CHAPTER V

MOTIONS IN SPACE

239. – After occupying ourselves in the preceding chapter with the motion of a gas, based on the assumption that its motion was exclusively rectilinear, we recall the equations of motion in three dimensions, in other words, the equations:

$$(1)\qquad \begin{cases} \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = X - \dfrac{\delta^2 x}{\delta t^2}, \\ \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = Y - \dfrac{\delta^2 y}{\delta t^2}, \\ \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} = Z - \dfrac{\delta^2 z}{\delta t^2}. \end{cases}$$

In chapter III, we saw that there is an apparent contradiction between these equations and the conditions at the wall. However, the discussion that was presented above in the case of rectilinear motion shows us how this difficulty may be clarified. The agreement between the two sets of conditions is maintained thanks to the production of discontinuities that arise on the wall and propagate into the body of the fluid. Parallel waves arise each time the accelerations of arbitrary order of the wall are different from the ones that result from the internal equations of motion, and will be of order equal to that of the accelerations for which this discord is meaningful. In the course of an arbitrary motion they are produced when the acceleration – or one of the derivatives of higher order – of the wall becomes discontinuous with respect to time.

We therefore study the propagation of a discontinuity in the gas upon supposing – to fix ideas – that it is of second order. Since the pressure is assumed to be a function of the pressure, the equations of motion may be written:

$$(1')\qquad \begin{cases} \dfrac{dp}{d\rho}\dfrac{\partial \log \rho}{\partial x} = X - \dfrac{\delta^2 x}{\delta t^2}, \\ \dfrac{dp}{d\rho}\dfrac{\partial \log \rho}{\partial y} = Y - \dfrac{\delta^2 y}{\delta t^2}, \\ \dfrac{dp}{d\rho}\dfrac{\partial \log \rho}{\partial z} = Z - \dfrac{\delta^2 z}{\delta t^2}. \end{cases}$$

On either side of a second order discontinuity the components of the acceleration take the two sets of values:

$$\left(\frac{\delta^2 x}{\delta t^2}\right)_1, \left(\frac{\delta^2 y}{\delta t^2}\right)_1, \left(\frac{\delta^2 z}{\delta t^2}\right)_1; \left(\frac{\delta^2 x}{\delta t^2}\right)_2, \left(\frac{\delta^2 y}{\delta t^2}\right)_2, \left(\frac{\delta^2 z}{\delta t^2}\right)_2,$$

and the derivatives of the density take the two sets of values:

$$\left(\frac{\delta\rho}{\delta x}\right)_1, \left(\frac{\delta\rho}{\delta y}\right)_1, \left(\frac{\delta\rho}{\delta z}\right)_1; \left(\frac{\delta\rho}{\delta x}\right)_2, \left(\frac{\delta\rho}{\delta y}\right)_2, \left(\frac{\delta\rho}{\delta z}\right)_2.$$

Both of them satisfy the preceding equations. Since the components of the force are supposed to be continuous, if one subtracts both sides of the relations thus obtained one will obtain:

$$\frac{dp}{d\rho}\left[\frac{\partial \log \frac{1}{\rho}}{\partial x}\right] = \left[\frac{\delta^2 x}{\delta t^2}\right],$$
$$\frac{dp}{d\rho}\left[\frac{\partial \log \frac{1}{\rho}}{\partial y}\right] = \left[\frac{\delta^2 y}{\delta t^2}\right],$$
$$\frac{dp}{d\rho}\left[\frac{\partial \log \frac{1}{\rho}}{\partial z}\right] = \left[\frac{\delta^2 z}{\delta t^2}\right].$$

Let λ, μ, ν be the components of the discontinuity *referred to the present state, which is taken to be the initial state,* and let θ be the velocity of propagation. The variations of the acceleration will be $\lambda\theta^2, \mu\theta^2, \nu\theta^2$. Those of the derivatives of log $1/\rho$ will be given by formulas (**63**) of no. **111**. One will have, upon always denoting the direction cosines of the normal to the discontinuity surface S by α, β, γ:

$$\text{(2)} \qquad \begin{cases} \alpha \dfrac{dp}{d\rho}(\lambda\alpha + \mu\beta + \nu\gamma) = \lambda\theta^2, \\ \beta \dfrac{dp}{d\rho}(\lambda\alpha + \mu\beta + \nu\gamma) = \mu\theta^2, \\ \gamma \dfrac{dp}{d\rho}(\lambda\alpha + \mu\beta + \nu\gamma) = \nu\theta^2. \end{cases}$$

λ, μ, ν are not simultaneously null, since otherwise the discontinuity would not be of second order, but of third order. Therefore if θ is different from 0 then the same is true for at least one of the right-hand sides of the preceding equations, and one sees that these right-hand sides are proportional to α, β, γ.

Therefore, *any second order discontinuity that propagates in a gas is, from* (**115**), *longitudinal.*

On the other hand, the quantity $\lambda\alpha + \mu\beta + \nu\gamma$, which, in the general case, represents the projection of the discontinuity onto the normal to the wave surface, is nothing but the magnitude of this discontinuity itself here, and, upon successively multiplying by α, β, γ one obtains the projections onto the coordinate axes, i.e., λ, μ, ν. Equations (**2**) thus reduce to:

(**3**) $$\theta^2 = \frac{dp}{d\rho}.$$

Therefore, *the velocity of propagation of the discontinuity, when referred to the actual state, has the value* $\sqrt{\frac{dp}{d\rho}}$.

240. – If one would like to find the velocity of propagation θ_0 when referred to an arbitrary initial state (a, b, c) then one must divide θ by the normal dilatation of the wave during its passage from that state to the actual state. Upon denoting the quadratic form that was introduced in no. **51** by $\varphi(a, b, c)$, the adjoint form to φ by Φ, and the density in the initial state by ρ_0, one will have ([1]):

(**4**) $$\theta_0^2 = \frac{\rho^2}{\rho_0^2}\frac{dp}{d\rho}\frac{\Phi(f_a, f_b, f_c)}{f_a^2 + f_b^2 + f_c^2}.$$

As for the velocity of displacement T, since it is related to θ by equation (**54**) in no. **100**, one has:

(**5**) $$T = \sqrt{\frac{dp}{d\rho}} + u\alpha + v\beta + w\gamma,$$

in which u, v, w are the components of the velocity.

241. – It remains for us to examine the hypothesis $\theta = 0$. Equations (**2**) then give $\lambda\alpha + \mu\beta + \nu\gamma = 0$. In other words, *the discontinuity is transversal.*

A gas will thus support:

1. Longitudinal discontinuities that propagate with the velocity $\sqrt{\frac{dp}{d\rho}}$.

2. Stationary transversal discontinuities.

242. – We assumed, to fix ideas, that the discontinuity was of second order. However, the results that we just obtained persist, in that they are essential for an order n that is greater than 2. Indeed, suppose – as we obviously have the right to do – that α is non-zero, and then differentiate equations ($\mathbf{1}'$) $n - 2$ times with respect to x. Only the

([1]) See the note on page ??.

terms that contain the partial derivatives of order n will be affected by the discontinuity. Now, on the right-hand side these terms are provided exclusively from the differentiation of $\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2}$, and in order to obtain the first one, one must apply all of the levels of differentiation to the factors $\frac{\partial \log \rho}{\partial x}, \frac{\partial \log \rho}{\partial y}, \frac{\partial \log \rho}{\partial z}$. Under these conditions, and if one refers to formulas (**57**), (**57′**), (**63**) of chap. II, one sees that the equations that one arrives at are nothing but relations (**2**), both sides of each equation being simply multiplied by α^{n-2}. Therefore, as in the foregoing, we may have, one the one had, longitudinal discontinuities that propagate with velocity $\sqrt{\frac{dp}{d\rho}}$, and, on the other hand, stationary transversal discontinuities.

Just as was the case with the Hugoniot remark, the same is true under more or less general conditions. We saw above that in certain cases p may be a function of not only ρ, but also a, b, c: This is the case, for example, when the gas is inhomogeneous at any moment, or when first order waves are produced.

What happens to equations (**1′**) under these conditions? One immediately sees (upon referring to equations (**1′**)) that they will be modified by the addition of terms ([2]):

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial p}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial p}{\partial c}\frac{\partial c}{\partial x}\right), \ldots$$

respectively.

Now, they contain only first order derivatives of x, y, z with respect to a, b, c, t, and, as a result, they suffer no discontinuity.

Therefore, formulas (**2**) persist, with the quantity $dp/d\rho$ being, of course, replaced with the *partial* derivative of p with respect to ρ. That derivative will therefore give the square of the velocity of propagation.

The same thing will again be true if the forces X, Y, Z depend upon the density (to the exclusion of its derivatives) or contain the first derivatives of x, y, z in an arbitrary fashion.

243. – We just saw that the velocity of propagation is expressed by a square root and is, as a consequence, given two signs. At first, it thus seems that the sense of this propagation is undetermined at an arbitrary instant.

Meanwhile, it is somewhat obvious *a priori* that this sense will not be completely arbitrary, since it might, for example, change quickly in the course of motion. Indeed, it is easy to see that for a given discontinuity *θ has a perfectly well-determined sign*. Indeed, that quantity must satisfy not only equation (**3**), but the compatibility conditions of no. **103**:

([2]) In these terms, the derivatives $\frac{\partial p}{\partial a}, \frac{\partial p}{\partial b}, \frac{\partial p}{\partial c}$ are deduced from the equation that gives p as a function of ρ, a, b, c, which are considered to be four independent variables.

$$
(\mathbf{6}) \qquad \begin{cases}
\left[\dfrac{\delta^2 x}{\delta a\,\delta t}\right] = -\lambda\alpha\theta, & \left[\dfrac{\delta^2 x}{\delta b\,\delta t}\right] = -\lambda\beta\theta, & \left[\dfrac{\delta^2 x}{\delta c\,\delta t}\right] = -\lambda\gamma\theta, \\
\left[\dfrac{\delta^2 y}{\delta a\,\delta t}\right] = -\mu\alpha\theta, & \cdots & \cdots \\
\cdots & \cdots & \left[\dfrac{\delta^2 z}{\delta c\,\delta t}\right] = -\nu\gamma\theta.
\end{cases}
$$

In the latter, it is the only unknown and is, as a result, given unambiguously since it appears to the first degree.

244. – If one has neither equations (**6**) nor (**2**), (**3**) then there is no compatibility. We then know that the discontinuity might not remain unique, and we may propose to study what will be produced under these conditions. However, before proceeding with this study, we must speak of the case of liquids.

For them, as we previously remarked (no. **136**), one may not have a normal discontinuity since it would influence the derivatives of the density.

On the other hand, we shall see that only a normal discontinuity may propagate. We may state this result in the following general form:

In a moving medium, if the components of the acceleration are equal, up to continuous quantities, to the partial derivatives (with respect to the actual coordinates) of the same everywhere continuous quantity Φ *then they may propagate only normal (second order) discontinuities.*

Indeed, the variations of the components of the acceleration are $\lambda\theta^2$, $\mu\theta^2$, $\nu\theta^2$ and must be equal to the variations of $\dfrac{\partial\Phi}{\partial x}, \dfrac{\partial\Phi}{\partial y}, \dfrac{\partial\Phi}{\partial z}$. Now, since Φ is assumed to be continuous, they must be proportional to α, β, γ, from the lemma of no. **73**. The same is therefore true for λ, μ, ν if θ is non-zero.

One thus sees that *the jump in acceleration is normal*, and that this result is obtained *without it being necessary to appeal to compatibility* or any other hypothesis than the continuity of Φ at the instant considered.

Now, if we admit that there is compatibility, with a velocity of propagation that is non-zero, then we know that the direction of the jump in acceleration is also that of the characteristic segment (λ, μ, ν).

The lemma that we just proved may be immediately applied to the case that occupies us now, with the quantity Φ being p/ρ here, by virtue of equations (**1**).

We remark that whether there is no compatibility or the discontinuity is of first order and not second, under only the condition that the pressure be everywhere continuous, the preceding reasoning shows that *the jump variation in the acceleration is a normal segment to the wave.*

245. – It is easy to generalize to a discontinuity of arbitrary order n. In this case, the variation of the acceleration of order n depends ([3]) on that of the derivatives of $\frac{\delta^{n-2} p}{\delta t^{n-2}}$. Now, the $(n - 2)^{\text{th}}$ derivatives of p with respect to x, y, z, which may be expressed as functions of the derivatives of order $n - 1$ of the coordinates, are continuous under the present hypothesis, and the same is true for the other $(n - 2)^{\text{th}}$ derivatives of p, by virtue of the fundamental proposition of no. **97**. One may thus apply the preceding lemma to $\frac{\delta^{n-2} p}{\delta t^{n-2}}$, and deduce from that that the variation of the acceleration of order n is a normal segment to the wave.

246. – Now, if one introduces the compatibility condition then one sees that the tangential component of the discontinuity and, as a consequence, all of the other ones are null if there is propagation.

We have therefore established that the motion of a liquid may present only discontinuities that are both stationary and tangential.

247. – The lemma that we just used is, moreover, likewise applicable to a gas, by taking Φ to be the quantity $\int \frac{dp}{\rho}$, which is a function of p. The previously stated fact that any discontinuity that propagates in a gas is normal is therefore, as one sees, a consequence of the fact that the acceleration is derived from a potential.

248. – Now take, as in chap. III, a liquid in which one give the positions and the velocities of the various molecules, and suppose that these givens present a second order discontinuity along a certain surface S, which will be, as a consequence, known at each point, as far as its derivatives of index zero $\frac{\delta^2 x}{\delta a^2}$, … and its derivatives of index one $\frac{\delta^2 x}{\delta a\, \delta t}$, … are concerned. We do not, moreover, assume the compatibility conditions are satisfied. However, by contrast, the identical conditions are necessary, since the discontinuity is of second order all along S. We thus have two given segments at each

([3]) On does not have $\frac{\delta^{n-2}}{\delta t^{n-2}}\left(\frac{\partial p}{\partial x}\right) = \frac{\delta}{\delta x}\left(\frac{\delta^{n-2} p}{\delta t^{n-2}}\right)$; however, the difference of these two expressions (as one sees by expressing the symbol $\frac{\delta}{\delta t}$ as a function of $\frac{\partial}{\partial t}$ and developing) consists of only the derivatives of the coordinates up to order $n - 1$ and the derivatives of the pressure up to order $n - 2$, all quantities being continuous under our hypotheses.

point of it, whose directions are not necessarily the same. Under these conditions, what will be the discontinuities that are experienced by $\frac{\delta^2 x}{\delta t^2}, \frac{\delta^2 y}{\delta t^2}, \frac{\delta^2 z}{\delta t^2}$?

Furthermore, in order to respond to this question, we accept the hypothesis whereby there are no cavities created in the interior of the fluid and, as a consequence, that the two regions situated on either side of S remain automatically contiguous to each other for the duration of the motion.

The question is noticeably simplified by reason of the particular physical properties of the fluid. Indeed, they will not preserve any trace of their initial state, as long as the density does not cease to be given by equation (**3′**) of no. **47**.

Furthermore, the restriction that is concerned with the choice of initial state in no. **45** ceases to be necessary. One may just as well substitute either of two initial states such that the derivatives of the coordinates of one with respect to the coordinates of the other present discontinuities or arbitrary singularities provided that the functional determinant of the old coordinates with respect to the new ones is continuous, along with its derivatives.

Now, in the present case the given positions of the molecules must obviously be chosen such that the density is constant.

Thus, whenever one has a discontinuity one may take the present state to be the initial state for all of the fluid, and consequently annul the segment that corresponds to the derivatives of index zero.

In order to see what the segment (λ_1, μ_1, ν_1) that corresponds to the derivatives of index one will be under these conditions, we must recall that the velocities are chosen such that the derivative of the density with respect to time is everywhere null. If we then refer to the calculation of the variation of that derivative *as we did in no.* **111** (cont.) (the considerations of no. **111** may not be invoked here since there is no compatibility) then we see that the segment (λ_1, μ_1, ν_1) must be tangent to the surface S.

As for the acceleration, *it exhibits no discontinuity* (if one always avoids the case in which the fluid contains cavities). Indeed, we have previously seen (no. **244**) that by virtue of the equations of motion such a discontinuity must be normal, and, on the other hand, we know that it must be tangential, since otherwise it will not persist when it propagates, which is impossible.

249. – However, one may go further and confirm not only that the accelerations of all orders are continuous, but also that the given discontinuity does not give rise to any absolute discontinuity during the course of the motion.

In order to see this, recall the considerations of no. **244**, from which it resulted that the jump acceleration is necessarily normal for the discontinuity considered. This conclusion persists *even when there is a jump in the velocity*.

Therefore, let ξ', η' be the curvilinear coordinates on the surface of the discontinuity, coordinates that define an arbitrary molecule of that surface that belongs to region 2. Let ξ, η be the curvilinear coordinates of the molecule that is in region 1 at the instant t_0, and which also coincide with the molecule (ξ', η') in region 2 at the instant t. ξ, η are functions of t for given ξ', η'. For these functions, the condition that the jump in acceleration be normal gives two second order differential equations that are obviously satisfied when ξ and η are constants ([4]). Moreover, this latter circumstance will necessarily be produced if the two derivatives $d\xi/dt$, $d\eta/dt$ are null at a given instant; we would like to establish this.

250. – It is easy to verify the existence of a motion with no absolute discontinuity by using simple examples of discontinuities that relate to vortices, i.e., transversal discontinuities that relate to the derivatives of the form $\frac{\delta^2 x}{\delta a\,\delta t}$, …

For example, take a motion in two dimensions that is defined by the two conditions: 1. Throughout the entire volume of a certain cylinder of revolution C whose axis is vertical it must reduce to a uniform rotation around that axis. 2. It must have null molecular rotation in the rest of space. The known methods of hydrodynamics show that under these conditions there exists a velocity potential that is equal to k arctan y/x, where k is a constant and the z-axis is C. The velocity will then be perpendicular to the plane through the point (x, y) and the axis, and inversely proportional to the distance $r = \sqrt{x^2+y^2}$. Each point that is exterior to C will thus describe a circumference and will turn through an angle equal to $\frac{k}{r^2}t$ during a time t.

Moreover, in order for the velocity to be continuous at the origin of the motion the constant k will have to be calculated in such a manner that the angular velocity at the surface of the cylinder is the same as that of the interior points.

Under these conditions, it is clear that the interior and exterior points that are in contact with each other will likewise be in contact at any instant.

By contrast, the surface of the cylinder will obviously be the site of a first order discontinuity that relates to the derivatives $\delta x/\delta a$, … Nonetheless, that discontinuity will not be physically appreciable. It will not exist at an arbitrary instant, considered in itself, but will relate uniquely to the way that the positions at two different instants compare to each other. In other words, a curve with a continuous tangent, such as the one that is represented in *fig.* 18, that traverses the surface of the cylinder will be replaced at the following instants by a curve that has the behavior that is represented in *fig.* 18 (cont.).

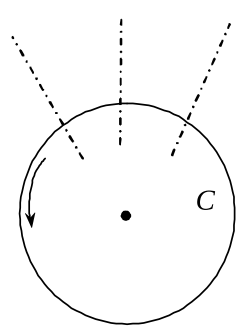

Fig. 18.

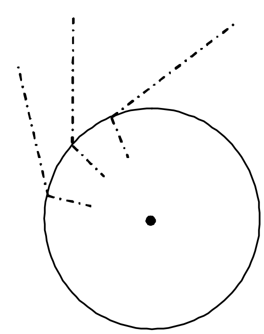

Fig. 18 (cont.)

([4]) See note III at the end of this volume.

In the general case, the existence of a discontinuity of this type result from the foregoing considerations; we know (no. **93**) that a stationary second order discontinuity that affects the derivatives of index one gives rise to a first order discontinuity that relates to the derivatives of index zero.

251. – Now return to the case of a gas. Again let the givens be the positions and velocities of the molecules with a second order discontinuity of at all of the points of a surface *S*, with the same conditions being satisfied, except for the compatibility condition. There will thus exist two segments (λ, μ, ν) and (λ_1, μ_1, ν_1) at each of its points, which correspond to the derivatives of index zero and one, respectively.

We first take a particular case, namely, the one where the segments are all normal to *S*. One may then determine two normal discontinuities, one of which propagates with the velocity θ that is given by formula (**3**) and the other of which propagates with the velocity $-\theta$, such that superposition produces the given discontinuity.

Indeed, let l and l' be the magnitudes of these two discontinuities, and let h and k be the magnitudes of the given segments (λ, μ, ν) and (λ_1, μ_1, ν_1), which are regarded as positive or negative according to their direction. It is clear that one must have:

$$\text{(7)} \qquad \begin{cases} h = l + l' \\ k = (l' - l)\theta \end{cases}$$

and that, conversely, if these two conditions are satisfied then the waves of magnitude l and l' are precisely the one that we seek.

252. – In order to treat the general case, it suffices to combine what we just said with the results that were obtained in the case of a liquid.

We are free to take the initial state as we like, provided that the density and its derivatives are continuous. We may, moreover, make that state coincide with the actual state in region 1, and define it in region 2 in the following manner:

Consider each point *M* of region 2 to be defined by its normal distance $Mm = \delta$ to *S* and the position of the point *m*. On the same normal to *S*, choose a new distance $Mm_0 = \delta_0$. We may obviously choose the latter as a function of the former and the position of *m*, in such a manner that if one imagines each molecule of region 2 to be transported from its true position *M* to the corresponding position M_0 then the density becomes continuous, along with all of its derivatives. It is the fictitious state thus obtained that we shall take to be the initial state. It is clear that the segment (λ, μ, ν) will then be normal to the discontinuity surface.

On the other hand, we may decompose the segment (λ_1, μ_1, ν_1) into its normal part and its tangent part. If we first abstract from the latter then we will be reduced to the case that we just studied, and we find two copies of the discontinuity that propagate in the opposite directions with the velocity θ.

Furthermore, it will suffice to add the discontinuity that is produced by the tangential component to the segment (λ_1, μ_1, ν_1) to these two waves; it is necessarily stationary.

One may apply the argument that was presented in the case of liquids to it without modification. The accelerations of all order will thus remain continuous when this third discontinuity no longer persists. The result that is produced will be a first order deformation of one of the regions with respect to the other one, just as we have been saying all along.

253. – It is in a completely analogous manner that one determines the state that is created at the point of contact with the wall when the normal acceleration of the wall disagrees with the one that results from the internal equations of motion, as we explained in nos. **139-140**. We must then deal with a normal discontinuity that propagates with the velocity $\theta = \sqrt{\frac{dp}{d\rho}}$ towards the interior of the fluid. The magnitude l of this discontinuity will be determined by the condition that $l\theta^2$ be equal to the difference of the two values of the normal acceleration. Having thus calculated l, all that remains for us to do will be to apply the formulas of ch. II to obtain the second order derivatives at the point of contact with the wall, since one knows the same values before the creation of the discontinuity.

254. – As we know, the most important results in hydrodynamics that have been obtained up till now relate to the conservation of vorticity, and consequently, the velocity potential, when it exists.

Now, the components of the vorticity are composed of the second order partial derivatives of x, y, z with respect to a, b, c, t. One must therefore demand to know whether the theorems that concern them do not break down when one passes our discontinuity.

The response is negative: it immediately results from this that hydrodynamic discontinuities are normal. Similarly, it will not affect the molecular rotation, whose variation is proportional (no. **114**) to the tangential component of the discontinuity.

255. – Furthermore, the same fact applies ([5]) to the consideration of the integral $\int u\,dx + v\,dy + w\,dz$, or *circulation,* which provides, as one knows ([6]), the simplest proof of the theorems that we shall discuss, which relate to the conservation of that integral in the course of motion when it is taken over a closed contour C.

The question is therefore that of knowing whether the integral in question, which necessarily keeps the same value as long as the contour C remains in one region or the motion is well defined, might change when the contour is traversed by a wave.

Now, during a time interval dt, the influence of one discontinuity is felt only on the arc s of C that exists between the two positions that are occupied by the wave at the

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([5]) We content ourselves by summarizing the logic of this argument in a manner that is completely analogous to the one presented later on in note III at the end of the volume.

([6]) THOMSON, *Cambridge Trans.*, 1869; BASSET, *Hydrodynamique,* t. I, pp. 70-73; DUHEM, *Hydrodynamique, Elasticité, Acoustique*, t. I, pp. 108-115; POINCARÉ, *Théorie des Tourbillons*, ch. I; APPEL, *Traité de Mécanique,* t. III, ch. XXXV, etc.

commencement and the end of that interval, arcs whose length is of order dt. On the other hand, if one writes the integral in the form:

$$\int \left(u\frac{dx}{dt} + v\frac{dy}{dt} + w\frac{dz}{dt} \right) d\tau \tag{8}$$

(τ being a parameter that defines a definite molecule of the curve C) then the expression $u\frac{dx}{dt} + v\frac{dy}{dt} + w\frac{dz}{dt}$ will not vary sharply on the wave, since it is of second order. The quantity by which it will be modified by the discontinuity at an arbitrary point of the arc s will therefore be of the same order as the arc itself, and the corresponding alteration of the integral (**8**) is of the same order as s^2, i.e., of dt^2. Therefore, the derivative of that integral will be null, as when the motion is continuous.

The fact that this line of argument is successful is surprising, given that it does not involve the direction of the discontinuity, and that, from the preceding no., the actual result will obviously cease to be true it that direction is not normal. However, one must observe (no. **247**) that the orthogonality that exists between the direction of the discontinuity and that of the wave surface amounts to the existence of an acceleration potential, which was used when one established the conservation of vorticity in the continuous motion.

256. – Other than acceleration waves, one may produce, as we have seen, first order waves, or *shock waves.* We have likewise confirmed that such waves might arise when the velocity of the wall does not present any sharp variation. It is easy to establish equations for the propagation of such waves that are completely analogous to the ones that we wrote in nos. **205-209** in the case of rectilinear motion.

Let (λ, μ, ν) be the characteristic segment of the discontinuity, the initial state being the one in region 1. The sharp variation of the velocity will be $(-\lambda\theta, -\mu\theta, -\nu\theta)$. On the other hand, let $[p] = p_2 - p_1$ be the variation of the pressure. Apply the theorem of the quantity of motion projected onto a small cylinder that exists between a portion S of the wave surface at the time t and the corresponding portion of the wave surface at the infinitesimally neighboring instant $t + dt$. Now, since this cylinder is considered in the state 1 of the medium, its height will be:

$$dn = q\, dt,$$

and its mass will be:

$$\rho_1\, \theta S\, dt\,.$$

We assume that S is very small, but that dt is negligible with respect to the dimensions of S. Thanks to that circumstance, we may neglect the pressures that act on the lateral surface of our cylinder by comparison to the ones that act on the bases. The effect of the forces X, Y, Z will be likewise negligible, as we saw in no. **205**. Therefore, if α, β, γ are the direction cosines of the normal to the wave, and they do not vary sharply, then, since our cylinder passes ([7]) from region 2 to region 1 during the time dt, and consequently, from the velocity ($u_1 - \lambda\theta$, $v_1 - \mu\theta$, $w_1 - \nu\theta$) to the velocity (u_1, v_1, w_1) under the action of the opposing normal pressures p_1 and p_2:

$$\rho_1\ \theta S\ dt\ \lambda\theta = -\ \alpha[p]\ S\ dt,$$
$$\rho_1\ \theta S\ dt\ \mu\theta = -\ \beta[p]\ S\ dt,$$
$$\rho_1\ \theta S\ dt\ \nu\theta = -\ \gamma[p]\ S\ dt.$$

This shows us, first of all, that the discontinuity is necessarily normal. Its magnitude l is:

(**9**) $$l = -\frac{[p]}{\rho_1\theta^2}.$$

The ratio of the densities is given by formula (**60**) of no. **109**. If we take into account the fact that the discontinuity is normal and its magnitude is l then we obtain:

(**10**) $$\frac{\rho_1}{\rho_2} - 1 = l.$$

One may eliminate l from these two equations, and one obtains:

(**11**) $$[p] = \rho_1\theta^2\left(1 - \frac{\rho_1}{\rho_2}\right) = \theta^2\,(\rho_2 - \rho_1).$$

This formula corresponds to the expression (**68**) that was obtained in no. **207** for the velocity of propagation. It nevertheless has a somewhat different form by reason of the fact that we have taken the initial state to be the actual state in region 1, which we did not do in the case of rectilinear motion.

257. – We must further write the equation of adiabaticity. If we adopt Poisson's law then this condition will be simply:

$$\frac{p_1}{\rho_1^m} = \frac{p_2}{\rho_2^m},$$

in which p_1 and p_2 are the two pressures.

If, on the contrary, we follow the path that was indicated by Hugoniot then we must directly write that the differential of the total work done by the pressure, when evaluated as we did in no. **209**, has the expression:

$$d\mathcal{T} = p_1(u_1\alpha + v_1\,\beta + w_1\,\gamma) - p_2\,[(u_1 - \lambda\theta)\,\alpha + (v_1 - \lambda\theta)\,\beta + (w_1 - \lambda\theta)\,\gamma]$$
$$= p_2\,l\theta - [p]\,(u_1\alpha + v_1\,\beta + w_1\,\gamma),$$

([7]) As in no. **205**, we assume that θ is positive in this argument, but the final result will be, of course, independent of this hypothesis.

and the sharp variation of the *vis viva* is equal to the variation of the internal energy. Now, this energy, which is, up to a factor of $\frac{1}{m-1}$, the product of the volume with the pressure, has the value $\frac{\theta p_1}{m-1} S\,dt$ in the state 1 and the value $\frac{\theta p_2}{m-1} S\,dt$ in region 2, where the volume is multiplied by $\frac{\rho_1}{\rho_2}$.

We thus have (upon suppressing the factor $S\,dt$).

$$p_2 l\theta - [p](u_1\alpha + v_1\beta + w_1\gamma) + \frac{\rho_1\theta}{2}[(u^2+v^2+w^2)] = \frac{\theta}{m-1}\left(p_1 - p_2\frac{\rho_1}{\rho_2}\right). \tag{12}$$

As in no. **209**, we must transform this equation in such a manner as to render it independent of the motion of the fluid. To that effect, we only have to use the previously-obtained equations:

$$\begin{aligned} u_2 &= u_1 - l\alpha\theta, \\ v_2 &= v_1 - l\beta\theta, \\ w_2 &= w_1 - l\gamma\theta, \end{aligned}$$

(in which u_1, v_1, w_1 and u_2, v_2, w_2 are the two velocities), which give us the variation of the *vis viva* per unit mass $[(u^2 + v^2 + w^2)]$.

Equation (**12**) thus becomes:

$$p_2 l\theta - [p](u_1\alpha + v_1\beta + w_1\gamma) + \frac{\rho_1\theta}{2}\{l^2\theta^2 - 2l\theta(u^2+v^2+w^2)\} = \frac{\theta}{m-1}\left(p_1 - p_2\frac{\rho_1}{\rho_2}\right),$$

and one sees precisely that the terms in $u_1\alpha + v_1\beta + w_1\gamma$ are eliminated, by virtue of (**9**). What remains (upon dividing by θ and then eliminating l and θ^2 by means of equations (**9**) and (**10**)) is:

$$\frac{p_1+p_2}{2}(\rho_1 - \rho_2) = \frac{1}{m-1}(p_1\rho_1 - p_2\rho_2), \tag{13}$$

i.e., the same equation that we obtained for the case of rectilinear motion in no. **209** (the quantities ω_1 and ω_2 that appear in that section are inversely proportional to ρ_1 and ρ_2).

258. – The proof in no. **254** that acceleration waves will not alter vorticial motion does not apply to shock waves. On the contrary, by conveniently modifying the argument of no. **255**, one may prove ([8]) that they are capable of giving rise to vortices when none existed prior to their passage.

([8]) See note III at the end of the volume.

CHAPTER VI

APLLICATIONS TO THE THEORY OF ELASTICITY

259. – In this chapter, we propose to study the propagation of waves, no longer in liquids, but in elastic solids. Contrary to what we did for liquids, for this study there is good reason to take the initial state to be no longer the present state, but a perfectly well-defined state of the body considered called the *natural state*. The initial state having been thus chosen, the internal tensions are functions of the *components of the deformation* ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 that were defined in no. **51**.

The distinction between the initial state and the present state does not have to play a role in the simplest case that one must study, the one in which one supposes: 1. that the body considered is homogeneous and isotropic; 2. that the deformations to which it is subjected are infinitely small.

In this case, the coordinates *a, b, c* of the initial state (i.e., natural state) essentially coincide with the coordinates *x, y, z, t* of the present state; one has:

$$x = a + \xi, \quad y = b + \eta, \quad z = c + \zeta,$$

in which ξ, η, ζ are assumed to be very small, along with their derivatives. Reduced to the infinitely small terms of first order, the components of the deformation will be:

$$(1) \qquad \begin{cases} \varepsilon_1 = \dfrac{\partial \xi}{\partial x}, & \varepsilon_2 = \dfrac{\partial \eta}{\partial y}, & \varepsilon_3 = \dfrac{\partial \zeta}{\partial z}, \\ \gamma_1 = \dfrac{\partial \eta}{\partial z} + \dfrac{\partial \zeta}{\partial y}, & \gamma_2 = \dfrac{\partial \zeta}{\partial x} + \dfrac{\partial \xi}{\partial z}, & \gamma_3 = \dfrac{\partial \xi}{\partial y} + \dfrac{\partial \eta}{\partial x}. \end{cases}$$

As we shall have occasion to recall a little later on, the equations of motion are deduced from the consideration of a certain function of the components of the deformation that is called the *elastic energy*. In the isotropic case that we now address, this quantity has an expression of the form:

$$(2) \qquad \iiint W \rho \, dx \, dy \, dz$$

$$(2') \qquad \begin{cases} 2\rho W = L(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + M(2\varepsilon_1^2 + 2\varepsilon_2^2 + 2\varepsilon_3^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2) \\ \qquad\quad = (L + 2M)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + M(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 4\varepsilon_2\varepsilon_3 - 4\varepsilon_3\varepsilon_1 - 4\varepsilon_1\varepsilon_2), \end{cases}$$

in which L and M are two constants ([9]) such that the quadratic form W is positive definite, i.e., subject to the inequalities:

$$M > 0\,, \qquad 3L + 2M > 0\,. \tag{3}$$

The equations of motion may be written:

$$\left\{\begin{aligned}\rho\frac{\partial^2\xi}{\partial t^2} &= M\Delta\xi + (L+M)\frac{\partial\sigma}{\partial x} + \rho X,\\ \rho\frac{\partial^2\eta}{\partial t^2} &= M\Delta\eta + (L+M)\frac{\partial\sigma}{\partial y} + \rho Y,\\ \rho\frac{\partial^2\zeta}{\partial t^2} &= M\Delta\zeta + (L+M)\frac{\partial\sigma}{\partial z} + \rho Z,\end{aligned}\right. \tag{4}$$

in which σ is the expression:

$$\sigma = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\,,$$

such that $1 + \sigma$ represents the dilatation ρ_0/ρ, and X, Y, Z are the given forces, which act on a unit mass.

260. – Equations (**4**) are, as one knows, of second order in ξ, η, ζ; they make known the components of the acceleration when ξ, η, ζ are given for each value of a, b, c, i.e., when one is given the positions of the molecules.

Now, experiments inform us that in order to determine the motion of an elastic body, one must give not only the positions and velocities of the molecules at the given instant, but also a set of boundary conditions, such as the motions of the different points of the surface of the body at every instant, or the pressures that are exerted on that surface at each instant.

Under these conditions, we again encounter the same difficulty as in the problem of hydrodynamics.

For example, we adopt the hypothesis that one gives the motion of each of the points of the surface. Moreover, we know the accelerations of these points, and the values found for these accelerations are completely independent of the interior equations. There will thus be no reason for them to agree with the ones that result from these equations. The contradiction is likewise more complete than the previous one, since these are the values of the accelerations themselves, and no longer just their normal components, which are given by the boundary conditions.

Since the present state essentially coincides with the initial state, let λ, μ, ν be the components of the discontinuity when referred to either of these states arbitrarily; let θ be

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([9]) We let L, M denote the coefficients that one usually calls λ, μ, since the latter letters are employed with a different significance here.

the velocity of propagation and let α, β, γ be the direction cosines of the normal to the wave surface. In equations (**4**), if we replace the sharp variations of the second order by their values derived from the formulas of no. **103**, then they become:

$$\text{(5)} \qquad \begin{cases} \rho\lambda\theta^2 = M\lambda + (L+M)\alpha(\lambda\alpha + \mu\beta + \nu\gamma) \\ \rho\mu\theta^2 = M\mu + (L+M)\beta(\lambda\alpha + \mu\beta + \nu\gamma) \\ \rho\nu\theta^2 = M\nu + (L+M)\gamma(\lambda\alpha + \mu\beta + \nu\gamma). \end{cases}$$

If we write these equations in the form:

$$\begin{aligned} (\rho\theta^2 - M)\,\lambda &= (L+M)\,\alpha\,(\lambda\alpha + \mu\beta + \nu\gamma) \\ (\rho\theta^2 - M)\,\mu &= (L+M)\,\beta\,(\lambda\alpha + \mu\beta + \nu\gamma) \\ (\rho\theta^2 - M)\,\nu &= (L+M)\,\gamma\,(\lambda\alpha + \mu\beta + \nu\gamma) \end{aligned}$$

then we see that they are entirely similar to equations (**2′**) of the preceding chapter. Consequently, from what was said there, they admit two types of solutions:

1. $\dfrac{\lambda}{\alpha} = \dfrac{\mu}{\beta} = \dfrac{\nu}{\gamma}$: the discontinuity is *longitudinal*. It propagation velocity will be given by the relation $\rho\theta^2 - M = L + M$, namely:

$$\text{(6)} \qquad \theta^2 = \frac{2M + L}{\rho}.$$

2. $\lambda\alpha + \mu\beta + \nu\gamma = 0$: the discontinuity is *transversal*. Its propagation velocity will be given by the relation $\rho\theta^2 - M = 0$, namely:

$$\text{(7)} \qquad \theta^2 = \frac{M}{\rho}.$$

The two values of θ thus obtained are, moreover, real, by virtue of the inequalities (**3**).

Therefore, solid isotropic bodies are capable of propagating two sets of waves with different velocities: one set is exclusively longitudinal, and the other is exclusively transversal. Just as we saw in no. **115**, the former are not accompanied by any variation of the instantaneous molecular rotation, and the latter are not accompanied by any variation of the derivatives of the density.

261. – If, at a particular instant, there exists an arbitrary difference between the accelerations of the surface that are derived from the internal equations and these same accelerations, as derived from the boundary conditions then this will give rise to two waves, one of which is longitudinal and the other is transversal, corresponding to the normal component and the tangential component of the segment that represents this difference, respectively.

On the other hand, if there exists a second order discontinuity along a well defined surface in the interior of the body at a particular moment and this discontinuity is absolutely arbitrary (the only restriction being the identical conditions) then it gives rise to *four* waves, one of which is longitudinal, the others of which are transversal; one propagates in one direction, while the others propagates in the opposite sense.

262. – If, instead of giving the positions of the points of the surface, one is given the tensions that act on those points at each instant then they may likewise have different values at the initial instant than the ones that were deduced from the components of the deformation at these same points. Under these conditions, one will likewise produce a wave. However, this time, it will be of first order, because a finite difference between the internal and external pressures will produce a sharp variation in the velocity. Such waves have been studied by Christoffel ([10]). Thanks to the hypothesis that the motions are infinitely small, that savant further obtained basically the same results as the ones that enable one to study acceleration waves.

263. – In the treatises on elasticity, one easily forms the equations of motion for the case of anisotropic bodies. We shall not develop the results for these bodies that correspond to the previous ones; indeed, we shall find them in the most general case of a finite deformation. Recall only that W is, moreover, a quadratic form in $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$, and that equations (**4**) are replaced by three second order equations that again give the projections of the acceleration as a function of the second derivatives (and also the first derivatives, if the body is not homogeneous) of ξ, η, ζ with respect to x, y, z.

In optics, when one looks for the vibratory states that satisfy the equations that we just wrote one confirms that *to any direction of a plane wave there correspond three directions of vibration, which are mutually orthogonal and are the three principal directions of a certain quadric* or *polarization ellipsoid*. One recovers precisely the same result when one adopt the viewpoint of Hugoniot: The calculation is completely analogous to the one that was presented above (no. **261**), or to the one that we made above (no. **267**).

264. – Now, we put aside the case of infinitely small deformations and propose to study elastic waves *with finite deformations* in a solid that is or is not isotropic.

The case in which such deformations exist has been envisioned by Boussinesq and Brillouin. In order to write the equations of equilibrium under these conditions one again starts with the consideration of *elastic energy*, i.e., a certain triple integral of the form:

$$(\mathbf{2}) \qquad \iiint W\rho\, dx\, dy\, dz = \iiint W\rho_0\, da\, db\, dc,$$

([10]) *Annali di Matematica*, series II, tome VIII, pp. 193; 1877.

in which $\rho\, dx\, dy\, dz = \rho_0\, da\, db\, dc$ is the mass element and W is a certain function of the six components $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$, of the deformation at each instant. This function contains the a, b, c explicitly, possibly from somewhere else, according to whether the body considered is heterogeneous or homogeneous in its initial state.

Since the system of independent variables that is composed of time and the *initial* coordinates a, b, c is the only one that is employed in this chapter, and consequently no confusion should arise in that regard, it will not be necessary for us to conform to the convention of no. **61**. We thus denote derivatives that are taken with respect to these variables by the symbol ∂, the sign δ being reserved for the components of the virtual displacements.

We write that the variation of the integral (**2**) for any system of virtual displacements (dx, dy, dz) that are communicated to the various points is equal to the work that corresponds to the given forces (acting on a unit mass) (X, Y, Z), namely, the expression:

$$\textbf{(3)} \qquad \iiint (X\,\delta x + Y\,\delta y + Z\,\delta z)\,\rho\,dx\,dy\,dz\,,$$

if the positions of the points of the surface are fixed, or to that expression, when it is combined with the work done by the external pressures, in the contrary case.

As in no. **47**, let $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ be the partial derivatives of x, y, z with respect to a, b, c. The variation of the integral (**2**) is (upon observing that the mass element $\rho\, dx\, dy\, dz$ does not vary):

$$\textbf{(9)} \qquad \begin{cases} \displaystyle\iiint\left(\frac{\partial W}{\partial a_1}\delta a_1 + \frac{\partial W}{\partial b_1}\delta b_1 + \frac{\partial W}{\partial c_1}\delta c_1 + \frac{\partial W}{\partial a_2}\delta a_2 + \cdots + \frac{\partial W}{\partial c_3}\delta c_3\right)\rho\,dx\,dy\,dz \\ \displaystyle = \iiint\left(\frac{\partial W}{\partial a_1}\frac{\partial(\delta x)}{\partial a} + \frac{\partial W}{\partial b_1}\frac{\partial(\delta x)}{\partial b} + \cdots\right)\rho\,dx\,dy\,dz. \end{cases}$$

According to the general rules of the calculus of variations, we must transform that expression by an integration by parts, or, more precisely, by Green's formula. We thus have one surface integral and one new volume integral:

$$\begin{aligned} -\iiint\Bigg\{ &\delta x\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial a_1}\right) + \frac{\partial}{\partial b}\left(\frac{\partial W}{\partial b_1}\right) + \frac{\partial}{\partial c}\left(\frac{\partial W}{\partial c_1}\right)\right] \\ &+ \delta y\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial a_2}\right) + \frac{\partial}{\partial b}\left(\frac{\partial W}{\partial b_2}\right) + \frac{\partial}{\partial c}\left(\frac{\partial W}{\partial c_2}\right)\right] \\ &+ \delta z\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial a_3}\right) + \frac{\partial}{\partial b}\left(\frac{\partial W}{\partial b_3}\right) + \frac{\partial}{\partial c}\left(\frac{\partial W}{\partial c_3}\right)\right]\Bigg\}\rho\,dx\,dy\,dz. \end{aligned}$$

In order for the sum thus obtained to be identically equal to the sum of the quantity (**8**) and the work done by pressure, it is necessary that there be equality for any δx, δy, δz, for both the surface integral and the volume integral individually. This gives us the internal equations of equilibrium, namely:

(10)
$$\begin{cases} \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_1}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_1}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_1}\right)=0 \\ \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_2}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_2}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_2}\right)=0 \\ \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_3}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_3}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_3}\right)=0, \end{cases}$$

whereas the equality of the surface integrals will provide (upon assuming that the exterior pressures were given) by the boundary conditions.

Finally, if we would like to pass from the case of equilibrium to that of motion, we must only substitute the principle of virtual work for Hamilton's principle: Moreover, this amounts (compare to what we said in chap. III) to making use of d'Alembert's principle and introducing inertial forces into the forces X, Y, Z. The equations on the surface will remain unaltered, whereas the internal equations will become:

(11)
$$\begin{cases} \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_1}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_1}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_1}\right)+X-\dfrac{\partial^2 x}{\partial t^2}=0 \\ \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_2}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_2}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_2}\right)+Y-\dfrac{\partial^2 y}{\partial t^2}=0 \\ \dfrac{\partial}{\partial a}\left(\dfrac{\partial W}{\partial a_3}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial W}{\partial b_3}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial W}{\partial c_3}\right)+Z-\dfrac{\partial^2 z}{\partial t^2}=0. \end{cases}$$

These equations are of second order, either with respect to t or with respect to a, b, c, since one must derive the terms $\dfrac{\partial W}{\partial a_1}, \dfrac{\partial W}{\partial b_1}, \dfrac{\partial W}{\partial c_1}$, which are certain functions of the first order derivatives. If the body is homogeneous then these quantities do not contain a, b, c, t explicitly, and consequently the equations contain only terms of second order. In addition, first order terms will enter into them in the contrary case.

265. - The case of hydrodynamics corresponds to the one in which W is a function of only density; in other words (the initial state being assumed to be homogeneous), of the functional determinant:

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

For $W = F(D)$ the derivatives that were previously considered are nothing but the products of $F(D)$ with the minors A_i, B_i, C_i of the previous determinant. The expression:

$$\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial a_1}\right)+\frac{\partial}{\partial b}\left(\frac{\partial W}{\partial b_1}\right)+\frac{\partial}{\partial c}\left(\frac{\partial W}{\partial c_1}\right)$$

that appears in the first equation (**10**) is therefore written:

$$F(D)\left(\frac{\partial A_1}{\partial a}+\frac{\partial B_1}{\partial b}+\frac{\partial C_1}{\partial c}\right)+F''(D)\left(A_1\frac{\partial D}{\partial a}+B_1\frac{\partial D}{\partial b}+C_1\frac{\partial D}{\partial c}\right).$$

The coefficient of $F(D)$ is null, as is well known from the theory of multipliers ([11]). That of $F''(D)$ may be written:

$$D\left(\frac{A_1}{D}\frac{\partial D}{\partial a}+\frac{B_1}{D}\frac{\partial D}{\partial b}+\frac{C_1}{D}\frac{\partial D}{\partial c}\right).$$

Now, this is nothing but $D\dfrac{\partial D}{\partial x}$, because the quantities $\dfrac{A_1}{D},\dfrac{B_1}{D},\dfrac{C_1}{D}$ are the partial derivatives of a, b, c with respect to x (x, y, z being taken to be independent variables). The equation:

$$DF''(D)\frac{\partial D}{\partial x}+X-\frac{\partial^2 x}{\partial t^2}=0$$

is identical to the first equation (**1**) of chap. V, by means of equation (**3′**) of no. **47** and the relation:

(**12**) $$p = -\rho_0 F'(D).$$

266. – Equations (**11**), which make known the components of the acceleration at each point by means of the partial derivatives of x, y, z relative to a, b, c at this point, suggest some remarks that are completely similar to the ones that we made for the equations of hydrodynamics (no. **139-140**) and for equations (**4**) (no. **260**). The agreement between the internal equations and the boundary conditions that must be established thus leads us once more to study the propagation of waves.

To that effect, we must first specify the equations of motion.

