 of the curve $(\mathbf{2'})$; the desired state will be represented by a third point A of the same curve. The differences $u - u_1$, $u - u_2$ will then have the hyperbolic distances $\overline{AA_1}$, $\overline{AA_2}$ for absolute values. If we always suppose that $\theta_1 < 0$, $\theta_2 > 0$ then u will be either external to u_1 and u_2 or between these two quantities, depending on whether p is between p_1 and p_2 or external to them. In the former case, the difference of the two given velocities u_1 and u_2 will be equal to the difference of $\overline{AA_1}$ and $\overline{AA_2}$, which are both less than $\overline{A_1A_2}$. In the latter case, $| u_1 - u_2|$ will be the sum of the distances $\overline{AA_1}$ and $\overline{AA_2}$, in which at least one of them is greater than $\overline{A_1A_2}$.

Therefore, *the first hypothesis necessarily corresponds to:*

$$| u_1 - u_2| < \overline{A_1A_2},$$

and the second one to:

$$| u_1 - u_2| > \overline{A_1A_2}.$$

Conversely, on the segment A_1A_2 on the curve $(\mathbf{2'})$ the difference $\overline{AA_1} - \overline{AA_2}$ will obviously take any value – positive or negative – that is less than $\overline{A_1A_2}$ in absolute value once and only once, and on the remaining arcs of that curve, the sum $\overline{AA_1} + \overline{AA_2}$ will take every value greater than $\overline{A_1A_2}$ once and only once.

The conic (**73**) is the locus of points such that the sum or difference of their hyperbolic distances to A_1 and A_2 has a given value; one might say that it has A_1 and A_2 for *hyperbolic foci*. They reduce to two straight lines when this given value is null or when it is equal to the distance $\overline{A_1A_2}$, just as when one considers ordinary distances instead of hyperbolic distances.

If the pressure p is outside of p_1 and p_2 then we know from no. **118** that it is necessarily greater than them when the given discontinuity is compressive, and that it is less than them when this discontinuity is dilative.

In the contrary case – i.e., the one where p is between p_1 and p_2 – the choice between the two points of intersection of the curve $(\mathbf{2'})$ with the conic (**73**) is made quite simply if one remarks that for $u_1 - u_2 > 0$, i.e. (no. **116**), if the discontinuity is compressive, the point A is *closer* to the point A_1 or A_2 that corresponds to the greater pressure; i.e., for $p > p_2$ the hyperbolic distance $\overline{AA_1}$ is smaller than the hyperbolic distance $\overline{AA_2}$ (because one then has $p < p_1$, $p_2 < p$; $u - u_1 = \overline{AA_1}$, $u - u_2 = \overline{AA_2}$). The contrary case is true for a dilative discontinuity.

Conversely, the point thus chosen will indeed satisfy the condition:

$$u_1 - u_2 = \pm\overline{AA_1} \pm \overline{AA_2},$$

in which the signs are precisely the ones that correspond to $\theta_1 < 0$, $\theta_2 > 0$.

215. – Nevertheless, one must note that the points A' and A'' definitely might not be the only points of intersection of the curve $(\mathbf{2'})$ and the conic (**73**). A' is indeed the one point of $(\mathbf{2'})$ that is situated on the arc C_1D_1; similarly, the point A'' is unique on the arc C_2D_2. However, there is nothing to say that there are no other points of intersection on the remaining arcs of the conic that correspond to velocities θ_1 and θ_2 in the same sense. It is likewise clear that such points will exist if, for example, the conic (**73**) is very close to the line A_1A_2.

What is more, it is obvious *a priori* that motions of this type must be produced. This is what happens when two first-order discontinuities proceed with velocities that are different, but have the same sense, and one overtakes the other.

If, to fix ideas, one supposes that the conic (**73**) is an ellipse then it is clear that the points of intersection may only be on the interior arc of C_1D_2 (*fig.* 15) that is situated above the line $A'A''$, and not on the arc that is above D_1C_2.

These new points, if they exist, will be at least two in number. One easily sees that they correspond to two intermediate motions for which the sense of the two velocities of propagation θ_1 and θ_2 is the same, as well as the order of magnitude of these two velocities (which amounts to saying that the two representative points are on the same side of the line C_1C_2). In both cases, the same vertex of our rectangle must be considered as representing the state of the region to the left.

The same considerations apply if, instead of looking for the states that immediately follow a given first-order discontinuity (without compatibility), one proposes to determine the states that immediately precede it. It is obvious that the region on the left must then correspond to the larger algebraic value of the velocity of propagation.

In particular, if, as we recently assumed, two discontinuities proceed in the same sense with different velocities and one overtakes the other, then the two new discontinuities that are created at this moment necessarily propagate in the contrary sense.

216. – We just found a case in which *several* possible motions that start from a given instant can correspond to a given state (position or velocity) of a fluid at that instant – at least theoretically.

Indeed, recall the motion that was envisioned in no. **201**. We saw that if the piston, after having attained a certain velocity u according to the law that was considered in that context, preserves this velocity, in turn, and exhibits a uniform motion then the representative surface of that motion is composed of two planar portions that are separated by a conical nappe, in such a way that up to a certain instant T there exist only two second-order discontinuities that reunite at the instant T into a first-order discontinuity.

Now, as one knows, the general equations of dynamics (in the absence of friction) possess the property of not changing under a change of t to $-t$; it is, moreover, easy to verify this fact for all of the equations that we just wrote.

Conversely, if we give ourselves a first-order discontinuity at the instant T that is defined precisely by the same elements as the ones that are established at that instant in the motion that we just discussed then we may assume that the ultimate motion is deduced from the one in no. **201** by changing t into $-t$. The first-order discontinuity is thereby resolved into a second-order discontinuity, as was indicated in no. **108**.

217. – One must nevertheless observe that the discontinuities that are susceptible to being resolved must therefore satisfy some very particular conditions. On the representative cone of motion that was studied in no. **201**, one has:

$$u + \chi(\omega) = \text{constant},$$

and, as a consequence, the quantity $u + \chi(\omega)$ must have the same value on either side of the discontinuity. It is clear that the same thing must be true each time that there exists a system of characteristics in a neighborhood of the conical point that intersect any regular line that issues from that point on the surface. Indeed, if one takes the partial differential equation in the form (**31**) then one sees that it admits the integral $\partial z/\partial u$ = constant, $\partial z/\partial \omega$ = constant, i.e., the *plane* that, after the Legendre transformation, corresponds to an arbitrary *point* of the space in which a, t, x play the role of coordinates. If one effects that same Legendre transformation on an integral surface at the a conical point then one will obviously have a transform that is tangent to the plane that corresponds to that conical point all along a line (since u and ω take an infinite of values at that point). That line will thus be a characteristic, and, as a consequence, it will be surrounded by infinitely close characteristics that satisfy the condition that we just spoke of.

It results from this, in particular, that for such a discontinuity the difference between the velocities is always less than the geometric mean of the dilatations, divided by ρ_0, and the pressure. Indeed, from the formulas (**7**), (**25**), this mean has the expression:

$$\sqrt{\frac{\omega_1 - \omega_2}{\rho_0}[\varphi(\omega_2) - \varphi(\omega_1)]} = \sqrt{(\omega_1 - \omega_2)\int_{\omega_1}^{\omega_2} \chi'^2(\omega) d\omega},$$

whereas the difference between the velocities is:

$$u_1 - u_2 = \chi(\omega_1) - \chi(\omega_2) = \int_{\omega_1}^{\omega_2} \chi'(\omega) d\omega.$$

The order of magnitude of the two quantities $u_1 - u_2$ and $\sqrt{\frac{1}{\rho_0}(\omega_1 - \omega_2)(p_2 - p_1)}$ is thus given by the Schwartz inequality (chap. I, no. **18**).