If we take into account the values of ε_i, γ_i that define formulas (**7**) of no. **51** then we see that one will have:

([11]) JORDAN, *Cours d'Analyse,* tome III, no. **44**, pp. 49.

$$\text{(13)}\qquad \begin{cases} \dfrac{\partial W}{\partial a_1} = a_1 \dfrac{\partial W}{\partial \varepsilon_1} + b_1 \dfrac{\partial W}{\partial \gamma_3} + c_1 \dfrac{\partial W}{\partial \gamma_2}, \\ \dfrac{\partial W}{\partial b_1} = a_1 \dfrac{\partial W}{\partial \gamma_3} + b_1 \dfrac{\partial W}{\partial \varepsilon_2} + c_1 \dfrac{\partial W}{\partial \gamma_1}, \\ \dfrac{\partial W}{\partial c_1} = a_1 \dfrac{\partial W}{\partial \gamma_2} + b_1 \dfrac{\partial W}{\partial \gamma_1} + c_1 \dfrac{\partial W}{\partial \varepsilon_3}. \end{cases}$$

When we substitute these values into the first equation (**11**), differentiation will give two types of terms. Indeed, in each term of the expressions that we just wrote, one may substitute either the first factor or the second one. In the former case, we obtain the three quantities:

$$\frac{\partial^2 x}{\partial a^2}\frac{\partial W}{\partial \varepsilon_1} + \frac{\partial^2 x}{\partial a\,\partial b}\frac{\partial W}{\partial \gamma_3} + \frac{\partial^2 x}{\partial a\,\partial c}\frac{\partial W}{\partial \gamma_2},$$

$$\frac{\partial^2 x}{\partial a\,\partial b}\frac{\partial W}{\partial \gamma_3} + \frac{\partial^2 x}{\partial b^2}\frac{\partial W}{\partial \varepsilon_2} + \frac{\partial^2 x}{\partial b\,\partial c}\frac{\partial W}{\partial \gamma_1},$$

$$\frac{\partial^2 x}{\partial a\,\partial c}\frac{\partial W}{\partial \gamma_2} + \frac{\partial^2 x}{\partial b\,\partial c}\frac{\partial W}{\partial \gamma_1} + \frac{\partial^2 x}{\partial c^2}\frac{\partial W}{\partial \varepsilon_3}.$$

Once more, let λ, μ, ν be the components of a discontinuity of second order, relative to the initial state (which is the state (a, b, c) this time, and not the present state); let α, β, γ be the direction cosines of the normal to the wave, and let θ be the velocity of propagation. If we consider the sharp variations of the second order derivatives that appear in the previous expression, and we replace them by their values found in no. **85** then we see that the discontinuity that results from the sum of these three expressions is λQ, when we let Q denote the quantity:

$$Q = \frac{\partial W}{\partial \varepsilon_1}\alpha^2 + \frac{\partial W}{\partial \varepsilon_2}\beta^2 + \frac{\partial W}{\partial \varepsilon_3}\gamma^2 + 2\frac{\partial W}{\partial \gamma_1}\beta\gamma + 2\frac{\partial W}{\partial \gamma_2}\gamma\alpha + 2\frac{\partial W}{\partial \gamma_3}\alpha\beta .$$

267. – Now, look at the result that was just obtained when one differentiates all of the second factors in the expression (**13**). Here, one encounters the derivatives of the components of the deformation with respect to a, b, c, derivatives whose variations were calculated in no. **113**. From what we found there, we introduce the quantities:

$$\text{(14)}\qquad \begin{cases} e_1 = L\alpha, & e_2 = M\beta, & e_3 = N\gamma, \\ g_1 = M\gamma + N\beta, & g_2 = N\alpha + L\gamma, & g_3 = L\beta + M\alpha, \end{cases}$$

in which one has:

(**15**) $L = \lambda a_1 + \mu a_2 + \nu a_3,\qquad M = \lambda b_1 + \mu b_b + \nu b_3,\qquad N = \lambda c_1 + \mu c_2 + \nu c_3,$

The sharp variations of the derivatives of the components of the deformation may then be written in the simple form:

$$\left[\frac{\partial \varepsilon_i}{\partial a}\right] = \alpha e_i,\ \left[\frac{\partial \varepsilon_i}{\partial b}\right] = \beta e_i,\ \left[\frac{\partial \varepsilon_i}{\partial c}\right] = \gamma e_i,$$
$$\left[\frac{\partial \gamma_i}{\partial a}\right] = \alpha g_i,\ \left[\frac{\partial \gamma_i}{\partial b}\right] = \beta g_i,\ \left[\frac{\partial \gamma_i}{\partial c}\right] = \gamma g_i, \qquad (i = 1, 2, 3).$$

From this, let us calculate $\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial \varepsilon_i}\right)\right]$. One has:

$$\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial \varepsilon_i}\right)\right] = \alpha\left(\frac{\partial^2 W}{\partial \varepsilon_i\,\partial \varepsilon_1} e_1 + \frac{\partial^2 W}{\partial \varepsilon_i\,\partial \varepsilon_2} e_2 + \frac{\partial^2 W}{\partial \varepsilon_i\,\partial \varepsilon_3} e_3 + \frac{\partial^2 W}{\partial \varepsilon_i\,\partial \gamma_1} g_1 + \frac{\partial^2 W}{\partial \varepsilon_i\,\partial \gamma_1} g_1 + \frac{\partial^2 W}{\partial \varepsilon_i\,\partial \gamma_1} g_1\right).$$

Consider the quadratic form:

$$\Psi(e_1, e_2, e_3, g_1, g_2, g_3) = \frac{\partial^2 W}{\partial \varepsilon_1^2} e_1^2 + \cdots + 2\frac{\partial^2 W}{\partial \varepsilon_1\,\partial \lambda_1} e_1 g_1 + \cdots \frac{\partial^2 W}{\partial \gamma_1^2} g_1^2 + \cdots + 2\frac{\partial^2 W}{\partial \gamma_1\,\partial \gamma_2} g_1 g_2,$$

in which the $\frac{6\times 5}{2}$ + 6 = 21 products of pairs and the squares of the six quantities e, g appear, each of these products having as its coefficient the second derivative of W with respect to the variables ε or γ, respectively, and the terms thus obtained must be doubled (as usual) if these variables are different, i.e., if they amount to a rectangular term.

One immediately sees how this expression comes about.

Indeed, the second differential of W is:

$$\frac{\partial W}{\partial \varepsilon_1} d^2\varepsilon_1 + \cdots + \frac{\partial W}{\partial \gamma_3} d^2\gamma_3 + \frac{\partial^2 W}{\partial \varepsilon_1^2} d^2\varepsilon_1^2 + \cdots + 2\frac{\partial^2 W}{\partial \gamma_1\,\partial \gamma_2} d\gamma_1\,d\gamma_2,$$

and contains, on the one hand, the second differentials $d^2\varepsilon_1$, …, $d^2\gamma_3$, and, on the other hand, the first differentials $d\varepsilon_1$, …, $d\gamma_3$. If one replaces them with e_1, e_2, e_3, g_1, g_2, g_3, respectively, in the terms that contain them then one obtains the quadratic form W.

Having thus introduced that form, the variation $\left[\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial \varepsilon_i}\right)\right]$ is obviously nothing but $\frac{\alpha}{2}\frac{\partial W}{\partial \varepsilon_i}$, and similarly, the derivative $\frac{\partial}{\partial a}\left(\frac{\partial W}{\partial \gamma_i}\right)$ is nothing but $\frac{1}{2}\alpha\frac{\partial W}{\partial g_i}$.

Substituting in the sum that we must evaluate, we find:

$$\frac{1}{2}\alpha\left(a_1\frac{\partial\Psi}{\partial e_1}+b_1\frac{\partial\Psi}{\partial g_3}+c_1\frac{\partial\Psi}{\partial g_2}\right)$$
$$+\frac{1}{2}\beta\left(a_1\frac{\partial\Psi}{\partial g_3}+b_1\frac{\partial\Psi}{\partial e_2}+c_1\frac{\partial\Psi}{\partial g_1}\right)$$
$$+\frac{1}{2}\gamma\left(a_1\frac{\partial\Psi}{\partial g_2}+b_1\frac{\partial\Psi}{\partial g_1}+c_1\frac{\partial\Psi}{\partial e_3}\right).$$

The coefficient of α_1 is:

$$\frac{1}{2}\left(\alpha\frac{\partial\Psi}{\partial e_1}+\beta\frac{\partial\Psi}{\partial g_3}+\gamma\frac{\partial\Psi}{\partial g_2}\right).$$

However, if one returns to the formulas (**14**) that define e_1, e_2, e_3, g_1, g_2, g_3 as functions of L, M, N then we see that this expression is equal to $\frac{1}{2}\frac{\partial W}{\partial L}$. Likewise, the coefficient of b_1 is $\frac{1}{2}\frac{\partial W}{\partial M}$, and that of c_1 is $\frac{1}{2}\frac{\partial W}{\partial N}$. The sum of the terms that contain a_1, b_1, c_1 explicitly is therefore:

$$\frac{1}{2}\left(a_1\frac{\partial\Psi}{\partial L}+b_1\frac{\partial\Psi}{\partial M}+c_1\frac{\partial\Psi}{\partial N}\right).$$

Finally, if we take into account the formulas (**15**) by which one defines L, M, N then we see that this expression represents:

$$\frac{1}{2}\frac{\partial W}{\partial\lambda}.$$

The desired equation and the two other analogous ones result from the last to equations of (**11**), which may thus be written:

$$(\mathbf{16})\qquad\begin{cases}\lambda\theta^2=\lambda Q+\dfrac{1}{2}\dfrac{\partial W}{\partial\lambda}\\[2mm]\mu\theta^2=\mu Q+\dfrac{1}{2}\dfrac{\partial W}{\partial\mu}\\[2mm]\nu\theta^2=\nu Q+\dfrac{1}{2}\dfrac{\partial W}{\partial\nu}.\end{cases}$$

They show that λ, μ, ν are proportional to the direction cosines of a principal direction of the quadric that is represented (λ, μ, ν being regarded as the coordinates, and $\alpha, \beta, \gamma, a, b, c, x, y, z, a_i, b_i, c_i$ being regarded as constant) by the equation:

$$(\mathbf{17})\qquad \Pi(\lambda, \mu, \nu) = Q(\lambda^2 + \mu^2 + \nu^2) + \Psi(e_1, e_2, e_3, g_1, g_2, g_3) = 1 .$$

This quadric is the *polarization ellipsoid*, which is analogous to the one that we spoke of in no. **263**.

268. – We thus find a result that is completely similar to the one that we already know in the case of infinitely small deformations, but which must be stated here in a form that is a little more precise since there is reason to distinguish between the initial state and the present state of the body considered. Since the segment (l, m, n) is, as we know, defined in the space of the present positions of the molecules, the statement is:

The same direction of a wave is capable of propagating three different directions of discontinuities, which are orthogonal to each other in the deformed medium.

269. – In addition, equations (**14**) tell us the values of the velocity of propagation. They are the square roots of the three roots of the equation in s relative to the quadric that we just spoke of. In order for them to be real it is necessary and sufficient that this quadric be a real ellipsoid.

We shall confirm that this condition is always satisfied in the cases that might present themselves.

In order to see what this circumstance is due to, first consider the case of liquids. We have seen that the velocity of propagation then has the quantity $dp/d\rho$ for its square. Now, the condition that this quantity be positive is nothing but the stability condition for internal equilibrium. It expresses that when a decrease in the volume is imposed on the gas, it produces an increase in the pressure, i.e., a change of the internal forces of such a nature as to oppose the modification that is so produced.

Conforming to what was established for the case of systems that depend upon a finite number of parameters, we assume that it is necessary for stability that the elastic energy actually have minima (instead of only having a null first variation), or, at the least, that its second variation must not become negative. By this means, we shall express the stability of the equilibrium of a fixed body in terms of all of the points on its surface, in the absence of forces X, Y, Z.

If we now apply the operation δ to the first variation (**9**) then there will be two types of terms under the $\iiint$ sign: The ones that one obtains by differentiating δa_i, δb_i, δc_i, and the ones that one obtains by differentiating the factors $\partial W/\partial a_i$, …

Just as it happens in all of the analogous cases in the calculus of variations, the first category of terms gives a null sum. Indeed, one may subject it to the same transformations as the first variation itself, which gives a result that is identical the one that we obtained previously, δx, δy, δz being simply replaced by $\delta^2 x$, $\delta^2 y$, $\delta^2 z$. Since these latter variations are null, like the first variations, on the surface (since its points are assumed to be fixed) the sum in question disappears, by virtue of equations (**10**).

What remains is the triple integral:

$$(\mathbf{18})\qquad \iiint\left[\delta a_1\cdot\delta\left(\frac{\partial W}{\partial a_1}\right)+\delta b_1\cdot\delta\left(\frac{\partial W}{\partial b_1}\right)+\cdots+\delta b_3\cdot\delta\left(\frac{\partial W}{\partial b_3}\right)+\delta c_3\cdot\delta\left(\frac{\partial W}{\partial c_3}\right)\right]\rho\,dx\,dy\,dz\,.$$

270. – The quantity under the ∭ sign is a quadratic form with respect to δa_i, δb_i, δc_i . If this form is positive definite for any value of *a*, *b*, *c* then the same is true for the preceding integral.

The converse is not exactly valid: Although the integral (**18**) must be essentially positive, it does not have to follow that the same thing is true for its differential element. However, we shall see, by contrast, that they must take only positive values whenever the variations δa_i, … have the form:

$$\begin{cases} \delta a_1 = \lambda\alpha, & \delta b_1 = \lambda\beta, & \delta c_1 = \lambda\gamma, \\ \delta a_2 = \mu\alpha, & \delta b_2 = \mu\beta, & \delta c_2 = \mu\gamma, \\ \delta a_3 = \nu\alpha, & \delta b_3 = \nu\beta, & \delta c_3 = \nu\gamma, \end{cases} \tag{19}$$

and this must be true for any $\lambda, \mu, \nu, \alpha, \beta, \gamma$. In other words, whenever the δa_i, … satisfy equations ([12]):

$$(\mathbf{19}') \qquad \delta a_1\, \delta b_2 - \delta b_1\, \delta a_2 = 0, \qquad \delta a_1\, \delta b_2 - \delta b_1\, \delta a_2 = 0, \ \ldots, \qquad \delta b_2\, \delta c_3 - \delta b_3\, \delta c_2 = 0 .$$

To that effect, we remark that the values thus written have an interpretation that one may immediately perceive. They coincide with the values of the sharp variations of the quantities a_i, b_i, c_i in a first order discontinuity that is defined on the wave surface considered.

In other words, in order to pass from the present position of the medium to an infinitely close position that corresponds to the variations (**19**), it will suffice to subject this medium to a deformation of the type that was considered in no. **56** and has (λ, μ, ν) for its characteristic segment.

Having said this, we make a small portion of the surface Σ, whose tangent plane has the direction cosines α, β, γ, pass through a particular interior point of our solid. If these quantities, when combined with three conveniently chosen values of λ, μ, ν, give values for the expressions (**19**) that make the differential element of (**18**) negative at the point considered then one may take Σ to be sufficiently small for the same circumstance to occur at all of the points of that portion of the surface.

Having thus chosen Σ once and for all, we consider it to be the base of a cylinder *C* of height *h*. If we suppose that its interior is subjected to a deformation of the type that was studied in no. **56**, the surface whose points remain fixed being Σ, and the characteristic segment being (λ, μ, ν) then the maximum displacement thus obtained will be of order *h*. It is easy to see that one may then determine the deformation of the rest of the solid in such a manner that: 1. The points of the exterior surface remain fixed. 2. The continuity of the displacement is conserved on the surface of the cylinder *C*, in other words, that δx, δy, δz do not change values for a point of that surface depending upon whether one

([12]) Conversely, one proves that if one obtains a positive definite form upon adding a linear combination of the left-hand sides of equations (**19′**) to the quadratic form that figures in the integral (**18**) then the integral is indeed a minimum (at least when one takes it over a sufficiently restricted volume). However, it remains for us to examine whether the sufficient condition thus formulated is equivalent to the necessary condition that was obtained in the text.

considers that points to be interior or exterior to C. 3. *δx, δy, δz and their first order partial derivatives* are everywhere (outside of C) quantities that are very small of order h.

Under these conditions, the integral (**18**), when taken over the exterior of C, will be of order h^2. On the contrary, for the interior of C (in which δa_i, ... have essentially the values that were determined in (**19**)), they will be negative and of order h. They will thus be negative when taken together.

In order for this to actually mean something, it must, consequently, be the case that the element in the integral (**18**) should not become negative when one gives a_i, b_i, c_i the values (**19**), as we have said.

271. – If we now substitute the values (**13**) for $\frac{\partial W}{\partial a_i}, \frac{\partial W}{\partial b_i}, \frac{\partial W}{\partial c_i}$ then we see that $\delta\left(\frac{\partial W}{\partial a_i}\right)$, for example, will contain two types of terms: ones that are obtained by taking the variations of the first factor, and which give:

$$\frac{\partial W}{\partial \varepsilon_1}\delta a_i + \frac{\partial W}{\partial \gamma_3}\delta b_i + \frac{\partial W}{\partial \gamma_2}\delta c_i,$$

and the ones that are obtained by applying the operation δ to the factors $\frac{\partial W}{\partial \varepsilon_1}$, …

Now, we know for example, that:

$$\delta\left(\frac{\partial W}{\partial \varepsilon_1}\right) = \frac{\partial^2 W}{\partial \varepsilon_1^2}\delta\varepsilon_1 + \frac{\partial^2 W}{\partial \varepsilon_1\,\partial \varepsilon_2}\delta\varepsilon_2 + \frac{\partial^2 W}{\partial \varepsilon_1\,\partial \varepsilon_3}\delta\varepsilon_3 + \frac{\partial^2 W}{\partial \varepsilon_1\,\partial \gamma_1}\delta\gamma_1 + \frac{\partial^2 W}{\partial \varepsilon_1\,\partial \gamma_2}\delta\gamma_2 + \frac{\partial^2 W}{\partial \varepsilon_1\,\partial \gamma_3}\delta\gamma_3.$$

If we consider the quadratic form $\Psi(e_1, e_2, e_3, g_1, g_2, g_3)$ that has been in question all along then it is clear that the preceding expression $\frac{1}{2}\frac{\partial W}{\partial \varepsilon_1}$ represents, provided that e_i, g_i are replaced by $\delta\varepsilon_i$, $\delta\gamma_i$, respectively. One therefore has:

$$\delta\frac{\partial W}{\partial a_i} = \frac{\partial W}{\partial \varepsilon_1}\delta a_i + \frac{\partial W}{\partial \gamma_3}\delta b_i + \frac{\partial W}{\partial \gamma_2}\delta c_i + \frac{1}{2}a_i\frac{\partial \Psi}{\partial(\varepsilon_1)} + \frac{1}{2}b_i\frac{\partial \Psi}{\partial(\gamma_3)} + \frac{1}{2}c_i\frac{\partial \Psi}{\partial(\gamma_2)}.$$

If one remarks that one has:

$$\begin{aligned}
\delta\varepsilon_1 &= a_1\,\delta a_1 + a_2\,\delta a_2 + a_3\,\delta a_3,\\
\delta\varepsilon_2 &= b_1\,\delta b_1 + b_2\,\delta b_2 + b_3\,\delta b_3,\\
\delta\varepsilon_3 &= c_1\,\delta c_1 + c_2\,\delta c_2 + c_3\,\delta c_3,\\
\delta\gamma_1 &= b_1\,\delta c_1 + c_1\,\delta b_1 + b_2\,\delta c_2 + c_2\,\delta b_2 + b_3\,\delta c_3 + c_3\,\delta b_3,\\
&\dots
\end{aligned}$$

and we find, in summation:

$$\frac{1}{2}\delta\varepsilon_1\frac{\partial\Psi}{\partial(\varepsilon_1)}+\frac{1}{2}\delta\varepsilon_2\frac{\partial\Psi}{\partial(\varepsilon_2)}+\frac{1}{2}\delta\varepsilon_3\frac{\partial\Psi}{\partial(\varepsilon_3)}+\frac{1}{2}\delta\gamma_1\frac{\partial\Psi}{\partial(\gamma_1)}+\frac{1}{2}\delta\gamma_2\frac{\partial\Psi}{\partial(\gamma_2)}+\frac{1}{2}\delta\gamma_3\frac{\partial\Psi}{\partial(\gamma_3)}$$
$$= \Psi(\delta\varepsilon_1,\ \delta\varepsilon_2,\ \delta\varepsilon_3,\ \delta\gamma_1,\ \delta\gamma_2,\ \delta\gamma_3)\ .$$

Finally, the quantity under the ∭ sign in the second variation will be:

$$(\mathbf{20})\qquad \begin{cases} \dfrac{\partial W}{\partial\varepsilon_1}(\delta a_1^2+\delta a_2^2+\delta a_3^2)+\dfrac{\partial W}{\partial\varepsilon_2}(\delta b_1^2+\delta b_2^2+\delta b_3^2)+\dfrac{\partial W}{\partial\varepsilon_3}(\delta c_1^2+\delta c_2^2+\delta c_3^2) \\ +2\dfrac{\partial W}{\partial\gamma_1}(\delta b_1\,\delta c_1+\delta b_2\,\delta c_2+\delta b_3\,\delta c_3)+2\dfrac{\partial W}{\partial\gamma_2}(\delta c_1\,\delta a_1+\delta c_2\,\delta a_2+\delta c_3\,\delta a_3) \\ +2\dfrac{\partial W}{\partial\gamma_3}(\delta a_1\,\delta b_1+\delta a_2\,\delta b_2+\delta a_3\,\delta b_3)+\Psi(\delta\varepsilon_1,\delta\varepsilon_2,\delta\varepsilon_3,\delta\gamma_1,\delta\gamma_2,\delta\gamma_3). \end{cases}$$

It is this quantity that must be positive when one gives δa_i, δb_i, δc_i, the values (**19**) in order for there to be stability.

The quantities $\delta\varepsilon_i$, $\delta\gamma_i$ take precisely the values e_i, g_i, that were defined by formulas (**14**), (**15**), and consequently the expression (**20**) will become identical to the left-hand side of (**17**). We thus obtain the desired conclusion precisely: from the stability of the internal equilibrium it results that the velocities of propagation for the various waves are real.

Moreover, if we assume that the expression (**20**) may not likewise be annulled under the indicated condition unless $\lambda = \mu = \nu = 0$ or $\alpha = \beta = \gamma = 0$, then these velocities of propagation always remain finite.

272. – In the case of hydrodynamics, in which $W = F(D)$, the element $[F'(D)\delta^2 D + F''(D)\delta D^2]\,\rho\,dx\,dy\,dz$ in the second variation will reduce to:

$$F''(D)(\delta D)^2\,\rho\,dx\,dy\,dz$$

and the quantity $\delta^2 D$ reduces, as is easy to insure, to a linear combination of quadratic forms that define the left-hand sides of equations (**19′**). The condition of stability is thus (as we stated in no. **131**) precisely $F'' > 0$ or $\dfrac{dp}{d\rho} > 0$.

273. – The foregoing considerations provide a simple interpretation for the left-hand side of equation (**17**).

Indeed, replace λ, μ, ν with $\lambda\,dt$, $\mu\,dt$, $\nu\,dt$ in formulas (**19**), in which we let dt denote the differential of a parameter. Under these conditions, the deformation will be infinitely small, and, just as we remarked in no. **113** (cont.), the increases in ε_i, γ_i will be precisely $e_i\, dt$, $g_i\, dt$. Suppose, in addition, and this is obviously compatible with the hypotheses that we just made, that the second derivatives of x, y, z – and consequently also those of a_i, b_i, c_i – with respect to t are null. For example, let x, y, z have the values:

$$x = x_0 + \lambda\, t\, f(a, b, c), \qquad y = y_0 + \lambda\, t\, f, \qquad z = z_0 + \lambda\, t\, f\,,$$

in which $f(a, b, c)$ represents the left-hand side of the equation for the wave surface, which is defined as we explained in no. **80**.

The coefficient of $t^2/2$ in the development of W will then be the quantity $\Pi(\lambda, \mu, \nu)$ that has occupied our attention.

If, as a consequence, we envision a small volume $d\tau$ around the point considered that we subject to the deformation that we just defined then the coefficient of $t^2/2$ in the value for the elastic energy thus generated will be the product of $\rho\, d\tau$ with the left-hand side of the equation for the polarization ellipsoid.

274. – We have seen above that in the case of infinitely small deformations of an isotropic body there exist two types of waves, ones that are exclusively longitudinal and ones that are exclusively transversal.

Does this theorem persist in the case of finite deformations?

This question may be regarded as a particular case of another more general theorem. Indeed, one knows that the optics of crystalline solids leads us to consider – and to the exclusion of all others – elastic media that are or are not isotropic, and for which an analogous decomposition into longitudinal and transversal waves has meaning.

The determination of the form of the function W for which this is the case is well known when one is concerned with infinitely small deformations, i.e., when one supposes that W is a quadratic form with respect to e_i, g_i. We propose to carry out this same determination in the general case.

The direction cosines of the wave surface in the deformed medium are proportional to the quantities l, m, n that are defined by the equations:

$$\textbf{(21)} \qquad \begin{cases} \alpha = la_1 + ma_2 + na_3, \\ \beta = lb_1 + mb_2 + nb_3, \\ \gamma = lc_1 + mc_2 + nc_3. \end{cases}$$

It must then be the case that equations (**16**) are satisfied when one replaces λ, μ, ν with l, m, n in them. Moreover, one must suppress the first terms of the left-hand sides in these equations, and write:

$$\text{(22)}\qquad \begin{cases} sl = \dfrac{1}{2}\dfrac{\partial\Psi}{\partial l} \\ sm = \dfrac{1}{2}\dfrac{\partial\Psi}{\partial m} \\ sn = \dfrac{1}{2}\dfrac{\partial\Psi}{\partial n}. \end{cases}$$

Since the terms lQ, mQ, nQ (which provide the quantity $Q(\lambda^2 + \mu^2 + \nu^2)$ that appears in Π) will only change the value of s by a quantity that is equal to Q without modifying the principal directions.

We observe that if l, m, n are given by relations (**21**) then the quantities L, M, N are nothing but α, β, γ. We carry out the calculation in such a manner as to introduce these quantities, to the exclusion of l, m, n. To that effect, we multiply equations (**22**), first by a_1, a_2, a_3, then by b_1, b_2, b_3, and finally by c_1, c_2, c_3, respectively. One will then obtain:

$$\begin{aligned} s\,(a_1 l + a_2 m + a_3 n) = sL &= \frac{1}{2}\left(a_1\frac{\partial\Psi}{\partial l} + a_2\frac{\partial\Psi}{\partial m} + a_3\frac{\partial\Psi}{\partial n}\right) \\ s\,(b_1 l + b_2 m + b_3 n) = sM &= \frac{1}{2}\left(b_1\frac{\partial\Psi}{\partial l} + b_2\frac{\partial\Psi}{\partial m} + b_3\frac{\partial\Psi}{\partial n}\right) \\ s\,(c_1 l + c_2 m + c_3 n) = sN &= \frac{1}{2}\left(c_1\frac{\partial\Psi}{\partial l} + c_2\frac{\partial\Psi}{\partial m} + c_3\frac{\partial\Psi}{\partial n}\right). \end{aligned}$$

If we now replace the derivatives $\dfrac{\partial\Psi}{\partial l}, \dfrac{\partial\Psi}{\partial m}, \dfrac{\partial\Psi}{\partial n}$ by their expressions with the aid of the derivatives taken with respect to L, M, N then we will have (in regard to the formulas that define ε_i, γ_i):

$$\begin{aligned} sL &= \frac{1}{2}\left[(1+2\varepsilon_1)\frac{\partial\Psi}{\partial L} + \gamma_3\frac{\partial\Psi}{\partial M} + \gamma_2\frac{\partial\Psi}{\partial N}\right] \\ sM &= \frac{1}{2}\left[\gamma_3\frac{\partial\Psi}{\partial L} + (1+2\varepsilon_2)\frac{\partial\Psi}{\partial M} + \gamma_1\frac{\partial\Psi}{\partial N}\right] \\ sN &= \frac{1}{2}\left[\gamma_2\frac{\partial\Psi}{\partial L} + \gamma_1\frac{\partial\Psi}{\partial M} + (1+2\varepsilon_3)\frac{\partial\Psi}{\partial N}\right]. \end{aligned}$$

We solve these equations for $\dfrac{\partial\Psi}{\partial L}, \dfrac{\partial\Psi}{\partial M}, \dfrac{\partial\Psi}{\partial N}$. This solution introduces the minors E_i, G_i of the determinant:

$$\text{(23)}\qquad D^2 = \begin{vmatrix} 1+2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1+2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1+2\varepsilon_3 \end{vmatrix}$$

with respect to the elements $1 + 2\,\varepsilon_i$, γ_i, respectively; the coefficients of sL, sM, sN in the values of $\frac{1}{2}\frac{\partial\Psi}{\partial L}, \frac{1}{2}\frac{\partial\Psi}{\partial M}, \frac{1}{2}\frac{\partial\Psi}{\partial N}$ are the quantities $\frac{E_i}{D^2}, \frac{G_i}{D^2}$. Upon introducing, instead of s, the number:

$$h = \frac{D^2}{s},$$

and upon letting Φ denote the form:

$$\Phi(p, q, r) = E_1\,p^2 + E_3\,q^2 + E_3\,r^2 + 2\,G_1\,qr + 2\,G_1\,rp + 2\,G_3\,pq\,,$$

i.e., the form that is adjoint to the one that gives the line element of the deformed medium, then the values in question will be:

$$\frac{1}{2}\frac{\partial\Psi}{\partial L} = \frac{1}{2}k\frac{\partial\Phi}{\partial L},$$
$$\frac{1}{2}\frac{\partial\Psi}{\partial M} = \frac{1}{2}k\frac{\partial\Phi}{\partial M},$$
$$\frac{1}{2}\frac{\partial\Psi}{\partial N} = \frac{1}{2}k\frac{\partial\Phi}{\partial N},$$

and the relations that we just wrote will be satisfied this time on the condition that one sets α, β, γ equal to L, M, N, respectively.

This substitution must be applied only after performing the differentiations. If, on the contrary, one immediately sets $\alpha = L$, $\beta = M$, $\gamma = N$ then one will introduce too many terms into the left-hand side as a result of the differentiations with respect to α, β, γ. However, the values of e_1, e_2, e_3, g_1, g_2, g_3, which only appear in Ψ, are symmetric with respect to the two systems of quantities L, M, N and α, β, γ. The terms thus introduced will be equal to the ones that originally existed, respectively, and will have the effect of doubling in value. We may thus replace the preceding equations with:

$$\text{(24)}\qquad \begin{cases} \frac{1}{2}\frac{\partial\Psi}{\partial L} = k\frac{\partial\Phi}{\partial L} \\ \frac{1}{2}\frac{\partial\Psi}{\partial M} = k\frac{\partial\Phi}{\partial M} \\ \frac{1}{2}\frac{\partial\Psi}{\partial N} = k\frac{\partial\Phi}{\partial N} \end{cases}$$

in which Ψ will now have the value that was obtained by replacing α, β, γ with L, M, N before any differentiation, i.e., by setting:

$$\text{(25)}\qquad e_1 = L^2, \quad e_2 = M^2, \quad e_3 = N^2, \quad g_1 = 2MN, \quad g_2 = 2NL, \quad g_3 = 2LM.$$

Along with relations (**24**), this time we are concerned with identities that are meaningful for all of the values of the independent variables *L, M, N* that figure in them. As one knows, these relations express the idea that Ψ is a function of Φ, and since the first of these two expressions is a homogeneous polynomial of fourth degree and the second one, a homogeneous polynomial of second degree, one necessarily has:

$$\Psi(L^2, M^2, N^2, 2MN, 2LN, 2LM) = h\,\Phi(L, M, N)^2 ,$$

in which h is independent of *L, M, N*.

275. – We must now demand to know what the quadratic form Ψ must be in order that it reduce to $h\Phi^2$ when one replaces e_i , g_i by the values in (**25**), respectively. This will be the case, not only if Ψ is equal to $h\Psi_0$, when one sets:

$$\Psi_0 = (E_1 e_1 + E_2\, e_2 + E_3\, e_3 + G_1\, g_1 + G_2\, g_2 + G_3\, g_3)^2 ,$$

but also if Ψ is equal to an arbitrary linear combination of $h\Psi_0$ and the six forms:

(**26**)
$$\begin{cases} 4e_2e_3 - g_1^2, & 4e_3e_1 - g_2^2, & 4e_1e_2 - g_3^2, \\ g_2g_3 - 2e_1g_1, & g_3g_1 - 2e_2g_2, & g_1g_2 - 2e_3g_3. \end{cases}$$

This condition is not only sufficient, but necessary. In order to convince oneself of this, it suffices to express directly that the form of fourth degree that is obtained by replacing by replacing the e_i, g_i in $\Psi - h\Psi_0$ with their values in (**25**) is identically null.

276. – Having thus obtained the expression for Ψ, it remains for us to return to that of W, for which it is clear that one therefore has a system of second order partial differential equations. The integration of this system is, moreover, completely elementary, and it will suffice for us to summarize the process.

Since the forms (**26**) lack terms in e_1^2, $e_1\, g_3$, $e_1\, g_2$ these terms must have values in Ψ that are proportional to the ones that they have in Ψ_0, and consequently the derivatives of W must satisfy the relations:

(**27**)
$$\frac{\partial^2 W}{\partial \varepsilon_1^2} : E_1^2 = \frac{\partial^2 W}{\partial \varepsilon_1\, \partial \gamma_3} : E_1 G_1 = \frac{\partial^2 W}{\partial \varepsilon_1\, \partial \gamma_2} : E_1 G_2 = h\, .$$

Since one has (from (**23**)):

$$E_1 = \frac{1}{2}\frac{\partial (D^2)}{\partial \varepsilon_1}, \qquad G_3 = \frac{1}{2}\frac{\partial (D^2)}{\partial \gamma_3}, \qquad G_2 = \frac{1}{2}\frac{\partial (D^2)}{\partial \gamma_2},$$

these relations show that the derivative $\frac{\partial W}{\partial \varepsilon_1}$ may be written:

$$\frac{\partial W}{\partial \varepsilon_1} = \text{funct.}\ (D,\ \gamma_1,\ \varepsilon_2,\ \varepsilon_3)\ .$$

By taking into account analogous relations that relate to the derivatives $\frac{\partial W}{\partial \varepsilon_2}, \frac{\partial W}{\partial \varepsilon_3}$, one will easily verify that one may write:

$$\frac{\partial W}{\partial \varepsilon_1} = a\ (1 + 2\varepsilon_2)\ (1 + 2\varepsilon_3) + a_1$$
$$\frac{\partial W}{\partial \varepsilon_2} = a\ (1 + 2\varepsilon_3)\ (1 + 2\varepsilon_1) + a_2$$
$$\frac{\partial W}{\partial \varepsilon_3} = a\ (1 + 2\varepsilon_1)\ (1 + 2\varepsilon_2) + a_3\ ,$$

in which a is a function of D, whereas a_1, a_2, a_3 may contain, in addition, γ_1, γ_2, γ_3, respectively.

On may thus introduce the other terms of the form Ψ, in turn: for example, one will write that the coefficient of $e_2\ e_3$ plus four times the coefficient of g_1^2 gives a sum that has the same value in Ψ as in $h\Psi$ (the former form (**26**) is eliminated in this combination).

One will thus easily arrive at the general expression for the function W, which is:

$$\text{(28)} \qquad \begin{cases} W = F(D) + a_{11}(\gamma_1^2 - 4\varepsilon_2\varepsilon_3) + a_{22}(\gamma_2^2 - 4\varepsilon_3\varepsilon_1) + a_{33}(\gamma_3^2 - 4\varepsilon_1\varepsilon_2) \\ +2a_{23}(2\varepsilon_1\gamma_1 - \gamma_2\gamma_3) + 2a_{31}(2\varepsilon_2\gamma_2 - \gamma_3\gamma_1) + 2a_{12}(2\varepsilon_3\gamma_3 - \gamma_1\gamma_2) + P, \end{cases}$$

in which the a_{ik} are constants and P is an arbitrary first degree polynomial in the ε_i, γ_i ([13]).

It is only when W has the preceding form that the polarization ellipsoid has an axis that is normal to the wave.

277. – The hypothesis that the solid is isotropic in its natural state expresses the idea that the properties of the body must not change when one performs an orthogonal coordinate transformation on a, b, c. The function W that represents the elastic energy must therefore not be modified by such a transformation.

([13]) h then has the value $\frac{1}{D}\frac{d}{dD}\left(\frac{F'(D)}{D}\right)$. Since the quantity $Q(\lambda^2 + \mu^2 + \nu^2)$ of no. **266-267** is, as one easily recognizes, proportional to Ψ if the a_{ik} are null, one confirms that the terms in $F''(D)$ disappear from the equation of the polarization ellipsoid, and one comes back to the precisely the same expression for the element of the second variation that was calculated in no. **271** that was obtained in no. **272** for the case of liquids.

Now, under this transformation the coefficients ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 of the quadric $\varphi = 1$ that was introduced in no. **51** vary. However, as one knows, three quantities remain invariant: They are the coefficients of the equation in *s* relative to that quadric, i.e., the expressions:

$$A = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$
$$B = (1 + 2\varepsilon_2)\,(1 + 2\varepsilon_3) - \gamma_1^2 + (1 + 2\varepsilon_3)\,(1 + 2\varepsilon_1) - \gamma_2^2 + (1 + 2\varepsilon_1)\,(1 + 2\varepsilon_3) - \gamma_3^2$$
$$D^2\ .$$

The isotropy of the body considered is expressed by the fact that *W* depends only upon the previous three quantities.

Now, it does not in any way result from this that *W* is necessarily of the form (**28**).

As a consequence, the conclusion that was established for the case of infinitely small deformations does not extend to finite deformations. For them, the waves that propagate inside an isotropic body are not, in general, longitudinal or transversal.

CHAPTER VII

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THE GENERAL THEORY OF CHARACTERISTICS

§ 1. – CHARACTERISTICS AND BICHARACTERISTICS

278. – We have seen that the propagation of waves in the rectilinear motion of a gas is related to the properties of the characteristics of second order partial differential equations in two independent variables.

In a completely analogous fashion, the study of waves in a three-dimensional space is not distinct from the theory of the theory of generalized characteristics, as Beudon ([14]) showed in the case of an arbitrary number of independent variables and extended to systems of several unknowns.

As in the case of two variables, this theory follows from the discussion of the Cauchy problem.

To fix ideas, take a second order equation that we suppose, in addition, to be linear with respect to the second derivatives, in such a way that one has the form:

$$\textbf{(1)} \qquad \sum_{i,k} a_{ik}\, p_{ik} + l = 0$$

in which p_{ik} denotes the partial derivative $\dfrac{\partial^2 z}{\partial x_i \partial x_k}$ of the unknown function z with respect to the independent variables x_i and x_k (which may or may not be different). We suppose that there are n of these independent variables $x_1, x_2, \ldots, x_n$, in such a way that the indices i and k take the values 1, 2, …, n independently of each other.

As for a_{ik} and l, they are functions of z, $x_1, x_2, \ldots, x_n$ and the first derivatives $p_1, p_2, \ldots, p_n$ of z with respect to $x_1, x_2, \ldots, x_n$.

279. – Consider the $n-1$-extended multiplicity – or *hypersurface* – that is represented by the equation:

$$\textbf{(2)} \qquad x_n = f(x_1, x_2, \ldots, x_{n-1}).$$

Let $P_1, P_2, \ldots, P_{n-1}$ be the partial derivatives of x_n with respect to $x_1, x_2, \ldots, x_{n-1}$ that are deduced from equation (**2**). If U is an arbitrary function of $x_1, x_2, \ldots, x_{n-1}, x_n$ and this latter quantity is replaced by its value in equation (**2**) then U will be a function of $x_1, x_2, \ldots, x_{n-1}$ on M_{n-1}. We let d denote the derivatives of U under this new hypothesis. It is clear that they are related to the former derivatives by the relations:

([14]) *Bull. Soc. Math. Fr.*, 1897, pp. 108-120.

(**3**) $$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + P_i \frac{\partial}{\partial x_n}, \qquad (i = 1, 2, \ldots, n\text{-}1).$$

For the function z one will therefore have:

(**4**) $$\frac{dz}{dx_i} = p_i + P_i\, p_n,$$

and, for $U = p_k$:

(**5**) $$\frac{dp_k}{dx_i} = p_{ik} + P_i\, p_{kn}, \qquad \begin{pmatrix} i = 1,2,\cdots,n-1 \\ j = 1,2\cdots,n \end{pmatrix}.$$

In a general manner, if we let the notation $p_{ik\ldots h}$ denote the derivative $\frac{\partial^\mu z}{\partial x_i \partial x_j \cdots \partial x_h}$ with respect to the μ variables (which may or may be different) x_i, x_j, …, x_h then one will have, for $U = p_{kh}$:

(**5′**) $$\frac{dp_{kh}}{dx_i} = p_{ikh} + P_i\, p_{knn},$$

and so on for the derivatives of all orders.

279 (cont.). – Having said this, we imagine that we are given the *Cauchy conditions* at every point of M_{n-1}, namely, the values of z and its first derivatives. Of course, they must satisfy the relation:

$$dz = p_1 dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n$$

on M_{n-1}, i.e., the relations (**4**) (in such a way that it will suffice to give z and p_n, in reality).

We seek to determine the second derivatives of z. They must verify equations (**5**), and it is easy to see that, in general, they will be determined once one has added equation (**1**). Indeed, if we first consider relations (**5**), in which the index k has the value n, then these relations will be give us:

(**6**) $$p_{in} = \frac{dp_n}{dx_i} - P_i\, p_{nn}.$$

On the contrary, if we suppose that k is different from n then we will have (upon permuting the indices i and k):

$$p_{ik} = \frac{dp_i}{dx_k} - P_k\, p_{in}\,,$$

and, on account of (**6**):

(**6′**)
$$p_{ik} = \frac{dp_i}{dx_k} - P_k\, \frac{dp_n}{dx_k} + P_i P_k\, p_{nn}\,.$$

All of the second derivatives are thus expressed as a function of p_{nn}. Finally, we substitute these expressions into the given equation; we will thus have a result of the form:

(**7**)
$$Ap_{nn} + K = 0,$$

in which A and K will have the values:

(**8**)
$$\begin{cases} A = \sum_{i,k=1}^{n-1} a_{ik} P_i P_k - \sum_{i=1}^{n-1} a_{in} P_i + a_{nn}, \\ \quad = \sum' a_{ik} P_i P_k - \sum' a_{in} P_i + a_{nn}\,, \end{cases}$$

(**8′**)
$$K = \sum' a_{ik} \left(\frac{dp_i}{dx_k} - P_k\, \frac{dp_n}{dx_i} \right) + \sum' a_{in}\, \frac{dp_n}{dx_i} + l,$$

in which the notation $\sum'$ denotes a summation in which one does not give the value n to the indices of the variables.

Suppose that A is different from zero. The preceding equation will determine p_{nn} for us, and, as a result, all of the second-order derivatives.

280. – We pass on to the calculation of the third derivatives. The relations (**5′**) permit us to calculate all of these derivatives as a function of only p_{nnn}. For this, we first make two, then one, of the indices i, k, and h equal to n. We will then have relations that are evidently distinguished from (**6**) and (**6′**) only by the index n that is added to each letter p, and which will give us, as a consequence:

(**9**)
$$\begin{cases} p_{inn} = \dfrac{dp_{nn}}{dx_i} - P_i\, p_{nnn}\,, \\ p_{ikn} = \dfrac{dp_{nn}}{dx_k} - P_k\, \dfrac{dp_{nn}}{dx_i} + P_i P_k\, p_{nnn}\,, \end{cases}$$

in which one will deduce the derivatives in which no differentiation index is equal to n by a third application of formula (**5′**).

On the other hand, we obtain relations between the desired derivatives by differentiating the given equation (**1**). However, it suffices to write just one of them. All of the others reduce to the first one by means of the relations (**5**), (**5′**). Now, if we let $\mathcal{F}$

denote the left-hand side of equation (**1**) then one may differentiate the equation $\mathcal{F} = 0$ on M_{n-1}, since it is satisfied at each point of M_{n-1}, and one will have:

$$0 = \frac{d\mathcal{F}}{dx_i} = \frac{\partial\mathcal{F}}{\partial x_i} + P_i \frac{\partial\mathcal{F}}{\partial x_n},$$

which shows precisely that the condition $d\mathcal{F} / dx_n = 0$ implies that $d\mathcal{F} / dx_i = 0$ for all values of i .

Now, if we differentiate equation (**1**) with respect to x_n then the result obtained will evidently be of the form:

(**10**) $$\sum a_{ik}\, p_{ikn} + l_1 = 0,$$

in which l_1 is a function of the x's , as well as z and only its first and second derivatives ([15]). If we then compare the system of linear equations (**9**), (**10**) to the system of equations (**1**), (**5**) then we see that they are identical, up to constant terms, once each unknown p_{ik} is replaced by p_{ikn} . As a consequence, when one expresses the latter as a function of p_{nnn} by means of relations (**9**), the equation for p_{nnn} will be:

(**11**) $$Ap_{nnn} + K_1 = 0,$$

in which:

(**11′**) $$K_1 = \sum' a_{ik} \left(\frac{dp_{in}}{dx_k} - P_k \frac{dp_{nn}}{dx_i} \right) + \sum' a_{ik} \frac{dp_{nn}}{dx_i} + l_1$$

is a function of the x_i, z, p_i, p_{ik} . The necessary and sufficient condition for conditions (**9**) and (**10**) to determine the third derivatives is therefore once more that $A \neq 0$.

The calculation of the fourth, fifth, etc., derivatives is completely analogous to the foregoing. For each order, one has an unknown that is determined by a first-degree equation in which the coefficient of that unknown is always the same quantity A. All of these unknowns are thus well defined, with only the condition that $A \neq 0$.

281. – One can arrive at the same result by a change of variable. Indeed, replace x_n with the new independent variable:

$$x'_n = x_n - f(x_1, x_2, \cdots, x_{n-1}).$$

The new equation in M_{n-1} will be $x'_n = 0$, and the partial differential equation, when referred to this new system of variables, will be $\mathcal{F}' = 0$. One may calculate all of the

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([15]) Similarly, l_1 will be *linear* with respect to the p_{ik} if the a_{ik} are independent of the first derivatives of z.

successive derivatives as functions of z and $\partial z / \partial x_n'$ if the equation $\mathcal{F}' = 0$ is soluble with respect to the derivative $\dfrac{\partial^2 z}{\partial x_n'^2}$.

Now, if one reverts to the old variables then it is obvious from the foregoing, and easy to verify directly, that the condition:

$$\frac{\partial \mathcal{F}'}{\partial\left(\dfrac{\partial^2 z}{\partial x_n'^2}\right)} \neq 0$$

thus obtained gives $A \neq 0$.

One thus arrives at the same conclusion as always. However, one may, moreover, obtain another one that is just as important. Indeed, from the proof of Kowalewski, one knows that if z and $\partial z / \partial x_n'$ are regular analytic functions of $x_1, x_2, \ldots, x_{n-1}$ for $x_n' = 0$, and the function $\mathcal{F}'$ is analytic and regular with respect to the quantities that figure in it, then the problem will admit a solution z that is analytic and regular in $x_1, x_2, \ldots, x_{n-1}, x_n'$. This result obviously carries over to the given system of variables. In other words, the successive derivatives whose calculation we just indicated *are the coefficients of a Taylor development that is convergent* for sufficiently small values of the arguments.

282. – Now suppose that one has the relation:

$$A = 0, \tag{12}$$

for any ([16]) hypersurface M_{n-1}.

Therefore, in order for the problem to be possible, or at least in order for there to exist a solution z that admits derivatives of all orders on M_{n-1}, it is necessary that the series of given values of $z, p_1, p_2, \ldots, p_n$ satisfies the condition $K = 0$, which may be written, upon replacing the p_i with their values as functions of p_n, as specified by (**4**) ([17]):

$$K = \Sigma' a_{ik}\left(\frac{d^2 z}{dx_i\, dx_k} - P_i \frac{dp_n}{dx_k} - P_k \frac{dp_n}{dx_i} - P_{ik}\, p_n\right) + \Sigma' a_{in} \frac{dp_n}{dx_i} + l = 0. \tag{13}$$

283. – On the contrary, if one considers a *given* solution of equation (**1**) for a moment, then the condition $A = 0$ is a first-order partial differential equation with respect to x_n, when considered to be a function of the $x_1, x_2, \ldots, x_{n-1}$. The multiplicities (**2**) that verify this equation will be called *characteristics* of the given equation.

([16]) We will not treat the case here in which A is null on just a subset of M_{n-1} (namely, on an n–2-times extended multiplicity that belongs to M_{n-1}) that corresponds to a singularity (compare ch. IV, no. **233**) if K is different from zero, and which we recall later on (nos. **316-318**) when K is null.

([17]) Of course P_{ik} denotes the derivative $\dfrac{d^2 x_n}{dx_i\, dx_k}$.

It is important to remark that in order to construct the characteristics, it does not suffice, in general, to be given just equation (**1**). The characteristics are defined only for a particular integral of that equation since the coefficients depend not only on the x's, but also on z and its derivatives. The only exception is for equations of a particular form in which the coefficients a_{ik} of the second-order terms are functions of only the x's.