On the other hand, the relation:

$$\textbf{(75)} \qquad u_1 - u_2 = \chi(\omega_1) - \chi(\omega_2)$$

must be completed with an inequality condition. In the motion that was studied in no. **201**, the two second order discontinuities that exist *before* the instant T must combine at that instant into a compressive discontinuity of first order. On the contrary, if one follows that motion in the opposite sense, as we just indicated, then the discontinuity of the first order that exists at the instant T and doubles *after* that instant into two second order discontinuities will be dilating. It is easy to see that this is inevitably the case for any analogous doubling. It suffices to further remark that the sign of the discontinuities (no. **116** and **213**) depends upon that of the product $(u_1 - u_2)(\theta_1 - \theta_2)$. Now, the sign of $u_1 - u_2$ or, from equation (**75**), that of $\chi(\omega_1) - \chi(\omega_2)$ is the same as that of $\theta_1 - \theta_2$, since we assume that the most compressed waves are the ones that propagate the fastest.

One sees from this that a discontinuity of order one that was created by the combination of two discontinuities of second order may not subsequently double into two discontinuities of second order since for this to happen it must be dilating and not compressing.

218. – We have assumed that there exists the relation:

$$\textbf{(66)} \qquad p_1 \omega_1^m = p_2 \omega_2^m,$$

between the pressures and the densities on either side of the discontinuity.

If this is not the case then, since (under the hypotheses that we have presently subjected ourselves to) the product $p\omega^m$ may not change on either side, it is impossible that at the following instants this discontinuity can be completely referred to the abscissa $a + \theta\, dt$, where we let a denote the abscissa of this discontinuity at the instant t (measured in the initial state) and assume that θ is non-zero. Necessarily, it will remain at this place no matter what sort of discontinuity it originally was, since a brief variation of the pressure may not exist since, as we have seen, when a discontinuity propagates, it is the densities that must remain different. Nevertheless, one must obviously not forget that this reasoning is completely theoretical; in reality, it will be impossible to assume that there is no exchange of heat between the pieces in contact. Their temperatures, and, as a result, their densities will thus tend to equalize.

219. – We will encounter this stationary discontinuity that thereby joins with two others when the relation (**66**) is satisfied if we take into account Hugoniot's objection. Indeed, in this manner of looking at things, it is clear that relation (**66**) no longer actually implies the existence of a unique intermediate state. We must assume that two intermediate states are created between the two given motions, which are situated on either side of the original discontinuity and are characterized by a unique pressure and velocity p, u, but with two different dilatations ω', ω''. We must then write the compatibility conditions of the state (p, u, ω') with the state (p_1, u_1, ω_1) and the state (p, u, ω'') with (p_2, u_2, ω_2), namely:

$$
(\mathbf{76}) \qquad \begin{cases} \theta_1 = \dfrac{p_1 - p}{\rho_0(u_1 - u)} \\[2mm] \theta_1 = \dfrac{u_1 - u}{\omega' - \omega_1} \\[2mm] \dfrac{1}{2}(p + p_1)(u_1 - u) = \dfrac{\theta_1}{m-1}(p_1\omega_1 - p\omega') \end{cases}
$$

$$
(76') \qquad \begin{cases} \theta_2 = \dfrac{p_2 - p}{\rho_0(u_2 - u)} \\[2mm] \theta_2 = \dfrac{u_2 - u}{\omega'' - \omega_2} \\[2mm] \dfrac{1}{2}(p + p_2)(u_2 - u) = \dfrac{\theta_2}{m-1}(p_2\omega_2 - p\omega''). \end{cases}
$$

When p_1, u_1, p_2, u_2, ω_2 are given, these equations must allow us to calculate p, u, ω', ω'' θ_1, θ_2.

To that effect, we first eliminate ω' from the last two equations (**76**):

$$
(\mathbf{77}) \qquad \frac{m+1}{2}\,p + \frac{m-1}{2}\,p_1 = \frac{\theta_1(p_1 - p)\omega_1}{u_1 - u} = \rho_0\theta_1^2\omega_1
$$

and likewise:

$$(77') \qquad \frac{m+1}{2}p+\frac{m-1}{2}p_2=\frac{\theta_2(p_2-p)\omega_2}{u_2-u}=\rho_0\theta_2^2\omega_2 .$$

Here, it will be convenient to take the unknowns to be θ_1 and θ_2. The elimination of p and u from equations (**77**), (**77′**) gives us:

$$(\mathbf{78}) \qquad \rho_0(\theta_1^2\omega_1-\theta_2^2\omega_2)=\frac{m-1}{2}(p_1-p_2)$$

$$(\mathbf{79}) \qquad \frac{\rho_0\theta_1^2\omega_1-mp_1}{\theta_1}-\frac{\rho_0\theta_2^2\omega_2-mp_2}{\theta_2}=\rho_0\frac{m+1}{2}(u_2-u_1) .$$

We suppose that the desired propagation of motion comes about in an opposite sense from the given motion. If, to fix ideas, we further assume that the state that is denoted by the index 1 is the one in the left-hand region then we must have:

$$(\mathbf{80}) \qquad \theta_1 < 0, \ \theta_2 > 0 .$$

Now, if one considers either θ_1 or θ_2 to be the given then equation (**79**) is of second degree its roots are always real and have opposite signs. One easily concludes that if we now consider θ_1 and θ_2 to be Cartesian coordinates then the cubic that is represented by that equation is composed of an odd branch ([72]) on which θ_1 and θ_2 have the same sign (and that we have, consequently, left aside), and two branches H_1, H_2, that are analogous to a hyperbola that is asymptotic to the axes and are situated, in one case, inside the angle that is defined by the inequalities (**80**), and the other, inside the opposite angle. Since the curve admits no tangent that is parallel to the axes, the absolute values of θ_1 and θ_2 vary constantly in the same sense on the odd branch and constantly in the opposite sense on H_1 or H_2.

Now, since equation (**78**) represents a hyperbola the inequalities (**80**) are satisfied on half of the branches. On the arc thus determined, the absolute values of θ_1 and θ_2 vary constantly in the same sense. It results from this that this arc cuts each of the two branches H_1 and H_2 of the cubic at one and only one point, which give a unique solution to the problem. Sadly, it is necessary to add that the study of the case in which the moving discontinuities are on the same side of the stationary discontinuity (a case that may present itself in theory, from what we saw in no. **215**, but which we will not discuss here) may not be carried out with the aid of the same calculations as in the preceding, and that the equations that one must write will be noticeably different.

One may likewise specify the way that the intermediate pressure p is situated with respect to the two pressures p_1 and p_2. One will easily respond to that question by making the point (θ_1, θ_2) vary on the hyperbola (**78**) and calculating p from the (consistent) equations (**77**), (**77′**) ; it is clear that p is increasing with $|\theta_1|$ and $|\theta_2|$. In order

([72]) One knows that this is what one calls a branch of the curve that is cut by any line at an odd number of points.

to see this, it will suffice to substitute the points that correspond to $p = p_1$ and $p = p_2$ in equation (**79**).

220. – If the gas is originally at rest, suppose that one briefly communicates a uniform motion to the piston with a given velocity V. One may propose to determine the motion that will give rise to these conditions.

As was first shown by Sébert and Hugoniot ([73]), and then Hugoniot himself in the cited memoir, the compatibility equations that we previously established permit us to solve this problem very simply.

Indeed, we shall see that under either the Riemann hypothesis or that of Hugoniot there will exist a motion of the form:

(**81**) $$x = \omega a + Vt,$$

(ω constant) which will be compatible with the state of rest; of course, the velocity of propagation θ is constant. During this motion the gas will indeed remain in contact with the piston since one will have $x = Vt$ for $a = 0$.

Moreover, the quantity k that figures in formula (**2′**) will be likewise constant if we take into account the objection of Hugoniot, all the while having a value that is different from the one that corresponds to rest under these conditions. Indeed, k depends only upon the elements of the discontinuity; however, these elements are constant here.

Since k is constant, the partial differential equation will have the form (**8**) and will be, as a consequence, satisfied by the linear expression (**81**).

First, start with the Hugoniot formulas: if p_0 is the original pressure in the rest state and p is the unknown pressure that exists when the motion begins then the compatibility equations will be:

(**82**) $$V + \theta(\omega - 1) = 0 \qquad \text{(kinematic condition)}$$

(**83**) $$p - p_0 = \rho_0\, \theta V \qquad \text{(dynamic condition)}$$

(**84**) $$\frac{p}{p_0} = \frac{m+1-(m-1)\omega}{(m+1)\omega-(m-1)}.$$

At this point, we remark that the solution will be a little more obvious if the given is, as in the problem in no. **190**, the pressure p (which is assumed to be constant and different from p_0). One will then have ω from equation (**84**), or from the Poisson equation, if one remains in the Riemannian viewpoint, since the two equations (**82**), (**83**) are then solved exactly as in no. **206**.

Return to the problem posed, in which the given is V and no longer p.

We then take θ to be the unknown; the preceding equations give:

(**85**) $$\theta^2 - \frac{m+1}{2}\theta V - \frac{mp_0}{\rho_0} = 0\,.$$

([73]) Sébert and Hugoniot, *C. R. Acad. des Sci.*, tome XCVIII, pp. 507; 25 February 1884.

The choice of unknown q offers the advantage of permitting us to immediately decide between the two roots of the preceding equation. Indeed, they are of opposite sign and if, as we always assume, the gas is situated on the positive a side then it is only the positive root that applies, since the negative root corresponds to the analogous motion that is generated by the same motion of the piston in a gaseous mass that is at rest on the other side of it.

We will thus have:

$$\theta = \frac{m+1}{4}V + \sqrt{\left(\frac{m+1}{4}\right)^2 V^2 + \frac{mp_0}{\rho_0}}\,.$$

Nevertheless, a further condition is necessary in order for the solution obtained to apply to the problem: It is necessary that one have $p > 0$. This condition is always satisfied for $V > 0$; however, in the contrary case, i.e., if the piston has a decompressive motion, one must then have:

$$q < \frac{p_0}{\rho_0 V}\,,$$

which gives:

(86)
$$V^2 < \frac{2p_0}{(m-1)\rho_0}\,.$$

For much larger values of – V the gas ceases to follow the piston, exactly as we saw in no. **192**, except that when the limiting velocity is eventually attained, its expression is $V = \frac{2\lambda}{m-1} = \frac{2}{m-1}\sqrt{\frac{mp_0}{\rho_0}}$, a quantity that is greater than the one that is given by formula (**86**).

One must also remark that in the case of a velocity that is briefly communicated the pressure and the absolute temperature can become null without that also being the case for the density.

221. – If one continues to apply the ideas of Riemann without taking into account the objection Hugoniot then one must replace equation (**84**) with:

(66′)
$$p\,\omega^m = p_0.$$

As before, this will represent a curve that one must intersect with the hyperbola $(p - p_0)(1 - \omega) = \rho_0 V^2$ that results from the elimination of θ from equations (**82**),(**83**), or rather, with the branch of that curve that corresponds to $\theta > 0$. One will further find one and only one solution, the point of the curve (**66′**) whose hyperbolic distance to the point $(1, p_0)$ is V.

The question will arise in a completely analogous manner whether the gas, instead of being originally at rest, can be animated with a motion of the form (**81**), with a dilatation

ω_0 and a velocity V_0. One must then seek a point on a curve that is analogous to (**66**) that is situated at the hyperbolic distance $(V - V_0)$ from the point (ω_0, p_0).

222. – One may easily deduce from the foregoing a measure of the resistance with which the gas opposes the motion of the piston.

To that effect, suppose that it is first placed between two masses of gas that are at rest and are both homogeneous, the one situated on the positive a side, and the other on the negative a side. If, under these conditions, we instantaneously give it the positive velocity V then there will arise, as we just saw, two waves that propagate in contrary senses. The one, which corresponds to the positive root θ_1 of the equation, will be a compression wave; the other, which corresponds to the negative root of the same equation, will be a dilatation wave. The corresponding pressures p_1 and p_2 are calculated immediately with the aid of equation (**83**), and they become:

$$(\mathbf{87}) \qquad p_1 - p_2 = \rho_0\, V(\theta_1 - \theta_2) = 2\rho_0\, V \sqrt{\left(\frac{m+1}{4}\right)^2 V^2 + \frac{mp_2}{\rho_0}}\,.$$

This quantity represents the desired resistance, being the resultant of the pressures that are exercised on the two faces of the piston.

The expression (**87**) will be roughly proportional to the velocity for small values of that variable, and to the square of the velocity when its value is large. This law is precisely analogous to the one that we observed in the motion of projectiles, but with a somewhat slower increase ([74]). This discordance should come as no surprise, and it is likewise natural that it is produced in the sense that we just described since the piston in our tube moves without completely turning back into the gas, whereas free air, which must slide laterally, obviously diminishes the resistance.

Meanwhile, still remaining in the viewpoint of rectilinear motion, the preceding considerations suggest two observations:

First of all, they must be modified if the velocity V exceeds the limit (**86**). Indeed, a vacuum is then created on the rear face of the piston; as a consequence, the (negative) pressure p_2 must be replaced by 0. The resistance is then:

$$R = p_1 = p_0 + \rho_0\, V\left(\frac{m+1}{4}V + \sqrt{\left(\frac{m+1}{4}\right)^2 V^2 + \frac{p_0}{\rho_0}}\right).$$

In the second place, it is more natural to assume that the piston acquires the velocity V gradually, and not instantaneously. One must then apply the formulas of nos. **141** and **182**, and not the ones that we just used. One must then calculate ω by the formula (**54**) (no. **182**), and take $p = \varphi(\omega) = p_0\,\omega^{-m}$, which gives:

([74]) As we already said in no. **212**, the resistance seems to have essentially the value that it takes when there is no depression behind; i.e., if one has $p_1 = p_0$ (as well as $\theta_1 = V$).

$$p = p_0\left(1+\frac{m-1}{2}\frac{V}{\lambda}\right)^{\frac{2m}{m-1}},$$

$\lambda = \sqrt{\dfrac{mp_0}{\rho_0}}$ again denotes the velocity of sound in the original state.

The same calculation for the backwards wave gives:

$$p = p_0\left(1-\frac{m-1}{2}\frac{V}{\lambda}\right)^{\frac{2m}{m-1}};$$

hence:

$$R = p_0\left[\left(1+\frac{m-1}{2}\frac{V}{\lambda}\right)^{\frac{2m}{m-1}} - \left(1-\frac{m-1}{2}\frac{V}{\lambda}\right)^{\frac{2m}{m-1}}\right],$$

in which the subtracted term must furthermore be replaced by 0 when V exceeds the limit that was found in no. **192**.

The resistance thus calculated increases noticeably faster than the square of the velocity.

Nonetheless, the preceding reasoning cannot be accepted without objection. Indeed, it assumes that the singularity of Riemann-Hugoniot is not produced. Now, the contrary hypothesis is much more likely given the conditions under which we operate – for example, the motion of projectiles. Moreover, one must admit that it will give rise to two discontinuities of first order, one of which moves forward and the other of which moves backward. The latter, by reflection on the piston, will produce a new wave with a positive velocity, which will propagate faster than the first one ([75]) and recapture it. At that moment, two new waves will be produced, and so on.

Hugoniot assumed that this exchange of waves finally succeeds in establishing a state that is identical with the one that is produced when the velocity V is communicated to the piston at once. Later on, we shall confirm, in a particular case, that things actually happen this way.

223. – We shall now begin the actual discussion of the Riemann-Hugoniot phenomenon.

We suppose, to simplify, that the gas is at rest in its primitive state, that the head of the wave is the first to present the singularity in question, and that the motion that is communicated by the piston to the part of that fluid that is nearest to it (which we assume is situated to the left) is analytic. First of all, we shall deduce the equation for this motion. To that effect, one must, as one knows, eliminate t_0 from equations (**59**) and (**60**).