As a first-order equation, the partial differential equation $A = 0$ itself admits characteristics ([18]) that are no longer $n-1$-times extended multiplicities, but *lines* (one-dimensional multiplicities) that are defined by the ordinary differential equations:

$$\frac{dx_1}{\left(\dfrac{\partial A}{\partial P_1}\right)} = \frac{dx_2}{\left(\dfrac{\partial A}{\partial P_2}\right)} = \cdots = \frac{dx_{n-1}}{\left(\dfrac{\partial A}{\partial P_{n-1}}\right)} = ds, \tag{14}$$

which further implies:

$$ds = \frac{dx_n}{P_1 \dfrac{\partial A}{\partial P_1} + P_2 \dfrac{\partial A}{\partial P_2} + \cdots + P_{n-1} \dfrac{\partial A}{\partial P_{n-1}}}. \tag{14'}$$

The lines likewise play an essential role in the present theory; we call them *bicharacteristics* – or *rays* – by reason of their physical significance, as we shall see later on.

Any characteristic hypersurface M_{n-1} is related to the bicharacteristics, with one of them passing through each point of M_{n-1}.

284. – The bicharacteristics cease to be defined in a case that we exclude – at least, for the moment: the case in which $\partial A / \partial P_i$ is null for all values that the index i may take.

If one considers $P_1, P_2, \ldots, P_{n-1}$ to be Cartesian coordinates and equation (**12**) to represent a surface, then, as we know, this case will correspond to the existence of a multiple point on the surface in question. By analogy, we say that M_{n-1} is a *multiple characteristic*, with its order of multiplicity being that of the point $(P_1, P_2, \ldots, P_{n-1})$ on the surface (**12**).

285. – Condition (**13**) already introduced the bicharacteristics. Indeed, the coefficient of dp_n / dx_i in that equation is:

$$-\sum{}' a_{ik} P_k + a_{in} = -\frac{1}{2}\frac{\partial A}{\partial P_i}. \tag{15}$$

One may set:

([18]) See GOURSAT, *Leçons sur l'integration des équations aux dérivées partielle du premier ordre.*

(**15′**)
$$\begin{cases} K = -\dfrac{1}{2}\sum{}' \dfrac{dp_n}{dx_i}\dfrac{\partial A}{\partial P_i} + L, \\ L = \sum{}' a_{ik}\left(\dfrac{d^2 z}{dx_i\,dx_n} - P_{ik}\,p_n\right) + l. \end{cases}$$

Therefore, if one is first given the distribution of values of z on the multiplicity (**2**), which is assumed to be characteristic ([19]), then condition (**13**) will give a linear partial differential equation that will determine p_n , and whose characteristics are the curves (**14**), precisely.

286. – We return to the Cauchy problem and suppose that $A = 0$ and the condition (**13**) is likewise verified. Thus, equation (**7**) no longer determines p_{nn} . As we already saw in the case of two variables, this quantity cannot be assumed to be completely arbitrary. Indeed, for $A = 0$ equation (**11**) is likewise impossible or undetermined, and the condition of possibility is:

$$K_1 = 0.$$

Now, if one operates on the expression (**11′**) for K_1 as we already did on the expression for K, i.e., if one replaces the p_{in} with their values as functions of p_{nn} that are obtained from (**6**), then one will obviously find:

(**16**)
$$\begin{cases} 0 = K_1 = -\dfrac{1}{2}\sum{}' \dfrac{dp_{nn}}{dx_i}\dfrac{\partial A}{\partial P_i} + L_1, \\ L_1 = \sum{}' a_{ik}\left(\dfrac{d^2 p_n}{dx_i\,dx_k} - P_{ik}\,p_{nn}\right) + l_1. \end{cases}$$

When p_{nn} is considered to be a function of $x_1, x_2, \ldots, x_{n-1}$ on the multiplicity M_{n-1}, it therefore satisfies a linear first-order partial differential equation.

The characteristics of this latter equation are nothing but the bicharacteristics that are situated on M_{n-1}.

If one lets ds denote the common value of the ratios (**14**) (in which s is a parameter that defines a variable point of the bicharacteristic) then equation (**16**) will become:

$$\frac{1}{2}\frac{dp_{nn}}{ds} - L_1 = 0,$$

which is, as one sees, a first-order differential equation in p_{nn} , when considered to be a function of s.

([19]) Nevertheless, one must observe that the characteristics may not be defined without being given the p_i , since the a_{ik} are not independent of these quantities.

Therefore, one may take p_{nn} arbitrarily at only *one* point of the bicharacteristic. In other words, if we trace a multiplicity M_{n-2} on M_{n-1} that meets each bicharacteristic at one and only one point then p_{nn} will be arbitrary only on M_{n-2} and not on M_{n-1}.

Once p_{nn} has been chosen, p_{nnn} will be determined not only by equation (**11**), but by the conditions that relate to the fourth derivatives. Now, the equations that determine them are identical with the ones that determine the third derivatives, up to a term that contains only the first and second derivatives, by replacing the p_{ik} with p_{ikn}, and the p_{ikn} with p_{iknn}. We will thus have a linear first-order partial differential equation for p_{nnn}, when considered on M_{n-1}, which is derived from (**16**) by the same substitution, except for the change in the term in which p_{nnn} is not differentiated (a term that will be linear, not quadratic, with respect to p_{nnn}); in this way, p_{nnn}, like p_{nn}, may be chosen arbitrarily at each point of each bicharacteristic.

It is clear that completely similar considerations may be applied to subsequent derivatives of all orders.

287. – We have supposed that the equation for M_{n-1} may be solved for x_n. If this equation is taken in an arbitrary form:

$$\Pi(x_1, x_2, \dots, x_n) = 0 \tag{2'}$$

then the partial derivatives $P_1, P_2, \dots, P_{n-1}$ of x_n with respect to $x_1, x_2, \dots, x_{n-1}$ may be expressed as functions of the partial derivatives $\pi_1, \pi_2, \dots, \pi_n$ of Π with respect to $x_1, x_2, \dots, x_{n-1}, x_n$ with the aid of the formulas:

$$P_i = -\frac{\pi_i}{\pi_n}, \qquad (i = 1, 2, \dots, n-1), \tag{17}$$

in such a way that the quantity A, which must be null in order for M_{n-1} to be characteristic, will be ([20]):

$$A = \sum_{i,k=1}^{n} a_{ik} \pi_i \pi_k . \tag{18}$$

One may, moreover, make this substitution in the series of calculations that led us to the equation of the characteristics. For example, consider relations (**5**); by means of the substitution (**17**) they become:

$$\pi_n \frac{dp_k}{dx_i} = \pi_n p_{ik} - \pi_i p_{kn} . \tag{19}$$

For arbitrary n, the same circumstance presents itself every time that the left-hand side of the equation is a linear function of the determinant:

([20]) This new quantity A is equal to the old one multiplied by π_n^2.

$$\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{vmatrix},$$

and its minors ([21]).

In particular, this relates to any equation that one deduces from an equation of the form (**1**) by a contact transformation. It is obvious *a priori* (compare ch. IV, no. **162**) that the preceding conclusions must persist for an equation so obtained, and, similarly, that *the characteristics and the bicharacteristics are preserved by the transformation.*

290. – As we saw in no. **161** in the case of two variables, condition (**12**) is the one that M_{n-1} must verify in order for the two integrals of the equation to be mutually tangent at all points of this multiplicity, at least when this contact is not of order infinity.

Moreover, this notion is equivalent to that of wave propagation when it is applied to the motions that may be considered as depending on only a single unknown function.

For example, consider the motions of a gas that are derived from a velocity potential Φ. The components of velocity then depend on the first derivatives of that potential, and the same is true for the pressure, from the equation ([22]):

$$V - \int \frac{dp}{\rho} = \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right]. \tag{23}$$

Suppose that two motions of this type present a discontinuity of order m ($m \geq 2$) between them. This order will also be that of the first derivatives of the potential that are discontinuous.

If x, y, z, t, Φ are considered to be five coordinates then each of the two motions will be represented by a surface in the space of five dimensions, two surfaces that have a common contact of order m. Moreover, both of them must satisfy the differential equation of motion, namely ([23]):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \Phi}{\partial z}\right) = 0, \tag{23'}$$

in which ρ must be replaced by its value from (**23**).

The contact multiplicity:

$$\varphi(x, y, z, t) = 0 \tag{24}$$

([21]) These equations have been studied in a general manner by Goursat (*Bull. Soc. Math. Fr.,* tome XXVII, pp. 1-34; 1899).

([22]) See, for example, KIRCHHOFF, *Mécanique,* 15th lesson.

([23]) KIRCHHOFF, *loc. cit.*

must therefore be a characteristic of that equation. Now, the second-order terms in that equation are:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\frac{\partial\Phi}{\partial x}+\frac{\partial}{\partial y}\frac{\partial\Phi}{\partial y}+\frac{\partial}{\partial z}\frac{\partial\Phi}{\partial z}\right)^2\Phi-\frac{dp}{d\rho}\Delta\Phi,$$

in such a way that we must have:

$$\textbf{(25)}\qquad \left(\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+v\frac{\partial\varphi}{\partial y}+w\frac{\partial\varphi}{\partial z}\right)^2=\frac{dp}{d\rho}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right].$$

This equation is equivalent to formula (**5**) (no. **240**), which gives us the displacement velocity of the wave.

The lines of reasoning by which we obtained these two formulas are, moreover, analogous, although this analogy does not seem to be complete, since we introduced only the velocity potential, instead of the coordinates *x, y, z, t,* which are considered to be functions of *a, b, c, t.* For example, consider equations (**20**) or (**20′**). They say that for two integrals that agree on the multiplicity (**2′**), with identity in their first derivatives, the differences between the second derivatives are like the squares of the products of pairs of partial derivatives in the left-hand side of (**2′**). This fact is nothing but the one we established in no. **97** ([24]).

291. – In order to apply the theory of characteristics to the study of the most general motions of a gas, it is necessary to extend it to the case of systems of equations, the number of them being assumed to be equal to the number of unknown functions. This is a case in which the theorem of Cauchy and Kowalewski continues to apply, at least when one assumes, on the one had, that all of the givens are analytic, and, on the other hand, that one excludes certain exceptional cases (the ones in which it is impossible to solve with respect to the highest-order derivatives that belong to the various desired functions, respectively) that will not occur in the problems that will occupy our attention.

Recall that, contrary to what happens for ordinary differential equations, the case of several partial differential equations that number as many as the unknown variables is essentially distinct from the case of one equation. It is impossible to reduce the one to the other by eliminating the one or more unknowns. Indeed, one does not therefore obtain a unique equation that would determine the remaining unknown, but a system of equations, the discussion of which will be, from the standpoint of the existence of and search for their common solutions, more complicated than that of the original system.

To fix ideas, we take the case that is presented most commonly in mechanics, that of three equations in three unknowns ξ, η, ζ, and we further suppose that the equations are second order and linear in the second derivatives p_{ik} of ξ, the second derivatives q_{ik} of η, and the second derivatives r_{ik} of ζ. They may therefore be written:

([24]) Nevertheless, the theory of characteristics does not dispense with the lemma of no. **72**, a lemma that was implicitly assumed in what we just said.

(**26**)
$$\begin{cases} \sum a_{ik} p_{ik} + \sum b_{ik} q_{ik} + \sum c_{ik} r_{ik} + l = 0, \\ \sum a'_{ik} p_{ik} + \sum b'_{ik} q_{ik} + \sum c'_{ik} r_{ik} + l' = 0, \\ \sum a''_{ik} p_{ik} + \sum b''_{ik} q_{ik} + \sum c''_{ik} r_{ik} + l'' = 0, \end{cases}$$

in which $a_{ik}, a'_{ik}, \cdots, c_{ik}, c'_{ik}, c''_{ik}; l, l', l''$ depend on the unknown functions, their first derivatives (those of ξ are denoted by $p_1, p_2, \ldots, p_n$, those of η by $q_1, q_2, \ldots, q_n$, and those of ζ by $r_1, r_2, \ldots, r_n$), and the independent variables, which are always the x's.

We further consider the multiplicity M_{n-1}, on which we suppose that we are given the values of ξ, η, ζ, and the first derivatives (or, more precisely, the p_n, q_n, r_n). Since the q_{ik}, r_{ik} satisfy equations that are completely similar to (**6**), (**6′**), one may apply the transformations that we performed in nos. **279**, **282** to the terms that contain them, and the given equations consequently take the form (compare (**7**), (**15**)):

(**27**)
$$\begin{cases} Ap_{nn} + Bq_{nn} + Cr_{nn} - \Sigma' \frac{1}{2} \frac{dp_n}{dx_i} \frac{\partial A}{\partial P_i} - \Sigma' \frac{1}{2} \frac{dq_n}{dx_i} \frac{\partial B}{\partial P_i} \\ \qquad - \Sigma' \frac{1}{2} \frac{dr_n}{dx_i} \frac{\partial C}{\partial P_i} + L = 0, \\ A'p_{nn} + B'q_{nn} + C'r_{nn} - \Sigma' \frac{1}{2} \frac{dp_n}{dx_i} \frac{\partial A'}{\partial P_i} - \Sigma' \frac{1}{2} \frac{dq_n}{dx_i} \frac{\partial B'}{\partial P_i} \\ \qquad - \Sigma' \frac{1}{2} \frac{dr_n}{dx_i} \frac{\partial C'}{\partial P_i} + L' = 0, \\ A''p_{nn} + B''q_{nn} + C''r_{nn} - \Sigma' \frac{1}{2} \frac{dp_n}{dx_i} \frac{\partial A''}{\partial P_i} - \Sigma' \frac{1}{2} \frac{dq_n}{dx_i} \frac{\partial B''}{\partial P_i} \\ \qquad - \Sigma' \frac{1}{2} \frac{dr_n}{dx_i} \frac{\partial C''}{\partial P_i} + L'' = 0, \end{cases}$$

in which A, B, C denote the quantities:

$$\begin{aligned} A &= \Sigma' a_{ik} P_i P_k - \Sigma' a_{in} P_i + a_{nn}, \\ B &= \Sigma' b_{ik} P_i P_k - \Sigma' b_{in} P_i + b_{nn}, \\ C &= \Sigma' c_{ik} P_i P_k - \Sigma' c_{in} P_i + c_{nn}, \end{aligned}$$

and $A', B', C', A'', B'', C''$ are completely analogous quantities that are formed from the second and third equations, and:

(**28**)
$$\begin{cases} L = \Sigma' a_{ik} \left(\frac{d^2 \xi}{dx_i dx_k} - P_{ik} p_n \right) + \Sigma' b_{ik} \left(\frac{d^2 \eta}{dx_i dx_k} - P_{ik} q_n \right) \\ \qquad \Sigma' c_{ik} \left(\frac{d^2 \zeta}{dx_i dx_k} - P_{ik} r_n \right) + l, \end{cases}$$

as well as the analogous quantities L', L'' are functions of p_n, q_n, r_n, and the distribution of values of ξ, η, ζ on M_{n-1}.

Moreover, the condition for the search for the second derivatives to be an impossible or undetermined problem is:

$$H = \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0. \tag{29}$$

As one sees, one therefore has a partial differential equation that is of first order, but sixth degree.

292. – If we first place ourselves in the most general case, the one in which the multiplicity M_{n-1} is *characteristic* – i.e., it verifies the equation $H = 0$ – then the minors of the determinant H are not all null at an arbitrary point of this multiplicity. This makes the condition for the system to be indeterminate (and not impossible) with respect to p_n, q_n, r_n unique, namely, a certain equation of the form:

$$\Sigma'\left(\lambda_i \frac{dp_n}{dx_i} + \mu_i \frac{dq_n}{dx_i} + \nu_i \frac{dr_n}{dx_i}\right) + \sigma = 0, \tag{30}$$

which is linear with respect to the derivatives of p_n, q_n, r_n taken on M_{n-1}.

This time, one may choose two of the three first derivatives p_n, q_n, r_n arbitrarily at each point of the multiplicity M_{n-1}, and then determine the third one with this condition. However, in this case, the characteristics of the first-order linear equations thus obtained are not in the least the analogs of the bicharacteristics that were always defined in the case of just one equation. They do not coincide with the lines that we shall encounter in the calculation of the third derivatives, and which will be the true bicharacteristics. Furthermore, the characteristics of the equation in r_n will not be the same as those of the equation in p_n or q_n .

In a word, since we stopped at the second derivatives the calculation presents itself in a very different manner depending on whether one is dealing with one or several equations.

293. – If we assume that condition (**30**) is verified then the solution of the system (**27**) will be indeterminate. If $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$ are the minors of H relative to the elements $A, B, C, A', B', C', A'', B'', C''$, respectively, in which α, for example, is assumed to be different from 0, then all of the solutions of that system can be summarized by the formula:

$$\begin{cases} p_{nn} = p_{nn}^0 + \alpha\rho, \\ q_{nn} = q_{nn}^0 + \beta\rho, \\ r_{nn} = r_{nn}^0 + \gamma\rho, \end{cases} \tag{31}$$

in which $(p_{nn}^0, q_{nn}^0, r_{nn}^0)$ is one of these solutions and ρ is an arbitrary parameter.

We pass on to the third derivatives. From the preceding calculations we performed, it will suffice to find p_{nnn}, q_{nnn}, r_{nnn}, which will be given by the equations:

$$\textbf{(32)}\qquad \begin{cases} Ap_{nnn} + Bq_{nnn} + Cr_{nnn} - \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\frac{\partial A}{\partial P_i} - \sum'\frac{1}{2}\frac{dq_{nn}}{dx_i}\frac{\partial B}{\partial P_i} \\ \qquad - \sum'\frac{1}{2}\frac{dr_{nn}}{dx_i}\frac{\partial C}{\partial P_i} + L_1 = 0, \\ A'p_{nnn} + B'q_{nnn} + C'r_{nnn} - \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\frac{\partial A'}{\partial P_i} - \sum'\frac{1}{2}\frac{dq_{nn}}{dx_i}\frac{\partial B'}{\partial P_i} \\ \qquad - \sum'\frac{1}{2}\frac{dr_{nn}}{dx_i}\frac{\partial C'}{\partial P_i} + L_1' = 0, \\ A''p_{nnn} + B''q_{nnn} + C''r_{nnn} - \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\frac{\partial A''}{\partial P_i} - \sum'\frac{1}{2}\frac{dq_{nn}}{dx_i}\frac{\partial B''}{\partial P_i} \\ \qquad - \sum'\frac{1}{2}\frac{dr_{nn}}{dx_i}\frac{\partial C''}{\partial P_i} + L_1'' = 0. \end{cases}$$

in which L_1, L_1', L_1'' denote the new terms that are quadratic in p_{nn}, q_{nn}, r_{nn} with known coefficients. The condition of possibility for this system is obtained by multiplying the first equation by α, the second one by α', and the third one by α''; it thus follows that p_{nnn}, q_{nnn}, r_{nnn} disappear, with the result that:

$$\textbf{(32)}\qquad \begin{cases} \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\left(\alpha\frac{\partial A}{\partial P_i} + \alpha'\frac{\partial A'}{\partial P_i} + \alpha''\frac{\partial A''}{\partial P_i}\right), \\ + \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\left(\alpha\frac{\partial A}{\partial P_i} + \alpha'\frac{\partial A'}{\partial P_i} + \alpha''\frac{\partial A''}{\partial P_i}\right), \\ + \sum'\frac{1}{2}\frac{dp_{nn}}{dx_i}\left(\alpha\frac{\partial A}{\partial P_i} + \alpha'\frac{\partial A'}{\partial P_i} + \alpha''\frac{\partial A''}{\partial P_i}\right), \\ \qquad - (\alpha L_1 + \alpha' L_1' + \alpha'' L_1'') = 0. \end{cases}$$

We need to substitute the values for p_{nn}, q_{nn}, r_{nn} that are given by formulas (**31**) in this equation. It is clear that we will thus obtain an (inhomogenous) linear first-order partial differential equation in ρ in order to determine ρ, in which the coefficient of $d\rho / dx_i$ is:

$$\frac{1}{2}\alpha\left(\alpha\frac{\partial A}{\partial P_i} + \alpha'\frac{\partial A'}{\partial P_i} + \alpha''\frac{\partial A''}{\partial P_i}\right) + \frac{1}{2}\beta\left(\beta\frac{\partial B}{\partial P_i} + \beta'\frac{\partial B'}{\partial P_i} + \beta''\frac{\partial B''}{\partial P_i}\right)$$
$$+ \frac{1}{2}\gamma\left(\gamma\frac{\partial C}{\partial P_i} + \gamma'\frac{\partial C'}{\partial P_i} + \gamma''\frac{\partial C''}{\partial P_i}\right).$$

However, from well-known identities, the condition $H = 0$ entails that:

$$\beta\alpha' = \alpha\beta', \quad \beta\alpha'' = \alpha\beta'', \quad \gamma\alpha' = \alpha\gamma', \quad \gamma\alpha'' = \alpha\gamma''.$$

We may therefore put α into the factor, and the coefficient of $1/2\alpha$, namely:

$$\alpha\frac{\partial A}{\partial P_i} + \alpha'\frac{\partial A'}{\partial P_i} + \alpha''\frac{\partial A''}{\partial P_i} + \beta\frac{\partial B}{\partial P_i} + \beta'\frac{\partial B'}{\partial P_i} + \beta''\frac{\partial B''}{\partial P_i} + \gamma\frac{\partial C}{\partial P_i} + \gamma'\frac{\partial C'}{\partial P_i} + \gamma''\frac{\partial C''}{\partial P_i},$$

is nothing but $\partial H/\partial P_i$.

Therefore, the characteristics of the equation in ρ are:

(**33**)
$$\frac{dx_1}{\dfrac{\partial H}{\partial P_1}} = \frac{dx_2}{\dfrac{\partial H}{\partial P_2}} = \cdots = \frac{dx_{n-1}}{\dfrac{\partial H}{\partial P_{n-1}}};$$

in other words, they are identical to those of equation (**29**).

One obviously recovers these same lines in the subsequent calculations of the higher-order derivatives. They are the ones that we call the bicharacteristics of the given system.

294. – The case that we now treat is that of the equations of hydrodynamics, at least as far as the propagation of discontinuities is concerned.

Indeed, as in the foregoing, it is clear, first of all, that the multiplicity $\mathcal{S}_0$ that expresses, as was explained in no. **96**, the propagation of a wave in time is necessarily a characteristic of the system of equations of the motion.

On the other hand, if two motions of a gaseous mass both propagate along a wave then we know that the discontinuity that exists between them is normal to that wave at each point. Therefore, if one is given one of the motions then the values of the second derivatives of the other one depend upon only one unknown at each point, namely, the magnitude of the discontinuity in question. This amounts to saying that the solution of the system (**27**) involves only one arbitrary unknown, and, as a consequence, that at least one of the minors of the determinant of H is different from zero.

If we take the equations of hydrodynamics in the Euler form then the independent variables are the present coordinates $x, y, z,$ and time t, and the equation for M_{n-1} must be written in the form:

$$\varphi(x, y, z, t) = 0.$$

The partial differential equation that the function φ satisfies – which will give $\dfrac{\partial\varphi}{\partial t}$ as a function of $\dfrac{\partial\varphi}{\partial x}, \dfrac{\partial\varphi}{\partial y}, \dfrac{\partial\varphi}{\partial z}$ – will therefore give us the displacement velocity of the wave. Effectively, if one forms the determinant H for the Euler equations, which are four first-order equations in four unknowns u, v, w, ρ then one comes down to equation (**25**), multiplied by a factor of $\left(\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+v\frac{\partial\varphi}{\partial y}+w\frac{\partial\varphi}{\partial z}\right)^2$ (which corresponds to stationary waves, which we shall discuss later on).

On the contrary, if we employ Lagrange variables and solve the equation for M_{n-1} for t, namely:

$$t = f(a, b, c), \tag{34}$$

then the characteristic equation will give us the velocity of propagation:

$$\theta = \frac{1}{\sqrt{f_a^2 + f_b^2 + f_c^2}}, \tag{35}$$

when referred to the initial state considered. We thus come down to formula (**4**) of no. **240**; indeed, we also have that the preceding calculations, when applied to equations (**1**) and (**3**) of ch. III, give ([25]), conforming to the formula in question, as well as relation (**35**):

$$\frac{1}{D^2}\frac{dp}{d\rho}\Phi(f_a, f_b, f_c) - 1 = 0. \tag{29′}$$

If the equation for M_{n-1} is taken in the form:

$$f(a, b, c) = 0, \tag{34′}$$

without being solved for t, then one will obtain the same equation (up to the replacement of the second term with f_t^2), multiplied by the factor f_t^2, which again corresponds to stationary waves.

295. – Moreover, under these conditions one sees quite well that the calculations by which one arrives at the result are not distinct from the ones that were carried out in chap. V. Indeed, one must write x for the unknown in equations of the type (**20′**), and analogous equations for y and z, in which the parameter λ will be replaced by μ or ν. Now, it immediately appears that one thus obtains the *kinematical compatibility conditions* that were the object of chap. II, and which we adjoined to the dynamical equations of motion ([26]).

([25]) To that effect (as we said in no. **124**), one must express the derivatives $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ with the aid of the derivatives with respect to a, b, c, and, on the other hand, take into account the remark that was made in the note on page (?).

([26]) Meanwhile, we must remark that the considerations of chap. II–V do not give the interpretation in a form that presents the terms as all known (independently of the p_{ik}) and, as a consequence, does not

296. – One will obtain the value of the velocity of propagation such as was given by formula (**3**) (no. **239**) upon taking the initial state to be the present state; moreover, since the form $\Phi(f_a, f_b, f_c)$ that figures in formula (**29′**) then reduces to $f_a^2 + f_b^2 + f_c^2$, we immediately obtain the tangent to the bicharacteristic at the instant considered, namely:

$$\frac{da}{f_a} = \frac{da}{f_b} = \frac{da}{f_c} = \frac{dt}{\sqrt{\dfrac{dp}{d\rho}(f_a^2 + f_b^2 + f_c^2)}}.$$

Therefore, *the bicharacteristic is normal to the wave when referred to an initial state that coincides with the present state at the instant and point considered.*

297. – If we pass from the equations of hydrodynamics to those of elasticity then we may likewise apply the foregoing considerations – at least when the coefficients of elasticity are completely arbitrary. Indeed, in general, the directions of the discontinuities that are compatible with a given wave surface are finite in number – equal to three – and each of them corresponds to a different velocity of propagation. In other words, when one gives the characteristic multiplicity $\mathcal{S}_0$ that represents the propagation of the wave, the direction of the discontinuity is determined. We may therefore reason as we did at the beginning of no. **294**.

298. – Things are otherwise in the case of an isotropic body whose deformation is assumed to be infinitesimal. Indeed, we have seen that the velocity of propagation in such a body has only two possible values (instead of three). The first corresponds to longitudinal waves, to which we can apply all of what we just said. On the contrary (with the notations of no. **260**), the other, which is equal to M/ρ, agrees with the transversal waves, and *an arbitrary transversal discontinuity may therefore propagate.* In other words, if we consider equations (**5**) of no. **260**, equations whose determinant is:

$$(\mathbf{36})\qquad \left\{\begin{aligned} &\begin{vmatrix} \rho\theta^2 - M - (L+M)\alpha^2 & -(L+M)\alpha\beta & -(L+M)\alpha\gamma \\ -(L+M)\alpha\beta & \rho\theta^2 - M - (L+M)\beta^2 & -(L+M)\beta\gamma \\ -(L+M)\alpha\gamma & -(L+M)\beta\gamma & \rho\theta^2 - M - (L+M)\gamma^2 \end{vmatrix} \\ &\qquad = (\rho\theta^2 - M)^2 (L + 2M - \rho\theta^2), \end{aligned}\right.$$

then *the factor $\rho\theta^2 - M$ will be common to this determinant and all of its minors.*

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permit us to find equations in which these terms intervene, such as equation (**30**) (no. **292**). There is a lacuna in all of this that will undoubtedly be interesting to fill.

Moreover, just as it results from the preceding developments – and as one immediately verifies – if one replaces the unknowns λ, μ, ν with $\frac{\delta^2\xi}{\delta t^2}, \frac{\delta^2\eta}{\delta t^2}, \frac{\delta^2\zeta}{\delta t^2}$, and the quantities α, β, γ, θ with:

$$\frac{f_x}{\sqrt{f_x^2+f_y^2+f_z^2}},\quad \frac{f_y}{\sqrt{f_x^2+f_y^2+f_z^2}},\quad \frac{f_z}{\sqrt{f_x^2+f_y^2+f_z^2}},\quad \frac{1}{\sqrt{f_x^2+f_y^2+f_z^2}},$$

in these linear equations then, up to terms that are independent of the unknowns, the equations thus obtained are nothing but the ones that one arrives at by substituting the second derivatives that are derived from (**6**), (**6′**) – i.e., equations (**28**) (with the equation of the wave being $t = f(x, y, z)$) – into the equations of motion themselves.

One therefore sees that the determinant *H, as well as those of all of its minors,* is null on transversal waves that propagate in isotropic elastic bodies. This is likewise obvious for the stationary transversal waves of hydrodynamics, as one confirms by performing the calculations of no. **294** without omitting these waves, i.e., on equation (**34′**), and not on the equation that is obtained by solving for *t*.

299. – It is therefore necessary to study, in turn, the systems for which these circumstances present themselves.

We then find ourselves in a previously excluded case (no. **284**) in the study of a single equation: that of a multiple characteristic. Indeed, it is clear that all of the quantities $\partial H/\partial P_i$ are null ([27]).

The preceding theories are, in general, invalid on a multiple characteristic. However, this is not the case if this characteristic nullifies all of the minors of the determinant *H*, and if its *rank* – i.e., the number of rows and columns that must be suppressed in the determinant in question in order to find a minor that is different from zero – is equal to its order of multiplicity. This is what is established in the work that was cited for Goursat ([28]) for the case of two independent variables.

Later on (no. **327**), we shall recover a result that is equivalent to the result that we just obtained for the case of arbitrary *n*. However, for our present objective we will be obliged to make an extra hypothesis.

Indeed, in the case that was envisioned in the preceding no., the double characteristics have the same degree of generality as the others; like them, they are defined by *just one* first-order partial differential equation.

We limit ourselves to the – obviously, very particular – case in which this condition is satisfied; more exactly, the one in which all of the minors of the determinant *H* are nullified, not only on the characteristic considered, but also on all characteristics that are infinitely close to the first.

([27]) This is because the factor $\rho\theta^2 - M$ figures as a square in expression (**36**).

([28]) *Equations aux dérivées partielles du second ordre,* tome II, note *xi*.

Therefore, recall the system of equations and the system (**27**) in p_{nn}, q_{nn}, r_{nn} that is a consequence of it, and suppose that the determinant H is null, along with all of its minors, and that this circumstance is true not only on M_{n-1}, but also on all of its neighboring characteristics.

The system (**27**) will then have two conditions of possibility, but if they are satisfied then the three equations that it refers to will reduce to just one, which will determine r_{nn}, for example, as a function of p_{nn} and q_{nn}; up to a known term, one will have:

$$r_{nn} = -\frac{A}{C}p_{nn} - \frac{B}{C}q_{nn}.$$

Likewise, equations (**32**) will have two conditions of possibility that we obtain, for example, upon multiplying the first of them by C'', and the third one by $-C$, and adding them, and then doing the same thing with the last two equations and the coefficients C'', $-C$. We thus find:

$$\sum{}'\frac{1}{2}\left(C''\frac{\partial A}{\partial P_i} - C\frac{\partial A''}{\partial P_i}\right)\frac{dp_{nn}}{dx_i} + \sum{}'\frac{1}{2}\left(C''\frac{\partial B}{\partial P_i} - C\frac{\partial B''}{\partial P_i}\right)\frac{dq_{nn}}{dx_i}$$
$$\sum{}'\frac{1}{2}\left(C''\frac{\partial C}{\partial P_i} - C\frac{\partial C''}{\partial P_i}\right)\frac{dr_{nn}}{dx_i} + CL_1'' - C''L_1 = 0,$$
$$\sum{}'\frac{1}{2}\left(C''\frac{\partial A'}{\partial P_i} - C\frac{\partial A''}{\partial P_i}\right)\frac{dp_{nn}}{dx_i} + \sum{}'\frac{1}{2}\left(C''\frac{\partial B'}{\partial P_i} - C\frac{\partial B''}{\partial P_i}\right)\frac{dq_{nn}}{dx_i}$$
$$\sum{}'\frac{1}{2}\left(C''\frac{\partial C'}{\partial P_i} - C'\frac{\partial C''}{\partial P_i}\right)\frac{dr_{nn}}{dx_i} + C'L_1'' - C''L_1' = 0,$$

(the third derivatives are eliminated by virtue of the relations:

$$\text{(37)} \qquad \alpha = \alpha' = \alpha'' = \beta = \beta' = \beta'' = \gamma = \gamma' = \gamma'' = 0).$$

If we replace r_{nn} with its value as a function of p_{nn}, q_{nn} then we will have two partial differential equations for them:

$$\text{(38)} \qquad \begin{cases} \sum{}'\frac{1}{2}\left[C''\frac{\partial A}{\partial P_i} - C\frac{\partial A''}{\partial P_i} - \frac{A}{C}\left(C''\frac{\partial C}{\partial P_i} - C\frac{\partial C''}{\partial P_i}\right)\right]\frac{dp_{nn}}{dx_i} \\ + \sum{}'\frac{1}{2}\left[C''\frac{\partial B}{\partial P_i} - C\frac{\partial B''}{\partial P_i} - \frac{B}{C}\left(C''\frac{\partial C}{\partial P_i} - C\frac{\partial C''}{\partial P_i}\right)\right]\frac{dq_{nn}}{dx_i} + \cdots = 0, \\ \sum{}'\frac{1}{2}\left[C''\frac{\partial A'}{\partial P_i} - C'\frac{\partial A''}{\partial P_i} - \frac{A}{C}\left(C''\frac{\partial C'}{\partial P_i} - C'\frac{\partial C''}{\partial P_i}\right)\right]\frac{dp_{nn}}{dx_i} \\ + \sum{}'\frac{1}{2}\left[C''\frac{\partial B'}{\partial P_i} - C'\frac{\partial B''}{\partial P_i} - \frac{B}{C}\left(C''\frac{\partial C'}{\partial P_i} - C'\frac{\partial C''}{\partial P_i}\right)\right]\frac{dq_{nn}}{dx_i} + \cdots = 0, \end{cases}$$

in which we have replaced all of the terms that do not contain derivatives of p_{nn}, q_{nn} by an ellipsis, terms whose form is not actually important.

300. – In appearance, these equations have a much more complicated form than the preceding ones, since we are in the presence of two partial differential equations in two unknowns p_{nn} , q_{nn} . Nevertheless, like the former they reduce to ordinary differential equations.

Indeed, if we take the relation $AC'' = CA''$ into account then the coefficient of dp_{nn}/dx_i in the first of them may be written:

$$\frac{1}{2}\left(C''\frac{\partial A}{\partial P_i} - C\frac{\partial A''}{\partial P_i} - A''\frac{\partial C}{\partial P_i} - A\frac{\partial C''}{\partial P_i}\right).$$

However, this is nothing but $\frac{1}{2}\frac{\partial}{\partial P_i}(AC'' - CA'') = \frac{1}{2}\frac{\partial \beta'}{\partial P_i}$.

Similarly, the coefficient of $\frac{dp_{nn}}{dx_i}$ in the second equation (**38**) is:

$$\frac{1}{2}\left(C''\frac{\partial A'}{\partial P_i} - C'\frac{\partial A''}{\partial P_i} - A''\frac{\partial C'}{\partial P_i} - A'\frac{\partial C''}{\partial P_i}\right) = -\frac{1}{2}\frac{\partial \beta}{\partial P_i},$$

(since one likewise has $AC' = CA'$); meanwhile, the analogous coefficients of $\frac{dq_{nn}}{dx_i}$ are $-\frac{1}{2}\frac{\partial \alpha'}{\partial P_i}, \frac{1}{2}\frac{\partial \alpha}{\partial P_i}$.

Now, we have supposed that the relations (**37**) are true, not only on M_{n-1}, but also on all of the infinitesimally close characteristics. This obviously demands that the expressions $\alpha, \alpha', \cdots$, which are polynomials in P_1, P_2, …, P_{n-1}, have a common factor H_1, since the characteristics in question are represented by the equation $H_1 = 0$. We may therefore set:

$$\alpha = H_1\mathcal{A},\quad \alpha' = H_1\mathcal{A}',\quad \alpha'' = H_1\mathcal{A}'',\quad \beta = H_1\mathcal{B},\quad \beta' = H_1\mathcal{B}',\quad \beta'' = H_1\mathcal{B}'',$$
$$\gamma = H_1\mathcal{C},\quad \gamma' = H_1\mathcal{C}',\quad \gamma'' = H_1\mathcal{C}'',$$

and we suppose that the quantities $\mathcal{A}$, …, $\mathcal{C}''$ are not always annulled at an arbitrary point of our characteristic.

Since H_1 is null, the derivative $\frac{\partial \beta'}{\partial P_i}$ reduces to $\mathcal{B}' \frac{\partial H_1}{\partial P_i}$, and an analogous reduction applies to $\frac{\partial \beta''}{\partial P_i}, \frac{\partial \alpha'}{\partial P_i}, \frac{\partial \alpha''}{\partial P_i}$. Our equations are then written:

$$\frac{\mathcal{B}'}{2}\sum{}' \frac{\partial H_1}{\partial P_i}\frac{dp_{nn}}{dx_i} - \frac{\mathcal{A}'}{2}\sum{}' \frac{\partial H_1}{\partial P_i}\frac{dq_{nn}}{dx_i} + \cdots = 0,$$
$$-\frac{\mathcal{B}}{2}\sum{}' \frac{\partial H_1}{\partial P_i}\frac{dp_{nn}}{dx_i} + \frac{\mathcal{A}}{2}\sum{}' \frac{\partial H_1}{\partial P_i}\frac{dq_{nn}}{dx_i} + \cdots = 0.$$

Therefore, if we consider the lines on M_{n-1} that are defined by the differential equations:

(**39**) $$\frac{dx_1}{\frac{\partial H_1}{\partial P_1}} = \frac{dx_2}{\frac{\partial H_1}{\partial P_2}} = \cdots = \frac{dx_{n-1}}{\frac{\partial H_1}{\partial P_{n-1}}} = ds$$

(s is an arbitrary parameter) then we may write our equations in the form:

$$\mathcal{B}\frac{dp_{nn}}{ds} - \mathcal{A}'\frac{dq_{nn}}{ds} + \cdots = 0,$$
$$-\mathcal{B}\frac{dp_{nn}}{ds} + \mathcal{A}\frac{dq_{nn}}{ds} + \cdots = 0.$$

These are two ordinary differential equations that define p_{nn} and q_{nn} as functions of s. We may therefore reach the same general conclusions as always. We may choose the values of p_{nn}, q_{nn} at a point of each of the lines that are defined by the differential equations (**39**), and these quantities will therefore be determined all along the line in question. These are the lines that we again call the *bicharacteristics* of the system.

In the case of transversal waves that propagate in isotropic solids, these bicharacteristics are still normal to the waves, since the equation of the characteristics is written:

$$H_1 = \rho f_t^2 - M(f_x^2 + f_y^2 + f_z^2) = 0.$$

301. – From what was said in no. **288**, it is clear that all of the results (such as those of nos. **291-293**, and the ones we just obtained) continue to be essential when the given equation is not linear with respect to p_{ik}, q_{ik}, r_{ik}. It will further suffice to differentiate these equations once with respect to x_n. The quantities a_{ik}, b_{ik}, c_{ik} will be replaced by the derivatives of the left-hand side with respect to p_{ik}, q_{ik}, or r_{ik}.

It is likewise clear that if the equation for M_{n-1} is considered in the form (**2′**) – not having been solved for x_n – then the characteristics will again be given by equation (**29**). A is replaced by the expression (**18**) (no. **287**); $A', A'', \cdots$ are replaced by analogous expressions. The bicharacteristics will be represented (under the hypotheses of no. **292**) by the equations:

$$\frac{dx_1}{\dfrac{\partial H_1}{\partial \pi_1}} = \cdots = \frac{dx_n}{\dfrac{\partial H_1}{\partial \pi_n}}.$$

302. – We return to the dynamical interpretation of the results that we just obtained.

In order to present our reasoning, we adopt the convention that we spoke of in no. **100** (cont.), i.e., that we trace the corresponding figures as if they were motions in the plane, with the surfaces of discontinuity being replaced by curves in the figures, the triply extended multiplicities by surfaces, etc.

Consider two motions of a second-order discontinuity (or one of order $m \geq 2$) along a surface, a subset of which that we denote by S_0, represents the initial state, and both of which satisfy the same system of dynamical equations – for example, the equations of hydrodynamics. Suppose that one is given the position of the surface S_0 at an instant t_0. The considerations of chapter V teach us to find the velocity of propagation at that instant at all points of the surface, or, what amounts to the same thing (no. **100** (cont.)), the angle that the hypersurface $\mathcal{S}_0$, which is related to S_0 as time varies, makes with the hypersurface $t = t_0$; as a consequence, one constructs the direction of $\mathcal{S}_0$ at this point. From what we saw in chap. V (no. **269-271**), this direction is always real in the case of the equations of hydrodynamics or elasticity. Two or more of them may exist; in that case, the compatibility conditions permit us to choose between them, as we explained in no. **243**.

303. – However, the considerations that were developed in the present chapter permit us to go much further. Indeed, if one of the two motions is completely known – the one that points towards the interior that relates to propagation – then we know a first-order partial differential equation (that of the characteristics) that $\mathcal{S}_0$ must satisfy.

Now, from the general theory of partial differential equations ([29]), such an equation, when combined with the condition that is satisfied by the surface S_0, completely determines the hypersurface in question.

To perform this determination effectively, it suffices ([30]) to possess *a complete integral* of the characteristic equation, i.e., (up to a restriction, upon which we do not insist here ([31])) an integral that depends on three arbitrary constants (in the case that interests us, which is the one in which the number of independent variables is four).

([29]) GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre,* no. **75**, pp. 189-191.

([30]) GOURSAT, *loc. cit.*

([31]) GOURSAT, *ibid.,* no. **43**, pp. 96.

We begin with a characteristic multiplicity that might not only play the role of a complete integral, but is also a bit more general, since it contains *four* constants, namely, the coordinates of an arbitrary point (a_0, b_0, c_0, t_0) of the space E_4. Let:

$$H(x_1, x_2, \ldots, x_n, P_1, P_2, \ldots, P_{n-1}) = 0 \tag{40}$$

be an arbitrary first-order partial differential equation that defines x_n as a function of x_1, x_2, …, x_{n-1}, in which P_1, P_2, …, P_{n-1} denote the first derivatives of x_n. For each system of values $(x_1^0, x_2^0, \cdots, x_n^0)$ of x_1, x_2, …, x_n, that equation gives a relation between P_1, P_2, …, P_{n-1}. For $n = 3$, the variables x_1, x_2, x_3 may be regarded as Cartesian coordinates, and the relation between P_1 and P_2 that is obtained represents a cone that must be tangent to the desired surface. In order to generalize to the geometry of n dimensions we may preserve the same geometric interpretation for all n and speak of the cone Γ that is represented by equation (**40**) at the point O, which has the coordinates $(x_1^0, x_2^0, \cdots, x_n^0)$.

To each direction (of the multiplicity M_{n-1}) that is tangent to this cone along a certain generator γ – i.e., to each system of values for P_1, P_2, …, P_{n-1} that satisfy the equation for the given values of x – the theory of first-order partial differential equations teaches us to associate a characteristic c of equation (**40**) that has the generator γ for its tangent at the given point. Any integral that passes through O, and for which P_1, P_2, …, P_{n-1} have the values considered at this point, necessarily contains the entire characteristic c.

The integral that we consider, with Darboux ([32]), which he called the *integral with singular point*, is nothing but the locus C of the different characteristics c that issue from the point O, and correspond to the various possible directions of γ. It obviously admits O as a conical point whose tangent cone is Γ. It is described by each integral that passes through O along the characteristic c.

In the case for which equation (**40**) is the one that defines the characteristics of an equation, or a system such as the ones that we studied in the foregoing, we give the hypersurface C that has O for its conical point the name of the *characteristic conoid* with vertex O, and the cone Γ is called the *characteristic cone* at this same vertex.

304. – Now if, in turn, the system in question is the one that regulates motion, in such a way that the independent variables are a, b, c, t, and one gives the position S_0 of a wave at the instant t_0 then in order to obtain the characteristic multiplicity $\mathcal{S}_0$ (*fig.* 19) that cuts $t = t_0$ along the surface S_0 – i.e., the multiplicity that figures in the progress of this wave – it will suffice to take the envelope of the characteristic conoids that have the different points (a_0, b_0, c_0, t_0) of the surface S_0 considered at the instant t_0 for vertices. This envelope will have several nappes, in general; however, as in no. **243**, if there is compatibility then the propagation takes place along just one of them, which is perfectly determined.

([32]) *Mémoire sur les solutions singuliéres des équations aux dérivées partielles du premier ordre*, no. **2**, pp. 34 (*Mém. des savants étrangers*, t. XXVII, 1880).

Let Σ be the surface (which is represented by a curve in *fig.* 19) along which the multiplicity $t = t'$ (which is represented by a plane that is parallel to $t = t_0$ in *fig.* 19) is cut by the characteristic conoid of vertex (a_0, b_0, c_0, t_0). The construction of $\mathcal{S}_0$ that we just indicated translates into geometrical language in the following manner: *If we are given the position S_0 of a wave at the instant t_0 then in order to obtain the position S_0' of that wave at an arbitrary final instant t', it suffices to take the envelope of all surfaces Σ that correspond to the different points of S_0.*

When the surface S_0 is infinitesimal and reduces to the unique point (a_0, b_0, c_0) the multiplicity $\mathcal{S}_0$ is nothing but the characteristic conoid itself. The surface S is therefore the one on which a discontinuity that is concentrated in the neighborhood of the point (a_0, b_0, c_0) for $t = t_0$ will be distributed at the instant t'.

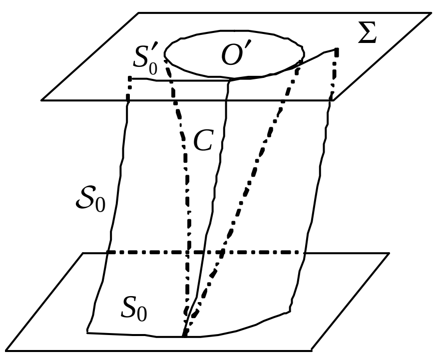

Fig. 19

305. – The waves that we encountered in chapters V and VI (nos. **239**, **271**) always had a real propagation velocity, and we were likewise led to assume (no. **271**) that these velocities are always finite.

As one immediately sees upon first referring to the case of motion in two dimensions, the condition that the velocities be real for any direction of the wave amounts to demanding that the multiplicity $t = t_0$ not be a secant to the characteristic conoid of vertex O, and the condition that these velocities always be finite expresses that they not be tangent to it; as a consequence, they are entirely external to it.

If this condition is satisfied then it is clear that the surface S cannot be extended indefinitely in any sense. In particular, the surface Σ that corresponds to the case in which S_0 reduces to the point O is always closed.

306. – Conversely, suppose we are given a surface S_0' at the time $t' > t$. First suppose that this surface reduces to a point O (*fig.* 20), and let Σ_0 be the surface of section of the characteristic cone C of vertex O of the multiplicity $t' = t$. If the surface Σ_0 is closed, as we said ([33]), then in order to determine the motion at O at the instant t' it will suffice to know the motion, not of all of the points of space, but just *the ones that are interior to* Σ_0 at the instant t_0. Indeed, we deduce that if

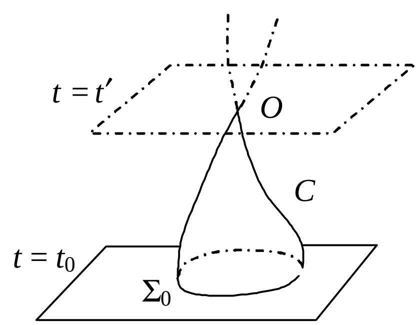

Fig. 20

([33]) If the characteristic conoid is comprised of several nappes, in such a way that Σ_0 is comprised of several closed nappes, then it is necessary to consider the most external of these nappes here, in such a way that C is *the nappe that is inclined towards the interior* of the characteristic that passes through Σ_0.

two motions coincide in the interior of Σ_0 for $t = t_0$ (although they might possibly be distinct outside of that surface) then they subsequently coincide in any region that is interior to the characteristic multiplicity defined by Σ_0, a multiplicity that is nothing but C.