The edge of regression of the developable surface thus obtained is defined (no. **194**) by the equation:

([75]) See below, no. **238**.

$$\frac{\partial a}{\partial t_0} = (t - t_0)\chi''(\omega_0)\frac{d\omega_0}{dt_0} - \chi'(\omega_0) = 0\,,$$

which is soluble in terms of t_0 in general (and we will not treat the exceptional case for which things are otherwise). Let t_0' be the function of t that, when substituted for t_0, satisfies the preceding equation. In equations (**59**) and (**60**), where t_0 is not equal to t_0', since one is no longer on the edge of regression, we set:

$$t_0 = t_0' + \tau.$$

a and x become functions of t and τ that, when ordered into powers of the latter variable, lack terms of first degree; thus:

(**88**) $$a = a_0 + a_2\,\tau^2 + a_3\,\tau^3 + \dots$$

(**89**) $$x = X_0 + x_2\,\tau^2 + x_3\,\tau^3 + \dots$$

a_0, X_0, a_2, a_3, …; x_2, x_3, … being functions of t such that the first two $a = a_0(t)$ and $x = X_0(t)$ give the equations of the edge of regression. All of these functions are, moreover, analytic.

Equation (**88**) permits us to develop t in powers of $\sqrt{a_0 - a}$, as long as a_2 is non-null at the origin, a hypothesis that we again distinguish ([76]).

If one substitutes this development in (**89**) then one obtains the value of x that corresponds to motion to the left. We denote this value by X. One will have:

(**90**) $$X = X_0 + (a_0 - a)\,X_1 + (a_0 - a)^{3/2}\,X_{3/2} + \dots$$

(if we let X_1, $X_{3/2}$, … denote analytic functions of t).

224. – We suppose that the origins of space and time have been transported to the place and instant at which the phenomenon originates. With these conditions, X_0 and a_0 are null with t; they begin with terms in λt, if we let λ denote the velocity of sound that corresponds to the primitive state of the fluid. Moreover, since the surface is tangent to the plane $x = a$, one has $X_1(0) = -1$.

We agree that the radical $(a_0 - a)^{1/2}$ in the preceding equation is taken to mean its positive value. If this is the case then the coefficient $X_{3/2}(0)$ must be positive. Indeed, since the motion of the piston is compressive one must have $X > a$, and this can be so for very small t and order up to at most the order of $a_0 - a$ only if $X_{3/2}(0) > 0$.

([76]) If the coefficient a_2 is different from zero then the same is true for x_2. This is because for $\frac{\partial a}{\partial t_0} = 0$ the quantity $\frac{\partial^2 x}{\partial t_0^2} = 2x^2$ is equal to $\omega_0 \frac{\partial^2 a}{\partial t_0^2}$, by virtue of the identity $\frac{\partial x}{\partial t_0} = \omega_0 \frac{\partial a}{\partial t_0}$. The coefficients a_2, x_2 are, moreover, negative in the case at hand, since the surface is situated to the left of the edge of regression.

225. – We shall now obtain the intermediate equation of motion that arises between the motion that was just defined and the part on the right that is at rest. We may, moreover, do this without determining, at the same stroke, the evolution of the two waves that propagate it. In other words, just like the equation of motion, it must be found in the domain in which it is defined.

This is the difficulty that was pointed out in no. **168**. However, it is particularly grave at this point. In the other questions of mechanics in which the desired motion is not represented by only one analytic equation in the entire body considered, the regions in which this motion has different expressions are generally known *a priori*. For example, such is the case for a wave *of second order* that propagates in a gas whose anterior motion is given, when *this motion appears only* in the expression for the velocity of propagation. As we have just seen, this is not the case in the present situation.

To simplify, we treat it without taking into account the objection of Hugoniot. We assume that when the discontinuity of first order is created it establishes a unique pressure, density, and velocity in the intermediate slice. It is then easy to see that the values of this pressure, density, and velocity may not be anything but the values that exist in the slice on the right (as a consequence, $u = 0$, $\omega = 1$), and that initially one has the same values in the slice to the left ([77]).

We must then determine:

1. The abscissa a_1 of the discontinuity between the desired motion and the motion to the left.

2. The abscissa a_2 of the discontinuity between the same motion and the part on the right that is at rest.

Since the two discontinuity waves propagate with an initial velocity that is equal to the velocity λ of sound that was introduced in no. **175**, a_1 and a_2 will have developments that begin with terms in $\pm \lambda t$; we write:

(**91**) $$a_1 = -\lambda t - \nu_{3/2}\, t^{3/2} - \nu_2\, t^2 \ldots$$

and:

(**92**) $$a_2 = \lambda t + \mu_2\, t^2 \ldots$$

by assuming in advance ([78]) (as the following calculations will verify) that a_2 contains only terms with fractional exponents in t.

3. The equation of motion of the intermediary slice.

The equations that these various unknowns will be, first, the partial differential equation:

([77]) Let p_0 be the original pressure, p, the pressure, and let u be the velocity that exists at the first moment in the intermediary slice. One must have both:

$$\frac{p - p_0}{u} = \theta_1\,, \qquad \frac{p - p_0}{u} = \theta_2\,,$$

where θ_1 and θ_2 denote the velocities of propagation of the wave.

However, this may be the case only if $p = p_0$, $u = 0$.

([78]) It is clear that we must first of all leave the exponents of t indeterminate, as well as the coefficients. The sequence of calculations will give the same values for these coefficients as the ones that we have assigned them here.

(**8**) $$\frac{\partial^2 x}{\partial t^2} = \psi\left(\frac{\partial x}{\partial a}\right)\frac{\partial^2 x}{\partial a^2},$$

which must be true in the entire intermediary slice.

Since the function ψ is given by relation (**41**), one will have, on setting:

(**93**) $$\frac{\partial x}{\partial a} = \omega = 1 + \varepsilon,$$

and, on taking into account the formula (**42**) that defines λ:

(**94**) $$\psi(1 + \varepsilon) = \lambda^2\left[1-(m+1)\varepsilon+\frac{(m+1)(m+2)}{2}\varepsilon^2+\ldots\right]$$

In the second place, one must have:

(**95**) $x = a$, for $a = a_2$

(**96**) $x = X = X_0 + (a_0 - a)\, X_1 + (a_0 - a)^{3/2}\, X_{3/2} + \ldots$ for $a = a_1$.

Moreover, the two discontinuities a_1 and a_2 must satisfy the compatibility conditions. We do not need to write the kinematical conditions, which are implicitly contained in conditions (**95**), (**96**).

The dynamical and physical conditions give (since we have made the Riemann hypothesis):

$$\theta = \sqrt{\frac{\varphi(\omega)-\varphi(\omega_1)}{\rho_0(\omega_1-\omega)}} = \pm\,\lambda\,\sqrt{\frac{1}{m}\frac{(1+\varepsilon)^{-m}-(1+\varepsilon_1)^{-m}}{\varepsilon_1-\varepsilon}}$$

(on setting $\omega = 1 + \varepsilon$, $\omega_1 = 1 + \varepsilon_1$) or:

(**97**) $$\theta = \pm\,\lambda\left[1-\frac{m+1}{4}(\varepsilon+\varepsilon_1)+\ldots\right].$$

ε_1 is null in the part that is at rest. On the contrary, in the motion to the left it has a value that is generally different from 0 and which must be calculated using equation (**90**).

One thus has the two supplementary conditions:

(**98**) $$\frac{da_1}{dt} = -\,\lambda\left[1-\frac{m+1}{4}(\varepsilon+\varepsilon_1)+\ldots\right]$$

(**99**) $$\frac{da_2}{dt} = \lambda\left[1-\frac{m+1}{4}\varepsilon+\ldots\right]$$

in which we intend that $a = a_2$ in equation (**99**), whereas, in equation (**98**) ε and ε_1 correspond to $a = a_1$.