Now, if S'_0 is an arbitrary closed surface, and no longer just a point, then what we just discussed will obviously still apply upon replacing the interior of Σ_0 with the domain that replaces the various surfaces Σ_0 that correspond to the different points of S'_0 or the interior of S'_0.

307. – When the coefficients a_{11}, a_{22}, … (no. **278**) of the highest-order derivatives are constant, in such a way that the characteristic equation does not explicitly contain the variables themselves, the characteristic cones that correspond to the different points of space are all equal.

Moreover, the characteristic conoid reduces to the characteristic cone. Indeed, the characteristic equation reduces to the characteristic cone. Indeed, the characteristic equation is verified when one gives constant values to all of that quantities that we have denoted by the letters P_i – which are $\frac{dt}{da}, \frac{dt}{db}, \frac{dt}{dc}$ here – which gives a linear function of a, b, c for t. The corresponding bicharacteristics are obviously straight lines ([34]), which are nothing but the generators of the cone Γ.

As for the characteristic multiplicities on which $\frac{dt}{da}, \frac{dt}{db}, \frac{dt}{dc}$ also reduce to constants, they are obviously the *plane waves*, which corresponds to the case in which the surface S_0 reduces to a plane, and for which, consequently, the same thing is true for the surfaces S'_0 that correspond to any final instant by means of the hypothesis of the constancy of the coefficients a_{11}, … that we adopted at the moment.

308. – When this hypothesis is satisfied, one gives the name of *wave surface* to the surface Σ that corresponds to $t' - t = 1$. Since the characteristic conoid is the envelope of the plane waves here, the wave surface may be considered to be the envelope of a plane S'_0, such that the distance from it to the parallel plane S_0 that is defined by O is equal to the velocity of propagation of a discontinuity that follows S_0.

On the contrary, when the coefficients of the higher-order derivatives are no longer constant, one defines the wave surface relative to an arbitrary definite point O by giving these coefficients the same value everywhere that they have at O; this amounts to substituting the tangent cone Γ for the characteristic conoid. The construction that we just indicated in the last section remains valid, moreover.

In all of the physical treatises, the equation of the surface thus generated is constructed for the cases of gaseous media, isotropic elastic media, and the luminous vibrations of crystalline media. In the first two cases, this surface reduces to a sphere. In

([34]) It is painfully necessary to recall that in the geometry of n dimensions, one uses the term "straight line" to refer to a one-dimensional multiplicity along which the n coordinates are linear functions of another.

the last one (which is nothing but that of an elastic medium that satisfies the particular hypotheses of nos. **274-276**), it is of fourth degree (Fresnel wave surfaces).

309. – The definition that we just gave for the wave surface permits us to confirm that the bicharacteristics, such as the one that we introduced in the foregoing, are nothing but the *rays* that one considers in physics.

Indeed, the direction of a ray is defined to be that of the line that joins the point O to the point of contact of the wave surface that relates to this point with the wave considered. Now, in our four-dimensional space this is represented by the multiplicity $\mathcal{S}_0$ (*fig*, 19) which is tangent to the characteristic conoid C along the bicharacteristic OO'.

To simplify, suppose that the coefficients of the higher-order derivatives are constant. The surface S (*fig.* 19) will then be homothetic with respect to the point O on the wave surface and the bicharacteristic OO', which will then be a straight line that is precisely the direction of the ray, as we shall indicate in an instant.

All of what we just said persists, moreover, when the coefficients are no longer constant; all that is necessary is to take an instant t' that is infinitely close to t_0. The equality of the bicharacteristics and the rays is thus established.

310. – In their present form, the preceding considerations do not permit us account for all of the physical properties of rays ([35]). Nevertheless, they do show that these lines play an essential role in the propagation of motion. This is further evidenced by the following proposition:

Suppose we are given an initial motion that satisfies the equations and a wave $\mathcal{S}_0$ (*fig.* 19) that propagates this motion, a wave that we furthermore consider to be determined by its position S_0 at a certain instant t_0. Let t' be the final instant when this wave attains a definite point O'. *The new motion at this point will depend exclusively upon the new motion that the point O* (*fig.* 19), *which is on the same bicharacteristic as O' at the instant t_0.*

Indeed, this is what results from the calculations that were done in nos. **293,** et seq. The latter show that if we know the multiplicity $\mathcal{S}_0$ and the elements of the discontinuity at just one point O then these same elements will be determined at all points of the bicharacteristic that issues from O.

In particular, if the discontinuity exists at the instant t_0 only for a small portion of the wave surface then it will exist only for a small portion of the surface S_0' at the instant t', namely, the one that is bounded by the same bicharacteristics as the first one.

311. – The result that we just stated persists in either of the two previously treated cases, namely, when the determinant does or does not have a minor that is different from 0, respectively. However, we have assumed that it is not necessary to work with the second case since the characteristic considered shares the property of annulling all of the

([35]) See below, nos. **350-351**.

minors of H with all of the infinitely close characteristics. Our reasons will be invalid if the characteristics that possess this property are particular ones, i.e., if the generators of the cone Γ that corresponds to these characteristics depend on the parameters less than the others at an arbitrary point do. In this case, nothing permits us to still assert the existence of the characteristics. Such singular characteristics undoubtedly deserve to be studied from the analytical viewpoint. They are well known in optics; they are what correspond to the phenomenon of *conical refraction.* Contrary to what is true for multiple characteristics in general ([36]), they are not related to singularities of the solutions (see below, no. **327**).

312. – The construction that was indicated in no. **304** further permits us to determine the wave in circumstances that are a little more complicated than the ones that we were recently faced with.

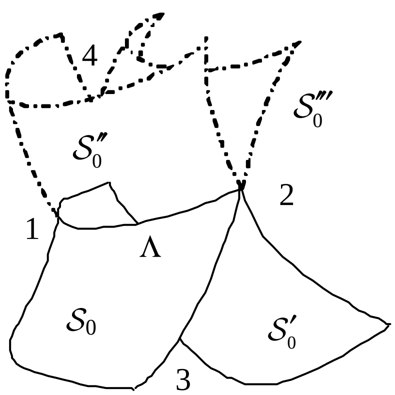

Fig. 21

For example, consider the *intersection* of two waves, i.e., the case in which two surfaces of discontinuity S, S' are originally completely separate from each other, and then they propagate in a gaseous medium, which we suppose, to simplify, to be indefinite, until they cross. This intersection defines a curve l that obviously varies with t. Upon once more employing the language of four-dimensional geometry and representing the wave surfaces by their positions S_0, S_0' on the initial state, one may say that the multiplicities $\mathcal{S}_0$, $\mathcal{S}_0'$ are generated by the surfaces S_0, S_0' as t varies and intersect along a twice-extended multiplicity Λ, whose t = const. sections are the successive positions of the curve l. As we have done before, it is easy to represent the analogous phenomenon in the case for which there are only two coordinates x, y, and the multiplicities $\mathcal{S}_0$, $\mathcal{S}_0'$ are surfaces that are traced out in a space of three dimensions (*fig.* 21). Λ will then be a curve traced in that space.

During the time when $\mathcal{S}_0$, $\mathcal{S}_0'$ are secant and after it, the successive positions of the curve l will obviously give rise to new waves, which are, in a sense, the continuation of the first two. It is clear that these new characteristic multiplicities $\mathcal{S}_0''$, $\mathcal{S}_0'''$ (*fig*, 21), which represent the progress of these waves, will be determined by the condition that they contain the multiplicity Λ, and that they are obtained, as a consequence, as envelopes of the characteristic conoids that have the different points of Λ for their vertices, precisely as we explained in the case where Λ corresponded to t = const. and reduced to a surface S_0.

Completely similar considerations apply to the intersection of a wave with a fixed or moving wall. The latter forms a hypersurface by the set of its positions for the different values of t, which will cut the wave along a multiplicity Λ that is of the same nature as the one that we have always denoted by that notation. It remains for us to pass a second

([36]) See, for example, the note on page (?).

characteristic (reflected wave) through Λ, which comes about by the same construction as the foregoing.

In this case, as in the preceding one, the multiplicity Λ is, from the way it was obtained, external to the characteristic cone that has an arbitrary point of Λ for its vertex, in such a way (compare no. **305**) that the new waves we obtain are real.

313. – Analytically speaking, the case of *refraction* corresponds to the case in which the space E_4 is divided into two regions in which the equations of the problem will have different forms. A wave that propagates into one of these two regions thus encounters their common boundary along a multiplicity Λ, through which a characteristic of the equations in the second region must pass. Nevertheless, this new characteristic (refracted wave) may be itself imaginary when the first wave is real.

It is clear that Huyghen's construction is only an application of this manner of operation.

314. – Finally, one often considers a wave that intersects itself; in other words, a wave surface that is originally devoid of singularities and acquires double lines ([37]) in the course of its propagation. Of course, that circumstance must not be confused with the phenomenon of Riemann and Hugoniot, which was studied in chapter IV; in general, it does not affect the regularity of the motion.

§ 2. – EXISTENCE THEOREMS

315. – In the foregoing, we confirmed that on a characteristic the derivatives of each order lead to an indeterminacy. It does not result from this that there exists an infinitude of integrals that solve the Cauchy problem, nor even that there exists only one.

For the case of a second order analytic equation in two independent variables, this fact was established by Goursat ([38]) as a consequence of the following theorem:

Being given one analytic partial differential equation in two independent variables, along with two concurrent analytic lines, each of which is tangent to one of the characteristics that issue from their point of intersection, the equation admits one (and only one) analytic integral that takes the given analytic values on the two given curves.

From this theorem, it easily follows that there exist an infinitude of analytic integrals that solve the Cauchy problem for one characteristic.

([37]) This is true in the general case when the curves that are parallel to a curve C have double points (even if C has none of them) when the distance becomes sufficiently large when referred to the concavity of C.

([38]) *Leçons sur les dérivées partielles du second ordre*, tome I, pp. 184-193.

316. – The theorem of Goursat has been generalized by Beudon, *loc cit.*, to the equation in an arbitrary number of variables that was treated in no. **278**, et seq.

We shall prove the result of Beudon by adopting some hypotheses that are a little more general. Indeed, already in the case of two variables it is not necessary that the *two* lines along which z is given be characteristics. As was shown by Picard ([39]) for linear equations that are or are not analytic, and then Goursat ([40]), upon assuming that the equations were analytic, but not necessarily linear, this property belongs to only one of the lines in question. A problem of this type was presented in no. **180** in the study of rectilinear motion in a gas.

We extend the theorem of Beudon in an analogous manner, by considering two multiplicities of dimension $n - 1$ that are not tangent to each other, and the first of which is tangent to one characteristic at a point that we take to be the coordinate origin. This property will persist, moreover (compare no. **162**), under a change of independent variables, by means of which we may assume that our two hypersurfaces have the equations $x_n = 0$, $x_{n-1} = 0$.

On each of them, we suppose that we are given a sequence of values of z such that:

$$\begin{cases} z = \psi(x_1, x_2, \cdots, x_{n-1}) & \text{for } x_n = 0 \\ z = \chi(x_1, x_2, \cdots, x_{n-1}, x_n) & \text{for } x_{n-1} = 0. \end{cases} \tag{41}$$

Of course, these values must coincide on the multiplicity (of dimension $n - 2$) that is common to the first two. We may then write:

$$\psi(x_1, x_2, \ldots, x_{n-2}, 0) = \chi(x_1, x_2, \ldots, x_{n-2}, 0) = \varpi(x_1, x_2, \ldots, x_{n-2}). \tag{42}$$

Since the partial differential equation is:

$$\mathcal{F} = 0,$$

the condition that $x_n = 0$ be tangent to one characteristic is expressed (no. **288**) by the condition:

$$\frac{\partial \mathcal{F}}{\partial p_{nn}} \neq 0.$$

By contrast, suppose that the equation is soluble with respect to p_{nn-1}. The condition $\partial\mathcal{F}/\partial p_{nn-1} = 0$ amounts to assuming that the multiplicity M_{n-2} that is defined by the equations $x_n = x_{n-1} = 0$, is not tangent to one bicharacteristic. If the contrary case is produced then the given values ψ, χ must verify new possibility conditions. Indeed, we have seen that the derivative of p_n along one bicharacteristic may be calculated as a function of the x, x_i, p_i. It follows that the value thus obtained at the origin must be equal to the one (namely, $\partial\chi/\partial x_n$) that one knows directly once p_n is given on the

([39]) In DARBOUX, *Leçons sur la Théorie générale des surfaces*, t. IV, note 1.

([40]) *Equations aux dérivées partielles du second ordre*, t. II, pages 303-308.

multiplicity M_{n-2}. One will likewise obtain another possibility condition by considering the derivatives of p_{nn}, and therefore, as a consequence, for each order of derivation.

317. – Therefore, let the second-order equation be solved with respect to p_{nn-1}:

$$p_{nn-1} = F(x_1, x_2, \ldots, x_n, z, p_1, \ldots, p_n, p_{11}, \ldots, p_{nn}), \tag{43}$$

and suppose that the function F is analytic and holomorphic with respect to the variables upon which it depends ([41]) in a domain that is composed of the values that these variables take at the origin, since the quantity $\partial F/\partial p_{nn}$ is null at this point.

We shall prove that *if the function ψ and χ are both analytic and holomorphic around the origin then the problem that was posed will admit one and only one holomorphic solution.*

We may, if we so desire, simplify the question by reducing ψ and χ to being null. To that effect, it will suffice for us to introduce, in place of z, the new unknown:

$$z' = z - \psi + \chi + \varpi,$$

($\varpi(x_1, x_2, \ldots, x_{n-2})$ being defined by equation (**42**)). We may likewise, upon subtracting from z the term $a x_n\, x_{n-1}$, where a is a suitable constant (which diminishes p_{nn} by this constant), we may arrange that F be null at the origin. Under these conditions, the function F will be represented by a converging development that can be ordered in powers of z, x_i, p_i, and p_{ik}, with the exception of p_{nn-1}, a development that lacks only the constant term and the term in p_{nn} .

318. – Whether or not this transformation has been performed, the givens of the problem are known from the values of all the derivatives of z at the origin.

First of all, when there is not both at least one differentiation with respect to x_n and at least one differentiation with respect to x_{n-1}, these values result from the differentiation of the equations in condition (**41**). They are null if one takes $\psi = \chi = 0$.

On the one hand, agree to say that one partial derivative of z is *anterior* to the other if:

1. It has lower total order.

2. When they have the same order, it is composed of fewer derivations with respect to x_{nn} .

3. When they have the same order and are composed of the same number of derivation with respect to x_{nn}, it has fewer derivations with respect to x_{n-1}.

Now, let $p_{nn-1ijk\ldots}$ (where the indices $i, j, k, \ldots$ have completely arbitrary values between 1 and n) be a derivative in which one has differentiated with respect to both x_n and x_{n-1}. We calculate the value of that quantity by applying the

([41]) The theorem that we have in view was, as we have said, established by Picard, independently of the hypothesis of analyticity, for the case of two variables. In the case where n is greater than 2, the extension to non-analytic givens – or rather, the question of knowing whether this extension is possible – presents new difficulties that have not been surmounted up till now.

operation $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\ldots$ to both sides of equation (**43**). All of the derivatives that appear in the right-hand side will obviously be anterior to the one that we seek, with the single exception of $p_{nnijk\ldots}$ However, the latter may be eliminated at the origin because it has the coefficient $\partial F/\partial p_{nn}$, a quantity whose initial value is null.

The right-hand side of the equation thus obtained is therefore composed only of quantities that are already known if we have chosen them carefully, which is obviously possible, since one never passes on to the calculation of one derivative without having performed all of the ones that are anterior to it.

The first conclusion is thus proved. It follows from it that if the problem admits one holomorphic solution then this solution is unique.

Moreover, we remark that:

1. All of the equations that result from the differentiation of (**43**) are thus utilized, in such a way that all of these equations are verified at the origin by the system of values of $p_{ijk\ldots}$ that was thus calculated.

2. This calculation involves only additions and multiplication.

By virtue of this last remark, we may apply the method of majorant functions. We replace the given developments of F, ψ, χ by other ones that majorize the first ones, respectively. If the problem, thus modified, has a holomorphic solution then we may conclude that the values of $p_{ijk\ldots}$ that correspond to the given problem furnish a convergent Taylor development, as well (which will necessarily satisfy the proposed equation, from the first of the two remarks that we just made).

As far as the given functions ψ and χ are concerned, we may suppose them to be null, as was just explained. With these conditions, each of them will admit as a majorant, any function that is represented by a development whose coefficients are positive.

As for the function F, since it lacks the constant term and the term in p_{nn}, it will admit, from a well-known remark, a majorant of the form:

$$\frac{M}{\left\{1-\dfrac{x_1+x_2+\cdots+x_n+z+\sum_{i=1}^{n}p_i+\sum' p_{ik}}{R}\right\}}-M\left(1+\frac{p_{nn}}{R}\right),$$

in which the sum Σ' refers to all of the second derivatives with the exception of p_{nn} .

Beudon, who assumed that $x_{n-1} = 0$ is a characteristic, further suppressed only the term in $p_{n-1\,n-1}$ in this expression. By reason of the presence of this term, we must now employ the artifice that was indicated by Goursat, which consists of remarking that the function F is *a fortiori* majorized if we replace x_n with x_n/λ in the denominator, where λ denotes an arbitrary positive number that is much smaller than 1. We are thus led to the equation:

(**44**)
$$p_{nn-1} = \frac{M}{\left\{1 - \dfrac{x_1 + \cdots + x_{n-1} + \dfrac{x_n}{\lambda} + z + \sum p_i + \sum' p_{ik}}{R}\right\}} - M\left(1 + \frac{p_{nn}}{R}\right),$$

and the theorem will be proved if we obtain a solution for this equation that is null at the origin along with its first and second derivatives, and they reduce, when both $x_n = 0$ and $x_{n-1} = 0$, to functions whose developments have all positive coefficients.

We seek such a solution by taking z to be a function of the two variables:

(**45**)
$$X = x_1 + x_2 + \ldots + x_{n-2}, \qquad Y = \lambda x_{n-1} + x_n .$$

Equation (**44**) will become:

$$\lambda \frac{\partial^2 z}{\partial Y^2} = -M\left(1 + \frac{1}{R}\frac{\partial^2 z}{\partial Y^2}\right)$$
$$+ \frac{M}{1 - \dfrac{1}{R}\left[X + \dfrac{Y}{\lambda} + z + (n-2)\dfrac{\partial z}{\partial X} + (1+\lambda)\dfrac{\partial z}{\partial Y} + C\dfrac{\partial^2 z}{\partial X^2} + (1+\lambda)(n-2)\dfrac{\partial^2 z}{\partial X \partial Y} + (1+\lambda^2)\dfrac{\partial^2 z}{\partial Y^2}\right]},$$

in which C is the numerical coefficient $C = (n-1)(n-2)/2$.

The right-hand side involves a term in $\partial^2 z/\partial Y^2$, namely, the term, $\dfrac{M}{R}\lambda^2 \dfrac{\partial^2 z}{\partial Y^2}$. We determine λ in such a manner that this term has a coefficient that is much smaller than the value of the right-hand side, namely:

(**46**)
$$\lambda < \frac{R}{M}.$$

We may then move the term in $\partial^2 z/\partial Y^2$ from the right-hand side to the left-hand, and the equation that is obtained will have of the form:

(**47**)
$$\lambda\left(1 - \frac{M\lambda}{R}\right)\frac{\partial^2 z}{\partial Y^2} = F_1\left(X, Y, z, \frac{\partial z}{\partial X}, \frac{\partial z}{\partial Y}, \frac{\partial^2 z}{\partial X^2}, \frac{\partial^2 z}{\partial X \partial Y}, \frac{\partial^2 z}{\partial Y^2}\right),$$

in which F_1 is holomorphic with respect to the variables that it depends upon around the null values of these variables, and its development has coefficients that are all positive and lack only the term in $\partial^2 z/\partial Y^2$.

The Cauchy-Kowalewsky theorem tells us that this equation admits an integral that is null and holomorphic for $Y = 0$, as well as its derivative with respect to Y. If one substitutes for X and Y their values (**45**) then one will have a holomorphic solution to equation (**44**). This solution, and consequently, the functions that it reduces to for $x_n = 0$

and $x_{n-1} = 0$, has, moreover, as one shows by calculation with the aid of equation (**47**) ([42]), a development with all positive coefficients, and its initial value is null, along with that of its first and second derivatives.

The theorem is thus proved.

319. – From the preceding proposition, one easily deduces what we have in mind, namely, the existence of an infinitude of holomorphic solutions for the Cauchy problem in the case of one characteristic.

Suppose further that the characteristic multiplicity has the equation $x_n = 0$. We may, in addition, suppose that the given values of z on this multiplicity are null, along with those of p_n and the ones that one deduces for p_{nn} . Indeed, it is clear that one is confronted with the case that is opposite to this one by a change of unknowns of the form:

$$z = z' + A + Bx_n + Cx_n^2. \tag{48}$$

(A, B, C being functions of $x_1, x_2, \ldots, x_{n-1}$) Under these conditions, equation (**43**) must be verified for any $x_1, x_2, \ldots, x_{n-1}$, while x_n and z are null, along with the p_i and p_{ik} .

However, the given multiplicity be a characteristic, and not only tangent to a characteristic at the origin, i.e., one must have, under these conditions, on the one hand $\partial F/\partial p_{nn} = 0$, and, on the other, equation (**16** cont.)(no. **288**), which reduces to $\partial F/\partial x_n = 0$ here.

This amounts to saying that any term in the development of F contains at least one of the quantities:

$$\begin{array}{ll} z,\ p_i & (i = 1, 2, \ldots, n) \\ p_{ik} & (i, k = 1, 2, \ldots, n-1) \\ p_{nh'} & (h' = 1, 2, \ldots, n-2) \\ x_n^2,\ x_n\, p_{nn},\ p_{nn}^2 & \end{array}$$

as a factor.

Therefore, let the holomorphic functions $\varphi_3, \varphi_4, \ldots$ of $x_1, x_2, \ldots, x_{n-2}$ be given arbitrarily, and consider the holomorphic solution of equation (**43**) that reduces to 0 for $x_n = 0$ and to:

$$\varphi_3 x_n^2 + \varphi_4 x_n^4 + \cdots \tag{49}$$

for $x_{n-1} = 0$, a solution whose existence was just established. It is easy to confirm that no matter what the functions $\varphi_3, \varphi_4, \ldots$ are, this solution solves our Cauchy problem; i.e., in addition to its values, those of its derivatives p_n and p_{nn} are null with x_n . To that effect, it suffices (since we are dealing with holomorphic functions) to assure that for this integral z all of the derivatives that contain one or two derivations with respect to x_n are null at the origin. Now, one verifies this without difficulty by repeating the calculations of the preceding no. by which one obtains these derivatives, but under the present hypotheses.

([42]) In order to effect this calculation, it is useless to solve equation (**47**) with respect to $\partial^2 z/\partial Y^2$, due to the fact that the coefficient of $\partial^2 z/\partial Y^2$ in the right-hand side is null at the origin.

The theorem is thus proved.

319 (cont.) **–** The expression (**49**) represents the most general value that may be taken on the multiplicity $x_{n-1} = 0$ by a holomorphic function z that is null, along with its first two derivatives with respect to x_{n-1} when $x_n = 0$.

Now, let us pass from the calculations that we just made to the ones that they correspond to when one does not perform the transformation (**48**). The values of z and its derivatives of the first two orders are no longer null for $x_n = 0$, but they must further verify: 1. Equation (**43**). 2. The condition on $\partial F/\partial p_{nn}$ that expresses that $x_n = 0$ is a characteristic. 3. The condition (**16** cont.), which is necessary for the existence of the third derivatives. Conversely, these conditions are the only ones that we have postulated in the argument of the preceding section.

They show, as a consequence, that *a distribution (on the multiplicity* $x_n = 0$) *of values for* p_n *,* p_{nn} *that satisfies the three conditions in question (when one gives the values* $y(x_1, x_2, \ldots, x_{n-1})$ *to* z*) will be the same ones that correspond to the solution of the problem that was treated in nos.* **316-318** *if they coincide with the ones that one deduces from the second condition* (**41**) *for any point of the intersection of the two multiplicities* $x_n = 0$ and $x_{n-1} = 0$.

When the equation is linear with respect to the p_{ik} *one may state the same property for a distribution of values for* p_n *that satisfies the same system of conditions, with the exception of* (**16** cont.), *which is replaced with the equation* (**13**) (no. **282**). This is true because one is reduced to the preceding statement upon determining p_{nn} by means of equation (**16**), combined with the condition that it coincide with the corresponding values of $\partial^2\chi/\partial x_n^2$ on the intersection of the two multiplicities.

320. **–** The proposition that was established in no. **318** is not just useful in the proof of the theorem in no. **319**. It is, in itself, susceptible to dynamical applications. The problem that it solves is, in particular, the one that one led to when one studies the phenomenon of the crossing of waves, as in no. **312**.

Prior to this crossing, the fluid is divided into three regions that are animated with distinct motions: We denote the propagation of the wave $\mathcal{S}_0$ by the index 1, the propagation of the wave $\mathcal{S}_0'$ by 2, and the intermediate motion by the index 3.

Suppose:

1. All three of these motions are devoid of rotation.

2. The are analytic, along with the multiplicities $\mathcal{S}_0$, $\mathcal{S}_0'$. The same will be true for the multiplicity A, as well as the waves $\mathcal{S}_0''$, $\mathcal{S}_0'''$ that are created, as we have seen, by the crossing of the first two, and propagate from A with the motions 1 and 2, respectively.

Having agreed upon these conditions, we shall show the existence of a fourth analytic motion that satisfies the hydrodynamical equations and agrees with 1 and 2 along the characteristics $\mathcal{S}_0''$, $\mathcal{S}_0'''$. It is by means of these conditions precisely that one determines the new intermediate motion that is created between the two corresponding waves.

It will suffice to calculate the velocity potential Φ of the desired motion. The function Φ must first satisfy equation (**23′**).

On the other hand, all of its first derivatives must be the same as the ones that correspond to the motions 1 and 2 on $\mathcal{S}_0''$ and $\mathcal{S}_0'''$, respectively, since the velocity and pressure remain continuous (the discontinuities being assumed to be of at least second order).

Now, we know that there exists a holomorphic function Φ that verifies equation (**23′**) and takes the same values as the velocity potential for motion 1 on $\mathcal{S}_0''$ and the same values as the velocity potential for motion 2 on $\mathcal{S}_0'''$.

Having thus chosen the velocity potential of the new intermediate motion, one will have continuity (upon crossing $\mathcal{S}_0''$ and $\mathcal{S}_0'''$), not only for the values of this potential, but also for those of its derivatives, as the conditions of our problem demand.

Indeed, the derivatives in question, which are deduced from motion 1, form a characteristic distribution on $\mathcal{S}_0''$. On the other hand, since equation (**23′**) is linear with respect to the second derivatives, the stated continuity will be valid upon extension to $\mathcal{S}_0''$ (by virtue of no. **319** cont.) if they exist at all points of Λ.

Now, at these points it happens that one can calculate the derivatives of Φ with the aid of the values of that function $\mathcal{S}_0$ for the motion 1, and on $\mathcal{S}_0''$ for the desired motion, values that one may consider to be given by the motions 3 and 2, respectively. On the other hand, as we are supposing, there will be continuity of the first derivatives for the original three motions (compare the note on page ?).

The motion that is deduced from a velocity potential that is calculated in the manner that we just described will therefore satisfy all of the conditions of the problem exactly.

321. – We now propose to generalize the proposition of nos. **316-318** to systems of several unknowns. Therefore, let ξ, η, ζ be one such system of unknowns. Furthermore, consider two secant multiplicities, which we may always assume to be given by equations of the form $x_n = 0$, $x_{n-1} = 0$, the first of which is tangent to a characteristic that is not multiple (no. **284**), and the second of which is arbitrary under the single condition that their intersection must not be tangent to a characteristic.

We suppose that the given system is analytic and regular and remains that way under the change of variables that we carried out in order to put the equations of our two multiplicities into that form. Under these conditions, if we seek the values of ξ, η, ζ that annul them at the origin, along with their first and second derivatives, then we must assume that the left-hand side of the equations are developable in increasing powers of x_1, x_2, …, x_n, ξ, η, ζ, p_i, q_i, r_i, p_{ik}, q_{ik}, r_{ik}. Moreover, if the terms in p_{nn}, q_{nn}, r_{nn} of these developments are:

$$\text{(50)} \qquad \begin{cases} Ap_{nn} + Bq_{nn} + Cr_{nn}, \\ A'p_{nn} + B'q_{nn} + C'r_{nn}, \\ A''p_{nn} + B''q_{nn} + C''r_{nn}, \end{cases}$$

and if one takes into account what we said in no. **301**, then the coefficients A, B, C, A', B', C', A'', B'', C'' are nothing but the initial values of the quantities that we have denoted by these names in no. **291**. The determinant H, which is equal to:

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}$$

at the origin, must be null, since $x_n = 0$ is tangent to a characteristic. In other words, we may form a linear combination of our three equations, such as terms of the form (**50**), that disappears completely, a combination that may replace one of the given equations – for example, the second one.

Therefore, assume that one has $A'' = B'' = C'' = 0$. The derivatives $\partial H/\partial P_i$ then reduce to:

(**51**)
$$\frac{\partial H}{\partial P_i} = \begin{vmatrix} A & B & C \\ A' & B' & C' \\ a_i & b_i & c_i \end{vmatrix},$$

in which a_i, b_i, c_i denote the coefficients of p_{ik}, q_{ik}, r_{ik} in the third equation. It then results from this that:

1. The determinants (**51**) are non-null, since our characteristic is simple ([43]).
2. In particular, the one that corresponds to $i = n - 1$ is different from zero, since the intersection of our two multiplicities is not tangent to a bicharacteristic.

322. – Under these conditions, we may perform a change of variables such that the two of them are replaced with the quantities:

(**52**)
$$\begin{cases} \xi_1 = A\xi + B\eta + C\zeta \\ \eta_1 = A'\xi + B'\eta + C'\zeta, \end{cases}$$

or, more generally, by the quantities:

([43]) If this is not true then the result that one arrives at will have a much different nature, as one shows immediately with the system:

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial x_n^2} = \dfrac{\partial}{\partial x_n} \psi(x_1, x_2, \cdots, x_n, \xi, \eta, \zeta, p_i, q_i, r_i) + M, \\ \dfrac{\partial^2 \xi}{\partial x_n \partial x_{n-1}} = \dfrac{\partial}{\partial x_{n-1}} \psi(x_1, x_2, \cdots, x_n, \xi, \eta, \zeta, p_i, q_i, r_i) + N, \\ \dfrac{\partial^2 \eta}{\partial x_n^2} = \Phi(x_1, x_2, \cdots, x_n, \xi, \eta, \zeta, p_i, q_i, r_i, p_{ik}, q_{ik}, r_{ik}) \end{cases}$$

(where M and N are given functions of x), a system that is impossible if one does not have $\partial M / \partial x_{n-1} = \partial N/\partial x_n$.

$$\textbf{(53)}\qquad \begin{cases} \mathcal{A}\xi + \mathcal{B}\eta + \mathcal{C}\zeta, \\ \mathcal{A}'\xi + \mathcal{B}'\eta + \mathcal{C}'\zeta, \end{cases}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ are arbitrary holomorphic functions that reduce to A, B, C, A', B', C' at the origin.

As for the third unknown, it will be an arbitrary function:

$$\zeta_1 = \psi(\xi, \eta, \zeta, x_1, x_2, \ldots, x_n),$$

such that one has:

$$\textbf{(54)}\qquad \frac{D(\xi_1, \eta_1, \zeta_1)}{D(\xi, \eta, \zeta)} = \begin{vmatrix} A & B & C \\ A' & B' & C' \\ \dfrac{\partial \psi}{\partial \xi} & \dfrac{\partial \psi}{\partial \eta} & \dfrac{\partial \psi}{\partial \zeta} \end{vmatrix} \neq 0.$$

at the origin.

Since one has:

$$\textbf{(55)}\qquad \begin{cases} \dfrac{\partial^2 \xi_1}{\partial x_n^2} = Ap_{nn} + Bq_{nn} + Cr_{nn}, \\ \dfrac{\partial^2 \eta_1}{\partial x_n^2} = A'p_{nn} + B'q_{nn} + C'r_{nn}, \\ \dfrac{\partial^2 \zeta_1}{\partial x_n^2} = \dfrac{\partial \psi}{\partial x_n} p_{nn} + \dfrac{\partial \psi}{\partial x_n} q_{nn} + \dfrac{\partial \psi}{\partial x_n} r_{nn} + \cdots, \end{cases}$$

the equality (**54**) expresses the idea that the third of the derivatives (**55**) is not expressed with the aid of the first two, and consequently that the given equations do not furnish the expression with the aid of derivatives that contain at least two differentiations with respect to x_n .

323. – Suppose that this change of unknowns has already been performed. The coefficients A, B' will then be equal to one, while B, A', C, C' will be null. Consequently, the functional determinant of the left-hand sides of our equations for p_{nn}, q_{nn}, r_{nn-1} will be equal to $\partial H/\partial P_{n-1}$ initially; i.e., it will be non-zero. One may thus solve these equations with respect to p_{nn}, q_{nn}, r_{nn-1} and write them in the form:

$$\textbf{(56)}\qquad \begin{cases} p_{nn} = F(x_i, \xi, \eta, \zeta, p_i, q_i, r_i, p_{ik}, q_{ik}, r_{ik}) \\ q_{nn} = \Phi(x_i, \xi, \eta, \zeta, p_i, q_i, r_i, p_{ik}, q_{ik}, r_{ik}) \\ r_{nn-1} = \Psi(x_i, \xi, \eta, \zeta, p_i, q_i, r_i, p_{ik}, q_{ik}, r_{ik}), \end{cases}$$

in which the right-hand sides do not contain p_{nn}, q_{nn}, r_{nn-1}, and one has:

$$\textbf{(57)} \qquad \frac{\partial F}{\partial r_{nn}} = \frac{\partial \Phi}{\partial r_{nn}} = \frac{\partial \Psi}{\partial r_{nn}} = 0.$$

at the origin.

We shall show that in order to determine a solution to such a system, one may be given:

1. For the unknowns ξ and η, the Cauchy conditions, namely, the values of these quantities and their first derivatives for $x_n = 0$.

2. For the unknown z, on the contrary, conditions that are analogous to the ones in no. **316**, namely, the values of that unknown itself on $x_n = 0$ and on $x_{n-1} = 0$ (values that must, of course, concur when x_n and x_{n-1} are both null).

The various givens will be assumed to be analytic, moreover.

We will obviously know the values at the origin of all the derivatives of ζ in which there is no differentiation with respect to both x_n and x_{n-1} and the derivatives of ξ, η in which there is at most one differentiation with respect to x_n .

In order to calculate the remaining derivatives, we further classify them in terms of their anteriority. The definition that is adopted for one derivative being anterior to another will be the same as before (no. **318**), with the additional convention that when two derivatives of the same order are composed of the same number of differentiations with respect to x_n and x_{n-1}, a derivative of ξ or η will be regarded as anterior to a derivative with respect to ζ.

The calculation will then be performed without difficulty by a method that is completely similar to the one in no. **318**. It will use all of the relations that result from the differentiation of the given equations.

In order to prove the convergence of the development that is thus obtained, one further assumes that all of the initial givens (the values of ξ, η, $\partial\xi / \partial x_n$, and $\partial\eta / \partial x_n$ for $x_n = 0$, the values of ξ for $x_n = 0$ and $x_{n-1} = 0$) are null, a result that one may always obtain by a change of unknowns.

Furthermore, since the operations that serve to obtain the successive derivatives at the origin here are composed exclusively of additions and multiplications, we may replace the various givens of the problem by majorants. For the initial null givens we may substitute other ones that are represented by developments with positive coefficients that are chosen entirely at our discretion so that their constant terms, as well as their terms of the first and second order are nonetheless null.

As for F, Φ, Ψ, their majorants will have the form:

$$\frac{M}{1-\frac{1}{R}\left[\sum x_i+\xi+\eta+\zeta+\sum(p_i+q_i+r_i)+\sum{}'(p_{ik}+q_{ik}+r_{ik})\right]}-M\left(1+\frac{r_{nn}}{R}\right)$$

(the denoted by Σ' refers to all of the second derivatives, with the except of p_{nn} , q_{nn} , r_{nn-1}), or, upon further replacing x_n with x_n / λ:

$$\frac{M}{1-\frac{1}{R}\left[\sum_{i=1}^{n-1} x_i + \frac{x_n}{\lambda} + \xi + \eta + \zeta + \sum (p_i + q_i + r_i) + \sum' (p_{ik} + q_{ik} + r_{ik})\right]} - M\left(1+\frac{r_{nn}}{R}\right).$$

If we seek solutions that depend upon the two quantities:

$$X = x_1 + x_2 + \ldots + x_{n-1}, \qquad Y = \lambda x_{n-1} + x_n$$

then these solutions will be determined by the equations:

$$\frac{\partial^2 \xi}{\partial Y^2} = \frac{\partial^2 \eta}{\partial Y^2} = \lambda \frac{\partial^2 \zeta}{\partial Y^2}$$

$$= \frac{M}{1-\frac{1}{R}\left\{\begin{aligned} & X + \frac{Y}{\lambda} + \xi + \eta + \zeta + (n-2)\frac{\partial(\xi+\eta+\zeta)}{\partial X} + (1+\lambda)\frac{\partial(\xi+\eta+\zeta)}{\partial Y} \\ & + \frac{(n-1)(n-2)}{2}\frac{\partial^2(\xi+\eta+\zeta)}{\partial X^2} + (n-2)(1+\lambda)\frac{\partial^2(\xi+\eta+\zeta)}{\partial X \partial Y} \\ & + (\lambda^2+\lambda)\left(\frac{\partial^2 \xi}{\partial Y^2} + \frac{\partial^2 \eta}{\partial Y^2}\right) + (\lambda^2+1)\frac{\partial^2 \zeta}{\partial Y^2} \end{aligned}\right\}} - M\left(1+\frac{r_{nn}}{R}\right),$$

which will satisfy (upon further setting $C = (n-1)(n-2)/2$) when $\xi = \eta = \lambda\zeta$:

$$\lambda \frac{\partial^2 \zeta}{\partial Y^2} =$$

$$-M\left(1+\frac{1}{R}\frac{\partial^2 \zeta}{\partial Y^2}\right) + \frac{M}{1-\frac{1}{R}\left\{\begin{aligned} & \left[X + \frac{Y}{\lambda} + (2\lambda+1)\left[\zeta + (n-2)\frac{\partial \zeta}{\partial X} + (1+\lambda)\frac{\partial \zeta}{\partial Y} + C\frac{\partial^2 \zeta}{\partial X^2}\right.\right. \\ & \left.\left. + (n-2)(1+\lambda)\frac{\partial^2 \zeta}{\partial X \partial Y}\right] + (\lambda^2(2\lambda+3)+1)\frac{\partial^2 \zeta}{\partial Y^2}\right] \end{aligned}\right\}}.$$

Now, in this latter equation if λ satisfies the inequality:

$$\lambda(2\lambda + 3) < \frac{R}{M}$$

then the term in $\partial^2\zeta / \partial Y^2$ will have a coefficient whose second member is less than the first one, which we may always arrange.

Furthermore, the reasoning becomes absolutely identical to the one that was made in the case of only one equation, and the existence of a holomorphic solution with positive coefficients is established.

324. – From this last result, one will deduce the existence of an infinitude of solutions to the Cauchy problem when the multiplicity $x_n = 0$ is a characteristic. In order to account for such a circumstance at all of the points of the multiplicity in question, and not just at the origin, one must express the notion that there exists at each of them a linear combination of the three given equations in which the derivatives with respect to p_{nn}, q_{nn}, r_{nn} have been eliminated. If (the equations always being holomorphic in their left-hand sides) we suppose, to fix ideas, that the minor α'' is non-zero then this linear combination may be substituted for the third given equation.

A completely analogous transformation will then be carried out on the unknowns: In the first two equations the derivatives with respect to p_{nn}, q_{nn}, r_{nn}, when considered at an arbitrary point of our multiplicity, will be holomorphic functions of x_1, x_2, x_{n-1}. Upon denoting these derivatives by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, we may take the combinations (**53**) to be two of our linear combinations.

We have thus reduced our equations to the form (**56**), assuming that the conditions (**57**) are verified at any point of the multiplicity $x_n = 0$, this time. On the other hand, we may assume, by means of a triple transformation that is analogous to (**48**), that the initial givens ξ, η, ζ, p_n, q_n, r_n are null on this multiplicity, along with the values that one deduces for p_{nn}, q_{nn}, r_{nn}. These null values will therefore verify the conditions (**53**), (**57**), and also the condition (**32**), which is $\partial\Psi / \partial x_n = 0$. In other words, each term of F or Φ must contain as a factor, one of the quantities:

(**58**)
$$\begin{cases} \xi, \eta, \zeta, p_i, q_i, r_i & (i = 1, 2, \cdots, n) \\ p_{ik}, q_{ik}, r_{ik} & \left(\begin{matrix} i = 1, 2, \cdots, n \\ k = 1, 2, \cdots, n-1 \end{matrix} \right\} \text{(except } r_{nn-1} \Big) \end{cases}$$

(**59**)
$$\begin{cases} x_n \\ r_n^2. \end{cases}$$

Each term of Ψ has as a factor, one of the quantities (**58**) or:

(**60**)
$$x_{nn}^2, \; x_n, \; r_{nn}, \; r_{nn}^2 .$$

It easily results from this that if one takes the initial givens to be:

1. On $x_n = 0$: ξ, η, ζ, p_n, q_n null,
2. On $x_{n-1} = 0$: ξ equal to the expression (**49**) (no. **319**),

then the quantities r_n, p_{nn}, q_{nn}, r_{nn} will be identically null with x_n, no matter what the values of φ_1, φ_2, … One may prove this, as in the preceding, by following the same sequence of calculations by which we obtained the successive derivatives.

324 (cont.) – It is clear that one may deduce consequences from the foregoing that are completely similar to the ones that were the objective of no. **319** (cont.). If we put ourselves, to simplify, in the case where the equations are linear with respect to the second derivatives then we may say that if a distribution of values for r_n on the multiplicity $x_n = 0$ (*combined with a given sequence of values for* ξ, η, ζ, p_n, q_n) *makes this multiplicity characteristic and satisfies equation* (**30**) (no. **292**) (*the condition for the existence of the second derivatives) then this distribution will be precisely the one that one obtains by solving the problem in no.* **323** *if this coincidence is true on the intersection of the two multiplicities* $x_n = 0$, $x_{n-1} = 0$.

325. – Like the theorem of nos. **316-318**, the one that we just proved in nos. **321-323** is susceptible to a simple hydrodynamic interpretation.

We saw above how, being given the initial motion of a gas and the motion of a wall, one may obtain the initial acceleration of the neighboring points of this wall. The new motion that is thus created propagates, moreover, as a wave whose partial differential equation (or, what amounts to the same thing, equation (**4**) of no. **240**) permits us to find the position at each instant once one has supposed that motion of a fluid beyond that wave is known (which furnishes the value of ρ).

Suppose that this latter motion is analytic, along with the motion of the wall. The same will then be true for the motion of the wave surface S. The motion that comes about between that surface and the wall must therefore be such that:

1. The fluid and the wall are in constant contact, i.e., for:

$$\psi_0(a, b, c) = 0 \tag{61}$$

(the equation of the surface in the initial state) one has:

$$\psi(x, y, z, t) = 0.$$

2. There is agreement along the wave between the new motion and the original one.

Take a new system of independent variables such that x_3 and x_4 are annulled – the one, along $\psi_0(a, b, c) = 0$, and the other, along the wave.

On the other hand, perform a change of unknowns such that the last one is replaced by the function $\psi(x, y, z, t)$. We then specify:

For $x_3 = 0$, the condition that this unknown be null.

For $x_4 = 0$, the condition that all of the unknowns have the same values as in the original motion, along with the first derivatives of both of them, which do not reduce to 0 with x_3. If this is the case then the coincidence between them will be established for the derivatives of the third unknown by reasoning that is completely similar to the one that was made above (no. **320**), in such a way that that the discontinuity will indeed be of second order, the only condition for this being that this coincidence exist at the points that satisfy both $x_3 = 0$ and $x_4 = 0$; i.e., that the normal velocity of the wall be initially equal to that of the neighboring molecules of the fluid. (It will suffice to apply the proposition that was stated in no. **324** cont.).

The problem thus posed falls within the category that was treated in no. **323**. It remains only for us to insure that:

1. The intersection of the two multiplicities ($x_3 = 0$, $x_4 = 0$) is not tangent to a bicharacteristic. – This is obvious, since that intersection corresponds to t = const. whenever t varies along the rays defined by the equations of no. **296**.

2. If A, B, C, A', B', C' have the significance that was indicated in no. **324** then one has the inequality (**54**). This amounts to saying that one may not form a combination of the equations of the problem that makes known the second derivative of ψ with respect to x_1 , or, what amounts to the same thing, the expression:

$$\frac{\partial\psi}{\partial x}\frac{\delta^2 x}{\delta t^2}+\frac{\partial\psi}{\partial y}\frac{\delta^2 y}{\delta t^2}+\frac{\partial\psi}{\partial z}\frac{\delta^2 z}{\delta t^2}.$$

However, in the contrary case the discontinuity that is compatible with these equations will be forced to be tangential and we know that this is not true.

The problem in analysis that we have been led to is therefore precisely the one that we recently solved. Moreover, the solution thus obtained will initially satisfy the principle of impenetrability (i.e., that a, b, c may be expressed as functions of x, y, z, t) when the normal velocity of the wall is less than the velocity of sound.

326. – By contrast, the problem of the crossing of waves that was treated in no. **320** under the hypothesis of a velocity potential is not, in general, immediately solved by considerations that are similar to the preceding ones.

Indeed, let two motions 1 and 2 (*fig.* 21) be given, so we seek a motion 4 that propagates into the first two along the waves $\mathcal{S}_0''$ and $\mathcal{S}_0'''$ (*fig.* 21), which intersect it along the multiplicity A.

By virtue of the preceding, we may, after performing a convenient change of unknowns that has the effect of substituting new unknown ξ, η, ζ for the x, y, z, determine them by the interior equations of motion and the following conditions:

1. On $\mathcal{S}_0''$, λ must take the same values as in the motion 1.

2. On the same multiplicity, the first derivatives of ξ and η will likewise have the values that result from motion 1.

3. On $\mathcal{S}_0''$, ζ will take the same values as in motion 2.

From these conditions, as before, the continuity of the derivatives of ζ upon crossing $\mathcal{S}_0''$ will result.

However, it remains for us to establish the continuity of ξ, η and all of the first derivatives upon crossing $\mathcal{S}_0'''$. This continuity does not in the least bit result from 3. Indeed, it entails five conditions that must be verified at each point of $\mathcal{S}_0'''$ and the unique differential equation that we know the existence of on that multiplicity entails simply the consequence that these five conditions reduce to four.

If one develops the right-hand sides of these four conditions in a Taylor series in increasing powers of $t - t_0$ (upon letting t_0 denote the value of t that corresponds to the point of Λ under consideration), and one equates the successive coefficients to 0 then one will have a sequence of compatibility conditions in all orders that must be satisfied at each point of the crossing of the two waves. Because of this, the new discontinuities will necessarily be more than two in number. For example, if one is dealing with a problem of hydrodynamics, in addition to the two waves $\mathcal{S}_0''$ and $\mathcal{S}_0'''$, one must add a stationary discontinuity that exists along the crossing surface, i.e., along the projection of Λ onto a plane t = const.

However, one must take into account the fact that under the conditions that we were subject to in no. **312** the discontinuities that exist between motions 1 and 2 are not arbitrary. Indeed, one supposes that before the production of the phenomenon that we are occupied with, there existed only two waves $\mathcal{S}_0$ and $\mathcal{S}_0'$, and a unique motion between them, viz., the motion 3. This amounts to saying that one has some compatibility conditions that are analogous to the ones that must be verified, but relative to the multiplicities $\mathcal{S}_0$ and $\mathcal{S}_0'$. It remains for us to investigate whether one can deduce the same conditions for $\mathcal{S}_0''$ and $\mathcal{S}_0'''$. Moreover, this is what one confirms without difficulty for the second order conditions, in general.

On the other hand, this is certainly true for the derivatives (of arbitrary order) with respect to just t, by virtue of the theorem to which we alluded in no. **240**, and to which we shall return in note $\mathfrak{m}$ at the end of this work.

327. – We just considered the case of a characteristic that annulled the determinant H without annulling its minors. Analogous results that relate to the contrary hypothesis (those of no. **299**) appear from it. It is clear that by means of a change of unknowns one may consider the given equations to be solved with respect to p_{nn}, q_{nn-1}, r_{nn-1}, the expressions thus obtained for these quantities being such that their derivatives with respect to q_{nn}, r_{nn} are null at the origin.

Under these conditions, one may give the values of the three unknowns and r_n for $x_n = 0$, as well as those of the first two unknowns for $x_{n-1} = 0$. The solution of the problem thus posed will be studied by procedures that are completely similar to the ones in no. **323**.

We observe that this result is independent of the hypothesis that was made in no. **299** for the neighboring characteristics to the one under consideration ([44]). Of course, it nevertheless supposes conditions of inequality that are analogous to the ones in no. **322**, but which no longer have the same geometric significance, since the bicharacteristics may no longer be defined.

§ 3. – THE CASE OF LINEAR EQUATIONS

328. – Among the systems of equations that belong to the category that we just considered, there is good reason to focus on the particular case of linear equations. These are the ones that one comes down to whenever one must study the most general motion of a body when one restricts oneself to infinitely small motions.

For example, this is the case when one is concerned with the simplest (next to the Laplace equation) and most important of these equations, namely:

([44]) The simultaneous vanishing of the minors of H may likewise be valid only at one and only one point of them, viz., the origin of the coordinates.

$$\frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} = \Delta\Phi, \tag{62}$$

where a is a given number that will, by virtue of the formulas that were established in the preceding, represent the velocity of propagation of a wave in a state of motion that is governed by that equation. This equation (with $a^2 = (dp/d\rho)_{\rho=\rho_0}$) is the one that equation (**23′**) of no. **290** (the equation of motion of a gas when this motion depends upon a velocity potential) reduces to when one supposes that the motion differs from rest infinitesimally, i.e., the derivatives of Φ are infinitely small, in such a manner that one may neglect the terms of second order in these equations.

329. – In a general manner, one immediately perceives that there is a noteworthy simplification to be found in the determination of the characteristics under the hypothesis that the equation is linear.

Indeed, the coefficients of a_{ik} are then functions of only the independent variables x_1, x_2, …, x_n and, contrary to what happens in the general case, no longer contain either the unknown function or its first derivatives. It then results (no. **283**) that the characteristics may be defined by abstraction from any well-defined solution of the equation. In particular, to each point (x_1, x_2, …, x_n) there corresponds a characteristic conoid that is perfectly well-defined once one has written the equation.

It is clear that whenever one solves one of the boundary-value problems for the equation that one poses in mechanics the formula for the solution must involve the characteristic conoid when it is real. Indeed, we have seen (no. **306**) that it suffices to be given the elements that determine that solution in the interior of the characteristic conoid that has a definite point O (*fig.* 20) for its vertex in order to know it at O.

330. – When the medium considered is unbounded and one is given the positions and velocities of the molecules in that space at a definite instant t_0 the determination of the ultimate motion leads to the Cauchy problem that we were occupied with in the preceding. The solution of that problem may be carried out in a large number of cases. Our intent is not to describe these solutions in detail ([45]). We content ourselves with only indicating the common principle upon which they rest, and which is nothing but a generalization of the Riemann method that we recalled in no. **171**.

([45]) See, especially, POISSON, *Mémoire sur l'intégration de quelque equations aux différences partielles et particuliérement de l'équation générale du mouvement des fluides élastiques* (read to the Ac. des Sc. on 19 July 1819); KIRCHHOFF, mécanique, lesson 23, pp. 314; *Zur Theorie der Lichtstrahlen*, Sitzungsberichte der K. Ak. der Wiss; 1882, pp. 641 et seq. (transl. by DUHEM, Ann. Ec. Norm. supérieure, 1886) and Optik: VOLTERRA, Att. Lincei, 1892 and Acta Math.; TEDONE, Att. Lincei, 1806; LE ROUX, Ann. Ec. Norm. 3rd series, t. XII, and Journ. de Mathém., 1898-1900; d'ADHÉMAN, Bull. Soc. Math. Fr. 1901 and C. R. Ac. Sc. 1902; COULON, Soc. Sc. Phys. et Nat. de Bordeaux, *passim* and thesis *sur l'intégration des équations aux dérivées partielles par la méthode des charactéristiques*, Paris, Hermann (1902).

First of all, it is easy to write, in the general case, the formula that corresponds to relation (**35**) of no. **171** for the equation in two variables with real characteristic (or to the analogous formula from potential theory). If:

$$\mathcal{F}(z) = \sum_{i,k} a_{ik}\, p_{ik} + \sum_i a_i\, p_i + lz = 0 \tag{63}$$

is the given linear equation, the a_{ik} , the a_i , and the l being given functions of x_1, x_2, …, x_n, then an obvious sequence of integrations by parts will permit us to write:

$$u\mathcal{F}(z) - z\mathcal{G}(u) = \frac{\partial M_1}{\partial x_1} + \frac{\partial M_2}{\partial x_2} + \cdots + \frac{\partial M_n}{\partial x_n}, \tag{64}$$

upon setting:

$$M_i = u\sum_k a_{ik}\, p_k - z\sum_h \frac{\partial}{\partial x_k}(a_{ik}u) + a_i uz, \tag{65}$$

$$\mathcal{G}(u) = \sum_{i,h} \frac{\partial^2}{\partial x_i \partial x_k}(a_{ik}u) - \sum_i \frac{\partial}{\partial x_i}(a_i u) + lu. \tag{66}$$

Equation $\mathcal{G}(u) = 0$ will be called the *adjoint* of the one that was proposed.

It is, moreover, clear that the preceding result is by no means peculiar to the case of a second order equation, and that one may obtain it for any order of equation that is proposed.

It further extends just as easily to a system of an arbitrary number p of equations in an equal number of unknowns upon introducing p new function u_1, u_2, …, u_n into the *adjoint system*, by which one may multiply the left-hand sides of the given equations.

Nevertheless, we confine ourselves to the case of just one second order equation. Likewise, for the sake of discussion we put ourselves in the case of one equation in three independent variables, but the reasoning will be, unless indicated to the contrary, true for any number of these variables.

331. – In order to solve the Cauchy problem that relates to our equation when the unknown and its derivatives are given on a certain multiplicity, we must suppose, conforming to the preceding, that this multiplicity is not tangent to a characteristic.

In general ([46]), when the Cauchy problem is posed in mathematical physics a more precise condition must be verified, viz., the one that we already encountered in no. **305**. Always placing ourselves in the case of three variables, the tangent plane to the multiplicity in question is *exterior* to the characteristic cone; a plane that is parallel to it will always cut this cone in a closed curve.

A completely analogous fact is true for equations in more than three independent variables. For example, in the most important one of them – viz., the equation in four variables (**62**) – the quadratic form that, when equated to 0, furnishes the equation of the characteristic cone is a sum of squares that all have the same sign, except for one, which

([46]) Cf. *infra*, no. **340**.

refers to the variable π_4 that corresponds to the variable t. Now, the Cauchy problem is then posed precisely relative to the multiplicity $t = 0$. It is cut by the characteristic cone or, more generally, *by the characteristic conoid, which has its vertex at an arbitrary exterior point on a closed multiplicity* (namely, it is generically a sphere).

The givens relative to the interior points of this closed multiplicity are, as we know, the only ones that figure in the determination of the value of the integral at the vertex of the conoid.

332. – Therefore, consider (in the case of three variables) a surface S that is situated in the manner that we just explained with respect to the characteristic conoid and along which we are given the values of the unknown and its first derivatives.

Let S_1 be another surface that, along with the first one, bounds a portion $\mathcal{T}$ of space. If the function u, which is a solution to the adjoint equation, is regular on it then, from Green's theorem ([47]), we may write, upon multiplying by the volume element and integrating over $\mathcal{T}$:

$$(\mathbf{67})\qquad \begin{cases} \displaystyle\iiint u\mathcal{F}(z)\,dx_1dx_2dx_3 \\ \displaystyle\quad = \iint M_1dx_2dx_3 + M_2dx_3dx_1 + M_3dx_1dx_2 \\ \displaystyle = \iint \pm\left[M_1\frac{D(x_2,x_3)}{D(\lambda_1,\lambda_2)} + M_2\frac{D(x_3,x_1)}{D(\lambda_1,\lambda_2)} + M_3\frac{D(x_1,x_2)}{D(\lambda_1,\lambda_2)}\right]d\lambda_1d\lambda_2. \end{cases}$$

(where the double integral is taken over S and then S_1, successively, and λ_1, λ_2 denote the curvilinear coordinates that are inscribed on these surfaces); or, if one prefers:

$$(\mathbf{68})\qquad \iiint u\mathcal{F}(z)\,dx_1dx_2dx_3 = \iint\left[\sum_i M_i\cos(N, x_i)\right]dS\,,$$

where dS denotes the surface element on S and then S_1, successively, and N denotes the corresponding normal that is directed *out of* $\mathcal{T}$. Nothing essential will change from the foregoing if the number n of independent variables is greater than three. The only difficulty that will present itself will be the introduction of the geometry of n dimensions. Instead of the surfaces S and S_1, one will have to consider n–1-fold extended multiplicities – or *hypersurfaces*. Formula (**67**) will become:

([47]) See PICARD, *Traité d'Analyse*, 2nd edition, t. 1, first part, Chap. IV, nos. 15 and 16, and chap. V, no. 8.

$$
\textbf{(67}'\textbf{)} \qquad \begin{cases} \iint \cdots \int uF(z)\, dx_1 dx_2 \cdots dx_n \\ = \iint \cdots \int \left(\sum_i M_i \pi_i \right) d\lambda_1 d\lambda_2 \cdots d\lambda_{n-1} \end{cases}
$$

(where the left-hand side is an n-fold integral and the one on the right-hand side is an n – 1-fold integral). In this formula, the quantities π_i are, up to sign, the functional determinants of any n – 1 of the x_i with respect to the n – 1 curvilinear coordinates λ_1, λ_2, …, λ_{n-1} that are chosen on the multiplicity S (or S_1). In other words, if one draws a line on it through each point, where s denotes the arc length, then the quantities π_i are defined by the condition that one have, for any such line:

$$
\textbf{(69)} \qquad \pi_1 \frac{\partial x_1}{\partial s} + \pi_2 \frac{\partial x_2}{\partial s} + \cdots + \pi_n \frac{\partial x_n}{\partial s} = \pm \frac{D(x_1, x_2, \cdots, x_n)}{D(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, s)}.
$$

These quantities may be considered to be the ones that we denoted by this nomenclature in no. **287**. They are proportional to the direction cosines of the normal to dS, or to the partial derivatives of the left-hand side $\Pi(x_1, x_2, \ldots, x_n)$ of the equation of the multiplicity.

If the normal N is directed into the domain $\mathcal{T}$, or if the function Π is positive on the exterior of that domain and negative in its interior, then the sign that one takes in equation (**67**) or equation (**69**) is the one that makes the π_i equal to the direction cosines or the partial derivatives that we just spoke of, up to the same *positive* factor.

We further let A denote the expression:

$$
\textbf{(18)} \qquad A = \sum a_{ik}\, \pi_i\, \pi_k\,,
$$

in such a way that the characteristics are defined by the equation $A = 0$. One will have:

$$
\textbf{(70)} \qquad \sum a_{ik}\, \pi_i = \frac{1}{2} \frac{\partial A}{\partial \pi_i}\,,
$$

and consequently:

$$
\textbf{(71)} \qquad \begin{cases} \sum_i M_i \pi_i = u \left[\sum_{ik} a_{ik} \frac{\partial z}{\partial x_i} \pi_k + \left(\sum_i a_i \pi_i \right) \right] - z \sum_{ik} \pi_k \frac{\partial}{\partial x_i}(a_{ik} u) \\ \qquad = \frac{1}{2} u \sum \frac{\partial z}{\partial x_i} \frac{\partial A}{\partial \pi_i} - \frac{1}{2} z \sum \frac{\partial u}{\partial x_i} \frac{\partial A}{\partial \pi_i} + Luz \end{cases}
$$

$$
\textbf{(72)} \qquad L = \sum a_{ik}\, \pi_i - \sum \pi_i \frac{\partial a_{ik}}{\partial x_k}.
$$

Now introduce, with d'Adhémar ([48]), the direction whose direction cosines are proportional to the quantities $\partial A / \partial \pi_i$, and which will be called the *conormal* to dS; in other words, the direction that is defined by the proportions:

$$\textbf{(73)} \qquad \frac{dx_1}{\frac{1}{2}\frac{\partial A}{\partial \pi_1}} = \frac{dx_2}{\frac{1}{2}\frac{\partial A}{\partial \pi_2}} = \ldots = \frac{dx_n}{\frac{1}{2}\frac{\partial A}{\partial \pi_n}} = \frac{1}{h} ds ,$$

in which s is a parameter and h is an arbitrary quantity that we may, for example, dispose of in such a manner the largest of the ratios π_1/h, π_2/h, …, π_n/h – and consequently, the largest of the ratios dx_1 / ds, …, dx_n / ds ([49]) – has an absolute value that lies between two finite, positive, non-zero limits (for example, if one takes dx_1 / ds, …, dx_n / ds to be the direction cosines of the aforementioned direction).

From its very definition, the conormal is ([50]) the conjugate diameter of the tangent plane to dS with respect to the characteristic cone (which is represented by the tangential equation $A = 0$).

It is tangent to the element dS when that element is characteristic, and only in this case (as one sees upon multiplying the terms of the fractions (**73**) by π_1, π_2, …, π_n, respectively, and then adding); it is nothing but the bicharacteristic direction that is tangent to that element.

By means of the preceding nomenclature and formula (**71**), equation (**67′**) may be written:

$$\textbf{(74)} \qquad \begin{cases} \displaystyle\iiint \cdots \int u \mathcal{F}(z)\, dx_1 dx_2 \cdots dx_n \\ \displaystyle = \iint \cdots \int \left[h\left(u \frac{dz}{ds} - z \frac{du}{ds} \right) + Luz \right] d\lambda_1 d\lambda_2 \cdots d\lambda_{n-1}. \end{cases}$$

333. – If we wish to determine the function u and the multiplicity S_1 in such a manner that the values of u and its derivatives on S_1 can be eliminated from this result then first of all we must have that if u and its derivatives are non-null on that same multiplicity ([51])

([48]) *C. R. Ac. Sc.*, 11 February 1901.

([49]) At least, if one supposes that the quadratic form A has a non-zero discriminant in the domain under consideration, and, in any case, on any simple characteristic.

([50]) See COULON, thesis, pp. 35.

([51]) On the other hand, u (if it not identically null) may not be annulled at the same time as its first derivatives on S_1 unless it is characteristic. Indeed, the solution to the Cauchy problem is unique for a non-characteristic multiplicity. This is what we established before upon supposing that the unknown is analytic and holomorphic. For u continuous and differentiable up to a certain order, but not analytic, the same fact will result from the extension (to the case of n independent variables) of a proof of Holmgren (See note 1 at the end of this work).

What finally remains is the case where S_1 is a singular multiplicity for u. However, as we shall verify later on (no. **342**), this case will no longer present itself (at least for the usual types of singularity) if S_1 is not characteristic.

then that multiplicity must be characteristic. On the other hand, if this is not the case then the preceding formula will contain, on the one hand, the values of z, and, on the other, those of its conormal derivative, which will be entirely independent of each other since the conormal will be exterior to the surface.

Suppose that S_1 is characteristic, and, first taking the case of of $n = 3$, refer S_1 to curvilinear coordinates, one of which, λ, is constant on the bicharacteristics, whereas the other s will define the position of a variable point on each of these curves, the derivatives dx_i / ds all being finite and not all infinitely small, from the convention that was made on h in the preceding no. Then, in the right-hand side of (**74**) the portion that relates to S_1, namely:

(**75**) $$\iint\left[h\left(u\frac{dz}{ds}-z\frac{du}{ds}\right)+Luz\right]d\lambda\, ds\,,$$

may be transformed by integration by parts into a simple integral:

(**76**) $$\int huz\, d\lambda$$

that is taken along the contour Γ in S_1, combined with the following one:

(**77**) $$\iint z\left[2h\frac{du}{ds}+u\left(\frac{dh}{ds}-L\right)\right]d\lambda\, ds\,.$$

We choose the function u in such a manner that it verifies, on each bicharacteristic, the differential equation:

(**78**) $$2h\frac{du}{ds}+u\left(\frac{dh}{ds}-L\right)=0\,,$$

which determines u by a quadrature, except for a constant factor that one may choose arbitrarily for each value of t.

All of this obviously persists for an arbitrary n. One will have only $n - 2$ coordinates λ (the coordinate s still being unique) and the contour Γ of S_1 will no longer be a curve, but an $n - 2$-fold extended multiplicity. The integral over S_1 reduces to an $n - 2$-fold integral:

(**76′**) $$\iint\cdots\int hus\, d\lambda_1 d\lambda_2\cdots d\lambda_{n-2}$$

that is taken over Γ, combined with an integral that is analogous to (**77**), which will disappear at the moment when we determine u by the differential equation (**78**).

334. – Up till now, we have allowed S_1 to be arbitrary. Now, suppose that one takes S_1 to be the characteristic conoid C that has a definite point O for its vertex.

In this case, it results from (**78**) that u must be infinite at O. Indeed, suppose, to fix ideas, that the parameter s has been chosen on each bicharacteristic in such a manner that it is annulled at that point. Thus, $x_1, x_2, \ldots, x_n$ must be equal to the coordinates of O for $s = 0$, no matter what the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$ are, and their derivatives with respect to these parameters are null as a consequence of the order of s. The functional determinants π_i of the $n - 1$ arbitrary coordinates x with respect to the $n - 2$ parameters λ and s are therefore of order s^{n-2}, and the same is true for L, as well as h, if (as we have agreed upon above) we take that quantity to be of order greater than π_i .

For example, for $n = 3$ it is clear that if the points of a cone are represented by their distance from the vertex and a parameter that defines the generator then the surface element of the cone will contain the first of these two quantities as a factor.

If h has order greater than π_i then the ration L/h is finite at O. The quadrature to which we are then led of the differential equation (**78**), namely:

$$u = e^{\int -\frac{1}{2h}\left(\frac{dh}{ds} - L\right) ds} = \frac{1}{\sqrt{h}} e^{\int \frac{L}{2h} ds},$$

then gives an infinite result of order $s(n - 2)/2$.

Under these conditions, in order to apply the fundamental formula we subtract from our volume integral the part that is immediately close to the point O. If we again put ourselves in the case of $n = 3$ then a small portion of the conoid C will thus be removed, a portion that is bounded by a curve γ (*fig.* 22). To fix ideas, one may assume that γ is the intersection of the conoid C with a sphere Σ with center O and very small radius.

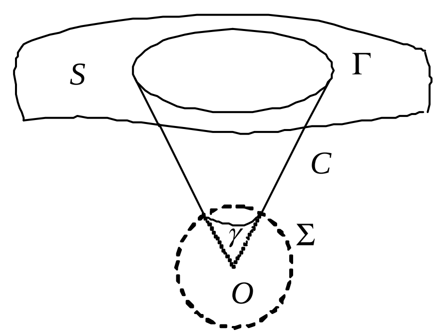

Fig. 22

S_1 will thus have two frontiers: its intersection Γ with S and the multiplicity γ. It is along these two frontiers that one must take the $n - 2$-fold integral (**76**), which reduces to (**74**) by means of the differential equation (**78**).

That integral is known along Γ since one knows z and its first derivatives.

One will thus have the expression for z by a natural generalization of the Riemann method if, the function u being regular throughout the volume of integration, with the exception of a neighborhood of O, and the radius of Σ tending to zero, the integral (**67′**), when taken over Σ, and added to the integral (**76′**), when taken over γ, reduces to z_0 .

335. - However, things do not happen exactly that way. For example, consider equation (**62**). In the category of equations that we are envisioning at the moment it was the first one for which the Cauchy problem was solved, thanks to the work of Poisson and Kirchhoff. The independent variables are thus four in number, the first three of which, which represent Cartesian coordinates in ordinary space, will be called x_1, x_2, x_3, whereas we will continue to denote the fourth one by t. We suppose that S has the equation $t = 0$, in such a way that one must be given conditions:

$$z = f$$
$$\frac{\partial z}{\partial t} = f_1$$

for $t = 0$, where f and f_1 are known functions of x_1, x_2, x_3 .

The method that is employed for expressing the value of z for $x_1 = x_1^0$, $x_2 = x_2^0$, $x_3 = x_3^0$, $t = t_0$ as a function of these givens consists of taking:

$$u = \frac{1}{r} F(r + at),$$

F being an arbitrary function, and r denoting the distance (in ordinary space) from the point (x_1, x_2, x_3) to the point (x_1^0, x_2^0, x_3^0):

$$r = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2}\ .$$

This quantity indeed satisfies the adjoint equation, which is identical to the one proposed here. It likewise verifies condition (**78**) for any function F. Indeed, on the characteristic cone it is proportional to $1/r$, and it precisely such a proportionality that suggests the differential equation (**78**).

However, this function is not uniquely singular (as the preceding theory demands) at only one point of the four-dimensional space. Indeed, it is infinite for $x_1 = x_1^0$, $x_2 = x_2^0$, $x_3 = x_3^0$ for any t, and not uniquely for the given value t_0 that corresponds to the vertex O of the characteristic cone. We must therefore subtract from our volume integral, not exclusively the immediate neighborhood of the vertex of the cone, but, for example, the set τ of points (x_1, x_2, x_3) that satisfy the inequality:

$$(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2 < z^2.$$

Conforming to the convention of no. **100** (cont.), that region is represented in figure 23 by the interior of a cylinder σ (to which it reduces if one considers only the coordinates x_1, x_2, and t, the variable x_3 having been suppressed).

On our cone, the frontier σ of the region τ will intercept the multiplicity γ (which will be nothing but the surface of a sphere of radius ε with $t = t_0 - \varepsilon/a$) and a multiplicity γ' (a sphere of radius ε with $t = 0$) on S.

Here, since the polynomial $\mathcal{F}(z)$ has the expression:

$$\mathcal{F}(z) = \Delta z - \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2},$$

and the characteristic conoid is:

$$(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2 - a^2(t - t_0)^2 = 0,$$

the corresponding bicharacteristics are nothing but the generators.

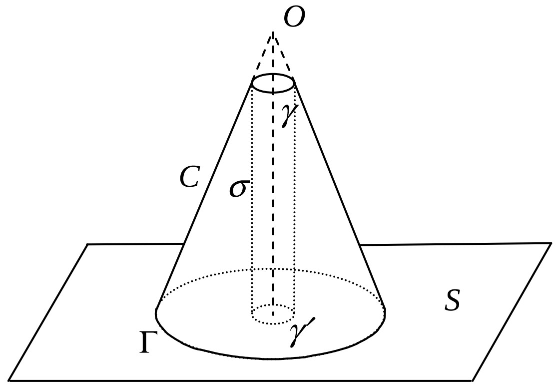

Fig. 23

We thus obtain the system of curvilinear coordinates on C that is required by the preceding argument by employing (in ordinary space) polar coordinates with origin O; i.e., by setting:

$$x_1 = x_1^0 + r \sin \lambda_1 \cos \lambda_2, \qquad x_2 = x_2^0 + r \sin \lambda_1 \sin \lambda_2, \qquad x_3 = x_3^0 + r \cos \lambda_1,$$
$$(0 \leq \lambda_1 \leq \pi, \quad 0 \leq \lambda_2 \leq 2\pi).$$

We may then take $s = r$, and we easily find that:

$$h = \frac{r^2 \sin \lambda_1}{a}.$$

Under these conditions, u being given the formula (**79**), the triple integral over S_1 will, by virtue of the calculations, reduce to the double integral:

$$\textbf{(80)} \qquad \iint \frac{r^2 uz}{a} \sin \lambda_1 \, d\lambda_1 \, d\lambda_2 = \iint \frac{r^2 uz}{a} \, d\Omega$$

($d\Omega = \sin \lambda_1 \, d\lambda_1 \, d\lambda_2$ being an element of a sphere of radius 1), which is taken over the multiplicities Γ and γ, successively.

The integral over S takes a particularly simple form when (as one supposes) this multiplicity has $t = 0$ for its equation. If we let $\varphi(r)$ and $\varphi_1(r)$ denote the mean values for $t = 0$ over a sphere with a center at (x_1^0, x_2^0, x_3^0) and radius r of the functions f and f_1 that z and $\partial z / \partial t$ reduce to for $t = 0$, namely:

$$\varphi(r) = \frac{1}{4\pi} \iint f(x_1^0 + r \sin \lambda_1 \cos \lambda_2, x_2^0 + r \sin \lambda_1 \sin \lambda_2, x_3^0 + r \cos \lambda_1) \sin \lambda_1 \, d\lambda_1 d\lambda_2$$
$$\varphi_1(r) = \frac{1}{4\pi} \iint f_1(x_1^0 + r \sin \lambda_1 \cos \lambda_2, x_2^0 + r \sin \lambda_1 \sin \lambda_2, x_3^0 + r \cos \lambda_1) \sin \lambda_1 \, d\lambda_1 d\lambda_2 ,$$

then this integral becomes:

$$\frac{1}{a^2}\iiint\left(u\frac{\partial z}{\partial t}-z\frac{\partial u}{\partial t}\right)dx_1dx_2dx_3=\frac{4\pi}{a^2}\int_\varepsilon^{at_0} r[F(r)\varphi_1(r)-aF'(r)\varphi(r)]\,dr\,,$$

or, by means of an obvious integration by parts:

$$\frac{1}{a^2}\iiint\left(u\frac{\partial z}{\partial t}-z\frac{\partial u}{\partial t}\right)dx_1dx_2dx_3=$$
$$=-4\pi t_0F(at_0)\varphi(at_0)+\frac{4\pi\varepsilon}{a}F(\varepsilon)\varphi(\varepsilon)+\frac{4\pi}{a^2}\int_\varepsilon^{at_0}F(r)[r\varphi_1(r)+a\frac{d}{dr}(r\varphi(r))]\,dr\,,$$

an expression in which the first term $-4\pi t_0\,F(at_0)\varphi(at_0)$ will annihilate the integral (**80**) over Γ precisely.

Now, if ε tends to zero then the second term of the preceding expression also becomes infinitely small, and the same is true for the integral (**80**) over γ, since r^2u tends to zero with r.

Finally, consider the integral over σ. This integral is:

$$\iiint\left(z\frac{\partial u}{\partial r}-u\frac{\partial z}{\partial r}\right)\varepsilon^2d\Omega\,dt\,,$$

where the double integral is taken over the surface of a sphere of radius ε ($\varepsilon^2\,d\Omega$ being the area element of that sphere). It will not have a quantity that is proportional to z_0 for its a limit, but rather (since $\partial u/\partial t$ has $-1/r^2\,F(at)$ for its principal part) the simple integral:

$$-4\pi\int zF(at)dt$$

taken from 0 to t_0, and for $x_1 = x_1^0$, $x_2 = x_2^0$, $x_3 = x_3^0$.

One ultimately finds that:

$$\int_0^{at_0}\frac{F(r)}{a}\left[r\varphi_1(r)+a\frac{d}{dr}(r\varphi(r))\right]dr-\int_0^{t_0}zF(at)\,dt=0.$$

However, this result simplifies considerably, thanks to the fact that the function F that appears in the preceding formulas is arbitrary. Indeed, upon exchanging r and at in the first integral, one may write:

(**81**) $$\int_0^{t_0}F(at)\left[t\varphi_1(at)+\frac{d}{dt}(t\varphi(at))-z\right]dt=0\,.$$

Now, it results from a classical argument from the calculus of variations that an equality of the form (**81**) might not be true for an arbitrary function F if one does not have for any value of *t:*

$$t[\varphi_1(at) + a\varphi'(at)] + \varphi(at) - z = 0 .$$

In particular, this is true for $t = t_0$, and one has:

$$z_0 = \varphi(at_0) + t_0\,[\varphi_1(at_0) + a\varphi'(at_0)] . \tag{82}$$

Here, one sees that the value of z_0 is expressed, not as a function of all of the values that z and $\partial z/\partial t$ take on S in the entire interior of the characteristic conoid, but only the values that are taken by these quantities *on* the conoid. This circumstance is due to the particular form of equation (**62**) and does not present itself for a second order equation that is taken at random ([52]).

336. – We remark that from the form itself of the solution that we just obtained it results that the method might not succeed in the original form that was indicated in no. **334**. Indeed, it will lead to an expression for the solution in the form that is analogous to the right-hand side of (**74**), i.e., in terms of the values of z and $\partial z/\partial t$ on the entire part S_0 of S that is interior to the characteristic conoid.

It is true that one may indeed transform the integral (**82**) into another one that is taken over all of S_0, but in order for this to be true it is necessary that the integration element contain the derivatives of z and $\partial z/\partial t$ with respect to the coordinates that are defined on S (in other words, with respect to x_1, x_2, x_3).

In a word, the right-hand side of formula (**82**) is irreducible to that of (**74**).

Thus, there might not exist a solution to equation (**62**) that verifies the various conditions that we postulated in no. **334**.

337. – These various results have been generalized to some very extensive categories of equations in the works that were cited above. We content ourselves by pointing out the simplest case, that of the equation:

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} - \frac{1}{n^2}\frac{\partial^2 z}{\partial t^2} = 0, \tag{83}$$

which is nothing but the analogue to (**62**) for the case of two dimensions, and for which the Cauchy problem (the multiplicity S always verifying the conditions that were imposed in no. **334**) was solved by Volterra. The function u that was chosen by the latter is then the one that is deduced from the quantity:

([52]) The supposition that one takes the multiplicity $t = 0$ for S is obviously not at all essential, and the preceding considerations persist, with results that are somewhat less simple, for an arbitrary S.

$$\text{(84)} \qquad \log\left(\frac{at+\sqrt{a^2t^2-x_1^2-x_2^2}}{\sqrt{x_1^2+x_2^2}}\right)$$

upon changing x_1, x_2, t into $x_1 - x_1^2$, $x_2 - x_2^2$, $t - t_0$.

As one sees, it admits the characteristic conoid as a singular surface. However, it is easy to see that this singularity does not compromise the application of our fundamental formula: The only part that one must subtract from the volume integral is again the one that is composed of the interior of a small cylinder that has as its axis, the line:

$$x_1 = x_1^2, \; x_2 = x_2^2 .$$

The formula (**74**) will then give us the simple integral:

$$\int z\,dt$$

taken along this line from $t = 0$ to $t = t_0$.

It only remains for us to take the derivative of that quantity with respect to t_0 in order to obtain the value of z_0 .

Like the solution of equation (**62**), the solution thus obtained is no longer expressible in the form of an integral that is taken over the part of S that is situated on the *surface* of a characteristic conoid; the values of the givens in all the interior of this cone necessarily figure.

By contrast, one may make several remarks concerning this solution that are completely similar to those of the preceding no. and deduce from this that the method might not succeed in the form that was described in no **334**.

338. – Furthermore, when we are concerned with the problem that we just spoke of, or the one that relates to equation (**62**), the preceding considerations up till now persist in the limiting case where S is characteristic. For example, one may take S to be a characteristic conoid, as long as one is nonetheless content to determine z in the *interior* of the this conoid.

In this case, since the conormal to S will be tangent to S, the knowledge of the values of z on the multiplicity in question will suffice, since it implies the knowledge of the conormal derivatives.

Thus, an integral of equation (**62**) or equation (**83**) is well-defined in the interior of a characteristic conoid when one gives its values on that conoid. In particular, *it cannot be annulled on the conoid* (except for the singular case, such as, for example, the one that we encountered for the expression (**84**) of the preceding no.) *without being identically null in the entire interior.*

This result obviously corresponds to the one that we found in no. **172** (ch. IV).

339. – We likewise remark that the method can be extended to the case where the linear equation has a right-hand side, i.e., where one no longer equates the left-hand side of equation (**63**) to zero, but to an arbitrary given function $\mathcal{F}$ of $x_1, x_2, \ldots, x_n$. Under these conditions, the n-fold integral that figures in left-hand side of equation (**74**) will no longer be null, but its value will be known. For equation (**62**), this will lead to completing formula (**82**) with the integral:

$$\iiint \frac{\mathcal{F}}{r} dx_1 dx_2 dx_3 \,,$$

taken over the characteristic conoid. In the case of equation (**83**), one will have to consider, not only the double integral that is taken over the characteristic conoid, but also a triple integral that is taken over the volume that is interior to that cone.

340. – In any case, a direct calculation will show that the expressions that are obtained by the preceding method indeed verify all of the required conditions, provided that the multiplicity S satisfies the hypotheses of no. **331**.

In this case, the solution to the Cauchy problem is thus always possible, regardless of whether the givens are or are not analytic.

This is no longer true if the hypotheses of no. **331** are not verified, such as when the multiplicity S cuts the characteristic cone that issues from one of its points. For example, this is the situation that presents itself in the generalization that was given by Kirchhoff ([53]) of the solution in no. **335** or in the analogous study that was carried out by Volterra ([54]) on equation (**83**).

Whenever S takes such a form, *a solution to the Cauchy problem ceases to be possible, in general*. That is what happened with the solutions that were given by Kirchhoff and Volterra: an infinitude of possibility conditions appeared. In reality, in the problems that they treated one may, as one easily confirms, give the Cauchy data – i.e., z and its first derivatives – on only a subset of S, since *only* z is given (as in no. **180-184** of chap. IV) on the other subset. The corresponding forms for S are, moreover, such that the proof of Cauchy-Kowalewski (relative to the existence of a solution for analytic givens) is no longer applicable.

However, just as in the case where this proof is possible – for example, in the context of equation (**62**), when one takes S to be the multiplicity $x_1 = 0$ – one confirms that the possibility of the solution to the problem ceases to be true, in general, with the analyticity of the givens if the condition of no. **331** is not met.

341. – We confine ourselves to the previous observations on the solution of the Cauchy problem, and now study another question that is closely related to the ones that were the object of the preceding chapter.

([53]) *Zur Theorie der Lichtstrahlen* and *Optik*.

([54]) *Sur les vibrations des corps élastiques isotropes*, no. 6 (Acta Math. t. XVIII).

We have confirmed that the waves by which the discontinuities propagate in a moving medium are nothing but the characteristics of the differential equations that determine these motions. To that effect, we are thus placed within the scope of the hypotheses that were formulated in no. **71**, and from which the quantities considered and their various derivatives must all tend to perfectly well-defined limits on each side of the discontinuity.

There is reason to demand that the analogous conclusions persist under the contrary hypothesis, i.e., upon assuming that there is not only a discontinuity between two compatible motions, but also a singularity of one of these motions in its own right, such that one of the unknowns or its derivatives becomes infinite. For example, this is what we are confronted with in the solution (**84**) to equation (**83**).

The results that we arrive at will be, moreover, important in that they permit us to relate the theory of waves that we have just described in the preceding chapters to the one that we encounter in various important branches of physics, particularly acoustics and optics.

One then knows that, instead of considering, as we have done, the propagation of the motion, properly speaking, i.e., the manner by which it *commences* at the various successive points of space, one suppose that this motion has already commenced and arrived at a sort of permanent state. Under these conditions, the wave surface, as we have envisioned it in the foregoing, is no longer applicable. However, on the other hand, the motion under study is not arbitrary: It is a periodic oscillation and the wave surface is then the locus of points in space where the phase of oscillation is the same. Of course, as in the foregoing, when one varies the time the set of wave surfaces in the space E_4 that correspond to the same phase is a triply-extended multiplicity that represents the progress of the wave and permits one to define the velocity of propagation.

We shall justify later on why one is led to the same waves as in the Hugoniot theory – namely, the characteristics – and we likewise justify the introduction of the *bicharacteristics* that were defined in the present chapter as possessing the fundamental properties of the *rays* that one considers in physics.

342. – Therefore assume, with Delassus ([55]), that a given second order linear equation:

$$(\mathbf{63}) \qquad \mathcal{F}(s) = \sum_{i,k} a_{ik}\, p_{ik} + \sum_i a_i\, p_i + lz = 0$$

possesses a solution of the form:

$$(\mathbf{85}) \qquad z = ZB(\Pi),$$

where Z, Π are regular functions – by which, I mean functions that are finite, continuous, and differentiable, – but where the function F admits a singularity for $\Pi = 0$. Substituting that quantity, it easily follows that:

([55]) *Annales Scient. de l'Ec. Norm. Sup.*, 3rd series, t. XIII, pp. 357, et seq., 1896.

$$\textbf{(86)} \qquad AZF''(\Pi) + \left(\sum_i \frac{\partial Z}{\partial x_i}\frac{\partial A}{\partial \pi_i} + MZ\right)F'(\Pi) + \mathcal{F}(Z)\cdot F(\Pi) = 0,$$

the π_i being the partial derivatives of Π, and A always being defined by equation (**18**) in no. **287**, while the F', F'' are the first and second derivatives of the function F, and one has that:

$$M = \sum_{i,k} a_{ik}\frac{\partial^2\Pi}{\partial x_i \partial \pi_i} + \sum_i a_i \pi_i = \mathcal{F}(\Pi) - l\Pi.$$

We shall not exactly leave the function F completely arbitrary: We suppose that this function is such that F' is infinitely large with respect to F and F'' is infinitely large with respect to F' for Π in the neighborhood of 0. This condition is satisfied for all of the usual forms for functions of one variable that are singular at the origin, such as:

$$F(\Pi) = \Pi^p \qquad (p, \text{ a non-positive integer})$$
$$F(\Pi) = \log \Pi,$$
$$F(\Pi) = \Pi^p \log \Pi.$$

Under these conditions, it is clear that the coefficient of $F''(\Pi)$ in equation (**86**) must be annulled with Π. One thus has (for $\Pi = 0$):

$$\textbf{(87)} \qquad A = 0$$

and *the singular multiplicity* $\Pi = 0$ *must be a characteristic* ([56]).

The will again be true when the desired integral is not composed of the expression (**85**) exclusively, but includes an additional arbitrary regular term.

343. – Conversely, being given a characteristic multiplicity $\Pi = 0$ – such that, as a consequence the left-hand side of equation is annulled with Π, and one has:

$$A = \Pi\mathcal{A}$$

(where $\mathcal{A}$ is a new regular quantity), we propose to find a solution of the given equation that has the form:

$$\textbf{(88)} \qquad z = ZF(\Pi) + z_1,$$

z_1 being a regular function.

To fix ideas, we take:

$$F(\Pi) = \log \Pi.$$

One will then have:

([56]) This conclusion will not be invalidated if Z goes to zero with Π. Indeed, in that case, it is convenient to restart the argument by replacing Z with $Z_1 = Z/\Pi$, and $F(\Pi)$ with $\Pi F(\Pi)$.

$$\frac{F''(\Pi)}{F'(\Pi)} = -\frac{1}{\Pi}.$$

As a consequence, upon assuming that the terms of order F' disappear, one will have (for $\Pi = 0$) the condition:

$$\sum_i \frac{\partial Z}{\partial x}\frac{\partial A}{\partial \pi_i} + (M - \mathcal{A})Z = 0\,. \tag{89}$$

This condition may be considered to be a first order linear partial differential equation that the function Z must satisfy. It is clear ([57]) that *the characteristics of that equation will be situated on our singular surface, and are nothing but the corresponding bicharacteristics.*

Consequently, one sees that condition (**89**) relates to the distribution of values for Z itself (and not its derivatives) on the hypersurface $\Pi = 0$. If one sets, as before:

$$\frac{dx_1}{\left(\dfrac{\partial A}{\partial \pi_1}\right)} = \frac{dx_2}{\left(\dfrac{\partial A}{\partial \pi_2}\right)} = \ldots = \frac{dx_n}{\left(\dfrac{\partial A}{\partial \pi_n}\right)} = ds \tag{14′}$$

then Z will have the value:

$$Z = Z_0 e^{\int(\mathcal{A}-M)\,ds}, \tag{90}$$

where Z_0 is a factor that is independent of s that one must choose arbitrarily at a point of each characteristic, moreover.

Z having been thus chosen (and assuming that it is regular, moreover), the left-hand side of the condition (**89**) will be annulled with Π. Consequently, it will have the form:

$$\Pi\mathcal{P},$$

$\mathcal{P}$ being a regular function.

As for the logarithmic terms, the necessary and sufficient condition for them to disappear is obviously that Z be itself a solution of the proposed equation.

This being the case, it will remain for us to determine z_1 from the equation:

$$\mathcal{F}(z_1) = -\mathcal{P}. \tag{91}$$

Since $\mathcal{P}$ is, as we have said, a regular function, we learn from the general theorems ([58]) that this equation admits a likewise regular solution.

To summarize, we see that it is necessary that we:

([57]) Compare, no. **332**.

([58]) At least, in the case where all of the calculations are analytic.

1. Choose the multliplicity $\Pi = 0$ to be a characteristic, conforming to the theorem of Delassus.

2. Calculate the distribution of values for Z on this multiplicity by means of equation (**89**), or, what amounts to the same thing, by means of formula (**90**).

3. Find a solution to the proposed equation that takes the values thus calculated for $\Pi = 0$.

4. Determine a regular function z_1 by means of equation (**91**).

We know, moreover, by this procedure, that if the calculations are analytic then the third operation is possible in an infinitude of ways.

344. – When the number of independent variables is two the logarithmic solution thus obtained plays a fundamental role in the study of the equation, and this is particularly true in the case where the characteristics are imaginary.

One may then ([59]), by a change of real variables, put the equation into the form:

$$\mathcal{F}(z) = \Delta z + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0, \tag{92}$$

Δ denoting the Laplace symbol in two variables $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$, while a, b, c are given functions of x, y. Under these conditions, the characteristics are $x - iy = \text{const.}$, $x + iy = \text{const.}$

We seek a uniform solution that becomes logarithmically infinite in a neighborhood of a given point (x_0, y_0).

If we suppose that a, b, c are analytic then nothing will stop us from applying the preceding reasoning to the multiplicities:

$$x - x_0 - i\,(y - y_0) = 0, \qquad x - x_0 + i\,(y - y_0) = 0;$$

and they will be the logarithms of $x - x_0 - i\,(y - y_0)$, on the one hand, and $x - x_0 + i\,(y - y_0)$, on the other. They will then have the form:

$$Z \log[x - x_0 - i\,(y - y_0)] + Z'\,[x - x_0 + i\,(y - y_0)] + z1.$$

However, one has:

$$\log[x - x_0 - i\,(y - y_0)] = \log r - i\omega,$$
$$\log[x - x_0 + i\,(y - y_0)] = \log r + i\omega,$$

upon denoting the distance between the two points (x_0, y_0) and (x, y) by r, and the angle that this distance makes with the x axis by ω. It is clear that the solution will not be uniform if ω does not indeed disappear, i.e., if one does not have:

([59]) PICARD, *Traité d'Analyse*.

$$Z' = Z.$$

Thus, we ultimately have to find a solution Z to equation (**92**) that is defined by the double condition of taking given values for $x - x_0 - i\,(y - y_0) = 0$, and other given values for $x - x_0 + i\,(y - y_0) = 0$, these values being, moreover, analytic. This is, as we have seen, the problem that was solved by Goursat and, for the case of a much larger number of variables, by Beudon.

We thus obtain a solution of the form:

$$\text{(93)} \qquad 2\,Z \log r + z_1,$$

i.e., one of the solutions whose existence was established by Picard in the special case where a and b are null ([60]). However, we have been obliged to restrict ourselves to equations with analytic coefficients. In regard to that, although we did not make that assumption in the developments of no. **343**, we are now applying them in the complex domain. On the contrary, the method of Picard, which is founded upon successive approximations, nowhere assumes the analytic nature of the coefficients.

Nonetheless, observe that just as it does for non-analytic a, b, c our method leads to the desired result in some very general cases. Indeed, if a, b, c admit derivatives up to a certain order p around (x_0, y_0) then the Taylor formula will be applicable to them around this point, i.e., one may represent them by polynomials α, β, γ at x, y, up to quantities that will be of order greater than p at $x - x_0$, $y - y_0$.

For the moment, we then replace a, b, c with α, β, γ. The equation will thus admit a solution z' of the form (**93**). The result of substituting z' in the given equation will be a quantity that is continuous and differentiable up to order $p - 1$. It only remains for us to augment z' with a quantity z'' that is defined by the equation:

$$\mathcal{F}(z'') = -\,\mathcal{F}(z'),$$

an equation such that the theorems of Picard ([61]) permit us to find a regular solution, once we have specified the order of differentiability.

The only question – which we will not, moreover, venture to elucidate – is that of knowing whether this order is the highest one possible when one is given the hypotheses that we made on the coefficients.

The integrals of the form (**93**) play the same role in the study of equation (**92**) that the function log r plays in the study of the Laplace equation. Indeed, consider the adjoint equation to (**92**), namely, the equation:

([60]) The preceding method was obtained in an independent manner by Hedrick (*Über den analytischen Character der Lösungen von Differentialgleichungen,* Göttingen 1901) and myself (see *Notice sur les travaux scientifiques de M. Jacques Hadamard*, February 1901, and also *Congrés international des Mathématiciens*) (Paris, 1900; Gauthier-Villars, 1902).

([61]) *Journal de Mathématiques,* 5th series, tome VI, 1900; pp. 138 et seq.

$$\mathcal{G}(u) = \Delta u - \frac{\partial}{\partial x}(au) - \frac{\partial}{\partial y}(bu) + cu = 0.$$

Here, the formula gives:

$$\iint_C [u\mathcal{F}(z) - z\mathcal{G}(u)]\,dx\,dy$$
$$= \int_S \left\{ u\frac{dz}{dN} - z\frac{du}{dN} - [a\cos(N,x) + b\cos(N,y)]\,zu \right\} ds\,,$$

S being the frontier of the planar domain $\mathcal{T}$, s being the arc length of S, and N being the normal to S.

However, equation $\mathcal{G}(u) = 0$ admits a solution of the form (**93**). Suppose that it chosen in such a manner that the coefficient $2Z$ is equal to 1 at the point (x_0, y_0) (which is possible, because from our calculations Z is necessarily non-zero at that point) and choose it for the function u.

Upon performing the integration, on the one hand, along an arbitrary curve ([62]) S that surrounds the point (x_0, y_0), and, on the other, a circle of very small radius having that point for its center, one finds exactly as in the theory of the ordinary logarithmic potential:

$$\frac{1}{2\pi}\int_S \left\{ u\frac{dz}{dN} - u\frac{du}{dN} + [a\cos(Nx) + b\cos(Ny)]\,zu \right\} ds = s(x_0, y_0).$$

This formula is completely analogous to the one that we recalled in ch. 1 (no. **1**). As is that context, we obviously may deduce the following consequences:

An equation (**92**) *with analytic coefficients admits only analytic solutions.*

If two solutions of an equation (**92**) *have analytic coefficients that are defined on one side of a line l or the other and take the same values on that line, while the same is true of their normal derivatives, then these functions are analytic continuations of each other.*

Finally, if one recalls that the regular term z that figures in the solution (**93**) may be modified by the addition of an arbitrary regular solution of the proposed equation then one will be led to determine such an additive term for the function u_1 in such a manner that the corresponding solution:

$$u = U \log r + u_1$$

of the adjoint equation is annulled on the contour S, or in such a manner that its normal derivative is constant there. u will then play the role of a true Green function for the solution of the Dirichlet problem or the Neumann problem relative to equation (**92**).

345. – One may demand to see what the calculations that we just performed become when the equation has real coefficients.

([62]) Or even a system of several curves, provided that the area that they bound includes the point (x_0, y_0).

Therefore, suppose that the variables are chosen in such a manner that these characteristics are x = const., y = const. The quantity:

$$\log r = \frac{1}{2}\left[\log[(x-x_0)-i(y-y_0)+\log[(x-x_0)+i(y-y_0)]\right]$$

must then be replaced (up to a factor of 2) by the logarithm of the product $(x - x_0)(y - y_0)$.

Now, since the characteristics are parallel to the axes, the equation has (no. **164**) the form:

$$\frac{\partial^2 z}{\partial x\,\partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0.$$

If it is to admit the solution:

$$z = Z \log\,[(x - x_0)(y - y_0)] + z_1$$

then the function Z, itself a solution of the equation, must verify, in addition, the relations (**89**) on the characteristics, which reduce to the form:

$$\frac{\partial Z}{\partial y} + aZ = 0$$

on the characteristic $x = x_0$ and:

$$\frac{\partial Z}{\partial x} + bZ = 0$$

on the characteristic $y = y_0$.

We may further take $Z = 1$ for $x = x_0$, $y = y_0$. We then see that the function Z *is none other than the Riemann function that was defined in* no. **171**.

346. – When one passes to the case of three independent variables, the important solutions to consider are no longer the ones that we just spoke of, but the ones that are infinite like $1/r$ in the environment of $r = 0$.