226. – In order to develop x into a series, we introduce, in place of a and t, the variables:

$$\begin{cases} \xi = a + \lambda t, \\ \eta = \alpha - a, \end{cases} \tag{100}$$

in which:

$$\alpha = \lambda\, t + (M_2 + \mu_2)\, t^2 + \dots = a_2 + M_2\, t^2 + M_2\, t^3 + \dots \tag{101}$$

denotes a development in unknown coefficients (except for the first one) in powers of t. Moreover, the variable ξ is introduced only in order to simply the calculations. As we have seen, the same is not true for the variable η, which plays a fundamental role in the development. We write:

$$x = a + F_{\frac{3}{2}} + F_2 + \dots = a + F\,, \tag{102}$$

in which the F_i are all homogeneous in ξ, η and of degree indicated by their indices.

Since one has:

$$\frac{\partial}{\partial a} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}, \qquad \frac{\partial^2}{\partial a^2} = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^2,$$
$$\frac{\partial^2}{\partial t^2} = \lambda^2 \frac{\partial^2}{\partial \xi^2} + 2\lambda\alpha' \frac{\partial^2}{\partial \xi \partial \eta} + \alpha'^2 \frac{\partial^2}{\partial \eta^2} + \alpha'' \frac{\partial}{\partial \eta}$$

(α', α'' denoting the first two derivatives of α with respect to t), equation (**8**) may be written:

$$\begin{cases} 4\lambda^2 \dfrac{\partial^2 F}{\partial \xi \partial \eta} = \left(\dfrac{\partial^2 F}{\partial \xi^2} - 2\dfrac{\partial^2 F}{\partial \xi \partial \eta} + \dfrac{\partial^2 F}{\partial \eta^2}\right)\left[\psi\left(1 + \dfrac{\partial F}{\partial \xi} - \dfrac{\partial F}{\partial \eta}\right) - \lambda^2\right] \\ \qquad -2\lambda(\alpha' - \lambda)\dfrac{\partial^2 F}{\partial \xi \partial \eta} - (\alpha'^2 - \lambda^2)\dfrac{\partial^2 F}{\partial \eta^2} - \alpha'' \dfrac{\partial F}{\partial \eta}. \end{cases} \tag{103}$$

In this equation, α', α'' may be replaced by their developments in t. However, they may likewise be developed in powers of the variable $\xi + \eta$, as a function of which, t may be expressed by means of solving the equation:

$$\xi + \eta = \alpha + \lambda t = 2\lambda t + (M_2 + \mu_2)\, t^2 + \dots + (M_h + \mu_h)\, t^h + \dots \tag{104}$$

In equation (**103**), thus written, an arbitrary term in the development of F (provided that it contains both ξ and η) will give a term on the left-hand side that is of degree at least as high as the one on the right-hand side.

We denote the values of ξ, η that correspond to $a = a_1$ by ξ_1 and η_1, namely:

(**105**)
$$\begin{cases} \xi_1 = -\nu_{\frac{3}{2}} t^{\frac{3}{2}} - \nu_2 t^2 \cdots \\ \eta_1 = 2\lambda t + \nu_{\frac{3}{2}} t^{\frac{3}{2}} + (\nu_2 + \mu_2) t^2 + \cdots, \end{cases}$$

and the values of the same variable for $a = a_2$ by ξ_2 and η_2, by:

(**106**) $\xi_2 = 2\lambda t + \mu_2\, t^2 + \ldots + \mu_h\, t^h + \ldots$
(**106′**) $\eta_2 = M_2\, t^2 + \ldots + M_h\, t^h + \ldots$

Equation (**96**) will thus be written:

(**107**)
$$F(\xi_1, \eta_1) = X_0 - a_0 + (a_0 - a_1)\,(1 + X_1) + (a_0 - a_1)^{\frac{3}{2}} X_{\frac{3}{2}} + \ldots$$

Equation (**95**) becomes:
(**108**)
$$F(\xi_2, \eta_2) = 0\,,$$

while (**98**) and (**99**) become:

(**109**)
$$\begin{cases} -\lambda - \dfrac{3}{2}\nu_{\frac{3}{2}} t^{\frac{1}{2}} - 2\nu_2 t - \cdots \\ = -\lambda\left[1 - \dfrac{m+1}{4}\left(\dfrac{\partial F}{\partial \xi_1} - \dfrac{\partial F}{\partial \eta_1} - X_1 - \dfrac{3}{2}(a_0 - a_1)^{\frac{1}{2}} X_{\frac{3}{2}} - \cdots\right) + \cdots\right] \end{cases}$$

(**110**)
$$\lambda + 2\mu_2 t + \ldots = \lambda\left[1 - \frac{m+1}{4}\left(\frac{\partial F}{\partial \xi_1} - \frac{\partial F}{\partial \eta_1}\right) + \cdots\right].$$

227. – Having said this, consider the terms of order – 1/2 in equation (**103**). They will be provided exclusively by the term $F_{\frac{3}{2}}$ in the development of F. One must therefore have $\dfrac{\partial^2 F_{\frac{3}{2}}}{\partial\xi\partial\eta} = 0$; hence:

(**111**)
$$F_{\frac{3}{2}} = K\eta^{\frac{3}{2}} + K'\xi^{\frac{3}{2}}.$$

The coefficients K and K' will be determined by the boundary conditions (**107**) and (**108**). First of all, for $a = a_2$, η is of order at least t^2, whereas ξ is of order t. Furthermore, condition (**108**) shows that K' must be null.

228. – On the contrary, for $a = a_1$, the quantity $a_0 - a$ has the principal part $2\lambda t$, and the same is true for η. A comparison of the terms of order 3/2 in x and X thus gives:

$$K = \pm X_{\frac{3}{2}}(0)\,.$$

We thus see that K is non-zero, in general. We will thus have a term in $\eta^{1/2}$, and, consequently, an edge of regression in the representative surface that corresponds to $\eta = 0$. One of the nappes that are separated by this edge must therefore be a subset of the region that was used (otherwise, as before, one will find two values of x for the same system of values for a and t); we agree that it is the one that is obtained by giving $\sqrt{\eta}$ its positive value.

If this is the case then the radical $\sqrt{\eta} = \sqrt{2\lambda t + \cdots}$ in condition (**107**) must thus receive its positive determination. Since the same thing is true for $\sqrt{a_0 - a}$, by virtue of the convention that was made in no. **224**, we must therefore write:

$$K = X_{\frac{3}{2}}(0)\,,$$

a positive quantity, as we remarked above.

229. – Now imagine equations (**98**) and (**99**) when one retains only terms of order 1/2. None of these terms exist in the quantity ε for $a = a_2$, since η_2 is of order higher than 1 in t. Therefore, they no longer exist in the left-hand side of equation (**99**), and consequently we see that a_2 indeed contains no term in $t^{3/2}$.

For $a = a_2$, some terms of order 1/2 appear in ε and ε_1; these terms are known, moreover. Indeed, we know the right-hand side of (**102**) up to terms of order 3/2, inclusive, and, on the other hand, the first part of the development of $\partial X / \partial a$ that depends upon $\nu_{3/2}$ (namely, the one that provides $(a_1 - a_0)^{3/2} X_{3/2}$) contains that quantity as a coefficient of the first power (at least) of t.

One confirms, moreover, that the terms in $t^{1/2}$ are destroyed in $\varepsilon + \varepsilon_1$ in such a way that one has $\nu_{3/2} = 0$.

230. – The determination of the terms of order 2 is completely analogous. Equation (**103**) thus gives:

$$4\lambda^2 \frac{\partial^2 F_2}{\partial\xi\,\partial\eta} = \frac{9}{8}(m + 1)\,\lambda^2 K^2,$$

because the only term of order zero that exists in the right-hand side of this equation is obtained by multiplying the factor $3/4\eta^{-1/2}$ (which is provided by $\dfrac{\partial^2 F}{\partial\eta^2}$) by $3/2(m + 1)$ (which is provided by $\psi\left(1 + \dfrac{\partial F}{\partial\xi} - \dfrac{\partial F}{\partial\eta}\right) - \lambda^2$). One will thus have:

$$F_2 = \frac{9}{32}(m+1)\, K_2\, \xi\eta + K_2\, \eta^2 + K_2'\xi^2\,,$$

in which K_2 and K_2' are coefficients to be determined. Equation (**108**) will give $K_2' = 0$, since the terms are all of order at least 3. Equation (**107**) will be composed of K_2 by examining terms of order 2 in t. Therefore, equation (**109**) determines ν_2.