This leads us to give the function F that was introduced in the foregoing the form $F = 1/\Pi$. However, it is easy to see that if the given equation and the characteristic $\Pi = 0$ are taken in an arbitrary manner then an integral of this type will not exist, in general.

In order to see this, it suffices to observe that in the expression:

$$z = \frac{Z}{\Pi} + z_1 ,$$

which is singular on the multiplicity $\Pi = 0$, the values that are taken by Z on that multiplicity determine only the values of the singularity, provided that upon adding to Z a regular function that is annulled with Π and that consequently has the form $\Pi\mathcal{Z}$ one modifies the expression z only in the regular quantity $\mathcal{Z}$.

Now, once the coefficient of F'' – i.e., of $2/\Pi^2$ – is annulled, thanks to a choice of Π, it remains for us to make the terms in $1/\Pi^2$ and $1/\Pi$ disappear. We will thus have *two* conditions on the partial derivatives, which both affect, as we have seen, and as one painlessly verifies directly, the distribution of values for Z along our multiplicity $\Pi = 0$. In general, these two partial differential equations will not have common solutions that are not identically null.

By contrast, there will exist solutions of the form:

$$Z \log \Pi + \frac{Z_1}{\Pi},$$

and this is likewise easy to deduce from what we previously obtained. Indeed, consider the characteristic that we start with to be part of a family of characteristics whose general equation is:

$$\Pi(x_1, x_2, \ldots, x_n, \lambda) = 0 .$$

For each value of λ, we may, by the preceding method, construct the solution:

$$Z \log \Pi + z_1 .$$

By differentiating the expression that is thus obtained with respect to λ we will have a new solution to the equation that may be written:

$$\frac{Z}{\Pi}\frac{\partial \Pi}{\partial \lambda} + \frac{\partial Z}{\partial \lambda} \log \Pi + \frac{\partial z_1}{\partial \lambda} .$$

This solution will thus have precisely the form that is demanded if one does not have $\partial\Pi / \partial\lambda = 0$.

For example, upon differentiating the solutions (**93**) of no. **344** with respect to x_0 and y_0 one will obtain:

$$\frac{P}{r^2} + Q \log r + z_1$$

(P and Q being regular functions, with $P(x_0, y_0) = 0$), which was likewise considered by Picard ([63]).

347. – The preceding results can be generalized to the case of an arbitrary order.

By contrast, they no longer persist if the characteristic considered is double (no. **284**), i.e., if it satisfies the conditions:

$$\frac{\partial A}{\partial \pi_i} = 0, \qquad\qquad (i = 1, 2, \ldots, n).$$

([63]) *C. R. Ac. des Sc.*, 1891.

Indeed, suppose that this is true, and also (as one may obviously do without diminishing the generality) that all of the multiplicities Π = const. are characteristic, in such a way that the quantity that was denoted above by $\mathcal{A}$ is identically null. Then, once the term in F' is annulled the term in F may disappear only for $Z = 0$, at least if one has for $\Pi = 0$ that:

$$M = 0.$$

For example, if one takes Π to be the variable x_0 then one must have:

$$a_n = 0$$

in equation (**63**).

By contrast, if this condition is verified then it is, in general, possible to construct integrals that present the indicated singularity. Thus, for the equation (in two independent variables):

$$\Delta\Delta z + a\frac{\partial}{\partial x}(\Delta z) + b\frac{\partial}{\partial y}(\Delta z) + c\frac{\partial^2 z}{\partial x^2} + 2d\,\frac{\partial^2 z}{\partial x\,\partial y} + e\frac{\partial^2 z}{\partial y^2} + 2f\,\frac{\partial z}{\partial x} + 2g\,\frac{\partial z}{\partial y} + hz = 0,$$

which satisfies the preceding condition, one easily proves by this procedure that there exist solutions of the form:

$$z = r^2 \log r \cdot Z + z_1,$$

which play the same role for this equation as the solutions (**93**) do for equation (**92**).

348. – One easily extends the preceding considerations to systems of equations. For example, suppose we have the linear system:

$$(\mathbf{94})\qquad \begin{cases} \sum a_{ik}\,p_{ik} + \sum b_{ik}q_{ik} + \sum c_{ik}r_{ik} + \sum a_i\,p_i + \sum b_i q_i + \sum c_i r_i + g\xi + h\eta + l\zeta = 0, \\ \sum a'_{ik}\,p_{ik} + \sum b'_{ik}q_{ik} + \sum c'_{ik}r_{ik} + \sum a'_i\,p_i + \sum b'_i q_i + \sum c'_i r_i + g'\xi + h'\eta + l'\zeta = 0, \\ \sum a_{ik}\,p_{ik} + \sum b''_{ik}q_{ik} + \sum c''_{ik}r_{ik} + \sum a''_i\,p_i + \sum b''_i q_i + \sum c''_i r_i + g''\xi + h''\eta + l''\zeta = 0, \end{cases}$$

where, as in no. **291**, the ξ, η, ζ are unknowns and the p, q, r are their first and second derivatives.

We seek a solution that is singular on the multiplicity $\Pi = 0$ in which the principal parts of the unknowns are:

$$\xi = \Xi\,F(\Pi), \qquad \eta = \mathrm{H}\,F(\Pi), \qquad \zeta = \mathrm{Z}\,F(\Pi),$$

respectively.

Since the function F is assumed to verify the same hypotheses as in no. **349**, the terms in F'' must disappear, and one will have, for $\Pi = 0$:

$$
(\mathbf{95}) \qquad \begin{cases} \Xi A + \mathrm{H} B + \mathrm{Z} C = 0, \\ \Xi A' + \mathrm{H} B' + \mathrm{Z} C' = 0, \\ \Xi A'' + \mathrm{H} B'' + \mathrm{Z} C'' = 0, \end{cases}
$$

where A is the expression $\sum a_{ik}\, p_i\, p_k$ and B, C, … are the analogous expressions that are formed as in no. **291**.

The determinant of these equations must consequently be null ([64]), in such a manner that *the singular multiplicity must again be a characteristic*.

If we place ourselves under the hypotheses of no. **292**, where the minors of the determinant in question are non-null, and we denote, as before, these minors by the notations α, β, γ, … then equations (**95**) give (for $\Pi = 0$):

$$
(\mathbf{96}) \qquad \Xi = \alpha\rho, \qquad \mathrm{H} = \beta\rho, \qquad \mathrm{Z} = \gamma\rho,
$$

ρ being indeterminate, by means of which the left-hand sides of these equations will be null with Π, and consequently of the form:

$$
K\Pi\rho, \quad K'\Pi\rho, \quad K''\Pi\rho,
$$

K, K', K'' being known regular functions.

Furthermore, let $F(\Pi) = \log \Pi$, in such a manner that one has $F'' / F' = -1/\Pi$. The disappearance of the singular terms in F' in the equation that one obtains by multiplying the first one by α, α', α'', respectively (in order to eliminate F'') furnishes an equation that is entirely analogous to the preceding one (**33**), up to the replacement of the quantities p_{nn} , q_{nn} , r_{nn} with Ξ, H, Z. Moreover, if we substitute for these latter quantities their values (**96**) then it is clear that we have an equation in ρ that is similar to the equation that was obtained in no. **293**, and to which one may apply all of the preceding conclusions that were established relative to the latter. Thus, thus equation will define the distribution of values of r on the singular multiplicity and will have the bicharacteristics of the system (**94**) for its characteristics.

One will thus determine the values of Ξ, H, Z on $\Pi = 0$.

349. – In order to see whether one may obtain a solution to the problem under these conditions, we suppose that the transformation that we described in no. **322**-**323** has been performed. In other words, our characteristics will be nothing but $x_n = 0$. Furthermore, one of our equations will no longer contain the second derivatives with respect to x_n; the other two will contain these derivatives, the one only in the term p_{nn}, and the other, only in the term q_{nn} ([65]).

([64]) There is no reason to be preoccupied with the case in which Ξ, H, Z go to zero with Π, for the same reason as in no. **342.** (See the note on page (?).)

([65]) More exactly, in order to obtain this result one must arrange for all of the multiplicities x_n = const. to be characteristic and perform a change of variables that is defined, not by formulas (**52**), but by formulas

If we group our various terms according to their order of differentiability with respect to x_n alone then we may write these equations in the form:

$$\frac{\partial^2\xi}{\partial x_n^2}+\varphi_1\left(\frac{\partial\xi}{\partial x_n}\right)+\psi_1\left(\frac{\partial\eta}{\partial x_n}\right)+\chi_1\left(\frac{\partial\zeta}{\partial x_n}\right)+\varphi_2(\xi)+\psi_2(\eta)+\chi_2(\zeta)=0,$$
$$\frac{\partial^2\eta}{\partial x_n^2}+\varphi_1'\left(\frac{\partial\xi}{\partial x_n}\right)+\psi_1'\left(\frac{\partial\eta}{\partial x_n}\right)+\chi_1'\left(\frac{\partial\zeta}{\partial x_n}\right)+\varphi_2'(\xi)+\psi_2'(\eta)+\chi_2'(\zeta)=0,$$
$$\varphi_1''\left(\frac{\partial\xi}{\partial x_n}\right)+\psi_1''\left(\frac{\partial\eta}{\partial x_n}\right)+\chi_1''\left(\frac{\partial\zeta}{\partial x_n}\right)+\varphi_2''(\xi)+\psi_2''(\eta)+\chi_2''(\zeta)=0,$$

where φ_1, ψ_1, $\chi_1, \varphi_1', \psi_1', \psi_1', \varphi_1'', \psi_1'', \chi_1''$ denote first order linear differential polynomials, each one of which defines a function in which appear only differentiations with respect to the variables other than x_n . φ_2, ψ_2, …, ψ_2'', χ_2'' are second order differential polynomials that are likewise devoid of differentiations with respect to x_n .

We must substitute the values:

(97) $$\xi = \Xi \log x_n + \xi_1, \qquad \eta = \mathrm{H} \log x_n + \eta_1, \qquad \zeta = \mathrm{Z} \log x_n + \zeta_1$$

for the unknowns in the equations that we just wrote. The factor log x_n must be treated as a constant for any differentiation with respect to a variable other than x_n , and one will obviously obtain the result:

(98) $$\begin{cases} -\dfrac{\Xi}{x_n^2}+\dfrac{1}{x_n}\left[2\dfrac{\partial\Xi}{\partial x_n}+\varphi_1(\Xi)+\psi_1(\mathrm{H})+\chi_1(\mathrm{Z})\right]+\cdots=0,\\ -\dfrac{\mathrm{H}}{x_n^2}+\dfrac{1}{x_n}\left[2\dfrac{\partial\mathrm{H}}{\partial x_n}+\varphi_1'(\Xi)+\psi_1'(\mathrm{H})+\chi_1'(\mathrm{Z})\right]+\cdots=0,\\ \qquad\qquad \dfrac{1}{x_n}[\varphi_1''(\Xi)+\psi_1''(\mathrm{H})+\chi_1''(\mathrm{Z})]+\cdots=0, \end{cases}$$

in which we have written only the terms in $1/x_n^2$ and $1/x_n$. The vanishing of the former in the first two equations shows that Ξ and H must be annulled with x_n ; this corresponds to formulas (**96**). Under these conditions, the vanishing of terms in $1/x_n$ in the third equation demands that one have, for Π = 0, that:

$$\chi_1''(\mathrm{Z}) = 0.$$

(**53**), the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ being equal *for any system of values for* x_1, x_2, …, x_n to the coefficients a_{nn}, b_{nn}, c_{nn}, a'_{nn}, b'_{nn}, c'_{nn} .

This is the partial differential equation that we recently found, which determines the distribution of values of Z on the multiplicity $x_n = 0$. If it is verified then our third equation (which reduces to the linear combination that we constructed in the preceding no. here) contains no other singular terms besides logarithmic terms.

Assume that this is the case for the remaining two equations. Since Ξ and H are initially null, one has:

$$\Xi = x_n\, \Xi_1, \qquad \mathrm{H} = x_n\, \mathrm{H}_1, \tag{99}$$

where Ξ_1 and H_1 are new regular functions. It then follows, upon equating to zero the terms in $1/x_n$ (which, by means of relations (**99**), are furnished either by the terms in $1/x_n$ or by the ones in $1/x_n^2$) in the first two of equations (**98**):

$$\Xi_1 + \chi_1(\mathrm{Z}) = 0,$$
$$\mathrm{H}_1 + \chi_1'(\mathrm{Z}) = 0.$$

Since Z is known for $x_n = 0$, *the two preceding relations make known the initial values of* Ξ_1 *and* H_1, i.e., of $\partial\Xi / \partial x_n$ and $\partial \mathrm{H} / \partial x_n$.

If we recall that Ξ, H, Z themselves must define a solution to the given system (in order to make the logarithmic terms disappear) then we see that we are led to determine one such solution once we are given the values of our three unknowns and the first derivatives of both of them on the characteristic $x_n = 0$. Now, we saw in no. **323** that to these givens we may add the values of Z on a multiplicity that is secant to the latter.

Therefore, there exist an infinitude of solutions of the form (**97**) that depend upon an arbitrary function of $n - 1$ variables and a second arbitrary function of $n - 2$ variables, since Z may be chosen arbitrarily at any point of an $n - 2$- fold extended multiplicity that is situated on our characteristic and cuts each bicharacteristic at one point.

350. – We are led to the waves that appear in the theory of periodic motions if we give the function F the form:

$$F(\Pi) = \sin \mu\Pi,$$

μ being an arbitrary parameter.

The function F thus chosen is holomorphic, in such a way that μ is always between +1 and –1. Properly speaking, it does not fall into the category that we just envisioned. Meanwhile, – and this is one notion that acquires considerable significance in many physical applications of analysis – since it is always regular in theory, it must be regarded as being *practically singular* when μ has large values. Indeed, it enjoys a certain number of properties that agree with those of functions provided with singularities. It is always continuous and never offers brief variations, in the absolute sense of the word. However, it meanwhile passes from the value +1 to the value –1 when its independent variable Π increases from the small quantity π / μ. It has a derivative that is never infinite, but the values of that derivative are very large with respect to those of the function, namely, of order μ; the second derivatives are likewise very large compared to μ^2, etc.

From this (restricting ourselves to the case of only one unknown), if we suppose that equation (**63**) has a solution of the form ([66]):

(**100**) $$z = \mathrm{Z} \sin \mu\Pi + z_1,$$

and if μ is very large then if the derivatives of Π, Z, and z (as well as those quantities themselves) are not very large then one must have that in the left-hand side of equation (**86**) the terms in F'', which are of order μ^2, and the terms in F', which are of order μ, are annulled separately. The first of these conditions gives:

(**12′**) $$A = 0.$$

and this time this will be true for any value of H, in such a manner that *the multiplicities* Π = const. *must be characteristics.*

The terms in F' give the condition (**89**) with $\mathcal{A} = 0$ as a consequence, upon introducing the bicharacteristic variable z that is defined by equation (**14′**) (no. **283**):

(**101**) $$\frac{d\mathrm{Z}}{ds} + M\,\mathrm{Z} = 0.$$

Conversely, suppose that the functions Π and Z are well-defined, the former by equation (**12**) and the latter by equation (**101**). The result of substituting the product Z sin $\mu\Pi$ in the given equation will then reduce to $Q \sin \mu\Pi$, upon setting:

$$Q = \mathcal{F}(\mathrm{Z}).$$

We thus have to determine z_1 by the equation:

(**102**) $$\mathcal{F}(z_1) = - Q \sin \mu\Pi.$$

Now, for the various second order partial differential equations (with real characteristics) that one must integrate (compare no. **339**), one confirms, for the equation whose left-hand side is:

$$\mathcal{F}(z) = \mathcal{F}$$

(where $\mathcal{F}$ is a given function of $x_1, x_2, \ldots, x_n$), the existence of solutions that are represented by sums of n-fold integrals of the form:

(**103**) $$\iint \cdots \int \mathcal{F}(x_1', \cdots, x_n')\, G(x_1, x_2, \cdots, x_n, x_1', \cdots, x_n')\, d\tau\,,$$

([66]) Nothing essential will be modified if we replace the term Z sin $\mu\Pi$ with the sum $\mathrm{Z}_1 \sin \mu\Pi + \mathrm{Z}_2 \cos \mu\Pi$

where G is calculated *a priori*, independently of the function $\mathcal{F}$, and where $d\tau$ is an element of the n-fold or n–1-fold extended multiplicity that is described by the point $(x'_1, x'_2, \cdots, x'_n)$.

In order to obtain a solution of equation (**102**) we must replace $\mathcal{F}$ by – Q sin $\mu\Pi$.

Supposing that the functions Q and G satisfy conditions that are analogous to those of Dirichlet, I would like to say that their total variation ([67]) over an arbitrary finite line does not become greater than these functions themselves. The theory of trigonometric series ([68]) then tells us that an integral of the form (**103**), when taken over the function QG sin $\mu\Pi$, *is very small with respect to the values of* Q *and the order of* Q/μ.

Finally, suppose that $Q = \mathcal{F}(\mathrm{Z})$ has the same order of magnitude as Z itself. One then sees that of the two terms of the expression (**100**), the latter is very small with respect to the former. Under these conditions, the solution therefore reduces approximately to the product Z sin $\mu\Pi$.

That quantity proves, from its form itself, that periodic oscillations become more rapid as μ becomes greater, and the points with corresponding phases are situated on the surfaces Π = const., i.e., on the characteristics.

351. – Now, return to the determination of Z. Once Π has been chosen, Z is subject to the differential equation (**101**).

Now, this only tells us the mutual ratios of the values of Z at the various points of the same bicharacteristic. It does not establish any relation between the values that are taken by this function on various bicharacteristics.

For example, suppose we are given an arbitrary region of n-dimensional space and consider the set of bicharacteristics that traverse that region. This is what one might call a *pencil* of bicharacteristics. *Nothing prevents us from supposing that* Z *is non-zero on the bicharacteristics of the pencil and null everywhere else.*

From the foregoing, the bicharacteristics are then precisely the only lines that possess this property.

Now, from this, we recognize precisely the essential character by which the rays intervene physically. It corresponds, moreover, to having at least one property of the solution (**100**) that μ is very large, and consequently, that z_1 is very small. Indeed, this is the case for oscillations that are extremely rapid, such as luminous vibrations, for which the propagation by rays is the neatest.

One must nevertheless observe that in the regions where Z varies rapidly the conclusions will be modified since z_1 will cease to be negligible (*diffraction*).

([67]) JORDAN, *Cours d'Analyse*, 2nd edition, tome 1, no. **67**, pages 54 et seq.

([68]) JORDAN, *Ibid.*, tome II, ch. IV; PICARD, *Traité d'Analyse*, 2nd edition, tome I, 2nd part, ch. IX.

NOTE I

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ON THE CAUCHY PROBLEM AND CHARACTERISTICS

While we established (in chap. VI) that if two integral surfaces of the same Monge-Ampère equation are tangent all along a line, which may only be a characteristic, our proof remains incomplete in one aspect: we have, in fact, omitted the case in which the contact is of infinite order. There is therefore reason to demand that, likewise on considering non-analytic solutions, the Cauchy problem is perfectly determined whenever the sequence of given values is not characteristic. If one is concerned with a linear equation with analytic and holomorphic coefficients then the solution has been obtained in a manner that is as general as possible by Holmgren ([69]), not only for a second order equation, but also for a linear system in an arbitrary number of equations.

Such a system may, as one knows, always be reduced to a form in which all of the equations are of first order. Moreover, if the multiplicities x = const. are not characteristic then these equations are soluble with respect to the derivatives relative to x, in such a way that they have the form:

(**1**) $$\mathcal{F}_i(x) = \frac{\partial x_i}{\partial x} - \sum_{k=1}^{n} A_{ik}(x, y)\frac{\partial z_k}{\partial y} - \sum_{k=1}^{n} B_{ik}(x, y) z_k = 0 \qquad (i = 1, 2, \ldots, n),$$

in which the quantities A_{ik}, B_{ik} are analytic and holomorphic functions of x and y. If the given line L (on which z_1, z_2, …, z_n are annulled) is not tangent to a characteristic at the point O, in whose vicinity we propose to study the system of function z, then one may assume that the y-axis is tangent at this point, since the equations preserve their previous form.

One may, by an obvious transformation, always exclude the case in which there is an inflection point at O, and assume consequently that the convexity of our line changes at a side that is determined by the y-axis; the choice of side is up to us. Suppose, to fix ideas, that we have chosen the one with positive x, or the right side.

The adjoint system to (**1**) is:

$$\mathcal{G}_i(x) = \frac{\partial u_i}{\partial x} - \sum_{k} \frac{\partial}{\partial y}(A_{ik}, u_k) - \sum_{k} B_{ik} u_k = 0$$

or:

(**2**) $$\mathcal{G}_i(x) = \frac{\partial u_i}{\partial x} - \sum_{k} A_{ik}\frac{\partial u_k}{\partial y} - \sum_{k} \beta_{ik} u_k = 0$$

([69]) *Ofversigt af Kongl. Vetenshaps. Akad. Förhandl,* 9 January 1904, pp. 91-103.

(**3**) $$\beta_{ki} = -B_{ki} - \frac{\partial A_{ki}}{\partial y}.$$

With the given system, this gives rise to the identity:

(**4**) $$\iint\left\{\sum_i [u_i\mathcal{F}_i(z) + z_i\mathcal{G}_i(u)]\right\} dxdy = \int\left(\sum_i z_i u_i\, dy - \sum_{ik} A_{ik}u_i z_k\, dx\right),$$

in which the double integral extends over an arbitrary area of the xy-plane, and the simple integral is along the contour of this area.

The functions β will be, like the A and B, analytic and holomorphic, and consequently developable into a Taylor series that is ordered in powers of $x - x_0$, $y - y_0$, where x_0, y_0 denote the coordinates of an arbitrary point near O. Moreover, the associated radii of convergence ([70]) of these developments do not go below a certain fixed limit when the point (x_0, y_0) varies in a neighborhood of O. Since the corresponding functions remain finite, any of the developments in question will admit a majorizing series of the form:

(**5**) $$\frac{M}{\left(1-\frac{x-x_0}{r}\right)\left(1-\frac{y-y_0}{r'}\right)},$$

in which M, r, r' are independent of x_0, y_0.

Under these conditions, if we give the values of u on the line $x = x_0$, namely, $u_i = f_i(y)$, these values being analytic and holomorphic and their developments in powers of $y - y_0$ admitting the common majorizing series:

(**6**) $$\frac{P}{1-\frac{y-y_0}{R}}$$

(P, R constant), then a classical argument relating to partial differential equations gives us ([71]) the existence of a holomorphic solution to the system (**8**) that takes the given values on the line $x = x_0$. Moreover, the developments of the functions u thus obtained converge for:

$$(x - x_0) < \rho, \qquad (y - y_0) < \rho';$$

ρ, ρ' depend on M, r, r', R, but not on P. Indeed, one may give the latter quantity an arbitrary value upon multiplying the values $f_i(y)$ by the same factor, which one finds in the values of the unknowns, and which does not modify the radii of convergence of their

([70]) See BOREL. – *Leçons sur les séries a termes positives,* pp. 86.

([71]) JORDAN. – *Cours d'Analyse,* t. III, chap. III. – GOURSAT, *Leçons sur l'intégration des operations dérivées partielles du premier ordre,*pp. 2-8.

developments. For example, ρ, ρ' might be greater than the values that they would take if one replaced all of the A_{ik}, β_{ik} by the function (**5**) and all of the f_i by the function:

$$\frac{1}{1-\frac{y-y_0}{R}}$$

(a case in which these values may be easily written, since the u are then obtained by direct integration).

Choose R arbitrarily, which will permit us to calculate ρ, ρ', since M, r, r' are known. Finally, give x_0 a value that is lower in absolute value than ρ, and such that the line $x = x_0$ intercepts our line at the endpoints of an arc PP' (*fig.* 24) that is completely situated in the domain in which the preceding considerations are valid.

Now suppose that the system (**1**) admits a solution such that all of the z are null on the arc, that solution being defined in the neighborhood of that arc, and, in particular, in the entire region between that arc and its chord. We apply formula (**4**) to the contour of the area thus defined by taking e to be the functions whose existence we just assumed.

As for the u, they will be defined in the following manner: Let $F_i(y)$ be the sequence of values that are taken by z_i along the line segment. We may find (for each value of i) a polynomial $f_i(y)$ that has, with F_i, a difference φ_i that is everywhere lower in absolute value than a number ε that is small as one desires.

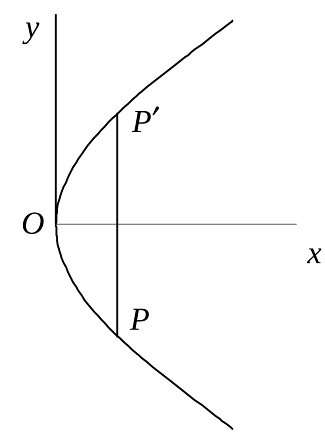

Fig. 24

We take the u_i to be a solution of the adjoint system such that the u_i reduce to $f_i(y)$ on the line. Since obviously the polynomials may always be regarded as admitting majorants of the form (**6**), the u_i will exist and will be analytic and holomorphic in all of our integration area.

In the right-hand side of (**4**), the integral over the arc PP' will be null, since all of the z are annulled along that arc. On the chord of PP', since dx is null the integration element reduces to:

$$\sum u_i\, z_i\, dy = dy \sum F_i\, f_i = dy\left[\sum F_i^2 + \sum F_i \varphi_i\right].$$

Let I be the integral $\int\left[\sum F_i(y)^2\right]dy$, H, the maximum of the modulus of F_i, and l, the length of PP'. It is clear that the integral considered will differ from I by a quantity that is less than $n\varepsilon Hl$. It may therefore be null only if one has:

$$\varepsilon < \frac{I}{nHl}.$$

Since ε is arbitrarily small, one may always assume that this inequality is satisfied, and consequently the formula will lead to a contradiction unless one does not have $I = 0$; i.e., unless not all of the F are identically null. It must therefore be the case that one has:

$$z_1 = z_2 = \ldots = z_n = 0 ,$$

at least everywhere to the right of L, since the abscissa x_0 is arbitrary, except for the condition that $x_0 < \rho$.

In order to establish the same conclusion for the points to the left of L, it will suffice to modify the sense of the convexity of that line by a change of variables.

The foregoing argument may be generalized to the case of an arbitrary number of variables. If one has, for example, three of them, then the Cauchy givens will relate to a surface, on which it will suffice to give (by a change of variables) curvatures that are of the same sign and different from zero.

On the other hand, one may reduce the case of an arbitrary equation to that of a linear equation by means of the following lemma:

Let $F(x_1, x_2, \ldots, x_m)$ *be a function that admits continuous partial derivatives up to a certain order* p *in a certain domain. If* $y_1, y_2, \ldots, y_m$ *denote a new sequence of variables that are the same in number as the former one then the difference:*

$$F(y_1, y_2, \ldots, y_m) - F(x_1, x_2, \ldots, x_m)$$

may be put into the form:

$$(y_1 - x_1)\,\varphi_1 + (y_2 - x_2)\,\varphi_2 + \ldots + (y_m - x_m)\,\varphi_m ,$$

in which $\varphi_1, \varphi_2, \ldots, \varphi_m$ *denote functions of* $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$ *that are continuous, along with all of their derivatives up to order* $p - 1$.

In order to prove this proposition, one begins by assuming that $m = 1$. One verifies without difficulty that for an F that is continuous, along with all of its first p derivatives, the function:

$$\frac{F(y_1) - F(x_1)}{y_1 - x_1}$$

is continuous, as well as its partial derivatives with respect to x_1 and y_1, up to order $p - 1$.

In order to pass to the general case, it will suffice to apply the conclusion thus obtained to each of the terms in the sum:

$$\begin{aligned}&[F(y_1, x_2, x_3, \ldots, x_m) - F(x_1, x_2, x_3, \ldots, x_m)]\\ &+ [F(y_1, y_2, x_3, \ldots, x_m) - F(y_1, x_2, x_3, \ldots, x_m)] + \ldots\\ &+ [F(y_1, y_2, \ldots, y_m) - F(y_1, y_2, \ldots, y_{m-1}, x_m)] .\end{aligned}$$

If F is analytic, as well, then the same thing will be true for $\varphi_1, \varphi_2, \ldots, \varphi_m$.

For $x_1 = y_1$, $x_2 = y_2$, …, $x_m = y_m$, these functions obviously have the value:

$$\varphi_i = \frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \ldots, m) . \tag{7}$$

Having said this, let:

$$\mathcal{F}(z) = F(x, y, z, p, q, r, s, t) = 0 \tag{8}$$

be a second order partial differential equation that defines z as a function of x and y, and let z and $z' = z + dz$ be two integrals of that equation that coincide along with their derivatives all along a certain line L. One will have:

$$\mathcal{F}(z + u) = \mathcal{F}(z) = 0 .$$

From the preceding lemma, the relation:

$$F\left(x, y, z+u, p+\frac{\partial u}{\partial x}, q+\frac{\partial u}{\partial y}, r+\frac{\partial^2 u}{\partial x^2}, s+\frac{\partial^2 u}{\partial x\,\partial y}, t+\frac{\partial^2 u}{\partial y^2}\right) - F(x, y, z, p, q, r, s, t) = 0$$

may be put into the form:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} + 2d\frac{\partial u}{\partial x} + 2e\frac{\partial u}{\partial y} + fu = 0 , \tag{9}$$

in which $a, b, \ldots, f$ are continuous differentiable functions of x, y, of z, u, and their derivatives; i.e. (if z and u are themselves assumed to be differentiable up to a certain order), continuous differentiable functions of x and y.

All of this therefore amounts to knowing whether that linear equation in u may admit a null integral, along with its first and second derivatives all along a particular line L without being identically null – or, at least, in all of a region surrounding L.

We remark that at any point where u is null, along with its first and second derivatives, one has, from relations (**7**):

$$a = \frac{\partial F}{\partial r}, \qquad 2b = \frac{\partial F}{\partial s}, \qquad c = \frac{\partial F}{\partial t}, \tag{10}$$

in such a way that at such a point the characteristics of equation (**9**) are tangent to those of the proposed ones.

The question will be resolved, moreover, by Holmgren's method of proof if equation (**9**) has analytic coefficients. However, we must not assume that same thing will be true if F itself is analytic. Indeed, as we saw at the beginning, we must assume that the integrals z and z' do not possess this property, which will not, moreover, contain the coefficients $a, b, c, \ldots$

It will therefore be necessary to extend the Holmgren argument to non-analytic linear equations. That extension has been made in only one case up till now: the one in which the characteristics of the equation (**9**) – and consequently those of the given equation – are real and distinct. Indeed, since the argument of no. **174** applies in this case, the Riemann function may always be constructed by the method of successive approximations ([72]). Our conclusion is therefore proved.

([72]) PICARD, in DARBOUX, *Leçons sur la théorie des surfaces,* tome IV.

NOTE II

ON SLIPS IN FLUIDS

In chap. V, we saw that, other than waves (which exist only in compressible fluids), arbitrary fluids – compressible or not – may present stationary discontinuities. One knows, moreover, that they may be *absolute*, i.e., that two portions of the fluid may slip over each other in the manner of two different bodies.

Ever since Helmholtz, who was the first to draw attention to that category of motion ([73]), they have played an important role in several hydrodynamical theories. Their existence is invoked in order to explain various paradoxical circumstances, such as the flow of liquids in the presence of angular walls, or the result that is known by the name of the *d'Alembert paradox* (the absence of resistance presented to a liquid by a solid that is symmetric with respect to a plane perpendicular to the direction of motion).

Nevertheless, all such explanations suffer a common objection, to which we have already alluded in the text (ch. V). Indeed, the slips that we just spoke of are possible, in the sense that nothing (in the absence of viscosity) opposes their *persistence* once they are produced between two arbitrary regions of the fluid. However, we have seen that their *creation* is impossible, at least, under the conditions that rational hydrodynamics demands.

If the slip velocity on a slip surface is null at a particular point at the instant t_0 then it will remain null between the same molecules for any later instant.

It is nevertheless essential to take into account the restriction that we have made on our statement for a moment. Indeed, one recalls that in the study of natural fluids there are cases that elude the argument that we shall present, since everything rests upon classical equations of hydrodynamics such as we wrote down in this text (ch. III and V), and consequently nothing precludes the production of absolute discontinuities in the course of motion.

They are the ones in which although the pressure vanishes, cavities are momentarily created in the fluid mass considered. In general, these cavities appear near eddies in which the molecules that belong to the different regions mix together in such a way that it becomes impossible to assume the hypothesis of continuity of no. **45** at any point.

What we shall therefore prove is simply that such a singularity (or all other analogous ones, provided that the hypothesis that served as the basis for rational hydrodynamics ceases to be valid ([74])), is necessary in order for a slip to be produced in an arbitrary time interval of motion if it did not exist before that interval.

([73]) *Monatsber. der Berl. Ac. der Wissensch.,* 23 April 1868.

([74]) The general equations of hydrodynamics are likewise modified in the case of friction. However, they may no longer be invoked in order to explain the creation of slips because they have the opposite effect of destroying that which made them exist to begin with.

The proof rests upon the fact that was stated in no. **244** that (under the fundamental hypotheses in question) to each instant of a relative slip the acceleration jump is normal to the surface S along which the discontinuity is defined.

We propose to construct the differential equations that express this condition.

Recalling the same notations as in no. **249**, we let ξ, η denote the curvilinear coordinates on S, which is regarded as being in its initial state S_0. The Cartesian coordinates x, y, z of a molecule of S *that belongs to the region* 1 will be functions of ξ, η, and time t:

$$\left\{\begin{aligned} x &= x(\xi,\eta,t), \\ y &= y(\xi,\eta,t), \\ z &= z(\xi,\eta,t). \end{aligned}\right. \tag{1}$$

The same will be true for the coordinates x', y', z' of a molecule that belongs to region 2. However, the expressions will be different in the two cases. Since there is a slip along S the molecule in region 1 that has the curvilinear coordinates ξ, η in the initial state will have, at the instant t, the coordinates in region 2 that were ξ', η' in the initial state (which are generally different from the former coordinates). If ξ' and η' are given then ξ and η will be functions of t, and it will suffice to substitute them in equations (**1**) in order to describe the motion of the molecule (x', y', z').

These are the functions that we must study.

The acceleration of the molecule (x, y, z) will be obtained by twice differentiating formulas (**1**) with respect to t, without varying ξ and η; it will have the components:

$$\frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2}.$$

On the contrary, the acceleration of the molecule (x', y', z') is obtained by substituting ξ, η for their values as functions of t; it will have the components:

$$\frac{d^2 x'}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial \xi}\frac{d^2\xi}{dt^2} + \frac{\partial x}{\partial \eta}\frac{d^2\eta}{dt^2} + \frac{\partial^2 x}{\partial \xi^2}\left(\frac{d\xi}{dt}\right)^2 + 2\frac{\partial^2 x}{\partial\xi\,\partial\eta}\frac{d\xi}{dt}\frac{d\eta}{dt} + \frac{\partial^2 x}{\partial\eta^2}\left(\frac{d\eta}{dt}\right)^2 + 2\frac{\partial^2 x}{\partial\xi\,dt}\frac{d\xi}{dt} + 2\frac{\partial^2 x}{\partial\eta\,dt}\frac{d\eta}{dt}$$

$$\frac{d^2 y'}{dt^2} = \frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial \xi}\frac{d^2\xi}{dt^2} + \frac{\partial y}{\partial \eta}\frac{d^2\eta}{dt^2} + \frac{\partial^2 y}{\partial \xi^2}\left(\frac{d\xi}{dt}\right)^2 + 2\frac{\partial^2 y}{\partial\xi\,\partial\eta}\frac{d\xi}{dt}\frac{d\eta}{dt} + \frac{\partial^2 y}{\partial\eta^2}\left(\frac{d\eta}{dt}\right)^2 + 2\frac{\partial^2 y}{\partial\xi\,dt}\frac{d\xi}{dt} + 2\frac{\partial^2 y}{\partial\eta\,dt}\frac{d\eta}{dt}$$

$$\frac{d^2z'}{dt^2}=\frac{\partial^2 z}{\partial t^2}+\frac{\partial z}{\partial \xi}\frac{d^2\xi}{dt^2}+\frac{\partial z}{\partial \eta}\frac{d^2\eta}{dt^2}+\frac{\partial^2 z}{\partial \xi^2}\left(\frac{d\xi}{dt}\right)^2+2\frac{\partial^2 z}{\partial \xi\,\partial \eta}\frac{d\xi}{dt}\frac{d\eta}{dt}+\frac{\partial^2 z}{\partial \eta^2}\left(\frac{d\eta}{dt}\right)^2$$
$$+2\frac{\partial^2 z}{\partial \xi\,dt}\frac{d\xi}{dt}+2\frac{\partial^2 z}{\partial \eta\,dt}\frac{d\eta}{dt}.$$

In order to obtain the components of the acceleration jump on the right-hand side of these expressions, it will suffice to shift the terms $\frac{\partial^2 x}{\partial t^2},\frac{\partial^2 y}{\partial t^2},\frac{\partial^2 z}{\partial t^2}$ to the left-hand side. We say that the segment thus obtained is normal to S by saying that it is perpendicular to the two directions:

$$\left(\frac{\partial x}{\partial \xi},\frac{\partial y}{\partial \xi},\frac{\partial z}{\partial \xi}\right)\text{ and }\left(\frac{\partial x}{\partial \eta},\frac{\partial y}{\partial \eta},\frac{\partial z}{\partial \eta}\right).$$

This gives the two relations:

$$\text{(2)}\quad\begin{cases}\dfrac{d^2\xi}{dt^2}\sum\left(\dfrac{\partial x}{\partial \xi}\right)^2+\dfrac{d^2\eta}{dt^2}\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta}\right)+\left(\dfrac{d\xi}{dt}\right)^2\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 x}{\partial \xi^2}\right)\\ \qquad+2\dfrac{d\xi}{dt}\dfrac{d\eta}{dt}\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 x}{\partial \xi\,\partial \eta}\right)+\left(\dfrac{d\eta}{dt}\right)^2\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 x}{\partial \eta^2}\right)\\ \qquad+2\dfrac{d\xi}{dt}\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 x}{\partial \xi\,\partial t}\right)+2\dfrac{d\eta}{dt}\sum\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 x}{\partial \eta\,\partial t}\right)=0,\\ \dfrac{d^2\xi}{dt^2}\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial x}{\partial \xi}\right)+\dfrac{d^2\eta}{dt^2}\sum\left(\dfrac{\partial x}{\partial \eta}\right)^2+\left(\dfrac{d\xi}{dt}\right)^2\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial \xi^2}\right)\\ \qquad+2\dfrac{d\xi}{dt}\dfrac{d\eta}{dt}\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial \xi\,\partial \eta}\right)+\left(\dfrac{d\eta}{dt}\right)^2\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial \eta^2}\right)\\ \qquad+2\dfrac{d\xi}{dt}\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial \xi\,\partial t}\right)+2\dfrac{d\eta}{dt}\sum\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial \eta\,\partial t}\right)=0,\end{cases}$$

in which the $\sum$ signs signify that one must replace x with y, and then z in the partial derivatives and then add the three expressions thus obtained.

Here, we see the introduction of the coefficients:

$$\text{(3)}\qquad E=\sum\left(\frac{\partial x}{\partial \xi}\right)^2,\qquad F=\sum\left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta}\right),\qquad G=\sum\left(\frac{\partial x}{\partial \eta}\right)^2,$$

of the linear element:

$$E\,d\xi^2 + 2F\,d\xi\,d\eta + G\,d\eta^2$$

of the surface S at the instant considered. They are the ones that appear as the coefficients of the second derivatives of ξ and η in the preceding equations.

On the other hand, their partial derivatives permit us to express the coefficients of $\left(\frac{d\xi}{dt}\right)^2, 2\frac{d\xi}{dt}\frac{d\eta}{dt}, \left(\frac{d\eta}{dt}\right)^2$, namely:

$$\sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\xi^2} = \frac{1}{2}\frac{\partial E}{\partial\xi}, \qquad \sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\xi\,\partial\eta} = \frac{1}{2}\frac{\partial E}{\partial\eta}, \qquad \sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\eta^2} = \frac{\partial F}{\partial\eta} - \frac{1}{2}\frac{\partial G}{\partial\xi},$$
$$\sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\eta^2} = \frac{\partial F}{\partial\eta} - \frac{1}{2}\frac{\partial G}{\partial\xi}, \quad \sum\frac{\partial x}{\partial\eta}\frac{\partial^2 x}{\partial\xi\,\partial\eta} = \frac{1}{2}\frac{\partial G}{\partial\xi}, \qquad \sum\frac{\partial x}{\partial\eta}\frac{\partial^2 x}{\partial\eta^2} = \frac{1}{2}\frac{\partial G}{\partial\eta}.$$

They likewise permit us to express two of the coefficients of $\frac{d\xi}{dt}, \frac{d\eta}{dt}$: those of $\frac{d\xi}{dt}$ in the first equation:

$$2\sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\xi\,\partial t} = \frac{\partial E}{\partial t},$$

and those of $\frac{d\eta}{dt}$ in the second one:

$$2\sum\frac{\partial x}{\partial\eta}\frac{\partial^2 x}{\partial\eta\,\partial t} = \frac{\partial G}{\partial t}.$$

However, the same is not true for the remaining two coefficients:

$$(\mathbf{4}) \qquad \sum\frac{\partial x}{\partial\xi}\frac{\partial^2 x}{\partial\eta\,\partial t}, \qquad \sum\frac{\partial x}{\partial\eta}\frac{\partial^2 x}{\partial\xi\,\partial t}.$$

Their sum may be calculated only with the aid of the coefficients (**3**); it is equal to $\partial F/\partial t$.

It is, moreover, evident, *a priori*, that one must introduce a distinct element of the form of the surface S in equations (**2**). Indeed, the motion of a molecule (x', y', z') may be regarded as the resultant of the motion of S, taken in region 1 (i.e., that of the molecule (x, y, z)), and the motion of (x', y', z') with respect to (x, y, z). The former of these motions may be regarded as the driving motion and the latter as the relative motion. Now, one knows that in the theory of relative motions, accelerations do not compose linearly as the velocities do. If, for example, the driving motion is that of a rigid system then one will have to take into account the complementary Coriolis acceleration, which depends upon the instantaneous rotation of the system. One must therefore attend to the intervention of a rotation of the type that is presently in question, and likewise the Coriolis theorem that we just alluded to in order to indicate that part of the rotation that will truly play a role. Indeed, if the rotation in question is tangent to S, since the same is true of the relative velocity, then the Coriolis acceleration (if one assumes that it is applicable) will give a normal complementary acceleration. As we are only interested in the vanishing of the *tangential* components of the acceleration jump, we will need to use only the normal component of the rotation.

It is easy to see that things happen essentially this way: It suffices to decompose the motion of S into a pure deformation and a rotation, as we did in nos. **51** and **62**. It is true that instead of a spatial deformation, here we only have a deformation of the surface. However, in order to reduce the latter case to the former one it suffices to imagine that the surface S drags along its normals such that they displace like rigid lines. One may then say, upon letting the symbol d denote the differentials that correspond to the displacements in space at the instant considered, letting u, v, w denote the components of the velocity of the point (x, y, z), and letting φ denote a quadratic form in dx, dy, dz that the equations of no. **62** may be written in the form:

$$du = \frac{1}{2}\frac{\partial\varphi}{\partial(dx)} + q\,dx - r\,dy\,,$$

$$dv = \frac{1}{2}\frac{\partial\varphi}{\partial(dy)} + r\,dx - p\,dy\,,$$

$$dw = \frac{1}{2}\frac{\partial\varphi}{\partial(dz)} + p\,dx - q\,dy\,,$$

$$\left(u = \frac{\partial x}{\partial t},\ v = \frac{\partial y}{\partial t},\ w = \frac{\partial z}{\partial t}\right).$$

Consequently, upon taking dx, dy, dz to be proportional to $\frac{\partial x}{\partial\xi}, \frac{\partial y}{\partial\xi}, \frac{\partial z}{\partial\xi}$, and then to $\frac{\partial x}{\partial\eta}, \frac{\partial y}{\partial\eta}, \frac{\partial z}{\partial\eta}$, in turn, one will have:

$$\frac{\partial u}{\partial\xi} = \frac{\partial^2 x}{\partial\xi\,\partial t} = \frac{1}{2}\frac{\partial\varphi}{\partial\left(\dfrac{\partial x}{\partial\xi}\right)} + q\frac{\partial z}{\partial\xi} - r\frac{\partial y}{\partial\xi}\,,$$

$$\frac{\partial v}{\partial\xi} = \frac{\partial^2 y}{\partial\xi\,\partial t} = \frac{1}{2}\frac{\partial\varphi}{\partial\left(\dfrac{\partial y}{\partial\xi}\right)} + r\frac{\partial x}{\partial\xi} - p\frac{\partial z}{\partial\xi}\,,$$

$$\frac{\partial w}{\partial\xi} = \frac{\partial^2 z}{\partial\xi\,\partial t} = \frac{1}{2}\frac{\partial\varphi}{\partial\left(\dfrac{\partial z}{\partial\xi}\right)} + p\frac{\partial y}{\partial\xi} - q\frac{\partial x}{\partial\xi}\,,$$

along with analogous equations in which ξ is replaced with η.

Now, multiply the first three of the equations by $\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}, \frac{\partial z}{\partial \eta}$, respectively, then the last three by $\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}, \frac{\partial z}{\partial \xi}$, and subtract the sum of the latter three products from the sum of the former three. The terms that depend upon the derivatives of φ are eliminated, and what remains is:

$$\sum \frac{\partial x}{\partial \eta}\frac{\partial^2 x}{\partial \xi\, \partial t} - \sum \frac{\partial x}{\partial \xi}\frac{\partial^2 x}{\partial \eta\, \partial t} = 2\begin{vmatrix} p & q & r \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{vmatrix} = 2\, R\sqrt{EG - F^2}\,,$$

in which R denotes the normal component of the rotation (p, q, r).

Thus equations (**2**) may finally be written:

$$(\mathbf{5})\qquad \begin{cases} E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{1}{2}\left[\frac{\partial E}{\partial \xi}\left(\frac{d\xi}{dt}\right)^2 + 2\frac{\partial E}{\partial \eta}\frac{d\xi}{dt}\frac{d\eta}{dt} + \left(2\frac{\partial F}{\partial \eta} - \frac{\partial G}{\partial \xi}\right)\left(\frac{d\eta}{dt}\right)^2\right] \\ \qquad\qquad + \frac{\partial E}{\partial t}\frac{d\xi}{dt} + \left(\frac{\partial F}{\partial t} - 2R\right)\frac{d\eta}{dt} = 0, \\ F\frac{d^2\xi}{dt^2} + G\frac{d^2\eta}{dt^2} + \frac{1}{2}\left[\left(2\frac{\partial F}{\partial \xi} - \frac{\partial E}{\partial \eta}\right)\left(\frac{d\xi}{dt}\right)^2 + 2\frac{\partial G}{\partial \xi}\frac{d\xi}{dt}\frac{d\eta}{dt} + \frac{\partial G}{\partial \eta}\left(\frac{d\eta}{dt}\right)^2\right] \\ \qquad\qquad + \left(\frac{\partial F}{\partial t} + 2R\right)\frac{d\xi}{dt} + \frac{\partial G}{\partial t}\frac{d\eta}{dt} = 0. \end{cases}$$

When the surface S is fixed, along with the molecules of the region 1 that are situated on that surface, the two equations that we just obtained reduce to the ones that define the motion of a point of S in the absence of accelerating forces; this is obvious *a priori* since the latter are obtained by expressing the fact that the acceleration is normal.

In any case, if the motion of the medium 1 is given then that of the molecule x', y', z' is determined by equations (**5**), which are of the same form as the equations of dynamics with two degrees of freedom, in the sense that the second derivatives of ξ and η are expressed by polynomials of second degree in the first derivatives ([75]).

On the other hand, since equations (**5**) are always soluble with respect to these second derivatives (since $EG - F^2$ is always positive) and admit the solution ξ = const., η =

([75]) If one substitutes a portion of the fluid with a solid wall that is animated with the same motion then the motion of the molecules of the fluid part will not change. Of course, it results from this that equations (**5**) are applicable to the surface motion of a fluid that is bounded by an arbitrary wall. This is true only in the case in which the motion of that wall is the one that it takes on when one assumes that the fluid has the same nature as the medium that it touches and is subject to the pressures of that medium.

const., it results from general theorems that relate to differential equations that ξ and η *are forced to be constant if their derivatives are null at some particular arbitrary instant* t_0*, i.e., if there is no point at which the velocity jumps at that instant.*

Moreover, this will be the case either at all points of the surface S – in which case, there will not be an absolute discontinuity, – or only at certain points of that surface, in which case, the molecules of region 1 that are situated at these points will coincide with the corresponding molecules of region 2 for all of the subsequent motion.

NOTE III

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ON THE VORTICES PRODUCED BY SHOCK WAVES

In nos. **254-255** we established that the presence of second order discontinuities does not invalidate the classical theorems of hydrodynamics that relate to the conservation of the velocity potential or vortices. We propose to investigate the effect that is produced in that regard when the discontinuity that is propagated is of first order. To that effect, we employ the integral:

$$\int u\,dx + v\,dy + w\,dz\,,$$

or *circulation*, which is taken around a closed contour C.

This contour being entirely arbitrary, we may suppose, to simplify, that during the instants when it traverses the wave surface it only encounters that surface at two points.

Therefore, let A, B be those two points at a particular instant t. Take the initial state to be the state of region 1 at that instant. Furthermore, let A', B' be the initial positions of the points of contact at the instant $t + dt$. These points will be determined by the new wave surface S_0', which is situated at a distance $\theta\,dt$ from the first one S_0.

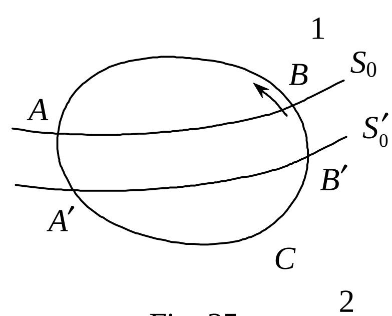

Fig. 25

In order to evaluate how much the circulation varies during the interval time dt, we consider separately:

1. The two arcs BA, $A'B'$ (*fig.* 25), the former of which belongs to region 1 during the interval of time considered and the latter of which belongs to region 2.

2. The two little arcs AA', $B'B$ that pass from one state to the other during the time dt.

We start with the simplest case: the one in which one does not take into account the Hugoniot objection, and which, consequently, the pressure and the density are related by Poisson's law, or, more generally, by a relation that has the form (**13**) of no. **131**.

The variation relative to the arc BA is given by the classical considerations that serve to establish the vorticity theorem ([76]). It is equal to the product of dt by the difference of the values that the quantity:

(**1**) $$Q = \frac{u^2 + v^2 + w^2}{2} + V - \int \frac{dp}{\rho}$$

([76]) KIRCHHOFF, *Mécanique*, 15th lesson.

takes at the points A and B, in which V is the ponderable force potential and the term that we denote by $-P$ is, as one knows, a function of ρ under our present hypotheses.

Likewise, the variation of the integral, when taken along the arc $A'B'$, is the product of dt with the difference of the values that the quantity Q takes at the points B' and A'.

The sum of these two terms gives (upon supposing that the contour is traversed in the sense of $A'B'BA$):

(2) $$dt\,(Q_A - Q_B + Q_{B'} - Q_{A'}) = dt\,[Q_A - Q_{A'} + (Q_B - Q_{B'})]\,,$$

in which one may abstract the term V in each of the differences $Q_A - Q_{A'}$, $Q_B - Q_{B'}$, since it is continuous during the passage of the wave.

We now occupy ourselves with the part that relates to the arc AA'. Let x_1, y_1, z_1, u_1, v_1, w_1 be the coordinates and the components of the velocity at a point of that arc in the state 1; let x_2, y_2, z_2, u_2, v_2, w_2 be the same quantities when considered in the state 2. One will have:

(3) $$\begin{cases} u_2 = u_1 - \lambda\theta, \\ v_2 = v_1 - \mu\theta, \\ w_2 = w_1 - \nu\theta, \end{cases}$$

and, from formulas (**9**) of no. **55**:

(4) $$\begin{cases} dx_2 = dx_1 + \lambda(\alpha dx_1 + \beta dy_1 + \gamma dz_1), \\ dy_2 = dy_1 + \mu(\alpha dx_1 + \beta dy_1 + \gamma dz_1), \\ dz_2 = dz_1 + \nu(\alpha dx_1 + \beta dy_1 + \gamma dz_1), \end{cases}$$

in which $\lambda, \mu, \nu, \theta$ always denote the components of the discontinuity and the velocity of propagation when referred to the initial state (i.e., the state of region 1) and α, β, γ denote the direction cosines of the normal to the wave.

It then follows, upon multiplying the quantities (**3**) by the quantities (**4**), respectively, that:

$$\begin{aligned} u_2\,dx_2 + v_2\,dy_2 + w_2\,dz_2 = u_1\,dx_1 + v_1\,dy_1 + w_1\,dz_1 &- \theta(\lambda\,dx_1 + \mu\,dy_1 + \nu\,dz_1) \\ &+ (\lambda\,u_1 + \mu\,v_1 + \nu\,w_1)\,(\alpha\,dx_1 + \beta\,dy_1 + \gamma dz_1) \\ &- (\lambda^2 + \mu^2 + \nu^2)\;\theta\,(\alpha\,dx_1 + \beta\,dy_1 + \gamma dz_1)\,. \end{aligned}$$

Since dt is regarded as infinitely small the integrals that are taken along the arc AA', for example, reduce to the corresponding differentials. The differential ($\alpha\,dx_1 + \beta\,dy_1 + \gamma\,dz_1$, which is the normal projection of the arc AA', is nothing but $\theta\,dt$, the distance between the two wave surfaces in the initial state. The expression $\lambda\,u_1 + \mu\,v_1 + \nu\,w_1$ reduces to the preceding one if we replace λ, μ, ν by their values $l\alpha$, $l\beta$, $l\gamma$ (in which l is the magnitude of the discontinuity). It will be equal to $l\theta\,dt$, whereas $\lambda\,u_1 + \mu\,v_1 + \nu\,w_1$ will represent lv_{1n} (in which, v_{1n} denotes, as in no. **103**, the normal component of the velocity in state 1). The variation of the integral relative to AA' will therefore be:

$$u_1\,dx_1 + v_1\,dy_1 + w_1\,dz_1 - (u_2\,dx_2 + v_2\,dy_2 + w_2\,dz_2)$$
$$= l\,\theta\,\{(l+1)\,\theta - v_{1n}\}\,dt\,,$$

The integral relative to BB' will be an analogous expression, but taken with the opposite sign.

However, in expression (**2**) we may also evaluate the difference of the values of $\frac{u^2+v^2+w^2}{2}$ at the points A and A' (or at the points B and B') with the aid of formulas (**3**). One must therefore have, as in no. **257**:

$$\left[\frac{u^2+v^2+w^2}{2}\right] = l\theta v_{1n} - \frac{l^2\theta^2}{2}\,.$$

The total variation of the circulation, i.e., the sum of the expressions (**2**) and (**5**), will then be equal to the product of dt with the quantity:

(**6**) $$P_2 - P_1 + l\theta^2\left(\frac{l}{2}+1\right),$$

relative to the point A, minus the analogous quantity relative to the point B.

Now use the formulas:

$$l = \frac{\rho_1}{\rho_2} - 1,$$
$$l = -\frac{1}{\rho_1\theta^2}\,(p_2 - p_1),$$

of no. **256**. The quantity (**6**) becomes:

(**7**) $$\Pi = P_2 - P_1 - \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right).$$

Under these conditions, it is clear that whenever the contour C completely passes through region 1 the circulation along the contour will be augmented by the curvilinear integral:

(**8**) $$\int \Pi\,dt\,,$$

in which t represents the instant at which an arbitrary point of the contour traverses the wave, and the corresponding quantity Π is calculated at the moment of this passage.

Suppose that the contour C is very small and very close to a particular point O of the surface S. Refer it to three rectangular axes $O\xi$, $O\eta$, $O\zeta$, the first two of which are tangent to S at O and the third of which is normal and directed towards region 2. t will then be a function of ξ, η, ζ, and the same will be true of Π. The integral (**8**) may then be written:

(**9**)
$$\int \Pi \left(\frac{\partial t}{\partial \xi} d\xi + \frac{\partial t}{\partial \eta} d\eta + \frac{\partial t}{\partial \zeta} d\zeta \right).$$

One knows that in order to obtain the components of vorticity it suffices to apply Stokes's theorem to the circulation along a closed contour in such a manner that it then takes the form:

(**10**)
$$\iint_\Sigma \; [p \cos(n, \xi) + q \cos(n, \eta) + r \cos(n, \zeta)] \, d\Sigma$$

(in which Σ is an arbitrary surface that contains the contour and is bounded by the contour, and n is the normal to an arbitrary point of Σ); the quantities p, q, r will then be the desired components. On will thus obtain additional components **p**. **q**, **r** of vorticity that are produced by the passage of the wave upon making the same calculations on the integral (**9**); one thus obtains:

$$\mathbf{p} = \frac{\partial}{\partial \eta}\left(\Pi \frac{\partial t}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta}\left(\Pi \frac{\partial t}{\partial \eta} \right) = \frac{\mathrm{D}(\Pi, t)}{\mathrm{D}(\eta, \zeta)}$$
$$\mathbf{q} = \frac{\mathrm{D}(\Pi, t)}{\mathrm{D}(\zeta, \xi)}$$
$$\mathbf{r} = \frac{\mathrm{D}(\Pi, t)}{\mathrm{D}(\xi, \eta)}.$$

Finally, if we take into account the fact that this contour is infinitely close to the origin then we must make $\frac{\partial t}{\partial \xi} = \frac{\partial t}{\partial \eta} = 0$ (since the ξ and η axes are tangent to Σ) and $\frac{\partial t}{\partial \zeta} = \frac{1}{\theta}$. Hence, we finally have the desired formulas:

(**11**)
$$\begin{cases} \mathbf{p} = \dfrac{1}{\theta} \dfrac{\partial \Pi}{\partial \eta} \\ \mathbf{q} = -\dfrac{1}{\theta} \dfrac{\partial \Pi}{\partial \xi} \\ \mathbf{r} = 0. \end{cases}$$

ξ and η may be considered to be the curvilinear coordinates on the surface S, where an arbitrary point that is close to O may be substituted for its projection onto the tangent plane at O. The values of **p** and **q** then depend uniquely upon the distribution of values of Π on S. It then results from formulas (**11**) that *a shock wave always creates vortices by its passage if the quantity* Π *is not constant on the wave at each instant.*

It clear, moreover, that Π will not be constant on any randomly chosen wave, at least when the relationship between pressure and density is not such that this quantity is identically null. However, this will be true, as is easy to assure, only in the case that we spoke of in no. **144**, where $1/\rho$ is a linear function of p.

In the foregoing, we have assumed the exactness of Poisson's law. If one takes the viewpoint of Hugoniot then the question loses all interest because the vorticity, after having been modified at the moment of passage of the wave, will continue to be altered in the continuous motion that follows. In fact, the quantity k that figures in the relation:

$$p = h\,\rho^m$$

will become variable after the discontinuity, the quantity dp/ρ will cease to be an exact differential, and the general theory of vortices will cease to be applicable.

It is, moreover, also easy to calculate the instantaneous variation of the vorticity in this case. One must nevertheless observe that this instantaneous variation must be combined with another continuous one. If we thus consider, as always, our contour C as taking a certain time τ to traverse the wave (which is, moreover, small, along with the dimensions of C) then the variation of the circulation along C that will be produced during the time τ will be the combined effect of the two cases that we just spoke of, and not just that of the instantaneous variation.

However, it is easy to discern the term that is provided by the latter, and which is due to the continuous variation. Indeed, the latter is of order $\Sigma\tau$, where Σ is the area that is bounded by the contour C. It will thus be infinitely small with respect to the instantaneous variation, which has the order of Σ.

In the expression $\Pi_A - \Pi_B$, which has always given us the elementary variation of the circulation, up to a factor dt, only one category of terms must be modified: the terms in P. Their totality $(P_1 - P_2)_A - (P_2 - P_1)_B$ must obviously be replaced by the difference of the values that one obtains by the integral $\int dp/\rho$ when one takes it from A to B on the part of the contour C that is situated in the state 1 or the part that is situated in the state 2.

Let the surface Σ be bounded by the contour C, and suppose, to fix ideas, that it is constantly composed of the same molecules. Let σ be the line BA, along which Σ cuts the wave surface S at the time t. We replace the difference of the two integrals that we just spoke of by the expression:

(12)
$$\int_\sigma \left(\frac{dp_2}{\rho_2} - \frac{dp_1}{\rho_1}\right).$$

Having done this, we alter the difference in question by a quantity of order Σ. When this quantity is used in the integral over t during the time interval τ the result will be of order $\Sigma\tau$, which must be negligible, from what we said above.

Let s be a parameter that corresponds to a variable point on σ and increases from B to A, for example, the arc length of σ when measured from the point B. We may take as curvilinear coordinates on Σ the instant t when an arbitrary molecule of that surface is reached by the wave and the value of s that is determined by the position of that molecule on the corresponding line σ. From the hypotheses that were made on the position of the $\xi\eta$-plane, one will have, approximately:

$$(\mathbf{13}) \qquad \frac{\partial \zeta}{\partial s} = 0\,, \qquad \frac{\partial \zeta}{\partial t} = 0,$$

and this will be true at any point of S and any instant t_1 that is later, but by a sufficiently small quantity, than the time interval t (and consequently the area Σ, which is completely situated in region 1, will be infinitely close to the wave).

On the other hand, the integral of the quantity (**12**), when taken over t, is obviously:

$$(\mathbf{14}) \qquad \iint_\Sigma \left(\frac{1}{\rho_2} \frac{\partial p_2}{\partial s} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial s} \right) ds\, dt\,.$$

Since p_1 and p_2 are determined for each molecule (at the moment of passage across the wave, as before) and are consequently functions of the coordinates ξ, η, ζ of that molecule at the instant t_1, one will have (since $\frac{\partial \zeta}{\partial s} = 0$):

$$\frac{\partial p_i}{\partial s} = \frac{\partial p_i}{\partial \xi} \frac{\partial \xi}{\partial s} + \frac{\partial p_i}{\partial \eta} \frac{\partial \eta}{\partial s} \qquad (i = 1, 2).$$

If we substitute these values in the integral (**14**) then it will suffice to employ the relations:

$$\frac{\partial \xi}{\partial s} ds\, dt = -\frac{d\Sigma}{\theta} \cos(n, \eta), \qquad \frac{\partial \eta}{\partial s} ds\, dt = -\frac{d\Sigma}{\theta} \cos(n, \xi),$$

which result from equations (**13**), in order to put them into the form (**10**), in which the coefficient $\cos(n, \xi)$ is $\frac{1}{\theta}\left(\frac{1}{\rho_2} \frac{\partial p_2}{\partial \eta} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial \eta} \right)$ and $\cos(n, \zeta)$ is $-\frac{1}{\theta}\left(\frac{1}{\rho_2} \frac{\partial p_2}{\partial \xi} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial \xi} \right)$. One thus has:

$$\mathbf{p} = \frac{1}{\theta} \left\{ \frac{1}{\rho_2} \frac{\partial p_2}{\partial \eta} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial \eta} - \frac{\partial}{\partial \eta} \left[\frac{p_2 - p_1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right] \right\}$$

$$\mathbf{q} = -\frac{1}{\theta} \left\{ \frac{1}{\rho_2} \frac{\partial p_2}{\partial \xi} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial \xi} - \frac{\partial}{\partial \xi} \left[\frac{p_2 - p_1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right] \right\}$$

$$\mathbf{r} = 0\,,$$

in which one must suppose that p_1, p_2, ρ_1, and ρ_2 are related by the dynamical adiabatic relation (**13**) of no. **257**. This relation will, moreover, permit us to put the preceding formulas into the form:

$$\mathbf{p} = -\frac{1}{(m-1)\theta}\left(\rho_2^{m-1}\frac{\partial k_2}{\partial \eta} - \rho_1^{m-1}\frac{\partial k_1}{\partial \eta}\right)$$
$$= -\frac{R}{(m-1)\theta}\left(T_2\frac{\partial \log k_2}{\partial \eta} - T_1\frac{\partial \log k_1}{\partial \eta}\right)$$

$$\mathbf{q} = \frac{1}{(m-1)\theta}\left(\rho_2^{m-1}\frac{\partial k_2}{\partial \xi} - \rho_1^{m-1}\frac{\partial k_1}{\partial \xi}\right)$$
$$= \frac{R}{(m-1)\theta}\left(T_2\frac{\partial \log k_2}{\partial \xi} - T_1\frac{\partial \log k_1}{\partial \xi}\right)$$

in which T_1, T_2 are the two absolute temperatures, as long as one has:

$$k_1 = \frac{p_1}{\rho_1^m}, \qquad k_2 = \frac{p_2}{\rho_2^m},$$

and R *denotes the constant that figures in the right-hand side of equation* (**5**) in no. **125**.

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NOTE IV

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ON REFLECTION IN THE CASE OF A FIXED PISTON

We saw in chapter IV (no. **180**) that if we desire to take into account both the pre-existing motion of the gas (this motion being arbitrary) and the motion of the piston one is led to a problem that is very different from the one that corresponds to the case in which only the initial motion is involved, and which has a much greater degree of difficulty, thanks to the circumstance that one must determine a solution to the Euler equation (equation (**46**), no. **175**) in terms of givens that relate to a an unknown line in the $\xi\eta$-plane.

Meanwhile, there is a particular case that is the exception to this, and in which the problem is solved without difficulty: the case in which the piston is immobile (or, more generally, animated with a uniform motion).

Indeed, one will then have that the quantity $u = (\xi + \eta)/2$ is null (or constant) at the extremity of the tube (for example, $a = 0$).

On the other hand, when u is constant and x is only constant initially one always has $x = ut$.

Under these conditions, *the quantity z that is defined by formula* (**30**) *of* no. **170** *will be null.*

We thus have to determine a solution z to the Euler equation under the following conditions:

1. x will be null for $\xi + \eta = 0$ (or for $\xi + \eta$ equal to a given constant $2v$).
2. The values of z will be known on a certain characteristic η = const., namely, the wave for which the desired motion agrees with the given initial motion, namely:

$$\eta = \eta_0 .$$

We said above (chap. VII) that such a problem is possible and well-defined provided that the preceding givens agree at the point of the $\xi\eta$-plane that is common to the preceding two lines, i.e., when one gives η the value η_0 and gives ξ the value $\xi_0 = 2v - \eta_0$.

In order to find the solution, trace out the second characteristic $\xi = \xi_0$ that passes through the same point, and which is nothing but the symmetric image of the first one with respect to the straight line Δ that is represented by the equation $\xi + \eta = 2v$. Consider the solution z of the Euler equation that takes the given values for $\eta = \eta_0$ and values that are equal, but with opposite signs, to the latter when $\xi = \xi_0$. By this, I intend to imply that z will have a value at the point (ξ_0, η) that is equal and opposite in sign to the one that it takes at the point $(2v - \eta, \eta_0)$ that is symmetric to the preceding one with respect to Δ.

From what we saw in no. **172**, if we therefore give z the values for $\xi = \xi_0$, on the one hand, and, on the other, for $\eta = \eta_0$, then we determine an integral of the Euler equation.

Now, it is clear that this integral takes values that are equal and of opposite sign at two *arbitrary* symmetric points, with respect to Δ, i.e., whose coordinates $\xi, \eta; \xi', \eta'$ are coupled by the relations:

$$\begin{cases} \xi' = 2v - \eta, \\ \eta' = 2v - \xi. \end{cases} \tag{1}$$

Indeed, the transformation thus defined does not change the partial differential equation, but changes the signs of the initial givens. Since they change sign when one passes from one side of Δ to the other, the integral z is null on Δ. *They represent the desired solution*, the solution that one determines by formula (**40**) of no. **172**.

It is easy to exhibit the phenomenon of reflection in the calculations that we just carried out. Indeed, let u', ω' be the values that are taken by u and ω by means of the formulas (**27**) of no. **170**, when one gives ξ the value ξ' and gives η the value η'. The transformation (**1**) that we have written will, for some instant, correspond to:

$$u' = 2v - u, \qquad \omega' = \omega.$$

Since the new value of z is $z' = -z$, the new values of $a = \dfrac{\partial z}{\partial \omega}$, $t = \dfrac{\partial z}{\partial u}$, and of $x = \omega a + ut - z$ will be:

$$a' = -a, \qquad t' = t \tag{2}$$

$$\omega' = 2vt - x. \tag{3}$$

Therefore, if the initial state of the given fluid is assumed to correspond to $a \geq 0$ then we consider a completely similar fluid mass to fill the region $a \leq 0$, and we impress upon this second medium a motion such that by means of relations (**2**) one obtains (**3**), and the totality of the real fluid and the fictitious one will form a single mass whose motion will satisfy the partial differential equation. This motion, which is calculated by starting with the initial state of the given fluid, as was explained in no. **179**, will itself satisfy the condition $x = vt$ for $a = 0$, in such a way that we may suppress the piston under these conditions.

Now, each molecule of the fictitious fluid is, at an arbitrary instant, symmetric to the corresponding molecule of the real fluid with respect to the barrier.

Of course, the solution thus obtained may be subject to the difficulty that was pointed out at the end of no. **179**, and gives rise to the singularities that were considered in nos. **194**, and following.

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Translator’s notes

Since this book is now over one hundred years old, it is more necessary than usual to justify why it would continue to be of interest in the current era, especially in light of how much the mathematics, physics, and engineering that pertains to its subject matter has advanced, if not exploded, since it was written. Therefore, in these notes concerning the translation, we shall first define the historical context of the book’s original conception and then point out the continuing influence that it had on the subsequent advances in the study of waves, in both the physical sense of that term and the mathematical one. After that, some observations will be made about the material in the individual chapters that might make that material more meaningful to modern readers.

§ 1. Biographical sketch. Jacques Hadamard began his long and prolific life on December 5 of 1865 in the town of Versailles outside of Paris. His father, Amédeé, was a schoolteacher at the local *lycée* whose career as a schoolteacher was not entirely stable, as the family did not stay in Versailles, but moved several times during Jacques’s childhood.

Jacques’s mother, Claire-Marie-Jeanne, supplemented the income of Amédeé by teaching piano. Although she was somewhat infamous amongst her students for her strictness – indeed, some students were reputed to have been seen weeping as they ascended the stairs to her apartment – her students did nonetheless include such celebrated musical figures as the composer Paul Dukas.

Jacques got most of his early schooling at the *Lycée Louis-le-Grand* in Paris, where his father eventually taught after Versailles. Interestingly, despite Jacque’s distinctions in adulthood as a world-renowned mathematician, in his childhood he apparently detested his arithmetic studies [77]. Indeed, one of his mathematics teachers later on, the celebrated mathematician Émile Picard, admitted that as a youth he himself only did his geometry homework to avoid punishment! Nevertheless, young Jacques must have eventually learned to focus on his studies, since he generally placed at or near the top of his school in his tests in numerous subjects, and was actually two years ahead of his other classmates when he graduated.

As a result of this superior scholasticism, he was able to get accepted at both the prestigious and highly competitive *École Polytéchnique*, which was more oriented towards educating engineers, and the *École Normale Supérieure*, which provided a more philosophically-inclined education for future professors and other teachers. He chose to enter the later institution, along with such distinguished fellow students as Paul Painlevé, Ernest Vessiot, and Eugene Cosserat. His teachers included some of the legendary figures of French mathematics, such as Charles Hermite, Camille Jordan, Pierre Bonnet, Gaston Darboux, Henri Poincaré, Paul Appell, Pierre Duhem, Edouard Goursat, and the aforementioned Picard. Hadamard completed a doctoral thesis entitled *Éssai sur l’étude*

[77] However, it is somewhat dubious that his daughter Jacqueline was being completely serious when she later insisted that he never learned to count past four (“…after that, there was just *n*…”)!

des fonctions données par leur developpement de Taylor in 1892 and defended it before a committee that was composed of Hermite, Picard, and Jules Joubert.

In that same year, he married Louise-Anne Trénel, and stayed married to her for the remaining sixty-eight years of her life until her death on July 6 of 1960. Her contribution to his life was inestimable, since this otherwise prolific contributor to the literature of mathematics was nonetheless hopeless at putting his thoughts down on paper and would dictate his research to Louise, while replacing all of the equations with the generic term "*poum,*" which he would fill in later. Moreover, his legendary absent-mindedness included the fact that he was equally hopeless at dressing himself, and could not tie a proper knot in his tie.

Jacques and Louise had five children, in the form of three sons, Pierre, Étienne, and Mathieu, and two daughters, Cécile and Jacqueline. However, Hadamard's family life was touched by tragedy during both World Wars, the first of which took the lives of Pierre and Étienne, while the Second World War took that of Mathieu.

Hadamard's career as a college professor began with various short-lived tutoring appointments, and he mostly taught at the university of Bordeaux until 1897, when he began with the faculty at the *Collège de France*, which is where he taught when this book was written. Later on, he accepted a position at the *École Polytéchnique* in 1912, and in that same year was inducted as a member of the prestigious French *Académie des Sciences*, when the death of Poincaré created an opening for a new member.

At a personal level, Hadamard had a special talent for languages, and even felt considerable pride in the fact that after teaching a seminar at a Spanish university for a semester, he had succeeded in giving his last lecture in Spanish, having learned enough of that language over the course of the seminar. Spending his childhood in a musical environment translated into the fact that as an adult he not only enjoyed playing the violin, but also had an informal home orchestra that sometimes included his colleagues, such as Einstein. One of Hadamard's other academic passions besides mathematics was botany, and he was known to have inadvertently abandoned one of his own children on a glacier while absorbed in the collection of specimens. For some years, he was also quite passionate about the politics of the Dreyfus affair.

In addition to Einstein, Hadamard's colleagues and acquaintances included not only mathematicians and physicists, but also philosophers and political leaders from all over the world. The mathematicians represented some of the most distinguished figures in their respective countries, and included the Russians, Andrei Kolmogorov, Alexander Liapunov, Vladimir Steklov, and Pavel Alexandrov, the Italians Francesco Tricomi and Vito Volterra, the Germans David Hilbert and Felix Klein, the Americans Norbert Wiener and George Birkhoff, the Englishman G. H. Hardy, and the Swede Gösta Mittag-Leffler. Along with the French mathematicians mentioned in the context of Hadamard's education, he was also acquainted with Henri Lebesgue and André Weil, as well as the physicist Paul Langevin. His non-mathematical associations included the French poet Paul Valéry, the French sociologist Émile Durkheim, and he was once introduced to prime minister Pandit Jawaharlal Nehru of India.

Hadamard's influence on the mathematics and mathematicians that followed him was incommensurable. For instance, his first student was Maurice Frechet, who eventually went on to be one of the founders of functional analysis. The mathematician Laurent Schwartz, who did much to define the theory of distributions, conceded his debt to the influence of Hadamard, as did Émile Borel, who had been a student at the *École Polytechnique* some five years behind Hadamard.

The methods of the present work by Hadamard were a seminal influence on much of the work that was done on wave theory later on, since they served to define the most useful way of characterizing a wave mathematically – namely, as a propagating discontinuity in the kinematical variables at some level of differentiability – as well as showing that such discontinuities could propagate only along bicharacteristic curves in characteristic hypersurfaces. The provenance of many subsequent publications on the subject of waves, especially those of the French school of partial differential equations, includes this work either explicitly or implicitly. We shall note some of those publications in the process of discussing the material contained in this book.

The evolution of Hadamard as a mathematician can be partially seen from a list of the books that he published:

1898	*Leçons de géométrie élémentaire: géométrie plane.*
1901	*La série de Taylor et son prolongement analytique*
	Leçons de géométrie élémentaire: géométrie dans l'espace
1903	*Leçons sur la propagation des ondes et les équations de l' hydrodynamique*
1910	*Leçons sur le calcul des variations*
1922	*Lectures on Cauchy's Problem in Linear Partial Differential Equations*
1926	*Cours d'analyse de l'École Polytechnique (v. 1)*
1930	*Cours d'analyse de l'École Polytechnique (v. 2)*
1945	*The Psychology of Invention in the Mathematics Field*
1965	*La théorie des équations aux dérivées partielles* (published posthumously)

Hadamard published over four hundred research papers on such diverse mathematical topics as analytic function theory, number theory, geometry, the calculus of variations, and partial differential equations, as well as on topics in physics such as mechanics, hydrodynamics, elasticity, and waves. Consequently, he was sometimes characterized as a "universal mathematician," in the sense of a mathematician who could make contributions to all of the branches of mathematics that were important in his era; indeed, the period that followed the turn of the Nineteenth Century was probably the last point in history when such an achievement was still humanly possible. However, one can see that the study of partial differential equations occupied the focal position in his research for more than sixty years.

Jacques Hadamard died on October 17 of 1963, two months short of his ninety-eighth birthday and three years after the passing of his wife. Strangely, for all of his prolific contributions, international renown, celebrity acquaintances, and honorary distinctions, the family tomb at *Père Lachaise* cemetery in Paris does not include an inscription for his name. Similarly, no street in the Latin Quarter bears his name, and it was more than thirty years after his death before a biography appeared. Perhaps the most lasting monument to his place in the history of mathematics will have to be the legacy of his contributions to the foundations of modern mathematics by way of publications such as the present one.

§ 2. The writing of this book. As one can see from the timeline of Hadamard's books that was given above, the publication of this work came at a rather early point in his mathematical career, certainly as far as his subsequent work on partial differential equations was concerned.

The inspiration for the study of the title topic itself came from lectures of Pierre Duhem on hydrodynamics, elasticity, and acoustics that were given during the years 1890 and 1891 while Hadamard was at the Collège de France. In those lectures, Hadamard was exposed to the fundamental work that had been done before by Riemann, Christoffel, Rankine, and Hugoniot on the propagation of shocks in elastic media, such as compressible gases.

Consequently, Hadamard gave a series of lectures on the mathematical aspects of the subject during the academic years from 1898 to 1900. A preliminary report on this work "Sur la propagation des ondes" was published in the *Bulletin de la Société Matematique de France* in 1901 prior to the publication of this book.

Now, shock waves pertain to discontinuities in the velocity of a wave across a surface. Hence, it is easy to see how this concept would suggest the generalization to discontinuities in other derivatives across surfaces, such as discontinuities in the acceleration, and this is one of the innovations that this book introduced.

The fact that such kinematical discontinuities propagate along bicharacteristics was established in the last chapter of this book, which Hadamard later regarded as something of a preliminary sketch of the more mathematically rigorous treatment that he gave the subject in his subsequent Yale lectures on the Cauchy problem in 1922. Hence, one can regard that later, primarily analytical, work on partial differential equations as a complementary continuation to the present work, and not as superseding replacement, since the present book is more concerned with establishing the foundations of wave theory in continuum mechanics, not analysis. As Hadamard points out in the Preface, his main intention was "to study how boundary conditions influence the motion of a fluid."

Another historical aspect of this book that is not entirely self-evident from reading it is the fact that before the book was written the mathematical study of partial differential equations had not yet matured into a particularly organized, rigorous state. Rather, it existed primarily in the form of descriptions of the methods of solution for various particular partial differential equations that grew out of specific problems in physics that seemed, at the time, otherwise unrelated. Hence, this work also served to address for the first time the problem of developing the theory of partial differential equations more generally as a problem in mathematical analysis, as well as a set of problems in applied mathematics.

In his 1960 work on the classical field theories, that most influential figure of modern continuum mechanics, Clifford Truesdell, listed what he considered to be the principal achievements of Hadamard with the publication of this book:

1. The basic lemma that distinguishes between the compatibility conditions for a kinematical variable at a discontinuity surface in general and the compatibility conditions for particular cases.

2. The recognition that there is more than one type of compatibility, such as geometric, kinematic, dynamic, energetic, and material (although this book only addresses the first two).

3. The classification of kinematical singularity surfaces and the construction of a general theory of such things.

4. The calculation of exact wave speeds in the case of finite elastic strains, when previously they were calculated only for infinitesimal strains. Moreover, he proved that they are all real and non-vanishing iff the equations of equilibrium for the strain are "strongly elliptic" partial differential equations (in a sense that we will clarify later).

5. The proof that "weakly singular" surfaces in gas dynamics still preserve the circulation of the velocity vector field. Consequently, such waves do not invalidate the theorem of Lagrange and Cauchy on velocity potentials.

6. The proof that an oblique shock wave in a gas generates vortices.

7. The first rigorous definition and analysis of stability in the context of elastic strains, along with the proof that when an elastic medium is in a state of stable equilibrium the inequality that defines strong ellipticity must still hold, except that one must replace the "≥" with a ">".

§ 3. Notes on Chapter I. It is almost a tradition of the pure mathematics community that the introductory chapter of any research monograph is composed of specialized results that are not used until much later in the study, and this book is no exception to that rule. Indeed, there is no loss of comprehension in beginning one's reading of the work with Chapter II, since the theorems of chapter I are used only occasionally in the remainder of the book.

The theorems of this chapter are primarily concerned with the Neumann problem for the Laplace equation, so their relationship with the problems of wave motion is somewhat peripheral and largely based in the consideration of standing-wave solutions to wave equations.

In a paper that he wrote in 1902, Hadamard introduced the concept of a well-posed boundary-value of Cauchy problem for a partial differential equation, by requiring that a well-posed problem admit a unique solution that depends continuously upon the boundary or Cauchy data. The problems concerning elliptic partial differential equations that are well-posed in the Hadamard sense are generally boundary-value problems.

The *Dirichlet problem* involves finding a solution ϕ to the Laplace equation:

$$0 = \Delta\phi \equiv \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j}, \quad i, j = 1, \ldots, n$$

in a region $V \subset \mathbb{R}^n$ with boundary ∂V when one is given the values of that solution on the boundary. By comparison, the *Neumann problem* involves being given the values of the normal derivative of the solution on the boundary. When the boundary in question has more than one connected component it is also possible to define mixed – or *Robin* – boundary-value problems.

The Dirichlet problem is often the most natural problem to pose in the context of electrostatics, since one can measure electric potential differences directly in the laboratory. However, in hydrodynamics, the solution that one is often seeking takes the form of a *velocity potential*, so since it is more natural to know the value of its gradient –

namely, the velocity vector field – on a boundary, one sees how the Neumann problem might be relevant to the subject of the present tome.

Furthermore, as long as ∂V divides the space into two distinct components that can identified as an "interior" region and an "exterior" one, one can distinguish between the *interior* boundary-value problem and the *exterior* problem. That is, one proposes to find a function that is defined on the chosen region that satisfies the given partial differential equation and agrees with a given function that is defined on the boundary.

A key result in the study of boundary-value problems for the Laplace equations is *Green's formula:*

$$\int_V (u\Delta v - v\Delta u)dV = \int_{\partial V} (u_n v - v_n u)dS\,,$$

which is valid for any pair of C^2 functions u and v on V. In this expression, $u_n = n^i\, \partial u / \partial x^i$ and $v_n = n^i\, \partial v / \partial x^i$ represent the directional derivatives of the functions u and v in the direction of the unit normal $\mathbf{n}$ to the boundary hypersurface.

One then looks for a *fundamental solution* $\gamma(x^i, y^i)$ to the Laplace equation, which then satisfies, by definition:

$$\Delta\gamma(x, y) = -\,\delta(x - y).$$

We point out that since the Dirac "delta function" $\delta(x - y)$ is not really a function, but the mythical kernel of the evaluation functional, the only way that this equation is rigorously defined is if both γ and δ are defined as two-point distributions. However, since this rapidly leads away from the presentation that is given in the book under discussion, we refer the curious to more modern literature on boundary-value problems, such as Stakgold [1967].

Hence, if one assumes that u satisfies the Poisson equation on V – i.e., $\Delta u = \rho$ – and $v = \gamma$ is a fundamental solution then Green's formula takes the form:

$$u(x) = -\int_V \gamma(x, y)\rho(y)\, dV - \int_{\partial V} \frac{\partial\gamma(x, y)}{\partial n_y} u(y) d_y S + \int_{\partial V} \gamma(x, y) u_n(y) d_y S\,.$$

For a harmonic function u, the source function ρ vanishes, along with the first integral. The second integral is referred to as a *single-layer surface potential*, while the final one is referred to as a *double-layer surface potential*.

Since one generally cannot specify both u and u_n on the same boundary components, one then makes one or the other integral vanish by specifying the boundary behavior of the fundamental solution γ. In the case of the Dirichlet problem, one is given u on ∂V so in order to make the integral involving u_n vanish, one specifies that γ must vanish on ∂V. The resulting fundamental solution, which is denoted by $G(x, y)$, is then called the *Green function* for this boundary-value problem. The solution to the Dirchlet problem is then given by:

$$u(x) = -\int_{\partial V} \frac{\partial G(x, y)}{\partial n_y} u(y)\, d_y S\,.$$

By comparison, for the Neumann problem one is given u_n on ∂V so one needs to make the other integral vanish, and one does this by specifying that $\partial\gamma/\partial n_y$ must vanish on the boundary. Such a fundamental solution is then called a *Neumann function*, which one denotes by $N(x, y)$, and the solution to the Neumann problem then becomes:

$$u(x) = \int_{\partial V} N(x,y)u_n(y)\,d_yS\ .$$

Note that finding a fundamental solution still amounts to a boundary-value problem in the Laplace equation, so in a sense there is only so much of a simplification. However, it turns out that the coefficients g^{ij} of the generalized Laplacian operator $g^{ij}\,\partial^2/\partial x^i\partial x^j$ are intimately based in the geometry of the space since they are the inverse matrices to the components of the metric tensor field $g = g_{ij}\,dx^i\,dx^j$ on it. Indeed, the fundamental solutions that one usually encounters in potential theory (such as Duff [1950]) are usually constructed from the geodesic distance function $s(x, y)$ that is associated with that metric, at least locally. That is, $s(x, y)$ equals the minimum length of the geodesics (in the sense of paths whose velocity vector fields are parallel translated) that connect x to y. Hence, since not every pair of points in a more general Riemannian manifold can be connected by at least one geodesic, one sees why such a construction is usually only local in scope.

In the case of electrostatics or Newtonian gravitostatics in three spatial dimensions, the fundamental solution is proportional to $1/r(x, y)$, where $r(x, y) = [(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2]^{1/2}$, which represents the potential at y due to a unit charge or mass at x, when one ignores the constitutive properties of the medium.

§ 4. Notes on Chapter II. The contents of this chapter constitute the fundamental basis for the remainder of the topics presented in the book, namely, the mathematical and physical nature of the compatibility conditions that must be satisfied by a wave propagating in an elastic medium when that wave is defined by a jump discontinuity in some kinematical derivative across a discontinuity hypersurface.

Section 1 of the chapter begins with a brief summary of the kinematics of deformation in continuum mechanics. For the benefit of modern readers, we now present the form that such a summary might take nowadays.

One begins by mathematical characterizing two elementary physical notions: an *(extended material) object* $\mathcal{O}$ and a *medium* M. Although there are many possible ways of axiomatizing both concepts, the one that we shall choose is that of making the medium M take the form of the vector space[78] $\mathbb{R}^n$, and the object $\mathcal{O}$ take the form of a differentiable (non-singular) cubic k-chain in M. That is, $\mathcal{O}$ is composed of a finite number N of embedded k-dimensional cubes $\sigma_i: I^k \to M$, $(a^1, \ldots, a^k) \mapsto \sigma_i(a^1, \ldots, a^k)$, $i = 1, \ldots, N$ that are "attached" to each other along their boundaries in a specified way. In particular, we are assuming that the maps σ_i are one-to-one, differentiable, and have a

[78] Although this vector space could be generalized to a differentiable manifold, since the material in the book under discussion is primarily non-relativistic and Euclidian in character, it would be something of a needless distraction to pursue that direction of inquiry in the present context. However, we shall still make occasional remarks about the use of manifold techniques more generally in what follows.

differentiable inverse when restricted to their images. For practical purposes, the values of k will usually be 1, 2, and 3, which corresponds to filaments, surfaces, and volumes. Rather than go into the details of the aforementioned construction, from now on, we will simply confine our attention to the individual k-cubes in M; i.e., $N = 1$. When the extended material object in question is a fluid, such as a stream in a given channel, the k-cubes can be thought of as "fluid cells."

In point mechanics, the object in question would be the embedding of a single point in M in the static case and a time interval $I = [0, 1]$ in the dynamic case.

Since we are restricting ourselves to $M = \mathbb{R}^n$, with $n = 2, 3$, or 4, we can express the embedding of a k-cube as a set of functional equations of the form:

$$x^i = x^i(a^1, \ldots, a^k), \qquad i = 1, \ldots, n,$$

in which the functions are continuously differentiable to some specified order.

We shall regard each individual embedding of an object $\mathcal{O}$ in M as a *state* of the object. In order to define a finite deformation of the object, we then need two states: an initial state $\mathcal{O}_0$ and a final state $\mathcal{O}_1$, which we then characterize by the embeddings $x_0^i(a^1, \cdots, a^k)$ and $x_1^i(a^1, \cdots, a^k)$, respectively. Often, the initial state takes the form of a "natural" state, which might be characterized by a state of stress equilibrium, but as long as one deals only with changes in physical properties between the initial and final states, it is not always necessary to start with the equilibrium state.

A *finite deformation* of the object $\mathcal{O}_0$ to the object $\mathcal{O}_1$ is then a diffeomorphism f: $\mathcal{O}_0 \to \mathcal{O}_1$, $x_0^i \mapsto x_1^i(x_0^1, \cdots x_0^n)$, $i = 1, \ldots, n$. That is, f is invertible, continuously differentiable, and has a continuously differentiable inverse.

Since we already have the x_0^i expressed as functions of the $a^1, \ldots, a^k$, we can also characterize the deformed coordinates x_1^i as functions of the form $x_1^i(a^1, \cdots, a^k)$. One can then distinguish two different ways of characterizing the deformation: the *Lagrangian* picture, which regards the coordinates x_0^i as the fundamental ones, and the *Eulerian* picture, which regards the coordinates x_1^i as fundamental. The Lagrangian picture is more convenient in solid mechanics, where one can speak of the initial state as being the "natural" state of the object, i.e., the state in which no external forces or moments are present, while the Eulerian picture is more convenient to fluid mechanics, where the natural state is not as well-defined, so one essentially uses the *present state* as a reference.

As long as the manifold M is a vector space, one can define the *displacement vector field* that is defined by the deformation f to be the following vector field on $\mathcal{O}_0$:

$$u^i(x_0^j) = x_1^i(x_0^j) - x_0^i.$$

One can also think of it as a vector field parameterized by the reference coordinates a^I, $I = 1, \ldots, k$ of the object itself:

$$u^i(a^1, \cdots, a^k) = x_1^i(a^1, \cdots, a^k) - x_0^i(a^1, \cdots, a^k) .$$

Corresponding to the two formulations, one can consider two sets of partial derivatives for the displacement vector field, namely:

$$\frac{\partial u^i}{\partial a^p} = \frac{\partial x_1^i}{\partial a^p} - \frac{\partial x_0^i}{\partial a^p} ,$$

and:

$$\frac{\partial u^i}{\partial x_0^j} = \frac{\partial x_1^i}{\partial x_0^j} - \delta_j^i ,$$

which refer to the differential of the embedding and the differential of the diffeomorphism, respectively. The first set of partial derivatives has the advantage that it is more intrinsic to the way that the deformation appears to the points of $\mathcal{O}$, but the second set has the advantage that it allows one to express the differential of the deformation in the form:

$$\frac{\partial x_1^i}{\partial x_0^j} = \delta_j^i + \frac{\partial u^i}{\partial x_0^j} .$$

Although one often regards the second term in the sum as an infinitesimal perturbation of the identity deformation, at this point, that approximation is not necessary. From now on, we write that term in the form $u_{i,j}$ and refer to it as the *displacement gradient*, although we have implicitly lowered the upper index using the Euclidian metric on $\mathbb{R}^n$, which is customary in non-relativistic continuum mechanics.

The displacement gradient can then be decomposed into a sum of a matrix of trace type, a traceless symmetric matrix $\overset{\circ}{e}_{ij}$, and an anti-symmetric matrix θ_{ij}:

$$u_{i,j} = \lambda\delta_{ij} + \overset{\circ}{e}_{ij} + \theta_{ij} ,$$

in which:

$$\lambda = \tfrac{1}{n}\,\mathrm{Tr}\, u_{i,j} = \tfrac{1}{n}\, u^i{}_{,i} , \qquad \overset{\circ}{e}_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) - \lambda\delta_{ij} , \qquad \theta_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}).$$

The scalar λ is called the *infinitesimal dilatation*, the matrix $\overset{o}{e}_{ij}$ defines the *infinitesimal (volume-preserving) shear*, and the matrix θ_{ij} gives the *infinitesimal rotation* that is associated with the deformation; we shall denote the matrix $\tfrac{1}{2}(u_{i,j} + u_{j,i})$ by e_{ij} .

In order to measure the deformation of an object due to a diffeomorphism, one introduces the concept of strain, which presumes the existence of a metric on the manifold M. Since we are dealing with the non-relativistic case in Hadamard's book, it is sufficient to regard M as $\mathbb{R}^n$ with the Euclidian metric $\delta_{ij}\, dx^i\, dx^j$; this has the advantage that one does not have to deal with the issue of whether the metric on M "changes" from point to point independently of the objects embedded in it, which necessitates the introduction of a connection and the consideration of its curvature.

Generally, strain is only introduced in the Lagrangian picture, since fluid media do not support strains; one can, however, make corresponding definitions in the Eulerian picture, though. The basic idea in measuring the deformation of an object is to compare the metric δ_{ij} that is defined on the initial state by restriction with the metric that one obtains by "pulling back" the metric, which also has the components δ_{ij}, on the final state along the deformation f. This gives the *Cauchy-Green finite strain tensor:*

$$2E_{ij} = \frac{\partial x_1^k}{\partial x_0^i}\frac{\partial x_1^l}{\partial x_0^j}\delta_{kl} - \delta_{ij} = 2e_{ij} + u_i^k u_j^k .$$

The matrix e_{ij} is then the same as above, and is also referred to as the *infinitesimal strain tensor.* Note that the strain is indifferent to the rotational part of the deformation, along with the addition of a spatially homogeneous displacement vector field. That is, rigid motions of an object do not produce strains. However, one should note that, so far, we have not specified that the deformation does not involve an inhomogeneous rotation – i.e., torsion.

If one wishes to go from statics to dynamics, one must introduce the concept of motion, which we take to mean a one-parameter family $\mathcal{O}_t$, $t \in [0, 1]$ of deformations such that $\mathcal{O}_0$ and $\mathcal{O}_1$ represent the initial and final state in the family. The family is also assumed to be continuously differentiable to some specified order, as well. Note that our previous choice of objects in the form of differentiable k-cubes is entirely consistent with this expansion of scope, as long as one regards the motion of a k-cube as a k+1-cube. We shall add the time parameter as the 0^{th} coordinate of the cube. Furthermore, we shall make the non-relativistic assumption that $M = \mathbb{R} \times \mathbb{R}^n$, which amounts to the Newtonian assumption that time works the same way for everyone; i.e., the time coordinate t is not deformed along with the spatial ones. Hence, a motion takes the functional form:

$$t = t_0 + \Delta t, \qquad x^i = x^i(t,\ x_0^i).$$

From this, we can associate a *velocity* vector field on the image of the k+1-cube $\mathcal{O}_t$ by way of the partial time derivative:

$$v^i(t, x^j) = \frac{\partial x^i}{\partial t}.$$

This can also be parameterized in terms of $(t, a^1, \ldots, a^k)$ by expressing the x^i as functions of $(t, a^1, \ldots, a^k)$.

One can analogously define the *velocity gradient* by the partial derivatives $\partial v^i / \partial a^p$ or $\partial v^i / \partial x^j$, in which we can include t as a coordinate in either case. Hence, the temporal part of the velocity gradient becomes the linear acceleration of the deformation.

One can also decompose the velocity gradient, in the form $v_{i,j}$, into the sum of a tracelike matrix, a traceless symmetric matrix $\overset{\circ}{\dot{e}}_{ij}$, and an anti-symmetric one ω_{ij}:

$$v_{i,j} = \rho\delta_{ij} + \overset{\circ}{\dot{e}}_{ij} + \omega_{ij} ,$$

in which:

$$\rho = \tfrac{1}{n} v^i_{,i} , \qquad \overset{\circ}{\dot{e}}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) - \rho\,\delta_{ij} , \qquad \omega_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}),$$

This time, we refer to ρ as the *rate of dilatation*, $\overset{\circ}{\dot{e}}_{ij}$ as the *rate of (volume-preserving) shear*, and and ω_{ij} as the *vorticity* of the deformation; we can also refer to $\overset{\circ}{\dot{e}}_{ij}$ as the *rate of strain*. It is no longer necessary to distinguish between finite deformations and infinitesimal ones, since all objects obtained by differentiation will be infinitesimal in character, although in the finite case one will have to say whether the resulting tensor is defined on the initial state or the present state..