On the contrary, condition (**110**) does not suffice to determine μ_2. Indeed, the terms in t contain the arbitrary coefficient M_2, which has played no role up to now, and which is introduced by the term $\dfrac{\partial F}{\partial \eta_2} = \frac{3}{2} K \eta_2^{\frac{1}{2}} + \dots$

Let:

(**112**) $$m_1 t + m_2 t^2 + \dots + m_h t^h + \dots$$

denote the *positive* square root in the development $\eta_2 = M_2\, t^2 + M_3\, t^3 + \dots$, in such a way that m_1 is the positive square root of M_2. The development (**112**) must then be substituted for $\eta_2^{\frac{1}{2}}$ in equation (**110**); we will obtain:

(**113**) $$\mu_2 = \frac{3\lambda(m+1)K}{16} m_1 + \frac{9}{128}\lambda^2 (m+1)^2 K^2 .$$

231. – We must therefore find a second relation between μ_2 and m_1; it will result from considering terms of order 5/2.

Consider the terms of order 1/2 in equation (**103**). Some of them are terms in $\eta^{1/2}$, but two other ones are terms in $\xi\eta^{-1/2}$; they are provided, on the one hand, by the product of $\dfrac{\partial^2 F}{\partial \eta^2} = \frac{3}{4}\eta^{-\frac{1}{2}} + \dots$ with:

$$\psi\left(1 + \frac{\partial F}{\partial a}\right) - \lambda^2 = \lambda^2 \left[-(m+1)\left(\frac{\partial F}{\partial \xi} - \frac{\partial F}{\partial \eta}\right) + \cdots \right]$$
$$= \frac{3}{2}\lambda^2 (m+1)\eta^{\frac{1}{2}} + \frac{9}{32}(m+1)^2 K^2 \xi + \dots$$

and with:

$$4\,\lambda\,(M_2 + \mu_2)\, t = 2\,(M_2 + \mu_2)\,(\xi + \eta) + \dots$$

on the other hand.

After one integration over ξ and η, these terms will contain only a factor of η with the power 1/2.

Now, that circumstance renders formula (**113**) inexact; indeed, the quantity $\dfrac{\partial F}{\partial a} = \dfrac{\partial F}{\partial \xi} - \dfrac{\partial F}{\partial \eta}$ will contain a term in $\xi^2 \eta^{-\frac{1}{2}}$ that will be of order 1 in t for $a = a_2$, since η is of order 2.

It is this inconvenience that we shall avoid by disposing of the arbitrary coefficient $M_2 = m_1^2$ in such a manner that it annuls the term in $\eta^{-\frac{1}{2}}$. We thus write:

$$(\mathbf{114}) \qquad 2(M_2 + \mu_2) = 2(m_1^2 + \mu_2) = \frac{9}{32}(m+1)^2 K^2 \lambda^2 ,$$

which, when combined with relation (**113**), permits us to determine m_1 and μ_2, this time. Upon eliminating the latter, it becomes:

$$m_1^2 + \frac{3}{16}\lambda(m+1)Km_1 - \frac{9}{128}(m+1)^2 K^2 \lambda^2 = 0 .$$

We know that we must take the positive root of this equation; we will thus have:

$$m_1 = \frac{3\lambda(m+1)K}{16}$$
$$\mu_2 = \frac{27}{256}\lambda^2 (m+1)^2 K^2 .$$

232. – Having thus calculated the first terms of our unknowns, we shall show, in a general manner, how one obtains the following ones:

Suppose that one knows:

The development of F as a function of ξ, η up to terms of order q, inclusive (q being either an integer or an integer + 1/2);

The development of a_1 as a function of t up to the same order;

The developments of α and a_2 only to order $q - 1/2$, and the knowledge of the development of $\alpha - a_2 = \eta_2$, equivalent to that of the development (**112**) up to terms in $t^{q-\frac{3}{2}}$.

We suppose moreover that:

1. The known part of the development of F contains no contribution from η with the exponent 1/2.

2. The quantity η is the only one that appears with fractional exponents, and that it does not enter into any of the known parts of the developments of α and a_2.

Under these conditions, we shall determine the terms of order $q + 1/2$ for F and a_1, and the terms of order q for α and a_2.

In the right-hand side of (**103**), all of the terms of order $q - 3/2$ are known, except for the ones that may be provided from the product of:

$$(M + \mu_q)\, t^{q-1} = (M + \mu_q)\left(\frac{\xi + \eta}{2\lambda} + \cdots\right)^{q-1}$$

(a quantity that comprises part of the development of α') with $\eta^{-\frac{1}{2}}$.

However, among those terms there is one that is in terms of $\xi^{q-1}\eta^{-\frac{1}{2}}$, with the coefficient $\frac{M_q+\mu_q}{(2\lambda)^{q-1}}$. We determine $M_q + m_q$ by the condition that this term destroys the similar term provided by $\frac{\partial^2 F}{\partial\eta^2}\psi\left(1+\frac{\partial F}{\partial\xi}-\frac{\partial F}{\partial\eta}\right)$. Moreover, it will give $M_q + m_q = 0$ when q is not an integer, since we suppose that the terms that are already known contain no fractional powers of ξ.

When $M_q + m_q$ is known, we know $\frac{\partial^2 F_{q+\frac{1}{2}}}{\partial\xi\,\partial\eta}$, and, as a consequence, F_q + 1/2 itself up to terms, one of which is in $\xi^{q+\frac{1}{2}}$ and the other of which is in $\eta^{q+\frac{1}{2}}$. The first of them will be determined by equation (**108**) and the second one, by equation (**107**); indeed, they give the only two terms in these equations that are further unknown ([79]) in $t^{q+\frac{1}{2}}$, respectively.

By means of these results, one knows, in the right-hand side of equation (**109**), all of the terms in $t^{q-\frac{1}{2}}$, and one has, consequently, the coefficient $\nu_{q+\frac{1}{2}}$.

In equation (**99**), one likewise knows all of the coefficients of t^{q-1}, except for the coefficient $q\mu_q$ on the left-hand side and the coefficient $\lambda\frac{m+1}{4}\frac{3K}{2}m_{q-1}$, which is provided in the right-hand side by the development of:

$$\lambda\frac{m+1}{4}\frac{\partial F}{\partial\eta_2}=\lambda\frac{m+1}{4}\left(\frac{3K}{2}\eta_2^{\frac{1}{2}}+\cdots\right).$$

One thus has the difference $q\mu_q - \frac{3\lambda K(m+1)}{8}m_{q-1}$. On the other hand, μ_q and m_{q-1} are known, since we have obtained $M_q + \mu_q$ (i.e., up to known terms, $2m_1m_{q-1} + \mu_q$). They are, moreover, null for non-integer q, since the calculations made in the right-hand side of equation (**110**) do not introduce fractional powers of t.

233. – We may thus indeed calculate all of the desired coefficients term by term, and we will have developments that formally satisfy all of the conditions of the problem. It will remain to be proved that these developments converge. However, this proof will be very difficult, if not completely impracticable, if we place ourselves at the viewpoint that we have adopted. In reality, it is only by putting the question into a completely different form that we will be able to treat it.

([79]) In the term $(a_0 - a_1)^4$ of the development of X, the term $\nu_q+\frac{1}{2}t^{q+\frac{1}{2}}$ of the development of a_1 gives a term in $t^{q+\frac{1}{2}+h-1}$. On the other hand, for $h = 1$, this term is multiplied by $1 + X$, which has no constant term.

As we have remarked, the development of x, when ordered in powers of ξ and $\sqrt{\eta}$ (to the exclusion of terms of first degree in $\sqrt{\eta}$) represents, when one assumes its convergence to have been proved, a surface that is an edge of regression. It is easy to see that any second order partial differential equation of the form (**17**) admits integral surfaces of this type. Indeed, in order to obtain one, it suffices to treat the Cauchy problem under the condition that the relation (**21**) is satisfied, but not the relation (**22**).

The considerations that were developed above (no. **159**) indeed show that the second derivatives are then infinite. Moreover, if one performs a change of variables in such a manner that the curve γ becomes the x-axis then it is easy to insure, at least formally, that z admits a development in powers of x and $\sqrt{y}$. More generally, suppose that condition (**21**) is satisfied at a *point* of the curve γ (to the exclusion of (**22**)). A calculation that is completely analogous to the one that was just carried out will provide a formal development of z that represents an edge of regression surface (this edge being tangent to γ at the point considered).

Finally, there is nothing to suggest that the developments thus obtained are convergent. One recognizes that the contrary is true if one effects a contact transformation. For example, perform a Legendre transformation. We must replace x, y, z, p, q by $p, q, px + qy - z, x, y$, and A, B, B', C, D with $D, B', B - C, A$. After this transformation, relation (**21**) will cease to be valid, unless one originally has, in addition, the following relation:

$$D(dp\,dx + dq\,dy) + B'\,dq^2 + 2C\,dp\,dq + B\,dp^2 = 0\,. \tag{115}$$

It is obvious *a priori* that this second relation is satisfied if one has (**22**), since the system of two equations (**21**) and (**22**) is invariant under a contact transformation. In order to verify this fact, it suffices to multiply equation (**21**) by $dp\,dq$, and equation (**115**) by $dx\,dy$, and add them. The relation that is obtained decomposes into equation (**22**) and the following one:

$$dp\,dx + dq\,dy = 0\,. \tag{116}$$

We exclude the case in which the relation (**22**) is satisfied; it is possible that one might then be dealing with a characteristic. The Legendre transformation will thus make the singularity disappear, except in the case in which one has (**116**). The transformed problem will have a regular solution, the representative surface of that solution having only the character of a developable surface at each of its originally singular points, i.e., it satisfies the condition $rt - s^2 = 0$ at each of its points, which is easy to insure.

Upon returning to the old system of variables, the singularity considered results from the formulas of no. **163**, which make known the effect of the transformation on the derivatives r, s, and t. An elementary calculation that is, moreover, completely analogous to the one that was carried out in no. **163** shows that this singularity is an edge of regression (around which the surface is represented by an equation that is analogous to (**90**)) that corresponds to the line that is the locus of the parabolic points under the Legendre transform.

What remains is the case in which one has (**116**). The Legendre transformation will not make the singularity disappear then. This is what we arrive at if the desired surface has the character of a developable surface in a neighborhood of its edge of regression. One may always avoid this circumstance by performing, at the outset, the transformation that consists of replacing the unknown function z by $z - F(x, y)$, F being an arbitrary function. p and q are then reduced by its derivatives, derivatives that one may obviously dispose of in such a manner that relation (**116**) ceases to be valid on γ. One thus sees that in all of the cases in which one has (**21**), but not (**22**), the Cauchy problem has a solution that is represented by a surface that is an edge of regression. It is, moreover, clear that conversely, any integral surface that has an edge of regression may be considered to be obtained in that fashion; it may be changed into a regular surface by a convenient contact transformation.

This will therefore be the case for the surface that we have used all along in order to develop the equation. The best method for studying that surface seems to be that of performing a contact transformation such that the surface (**90**) and the desired surface are replaced by regular surfaces. The question, thus transformed, will then be of the kind that one might apply the method of majorizing functions to. However, a new analysis will be necessary to that effect, since this question has not appeared in any of the problems that we have treated up till now. It leads to further generalizations by starting with the following problems, which comprise all of the particular cases and whose study offers considerable interest in its own right:

Given five partial differential equations:

$$F = 0,\ \ f_1 = 0,\ \ f_2 = 0,\ \ f_3 = 0,\ \ f_4 = 0\ ,$$

find an integral surface of first equation on which there exists a line l on which one has both $f_1 = 0,\ f_2 = 0$, *and a line* l' *on which one has both* $f_3 = 0,\ f_4 = 0$ (these givens being assumed to be such the various conditions must be satisfied together at the origin, through which the lines $l,\ l'$ must pass).

In a word, one does not know any line here through which the desired surface must pass; one only knows that along its (unknown) intersection with the surface (**90**) the singular coefficients of its tangent plane must satisfy equation (**98**) and that an analogous equation must be valid on its intersection with the surface $x = a$.

234. – Without stopping to examine whether one may define the contact transformation in such a manner that these conditions become *pointlike,* in such a way that it results in the knowledge of two lines that are situated on the transformed surface, we remark that the question may be posed in a somewhat different fashion if it is not on the first wave that the phenomenon is produced to begin with, which might be the case, for example, if one commences by giving the piston a negative acceleration in order to change the sign of the acceleration later on. In this case, the edge of regression of the surface that represents the motion to the left will have a point of regression, in such a way that the development (**90**) and the desired development must be consequently modified.

It is likewise clear that the question becomes notably more complicated when one takes into account the objection of Hugoniot. Indeed, not only will one have two new surfaces to find, and not just one, since a stationary discontinuity will be established at the point of origin for the phenomenon that affects the dilatations, but, as we said in no. **211**, none of these surfaces will satisfy the partial differential equations (**8**). This equation will be replaced by an equation of the form (**6**) in which the value of *k* will be not only a function of *a*, but an unknown function of that quantity, a function whose form will depend on various quantities that figure in equations (**91**) and the following ones.

By contrast, while remaining at the Riemannian viewpoint, note that one might hope to simplify the question by giving *m* the value 1.4, for which (no. **175**), equation (**8**) may be explicitly integrated.

235. – From the preceding, the Riemann-Hugoniot phenomenon gives rise to two waves that propagate in opposite senses. As we have already remarked (no. **222**), by reflecting two waves that propagate with different velocities from the piston and recombining them one thus obtains any of a series of new states for the fluid. Must one, with Hugoniot, assume that all of these states tend to a common limiting state, viz., the one that one obtains by briefly communicating the velocity *V* to the piston, which it acquires in reality by a gradual acceleration?

Obviously, one may answer this question in a general fashion only by first making a profound study of the first motion that gives rise to the Riemann-Hugoniot phenomenon as a result, which the foregoing method does not permit us to do. We thus content ourselves by responding to the question in a case in which this prior study has been done completely, the one that was considered in no. **202**, and in which the acceleration law is such that all of the successive waves that are born by contact with the piston are recaptured at the same point. Moreover, we do not take the objection of Hugoniot into account, and assume that Poisson's law is always applicable.

Under these conditions, we know that at the instant *T* at which the waves recombine, a first order singularity is formed. If, after having attained the velocity *V* by accelerating its motion according to the law that was indicated in no. **202**, the piston moves uniformly with that velocity then the motions between which the discontinuity exists will both be represented by equations of the form (**81**) (ω being calculated by equation (**54**) for the motion on the right, and equal to 1 for the motion on the left). At the same time, one may assume that the intermediate motion obeys an equation of the same form, with a velocity u_1, a dilatation ϖ_1, and a pressure *q*, which is obtained as was indicated in no. **213-214**.

As was confirmed above (no. **217**), the pressure q_1 will consist of the pressure p_1 of the motion on the left and the original pressure p_0. On the contrary, u_1 will be not only positive, but greater than *V*. Since the intermediate state of the fluid will be represented by a point of the curve $(\mathbf{2'})$ – a point that we denote, to abbreviate, by the letter q_1 that represents the pressure – which will be between the point p_0 that corresponds to the rest state and the point p_1 that corresponds to the motion to the left, the point q_1 (*fig.* 17) will be, moreover, determined by the equation:

$$\overline{q_1 p_0} - \overline{q_0 p_1} = V$$

in which $\overline{q_1 p_0}, \overline{q_0 p_1}$ denote the *hyperbolic distances* that were defined in no. **214**.

When the retrograde wave, by which the state (q_1, ϖ_1) propagates into the state (p_1, ω_1), reaches the piston, it gives rise, by reflection, to a new state (p_2, ω_2), which is defined by the double condition of being compatible with the first intermediate state and corresponding to a velocity that equals V. The propagation velocity must be positive, one sees, as it was pointed out in no. **221** that the pressure p_2 is less that q_1 and that, on the other hand, the hyperbolic distance $\overline{q_1 p_2}$ is equal to $u_2 - V$, i.e., to $\overline{q_1 p_1}$.

One sees that this point p_2 may be considered to be, in a certain sense, the *symmetry* of p_1 with respect to q_1, in such a way that the reflection translates into a certain reversal in the pressure differences here.

236. – Let P be the point of the curve $(\mathbf{2}')$ such that $\overline{Pp_0}$ is equal to V, the being pressure P being assumed to greater than p_0.

I say that p_2 is less than P.

This results from the following lemma that relates to hyperbolic distances:

Let $p_1\, p_2\, p_3$ be a triangle such that the hyperbolic lengths of all three of its sides are real. Then *the greatest of these lengths will be greater than or equal to the sum of the other two,* the equality being valid only if the three points are in a straight line.

Indeed, suppose, to fix ideas, that $p_1 > p_2 > p_3$, and, consequently, that $\omega_1 < \omega_2 < \omega_3$, in such a way that that the greatest of the three hyperbolic lengths will be $\overline{p_1 p_3} = \sqrt{\frac{1}{\rho_0}(p_1 - p_3)(\omega_3 - \omega_1)}$. The inequality to be proved can then be written (after taking the square):

$$\frac{1}{\rho_0}\,[(p_1 - p_2) + (p_2 - p_3)]\,[(\omega_1 - \omega_2) + (\omega_2 - \omega_3)]$$
$$> \frac{1}{\rho_0}\left(\sqrt{p_1 - p_2}\sqrt{\omega_2 - \omega_1} + \sqrt{p_2 - p_3}\sqrt{\omega_3 - \omega_2}\right)^2$$

and, in that form, it results from the well-known Lagrange identity, when it is applied to the four quantities $\sqrt{p_1 - p_2}, \sqrt{p_2 - p_3}, \sqrt{\omega_2 - \omega_1}, \sqrt{\omega_3 - \omega_2}$.

By virtue of this same identity, the inequality is replaced by an equality only if one has that:

$$\sqrt{p_1 - p_2}\sqrt{\omega_2 - \omega_1} - \sqrt{p_2 - p_3}\sqrt{\omega_3 - \omega_2} = 0\,,$$

which is the condition for the three points to be in a straight line.

Our conclusion is therefore proved. Of course, it may be further stated as: *Each of the hyperbolic lengths of the sides of the triangle, with the exception of the largest one, is less than the difference of the other two.*

237. – Having established this, consider the triangle $p_0\, q_0\, p_2$. In this triangle, one has $\overline{q_1 p_0} - \overline{q_1 p_2} = V = \overline{Pp_0}$. Suppose we are given the same situation at the three vertices of the triangle, respectively, which shows that the third side $\overline{p_2 p_0}$ is less than $\overline{Pp_0}$, which then entails precisely $p_2 < P$.

238. – The wave thus created by reflection on the piston will propagate with a certain velocity that is certainly greater than that of the discontinuity that exists between the state (q_1, ϖ_1) and original rest state. Indeed, the velocities of propagation of these discontinuities depend upon the angular coefficients of the chords that join the representative points of the states, between which they are defined. At the same time, by reason of the convexity of the curve $(\mathbf{2'})$, these velocities will increase with pressure. Now, the pressure p_2 is greater than the pressure p_0.

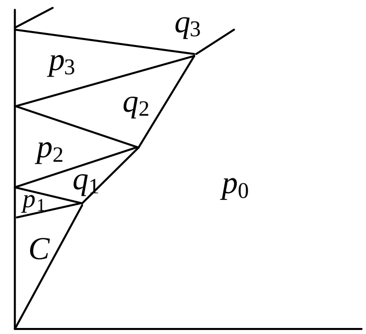

Fig. 16

Under these conditions, the new wave will certainly recapture the original one; since a and t are considered to be the plane coordinates (as we already did in *fig.* 10), the evolution of these two waves will be as represented in *fig.* 16. At their point of intersection a new intermediate state will be formed that is characterized by a pressure q_2, a dilatation ϖ_2, and a velocity u_2.

Since $\overline{p_2 p_0}$ is less than V this time, q_2 will be found between p_2 and P, and will be determined by the condition:

$$\overline{q_2 p_0} + \overline{q_2 p_2} = V .$$

Now, let p_3 and ω_3 be the pressure and the dilatation that come about by reflection when the retrograde wave (q_2, ϖ_2, u_2) encounters the piston. p_3 will be greater than q_2 (because u_2 is less than V) and one will have:

$$\overline{q_2 p_3} = V - u_2 ,$$

in such a way that $\overline{q_2 p_3}$ is equal to $\overline{q_2 p_2}$ (*fig.* 17).

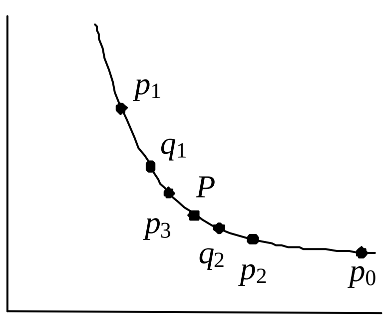

Fig. 17

The pressure p_3 is greater than P. One can see this in the triangle $p_0\ q_2\ p_3$, in which the greatest side is $\overline{p_0 p_2}$, while the sum of the two other sides is equal to $\overline{Pp_0}$.

At the same time, the same sequence of phenomena will start over. For the same reason as all along, the wave that propagates the new pressure p_3 will rejoin the one that propagates the pressure q_2 and, at their point of intersection, will give rise to a new pressure q_3 that lies between q_1 and P; by reflection on the piston they will generate a pressure p_4 that lies between p_3 and P, and so on.

The pressures p_1, p_3, …, p_{2n-1}, … are greater than P and decreasing; they therefore tend towards a limit, and the same is true for q_1, q_3, …, q_{2n-1}, … Parallel to this, p_2, p_4, …, p_{2n}, … are increasing and remain less than P; they tend towards a limit, as well as q_2, q_4, …, q_{2n}, …

Finally, we shall confirm that all four of these limits are equal to P.

Indeed, the triangle $P\,p_1\,q_1$ first gives us:

$$\overline{Pq_1} + \overline{q_1 p_1} = \overline{Pq_1} + \overline{q_1 p_2} < \overline{Pp_1} ;$$

the triangle $P\,q_2\,p_3$ then gives:

$$\overline{q_2 p_3} - \overline{Pq_2} = \overline{q_2 p_2} - \overline{Pq_2} < \overline{Pp_3} .$$

Upon subtracting the two inequalities side-by-side, one obtains:

$$\overline{Pq_1} + \overline{Pq_2} + \overline{p_2 q_1} - \overline{p_2 q_2} < \overline{Pp_1} - \overline{Pp_3} .$$

In other words, the quantities $\overline{Pq_1}$, $\overline{Pq_2}$ are less than the difference $\overline{Pp_1} - \overline{Pp_3}$. In a general manner, the quantities $\overline{Pq_{2n-1}}$, $\overline{Pq_{2n+1}}$ are less than $\overline{Pp_{2n-1}} - \overline{Pp_{2n+1}}$. They therefore tend towards 0 when n increases indefinitely, from which, it results that q_{2n-1} and q_{2n} tend towards P. Moreover, the relation:

$$\overline{q_i p_{i+1}} = \pm(\overline{Pp_0} - \overline{q_i p_0})$$

shows that the same is true for p. Now, from the considerations of no. **221**, the pressure P is the one that is established if the piston passes without transition from the velocity 0 to the velocity V. Conforming to the viewpoint of Hugoniot, we thus confirm that this pressure is indeed the same one that is finally produced by the process of successive reflections.