In fluid media, which do not support strains, one usually starts with the flow velocity vector field $\mathbf{v}(t, x^i)$ in the space of the motion. The flow is said to be *irrotational* if the vorticity vanishes. In that event, the velocity vector field (or really, the *covelocity 1-form* $v = v_i\, dx^i$) will admit a global velocity potential ϕ when M is simply connected [79]. Hence, $\mathbf{v}$ will be the gradient of ϕ:

$$\mathbf{v} = \nabla\phi.$$

When M is not simply connected, $\mathbf{v}$ will admit velocity potentials that are defined only in some neighborhood of each point, since every point of any manifold will have a simply connected neighborhood.

Section 2 of this chapter is concerned with the nature of kinematical discontinuities in the form of jump discontinuities in some level of kinematical derivative across a particular hypersurface. This material is essential to all of the material that follows in the book, since it is in this context that Hadamard gives a rigorous definition to the notion of a wave itself.

As Hadamard describes the provenance of the topics, this section was based in earlier work of Riemann, Christoffel, and Hugoniot. In 1860, Riemann established the main results on compatibility conditions associated with the discontinuity in case of shock waves in one-dimensional gases, where a shock wave represents a discontinuity in the velocity vector field. In 1877, Christoffel extended these results to three dimensions, and in 1887, Hugoniot, apparently unaware of the work of Riemann and Christoffel, made a more general study for higher-order discontinuities, in which he explicitly introduced the idea that the conditions obtained represented compatibility conditions for the discontinuities.

[79] A topological space is called *simply connected* when every continuous loop in that space can be continuously deformed to a constant loop; i.e. to some point of the space. For instance, any vector space is simply connected, but all that it takes to render a plane non-simply connected is to remove a single point from it.

Since the discontinuities are assumed to be defined only on an isolated hypersurface S in M, one represents S by the zero hypersurface of some continuously differentiable function f:

$$f(t, x^i) = 0.$$

This hypersurface is, moreover, assumed to divide M into two disjoint regions M_1 and M_2, which one may regard as the disturbed and undisturbed regions.

The class of functions Φ on M that one considers are functions that are smooth on both components M_1 and M_2 and approach finite limits $\Phi_1(x)$ and $\Phi_2(x)$ on the points $x \in S$. Hence, the *jump discontinuity* in Φ across S:

$$[\Phi](x) \equiv \Phi_2(x) - \Phi_1(x)$$

defines a smooth function on S. Hence, even though the singularity is in the function Φ and not the surface S, nowadays one refers to S as a *singular surface.*

This same process of definition can be applied to the partial derivatives of Φ in an analogous manner:

$$\left[\frac{\partial \Phi}{\partial x^i}\right] \equiv \left[\frac{\partial \Phi}{\partial x^i}\right]_2 - \left[\frac{\partial \Phi}{\partial x^i}\right]_1$$

One finds that the jump in any derivative is not arbitrary, but must satisfy *geometrical compatibility conditions* that are based in the assumption that the jump in Φ is smoothly distributed across S and not only over some lower-dimensional subset, such as a set of isolated points or curves. One sees that this will the case iff there is no jump in the specified derivative when you go from one point of S to another.

Here, it helps to know that nowadays (cf., e.g., Truesdell and Toupin [1960]) the discussion in no. **72** of Hadamard's book gets phrased as *Hadamard's lemma:* If Φ is smooth on either side of S and its restriction to S is smooth then for any continuously differentiable curve $x^i(s)$ on S the derivative of either Φ_1 or Φ_2 along $x^i(s)$ with respect to s is the directional derivative one usually computes:

$$\frac{d\Phi_{1,2}}{ds} = \frac{dx^i}{ds}\frac{\partial \Phi_{1,2}}{\partial x^i}.$$

(The only analytical detail to be resolved in this is the passage to the one-sided limits on either side of S.)

By taking the difference of these latter two equations, one finds the useful consequence that the derivative of $[\Phi]$ with respect to s is also the tangential projection of $[\partial\Phi/\partial x^i]$:

$$\frac{d[\Phi]}{ds} = v^i\,[\Phi_{,i}] = [v^i\,\Phi_{,i}].$$

That is, the derivative of the jump equals the jump of the derivative.

One can extend this result to partial derivatives with respect to any coordinate system ξ^a, $a = 1, \ldots, \dim S$ on S:

$$\frac{\partial[\Phi]}{\partial\xi^a} = \frac{\partial x^i}{\partial\xi^a}[\Phi_{,i}] = \left[\frac{\partial x^i}{\partial\xi^a}\Phi_{,i}\right].$$

This has the immediate consequence that if Φ is continuous across S – so $[\Phi] = 0$ – then since all of the vectors that are described by the coordinate derivatives will be tangent to S, one must conclude that any jump discontinuity in $[\Phi_{,i}]$ will have to be normal to S; of course, this is consistent with the notion that compatibility is based in the assumption that the jump discontinuities on Φ are smoothly distributed across S.

Although Hadamard does not mention the fact in his book, one can solve the previous formula for $[\Phi_{,i}]$ by projecting it into its normal and tangential components:

$$[\Phi_{,i}] = \left[n^j\Phi_{,j}\right]n^i + g^{ab}\frac{\partial x^i}{\partial\xi^a}\frac{\partial[\Phi]}{\partial\xi^b},$$

in which we have introduced the unit normal vector field $n^i = f_{,i} / \| f_{,j} \|$ to S and the metric tensor for it (i.e., the first fundamental form):

$$g_{ab} = \delta_{ij}\frac{\partial x^i}{\partial\xi^a}\frac{\partial x^j}{\partial\xi^b};$$

the matrix g^{ab} is then the inverse to this matrix.

With this formula, we see that when $[\Phi] = 0$ there must be a function λ on S such that:

$$\left[\frac{\partial\Phi}{\partial x^i}\right] = \lambda n_i,$$

namely:

$$\lambda = [n^i\,\Phi_{,i}].$$

Apparently, this result had been previously obtained by Maxwell in the context of electromagnetism, although Hadamard does not mention this fact.

This same logic can be applied to all higher-order derivatives analogously if one assumes that the derivatives up to the specified derivative are continuous across S. Hadamard defines the *order* of a discontinuity to be the smallest order of derivative that exhibits a discontinuity, and its *index* to be the order smallest *time* derivative that is discontinuous.

For instance, if the first derivative to have a jump discontinuity is in second order then one has the compatibility condition:

$$\left[\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right] = \lambda n_i\, n_j, \qquad \lambda = [n^i\, n^j\, \Phi_{,i,j}].$$

In § 3 of this chapter, Hadamard introduces the notion of *kinematical compatibility conditions,* in addition to the purely geometrical ones of the previous section. They are based in the fact that, so far, there is nothing in the previous compatibility conditions for a discontinuity that would prevent the two regions from drifting apart in time or intermingling with each other. Hence, one would probably wish to add this further constraint, although, one would probably still wish to allow for sliding contact at S.

This restriction would imply that the normal components of the kinematical variables must be continuous across S. Hence, discontinuities can appear only in the tangential components.

One can then distinguish two types of discontinuity surfaces: *stationary discontinuities,* for which the function $f = f(x^i)$ that defines S is independent of t, and *propagating discontinuities,* for which $f = f(t, x^i)$ depends on t. More commonly, one refers to the propagating discontinuities as *waves*.

One notes that the stated conditions on the normal derivatives work only for the case of stationary discontinuities. For a wave, the first order of derivatives in which a discontinuity can appear must have index 0; that is, the time derivatives must be continuous. An example of a stationary discontinuity of order one is given by *vortex sheets*, which are surfaces across which the flow velocity vector field of a moving fluid has a jump in its transverse components.

Previously, for the case of waves, Hugoniot had called the motions in both regions M_1 and M_2 *(kinematically) compatible* iff S remains unique for all t. By this, he meant that if the intersection of S with the constant t hyperplane consists of a single connected component for one value of t then this will be true for all others; i.e., it will not undergo any "topology-changing processes." As pointed out in Truesdell and Toupin [1960], kinematically incompatible motions do not have to result in a splitting of the one surface into more than one surface, but might also involve the disappearance of the surface. Hence, the latter authors regard kinematical compatibility as involving the persistence of the surface, and therefore the discontinuity itself, in time.

One finds that propagating discontinuities can never give rise to a discontinuity of order zero. Hence, the discontinuous kinematical derivatives for a wave must be time derivatives. When the discontinuity is in the normal component of the first time derivative, one refers to the wave as a *shock wave* and when it is in the second time derivative, one calls it an *acceleration wave*. A common way of producing acceleration waves is by means of forcing functions that take the form of impulse or step functions in time. Kinematical compatibility also implies that one cannot have one derivative of index 0 being discontinuous without all other derivatives of the same order being discontinuous, as well.

When S is defined implicitly by a level hypersurface for a function such as $f(t, x^i)$, one can define its *slowness covector* to have the components:

$$s_0 = 1, \qquad s_i = -\frac{f_{,i}}{f_{,t}},$$

which makes $f_{,t}\, s_\mu = f_{,\mu}$, $\mu = 0, \ldots, n$ so the slowness covector is collinear with the normal covector.

The *speed of propagation* of S is one over the norm of the spatial part s_i of its slowness covector:

$$v = \left| \frac{f_{,t}}{\sqrt{f_{,1}^2 + \cdots + f_{,n}^2}} \right|.$$

Hadamard points out that using this definition of v means that one will generally obtain different values for v depending upon whether one uses spatial coordinates for S that pertain to the initial or the present state of S. The difference between them is then due to the relative velocity of displacement of the two. Of course, this "addition of velocities" is valid only in the context of non-relativistic motion, so, for instance, S cannot represent a moving electromagnetic wave surface.

One defines the *velocity* vector field v^μ, $\mu = 0, 1, \dots, n$ of S to be:

$$v^0 = 1, \qquad v^i = v\, n^i = \| f_{,i} \|^{-2} f_{,t} f_{,i}$$

with the unit normal vector field n^i as above. This makes:

$$s_\mu v^\mu = 1 + s_i v^i = 1 - \frac{f_{,i}}{f_{,t}} \frac{f_{,t} f_{,i}}{\| f_{,i} \|^2} = 0.$$

and the velocity vector field of S is tangential to S, at least when one looks at S as a hypersurface in spacetime. However, when one regards the spatial parts s_i and v^i at a particular instant of t, one sees that $s_i v^i = -1$, so the velocity can have parts that are both normal and tangential to S.

Since the time component v^0 is always equal to unity, it cannot suffer any discontinuities, and one finds that when there is a jump discontinuity $[v^i]$ in the spatial part of the velocity, the last formula implies, by subtraction, that:

$$f_{,i} [v^i] = 0.$$

That is, the jump in the velocity must be tangential to S. One similarly deduces that any jump in acceleration must also be tangential.

In addition to its implicit definition by means of f, the moving singular surface S (or at least a portion of it) can also be defined explicitly by embedding some subset $\mathcal{O}$ in $\mathbb{R} \times \mathbb{R}^{n-1}$, which we parameterize by the coordinates (t, a^I), $I = 1, \dots, n-1$; as pointed out above, the coordinates a^i might possibly represent the coordinates of the initial state of S at $t = 0$. Hence, the points of S will have coordinates of the form $(t, x^i(t, a^I))$. The fact that we are assuming an embedding implies that the differential (i.e., Jacobian) matrix $\partial x^i / \partial a^I$ has maximal rank $n - 1$.

One can define the *displacement* (or convected, material, etc.) derivative of any function Φ on S with respect to t to be its total derivative:

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial x^i}{\partial t} \frac{\partial \Phi}{\partial x^i} = \Phi_{,t} + u^i\, \Phi_{,i}\,,$$

which is also the directional derivative of Φ in the direction $(1, u^i)$. This time, we have defined the velocity vector field u^μ of S by:

$$u^0 = 1, \qquad u^i = \frac{\partial x^i}{\partial t}, \;= 1, \ldots, n.$$

Suppose that S is defined by both a function f and an embedding $(t, x^i(t, a^I))$, so we have $f(t, x^i(t, a^I)) = 0$. Taking the displacement derivative of f, which must necessarily vanish, gives:

$$0 = f_{,t} + u^i f_{,i} = f_{,\mu} u^\mu.$$

Since we also have that $f_{,\mu} v^\mu$. vanishes, this gives us that $f_{,i} u^i = f_{,i} v^i$, but we cannot actually conclude that $u^i = v^i$, since they could differ by a vector tangent to the spatial part of S. Hence, we only have that the normal components of u^i and v^i agree; i.e., $u^n = v$.

Now, if Φ is singular across S, an application of the Hadamard lemma gives:

$$\frac{d}{dt}[\Phi] = [\Phi_{,t}] + u^i [\Phi_{,i}],$$

so if Φ is continuous across S then this gives:

$$[\Phi_{,t}] = - u^j [\Phi_{,j}] = - u_n [n^j \Phi_{,j}],$$

which makes the discontinuity in the time derivative of Φ equal to the projection of the discontinuity in the gradient in the direction of u^i.

By iteration, one obtains:

$$[\Phi_{,t,t}] = - u^j [\Phi_{,j}] = - u_n [n^j \Phi_{,t,j}],$$

etc.

In particular, when $\Phi = x^i$ for a choice of i, one has:

$$\left[\frac{\partial x^i}{\partial t}\right] = - u_n n^i, \qquad \ldots, \qquad \left[\frac{\partial^k x^i}{\partial t^k}\right] = (- u_n)^k n^i.$$

This shows what Hadamard concludes in this section: A kinematical discontinuity is completely determined by the knowledge of only the number u_n and the unit vector n^i.

One can also apply Hadamard's lemma to the components of the deformation gradient $u^i_{,j}$. In particular, one can derive compatibility conditions for the jumps in dilatation, shear, and rotation that follow from the discontinuities in u^i across S.

One starts with the fact that if the displacement vector field has a first-order discontinuity across S then for each i the jump $[\partial u^i / \partial x^j]$ will be a normal vector to S. In fact:

$$\left[\frac{\partial u^i}{\partial x^j}\right] = \bar{n}^i n_j,$$

where the unit vector $\bar{n}^i$ is defined by the jump in v^i:

$$\bar{n}^i = -1/u_n\,[v^i].$$

Hence, taking the trace of the above equation gives us that the jump in the infinitesimal dilatation ε is:

$$[\varepsilon] = \left[\frac{\partial u^i}{\partial x^i}\right] = \bar{n}^i n_i = \cos(\angle\bar{\mathbf{n}}, \mathbf{n});$$

that is, it is the cosine of the angle between the two unit vectors. This means that a transverse (i.e., tangential) jump in **v** will not affect the dilatation.

This makes the finite dilatation take the form:

$$\lambda = 1 + \varepsilon = 1 - \frac{[v_n]}{v}.$$

Although one can express this as a ratio of densities, since we are still concerned with only kinematics, it would be slightly premature to introduce dynamical considerations, such as mass, at this point.

The jump in the infinitesimal shear takes the form:

$$2[\gamma_{ij}] = \bar{n}_i n_j + \bar{n}_j n_i,$$

and the jump in the infinitesimal rotation takes the form:

$$2[\theta_{ij}] = \bar{n}_i n_j - \bar{n}_j n_i = \varepsilon_{ijk}(\bar{\mathbf{n}}\times\mathbf{n})_k.$$

Thus, longitudinal (i.e., normal) jumps in **v** will not affect the infinitesimal rotation.

One derives analogous results for the jump in the velocity gradient relative to the jump in the acceleration vector.

In the general case when the motions in two regions might not be compatible, Hadamard says that when the tendency is for the regions to interpenetrate one calls the discontinuity *positive* or *compressive*, and when they tend to separate, he calls them *negative* or *dilative.*

This clearly relates to the jump in the normal component of the velocities or accelerations of the two regions, depending upon the order of the discontinuity. Furthermore, one essentially looks at the sign of the scalar product of the jump vector field with the normal vector field to S. If the sign is negative then the normal component of the jump is oppositely directed to the normal vector field and the discontinuity is compressive; conversely, if the sign of the scalar product is positive then the discontinuity is dilative. (One notes that the signs of the scalar product are then opposite to the sign of the discontinuity in both cases.)

This definition of the sign of a discontinuity is meaningless in the case of stationary discontinuities, which are neither compressive nor dilative.

When a first-order discontinuity satisfies the kinematical compatibility conditions, one can relate the sign of the discontinuity to the jump [ε] in the infinitesimal dilation, as one might suspect. It is negative for a compressive discontinuity and positive for a dilative one, so again the sign convention of the definition is opposite to the natural one. One finds that the tendency of a compressive (dilative, resp.) discontinuity is to make its motion evolve into the region of lower (higher, resp.) density.

One can obtain analogous conditions for higher-order discontinuities.

Hadamard concludes this section with some remarks on the splitting of one singular surface into two singular surfaces that move in opposite directions in the case of incompatible motion.

In § 4, he points out that, so far, the compatibility conditions involved relations between discontinuities of order n and derivatives of other things that were also of order n. The question arises whether one might find relations between discontinuities of order n and derivatives of things of order higher than n. In particular, one might consider higher derivatives of the function f that defines the hypersurface S.

One finds that nowadays it would be more illuminating to consider how the differential geometry of the surface S affected the compatibility relations for its motion. For instance, in addition to the normal vector field that is defined by the gradient (or differential) of f and the first fundamental form that one obtains from the restriction of the background metric on the ambient space, one could also consider the second fundamental form that derives from the "covariant" derivative of the normal vector field, using the Levi-Civita connection that is obtained from the first fundamental form, and the curvature of that connection, which pertains to the second covariant derivative operator.

We will not go further in these directions here, since they rapidly leave the scope of Hadamard's treatise, but one can confer more modern books on continuum mechanics that also treat the problem of compatibility, such Truesdell and Toupin [1960], Thomas [1961], and Eringen [1962].

§ 5. Notes on Chapter III. The basic content of the third chapter of Hadamard's book is a summary of elementary hydrodynamics, as he intends to use it in the subsequent chapters. As he does not introduce any personal innovations of his own in this chapter, one understands that it is included largely for the sake of completeness in the presentation. He also defines a problem that eventually evolves into the more general mathematical theory of characteristics, namely, the problem of whether one can derive the values of the initial acceleration on a Cauchy surface when one is given the Cauchy data – viz., the initial position and normal velocity of the surface.

In the first section of this chapter, Hadamard discusses the issues that are associated with the equations of motion for moving fluids and the equation of state that must be added to make the system of equations well-determined.

Equations of motion in mechanics can be derived by starting with various fundamental hypotheses. The two most common ones are balance (or conservation) laws and variational principles.

The concept of a balance law assumes the general case of an open system and then expresses the time rate of change of the total value of some system parameter in terms of a sum of incoming and outgoing flow rates, suitably signed. By contrast, the concept of a conservation law assumes the more idealized case of a closed system in which the incoming and outgoing rates vanish, so the equation of conservation takes the form of the vanishing of a time derivative or, equivalently, some algebraic condition on the quantity that is conserved. One sees that there is a subtle difference between the steady state of an open system, which means that the incoming sum equals the outgoing sum, and the equilibrium state of a closed system, which says that all of the incoming and outgoing rates vanish.

In the case of the mechanical systems of interest to the present book the state variables of the system consist of the total mass and total linear momentum. The balance laws then consist of conservation of total mass M_{tot} (assuming no mass is coming in or going out):

$$\frac{dM_{tot}}{dt} = 0,$$

and Newton's second law of motion:

$$\frac{d\mathbf{P}_{tot}}{dt} = \sum \mathbf{F}_{comp.} - \sum \mathbf{F}_{tens.} \,.$$

That is, the time derivative of the total momentum $\mathbf{P}_{tot}$ is the sum of the forces of compression minus the sum of the forces of tension. Of course, for a point mass, the distinction between a compression and tension is meaningful only for a given force vector, since it amounts to a change to the opposite direction.

When the system in question is a fluid that is confined to some region of space, the total quantities must resolve into integrals of densities in each case: The total mass resolves to a mass density $\rho(t, x^i)$, and the total linear momentum resolves to a linear momentum density $\rho v^i(t, x^j)$, where v^i represent the components of the flow velocity vector field. The forces, however, take two forms: bulk forces $f^i(t, x^j)$, which act on the mass elements of the fluid independently of the fluid properties, and pressure $p(t, x^j)$, which acts on the mass element as a consequence of the neighboring fluid elements. The most common examples of bulk forces are gravity and electric or magnetic forces that might act on charged fluids, such as electrolytes and plasmas.

As mentioned previously, one has two ways of defining the time derivative: the Lagrangian description and the Eulerian one. In the Lagrangian picture, the motion takes the form $x^i = x^i(t, a^j)$, where the components a^j coordinatize the initial state, and the time derivatives are simply the partial derivatives with respect to t. Hence, the equations of motions take the form:

$$\text{(EOM}_\text{L}) \qquad \rho(t, a^i) = \left|\frac{\partial x^i}{\partial a^j}\right|_{(t,a^k)} \rho_0(a^k)\,, \qquad \rho\frac{\partial v_i}{\partial t} = f_i - p_{,i}\,.$$

The Eulerian picture, which is more convenient for the purposes of modeling fluid motion, follows the present state of a fluid cell, so each point of the fluid lies on a path line $x^i(t)$ and the appropriate time derivatives are the convected (material, Lie, total, etc) derivatives:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}\,.$$

The equations of motion are then:

$$\text{(EOM}_\text{E}\text{)} \qquad \frac{d\rho}{dt} = 0, \qquad \rho\frac{dv_i}{dt} = f_i - p_{,i}\,.$$

So far, either of these systems represents four equations (EOM_L) or (EOM_E) for five unknowns, namely, x^i, ρ, p. Hence, one must append another equation to make the system well-determined, and this supplemental equation usually takes the form of an *equation of state* for the fluid itself, which might take the forms:

$$\rho = \rho(p) \qquad \text{or} \qquad F(\rho, p) = 0.$$

In the first case, one now refers to such a fluid as *barotropic*.

The determination of an equation of state for a medium is basically a thermodynamic problem, since one usually settles on a form for this equation by first making thermodynamic assumptions about the gas in question. More generally, the thermodynamic state of the gas at a point can be described by the (intensive) variables of pressure p, temperature T, specific volume v or density $\rho = 1/v$, specific internal energy ε, and specific entropy s, which are all function of (t, a) in the pipe. In general, an equation of state amounts to a choice of hypersurface in the five-dimensional region of the space of all $(p, T, \rho, \varepsilon, s)$ that is defined by the bounds that one places on each physical variable. Hence, the equation of state at issue is one that assumes that the temperature, specific internal energy, and specific entropy are either constant throughout the motion or do not affect the relationship between pressure and density.

The simplest example of an equation of state is given by an *incompressible fluid*, for which:

$$\rho = \rho_0 \qquad (\text{ = constant}).$$

The equation of state for an ideal gas is given by Mariotte's law $p = K\rho = K'/\omega$, which is a special case of the more general Poisson law:

$$p(\rho) = K\rho^m = K'\omega^{-m},$$

which is also referred to as the *Poisson adiabatic*. Such an equation of state comes about when one considers a gas with vanishing heat conductivity, but which contracts or expands adiabatically (i.e., with no change in heat content) under isothermal conditions The exponent m is called the *adiabatic index* (see Landau and Lifschitz [1987]) and is equal to the ratio c_p / c_v of the specific heat of the gas at constant pressure to the specific heat of the gas at constant volume. For a monoatomic gas, it equals 5/3 and for diatomic gases, it equals 7/5, but in any case, it is greater than 1. Nowadays, one also refers to gases that obey this law as *polytropic*.

In general, the implicit form of an equation of state will then be:

$$F(\rho, p, T) = 0.$$

The assumption that F is not also a function of the velocity, or any higher kinematical variables is equivalent to the existence of thermodynamic potential ϕ for the force $f_i - p_{,i}$, which is the difference between an external force potential and the pressure:

$$\phi(x^i) = \int_V \rho \, \Phi(t, \rho) \, dV ,$$

for some appropriate function Φ.

From the variational principle that the variation of this thermodynamic potential must vanish for every "acceptable" variation of the state of the fluid cell, one derives the equation of state:

$$p = \rho^2 \frac{\partial \Phi}{\partial \rho} .$$

From the preceding remarks, the question arises whether the relation $p = A\rho^m$ must necessarily follow from an equation of state of the form $F(\rho, p, T) = 0$ under the assumption of adiabatic compression or relaxation. Hadamard gives a demonstration that thermodynamic consideration involving the specific heat of the fluid show that the existence of a thermodynamic potential does indeed establish that outcome.

However, his argument excluded the possibility that the velocity might have jump discontinuities inside the gas, which was necessary for the definition of the variation of kinetic energy.

If the fluid is neither incompressible nor an ideal gas then one wonders if one can still use the equation of state $p = A\rho^m$ under Duhem's assumption of the existence of a thermodynamic potential. This question can be resolved in the affirmative when the velocity is continuous, and under the general assumption in the isothermal case that the pressure is an increasing function of density; this amounts to the statement that the equilibrium state of the fluid is stable.

In the second section of this chapter Hadamard addresses the issue of initial and boundary conditions for the system of partial differential equations that dictate the time evolution of the fluid state. Indeed, one sees that since fluids are usually confined to pipes and channels (but not, for instance, in the case of interstellar gases), one must define boundary conditions for the state variables even in the case of the Cauchy – or *initial value* – problem, in which one defines the initial (i.e., $t = t_0$) values of the state variables and their time derivatives.

The boundary surfaces can be both fixed and moving, so one must assume that the equation of the boundaries and their motions are givens. Further, one assumes that the pressure is given on these surfaces.

The problem arises: Can one derive the initial values of the acceleration from the Cauchy data and the equations of motion or can that information be specified arbitrarily? Indeed, this is a question of fundamental importance in the eyes of the theory of characteristics, which the book eventually converges to. In particular, one assumes that one is given:

1. The forces that act on the fluid.
2. The points of the initial surface and their velocities.
3. The motion of the walls and the initial pressures on the free surfaces, along with their time derivatives.

One first finds that there is a fundamental difference between the special cases of a liquid – i.e., an incompressible fluid – and a gas, or compressible one. In the incompressible case, the equations of motion simplify to:

$$v^i_{,i} = 0, \qquad \rho\frac{dv^i}{dt} = -p_{,i} + f_i.$$

The first one now says that the flow generated by the velocity vector field must be volume-preserving, not just mass density-preserving.

Taking the divergence of both sides of the second equation, while inverting the order of time and space differentiation, gives:

$$\Delta p = f^i_{,i}\ .$$

Of course, the resulting equation is simply Poisson's equation for the pressure function. As Hadamard points out, if one is given the normal derivative dp/dn of p on the boundaries then this defines a Neumann problem. Hence, one can resort to the methods of chapter I to solve it. However, this assumes a necessary and sufficient condition of possibility, namely, that the second time derivative of any elementary fluid volume must be zero.

Once one has solved the Neumann problem for p, one can deduce the initial acceleration from the equations of motion.

Once again, the argument presented by Hadamard assumes that the initial accelerations are continuous over the initial surface. Hence, one wonders if the results remain valid for the discontinuous case of acceleration waves. One finds that as long as one assumes that the singular surface satisfies the compatibility conditions, nothing changes, although this still assumes the continuity of the initial velocity and its initial gradient. Hadamard promises to return to the question of whether this restriction is necessary in chapter V.

When there is a free surface, along with fixed surfaces, the problem of solving for p becomes a mixed problem; one specifies p on the free surface and dp/dn on the wall. One also assumes that the pressure is positive. Hence, one is ruling out the possibility of "cavitation," which involves the formation of bubbles (i.e., topological "point defects") in the fluid in regions of negative pressure.

When one is concerned with a gas, matters are somewhat more involved. If one uses the equation of state for density as a function of pressure then, since the initial density is given, the equations of motion give the initial accelerations.

Now, say the initial surface is given by an equation of the form $f(t, x^i) = 0$. Two time differentiations give an equation of the form:

$$0 = f_{,i}\ddot{x}^i + \left(\frac{\partial}{\partial t} + v^i_{,i}\right)^2 f\ .$$

However, this represents a constraint on the initial acceleration that contradicts the freedom to deduce it independently of the choice of surface.

In the chapters that follow Hadamard proposes to attempt to resolve this contradiction.

§ 6. Notes on Chapter IV. In this chapter, Hadamard first discusses the Riemann method for modeling the propagation of discontinuity waves in one-dimensional gas dynamics and then discuss the form that Hugoniot gave it by making more definitive thermodynamic assumptions about the gas in question.

The general picture that recurs throughout the chapter is that of a gas in a cylindrical pipe that is closed at each end by means of pistons that can be given pre-assigned motions that serve as the sources of disturbances in the gas or simply left stationary, so they might serve as potential sources of reflection for the waves that propagate. The longitudinal dimension of the pipe is described by the variable a, which ranges from 0 to the length l of the pipe.

In order to make the problem essentially one-dimensional, one assumes that the dynamical variables of density ρ and pressure p are constant across any perpendicular cross-section of the pipe, although they may still vary with a, as well as time t. The wave function at issue is defined by the position $x(t, a)$ of a molecule of the gas that started out at the position a when t was 0; one then sees that $x(0, a) = a$. Equivalently, one could consider the displacement $u(t, a) = x(t, a) - a$ to be the wave function, as long the first partial derivative $u_a = x_a - 1$ did not enter explicitly. One can also introduce the dilatation $\omega(t, a) = \rho / \rho_0 = \partial x / \partial a$, which allows one to equate a dynamical variable with a kinematical one.

The one-dimensional equation of motion for the gas, in the absence of external forces, takes the general form:

$$\frac{\partial^2 x}{\partial t^2} + \frac{1}{\rho_0}\frac{\partial p}{\partial a} = 0.$$

In order to proceed, one must be more specific about the nature of the gas. In particular, one must specify an equation of state $p = p(\rho)$. One can also express this as a differentiable function $p = p(\omega)$, which one restricts to be monotone, so $dp/d\omega > 0$.

By the chain rule:

$$\frac{\partial p}{\partial a} = \frac{\partial \varphi}{\partial \omega} + \frac{\partial \omega}{\partial a}\frac{\partial \varphi}{\partial \omega} = \frac{\partial \varphi}{\partial a} + \varphi'(a, \omega)\frac{\partial^2 x}{\partial a^2},$$

in which we are allowing K' to vary along a, and the equation of motion becomes:

$$\frac{\partial^2 x}{\partial t^2} - c^2(a, \omega)\frac{\partial^2 x}{\partial a^2} = \frac{\partial \varphi}{\partial a},$$

in which we are deviating slightly from Hadamard's notation by denoting the speed of propagation with a c, instead of a ψ:

$$c^2(a, \omega) = -\varphi'(a, \omega)/\rho_0 .$$

Since ω is itself a partial derivative of x, one sees that the equation of motion is generally a nonlinear one-dimensional wave equation. In the years since Hadamard wrote this book, much progress has been made in elaborating on the general theory of nonlinear one-dimensional wave equations, as well as the nature of various specific cases that grew out of fundamental problems in continuum mechanics. Clearly, when K' is assumed to be constant along a, everything comes down to the nature of c as a function of ω, which then reverts to the thermodynamical nature of the gas.

In § 1, Hadamard treats the simplest case, where c is constant, along with K'. (His notation for c is θ, in this case.) The resulting wave equation is linear and can be solved in the general form given by d'Alembert:

$$x(t, a) = \tfrac{1}{2}[f_1(a + ct) + f_2(a - ct)],$$

in which the functions f_1 and f_2 can be determined by specifying the Cauchy data $x_0(a) = x(0, a)$ and $[\partial x/\partial t]_0(a) = [\partial x/\partial t](0, a)$. They represent travelling waves whose shape is defined by the initial functions $f_1(a)$ and $f_2(a)$ that travel with constant speed in the $-\, a$ direction for f_1 and the $+\, a$ direction for f_2.

Something that Hadamard returns to throughout this chapter that is not commonly discussed in the modern treatments of gas dynamics is his geometric picture of any solution $x(t, a)$ to the wave equation as defining a surface in $\mathbb{R}^3$ that one obtains by considering all points of the form $(t, a, x(t, a))$ such that t and a are constrained by the physical considerations; e.g., $t \geq 0$, $a \in [0, l]$.

In addition to initial values, we might also realistically wish to specify the *boundary* conditions on x, in the form of specifying the time functions $x(t, 0)$ and $x(t, a)$. For instance, one of the pistons – say, the one at $a = l$ – might be held stationary (so $x(t, a) = 0$) and one at $a = 0$ might be given a specified motion as a means of originating the disturbance that propagates away from it. Hence, we are really dealing with something slightly different from the Cauchy problem, namely, the mixed initial-boundary value problem.

The question that eventually leads one to consider the theory of characteristics of wave equations more generally is the problem of whether the initial value $[\partial^2 x/\partial t^2]_0(a)$ of the acceleration can be determined uniquely when one is given the equation of motion and a set of Cauchy data. The well-known answer is that this is possible iff the Cauchy data is not specified on a characteristic curve in the tx-plane. For the elementary case at hand, these are the pairs of lines that are defined by the ordinary differential equations:

$$\frac{dx}{dt} = \pm\, c.$$

That is, when c is constant the characteristics are the lines $x(t) = a \pm ct$ that figured in the d'Alembert solution; hence, the functions f_1 and f_2 must be constant on the characteristic lines. Geometrically, this means that the surfaces defined by these solutions will be cylinders that are generated by the lines $(t, l - ct, 0)$ and $(t, ct, 0)$, respectively.

One is cautioned that in spatial dimensions higher than one the characteristic equation that defines the characteristic hypersurface will be a first-order *partial* differential equation, not a first order ordinary one. This is further confused by the fact that the bicharacteristic curves that represent the rays of geometrical optics – or geometrical acoustics, in the present case – must lie in the characteristic hypersurface. Hence, in the one-dimensional case, the characteristic hypersurface and the bicharacteristic curves are essentially the same thing; we shall return to this in our discussion of the final chapter.

A subtle and powerful link between characteristics and the propagation of discontinuities is given by the fact that discontinuities in the initial acceleration can only exist across characteristics. Consequently, discontinuities can propagate only along characteristics. In terms of the motion of the gas, this means that acceleration waves must propagate with the characteristic speed c.

If one uses the d'Alembert solution for x, one can express the jumps in the partial derivatives of the dilatation $\partial x/\partial a$ in terms of f_1 and f_2:

$$\left[\frac{\partial^2 x}{\partial a^2}\right] = [f_1''] + [f_2''], \qquad \left[\frac{\partial^2 x}{\partial t \partial a}\right] = c([f_1''] - [f_2'']),$$

and solving for $[f_1'']$ gives:

$$[f_1''] = \left[\frac{\partial^2 x}{\partial a^2}\right] + \frac{1}{c}\left[\frac{\partial^2 x}{\partial t \partial a}\right].$$

If the jump discontinuity is in $[f_2'']$ then this must vanish, so:

$$\left[\frac{\partial^2 x}{\partial t \partial a}\right] = -c\left[\frac{\partial^2 x}{\partial a^2}\right].$$

In § 2, Hadamard comes back to the more general case of constant K', non-constant c. From the equation of motion, one gets the compatibility condition for an acceleration wave:

$$\left[\frac{\partial^2 x}{\partial t^2}\right] = -c^2\left[\frac{\partial^2 x}{\partial a^2}\right].$$

The characteristics are still defined by the same ordinary differential equation, but c is not assumed to be a constant function of (t, a), so the characteristic curves do not have to be straight lines, anymore.

Furthermore, the surface in $\mathbb{R}^3$ that a solution defines does not have to be a cylinder, but will be, more generally, a *developable surface* [80]. Such a surface is the envelope of some one-parameter family of planes, such as a cylinder or a cone. Except for these two examples, every other developable surface can be represented as the *tangent surface* to

[80] Although numerous modern treatments of the geometry of surfaces exist, a reference on developable surfaces that is perhaps closer in spirit to the discussion of Hadamard is the older one by Eisenhart [1940]. In particular, the term "edge of regression" seems to be an older terminology.

some differentiable curve. That is, it is the surface that is swept out by the tangent line as it moves along the curve. Generally, such surfaces will have two sheets, one of which is swept out by the forward ray and the other of which is swept out by the backward ray. They intersect along the curve itself, which was once referred to as the *edge of regression* of the surface, since a transverse planar section of the surface would appear to be a cusp.

If a second-order discontinuity does not satisfy the compatibility conditions above then it will tend to split into a pair of discontinuities that separate from each other with speeds of $\pm c$. One must note that acceleration waves are not possible in fluid media, only stationary second-order discontinuities.

The Riemann problem for one-dimensional gas dynamics is then defined by starting with the present physical scenario and assuming that the gas is kept at constant temperature, with a pressure that varies with length only by way of the density, and that as one crosses a second-order discontinuity the entropy of the gas remains constant. One then poses the Cauchy problem for the nonlinear wave equation that we have been considering.

The method that Riemann used to solve the problem thus posed involved the introduction of new coordinates for the wave function, as well as transforming the wave function itself, in such a way that the resulting form of the wave equation is linear and the Cauchy problem can be solved by the method of Green functions.

In order to get a better sense for the nature of this transformation, one must return to the geometry of the solution to the generalized wave equation as a surface in the space of variables (t, a, x) and extend this to the space of variables (t, a, x, u, ω). For a solution, one not only has $x = x(t, a)$, but one also has $u(t, a) = x_t$, $\omega(t, a) = x_a$, in which the subscripts refer to partial differentiation. This five-dimensional space projects onto the two-dimensional space of all (t, a) in the usual way: $\mathbb{R}^5 \to \mathbb{R}^2$, $(t, a, x, u, \omega) \mapsto (t, a)$, so a solution represents a "section" of this projection, which we may write in the form $(t, a, x(t, a), u(t, a), \omega(t, a))$, namely, a section of the form $(t, a, x(t, a), x_t(t, a), x_a(t, a))$.

It happens that nowadays the general theory of spaces of the form of this five-dimensional space in question has developed considerably since the time of Hadamard, based on notions that were first suggested by Charles Ehresmann in the 1930's, and which go by the name of the geometry of *jet manifolds* (see Saunders [1989]). In the case at hand the relevant definition is that of a *1-jet* of a differentiable function, such as f, on $\mathbb{R}^2$ at a point of $\mathbb{R}^2$. By definition, the 1-jet $j_x^1 f$ of f at $x \in \mathbb{R}^2$ is the set of all differentiable functions that are defined in some neighborhood of x and have the same values as f at x, along with the same values of their partial derivatives. A section of the form $j^1 f(t, a) = (t, a, f(t, a), f_t(t, a), f_a(t, a))$ is referred to as the *1-jet prolongation* of the differentiable function f on $\mathbb{R}^2$ to a section of the aforementioned projection. However, not all sections of that projections are representable as 1-jet prolongations, only the *integrable* ones, by definition.

The way that all of this relates to partial differential equations is that one can represent a first order partial differential equation for the function x as a hypersurface in $\mathbb{R}^5$ (i.e., the space of 1-jets of differentiable functions on $\mathbb{R}^2$) for some function F [81]:

$$F(t, a, x, u, \omega) = 0.$$

A *solution* to this partial differential equation is then a differentiable function $x(t, a)$ on $\mathbb{R}^2$ whose 1-jet prolongation $j^1 f$ maps $\mathbb{R}^2$ into this hypersurface.

One can represent a second-order partial differential equation for the function x as a system of first-order partial differential equations on $\mathbb{R}^5$. For instance, the wave equation we are considering becomes the system of four equations:

$$\frac{\partial x}{\partial t} = u, \qquad \frac{\partial x}{\partial a} = \omega, \qquad \frac{\partial u}{\partial t} - c^2 \frac{\partial \omega}{\partial a} = 0, \qquad \frac{\partial \omega}{\partial t} - \frac{\partial u}{\partial a} = 0,$$

the last of which is redundant, as it can be obtained by differentiating the first two. This system can be put into the standard "conservation law" form that one often encounters nowadays (see, e.g., Jeffrey and Taniuti [1964]):

$$X_t - AX_a = 0,$$

by setting:

$$X = \begin{bmatrix} u \\ \omega \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & c^2 \\ 1 & 0 \end{bmatrix}.$$

One notes that the eigenvalues of A will be $\pm c$, while the corresponding eigenvectors will be along the lines generated by $u \mp c\omega$, respectively.

Now, let us return to the Riemann method of solution for the Cauchy problem. One introduces the *Riemann variables (i.e., the Riemann invariants):*

$$\xi = u + \chi, \qquad \eta = u - \chi,$$

in which:

$$\chi = \int \frac{d\omega}{c(\omega)} = \int \frac{d\rho}{\rho_0 c};$$

one also encounters the notation $J_\pm = u \pm \chi$.

Since one has:

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial a}\right] J_\pm = 0$$

[81] Some good references on the subject of how jet manifolds relate to differential equations, both ordinary and partial are Olver [1993] and Arnol'd [1988].

one sees that the Riemann variables are constant for a *simple wave* that propagates to the left in the case of the positive sign and to the right in the opposite case. Such a wave takes the form:

$$x = f(v) + (v \pm c)t$$

for a suitable function $f(v)$. Hence, they generalize the d'Alembert solutions to the nonlinear case.

In order to convert the wave equation into Riemann variables, one must also perform a *Legendre transformation* on the wave function by defining:

$$z(u, \omega) = ut + \omega a - x.$$

Hence, one can regard the pair of variable (u, ω) as the *canonically conjugate variables* to the pair (t, a), in the same way that conjugate momenta are associated with generalized coordinates in Hamilton mechanics. Once again, this is no coincidence, and relates to the fact that the aforementioned first-order partial differential equation on $\mathbb{R}^5$ has characteristic equations that reduce to the Hamilton equations for the Hamiltonian function F when it does not depend upon f explicitly. We shall clarify these remarks later when we discuss Chapter VII, as they bear upon the nature of bicharacteristic curves.

Geometrically, we are replacing the representation of a wave function as a surface in the space of all (t, a, x) with a subset of the space of all (u, ω, z). However, one must be careful since the transformation does not have to take a surface in the space of (t, a, x) to another surface in the space (u, ω, z). It can map some developable surfaces to curves by projecting all of the points along a generating line to a point of the resulting curve in (u, ω, z)-space. Physically, the points of the edge of regression for the developable surface in (t, a, x)-space represent the point in time when a later wave overtakes an earlier one; this is, of course, possible only when the speed of propagation is not constant.

However, the Legendre transformation can sometimes be used to remove singularities in the surface. This does not work for an edge of regression, though; it amounts to a non-removable singularity.

One sees that the resulting Cauchy problem in (u, ω, z)-space might very well be inequivalent to the original one in (t, a, x)-space. As Hadamard points out, Hugoniot only solved the problem in the case where the gas in question was initially at rest; i.e., its flow velocity u was initially zero.

After converting to Riemann variables and performing a Legendre transformation the nonlinear wave equation now takes the linear form:

$$\frac{\partial^2 z}{\partial\xi\partial\mu} - f(\xi - \eta)\left(\frac{\partial z}{\partial\xi} - \frac{\partial z}{\partial\eta}\right) = 0$$

with:

$$f[2\chi(\omega)] = \frac{1}{4}\frac{\chi''(\omega)}{\chi'^2(\omega)}.$$

One can then solve the Cauchy problem for the function z by the method of Green functions, in principle.

The cases that were treated by Hadamard involved gases that obeyed the Poisson adiabatic and its special case of Mariotte's law. In the former case, the Green function involved hypergeometric functions, while in the latter, it involved Bessel functions.

One finds that there is no symmetry in the nature of the solutions that one obtains by using an initial compressive pulse of the piston versus an initial decompressive pulse. In particular, an extremely fast compressive pulse can produce a shock wave, but an extremely fast decompression can lead to "cavitation," or the formation of a partial vacuum between the piston and the gas. By definition, a compressive pulse produces a *compression wave* while a decompressive pulse produces a *rarefaction wave*.

In the third section of this chapter Hadamard focuses on an important consequence of the work that was done by Riemann and later Hugoniot, which he calls "the Riemann-Hugoniot phenomenon." What it represents, as a natural phenomenon, is the possibility that when two consecutive acceleration waves in a gas whose speed of wave propagation is not constant are moving in such a manner that the later one overtakes the earlier one they can combine to produce a shock wave. That is, two moving second-order discontinuities can combine to produce a moving first-order discontinuity. Indeed, this is essentially how the expanding shock wave forms in an explosion. Hence, there is ample experimental evidence for the phenomenon.

The point of departure between the earlier work of Riemann and the later work of Hugoniot was in the thermodynamical assumptions about the gas. Riemann assumed that there is no change in the entropy of the gas from one side of the discontinuity to the other, while Hugoniot assumed that there would be an increase in entropy. Although both theories produced qualitatively correct results, the assumption of Hugoniot proved to be more consistent with experimental measurements.

In order to describe this scenario mathematically, Hadamard first considers the case of a gas that is initially at rest and has a constant value for c. The piston that initiates the wave in the pipe is given a differentiable motion $x_0(t)$. The resulting motion $x(t, a)$ is differentiable and single-valued, but if one considers the map $x_t : \mathbb{R} \to \mathbb{R}$, $a \mapsto x(t, a)$ for each value of t, then one finds that it is locally invertible iff the dilatation $\omega = \partial x/\partial a$ does not vanish anywhere; equivalently, the dilatation cannot change sign. The first time that ω vanishes will be at the point of contact with the piston.

When c is not constant, but varies with ω, the situation is more complicated. To begin with, one must choose an equation of state. The choice that Riemann made was based on the Poisson adiabatic. Consequently, ω vanishes only if c is infinite.

In this situation one can have two types of singularities: points where ω vanishes and points along the edge of regression of the developable surface that represents the solution $x(t, a)$ in the space of (t, a, x). At such points, as mentioned above, one is dealing with a later wave overtaking a previous one. If one is using the Poisson law then the fastest waves are the ones that are compressed the most.

Hence, assume that $\chi'(\omega)$ is decreasing (i.e., $\chi''(\omega) < 0$). This implies that the piston must have a positive acceleration, which says that one is dealing with compression waves. If the piston had a negative acceleration then one would be dealing with rarefaction waves, which cannot cross each other in the manner that is intended. One finds that when the successive compression waves intersect a shock wave is produced.

Riemann's compatibility conditions for the shock wave were then:

$$\text{Kinematical:} \quad [u] = -\,c[\omega],$$
$$\text{Dynamical:} \quad [p] = +\,c[u],$$

in which u is the velocity of the wave. These, in turn were based on:

$$[u]^2 = -\,1/\rho_0\,[p][\omega],$$

which followed from the Poisson adiabatic. This included the basic assumption that the entropy did not change across the shock discontinuity.

From the above, one sees that the speed of propagation can be obtained from:

$$c = \sqrt{\frac{-[p]}{\rho_0[\omega]}}\,.$$

Hugoniot's objection to the foregoing was based in the observation that compression and dilatation makes the Poisson adiabatic inappropriate to the problem. Instead of the Poisson law $p_1\omega_1^m = p_2\omega_2^m$ for polytropic gases, where the subscripts refer to the values on either side of the singular surface, he introduced the *Rankine-Hugoniot adiabatic:*

$$\varepsilon_2 - \varepsilon_1 = \tfrac{1}{2}(p_1 + p_2)(u_1 - u_2) = \frac{c}{m-1}(p_1\omega_1 - p_2\omega_2),$$

in which ε represents the specific internal energy of the gas. This adiabatic results in an increase of the entropy across the discontinuity and also leads to the *Rankine-Hugoniot* compatibility conditions for a shock wave:

$$[v]^2 = -\,[\rho][u], \qquad [p] = -\,[\rho][v]^2,$$

in which $v = 1/\rho$ is the specific volume of the gas.

One finds that the numerical agreement between the theory and the experiments is much better when one uses the Hugoniot adiabatic instead of the Poisson one.

If one considers the opposite phenomenon to the Riemann-Hugoniot phenomenon, namely, a single shock wave splitting into a pair of acceleration waves, one finds that this is only possible if the initial shock wave is dilative, since the Riemann-Hugoniot process produces a compressive one.

An elementary form of the problem that we have considering more generally is: Suppose that the gas is initially at rest and the piston is given a uniform rectilinear motion $x_0(t) = Vt$, with V constant. Find the resulting motion of the gas.

As Hadamard points out, Sébert and Hugoniot showed that the solution follows directly as long one satisfies compatibility conditions for the motion of the piston that take form:

Kinematical: $V + c(\omega - 1) = 0$,
Dynamical: $p - p_0 = \rho_0 cV$,

$$\frac{p}{p_0} = \frac{(m+1)-(m-1)\omega}{(m+1)\omega-(m-1)}.$$

If V is given, instead of p, then c is unknown, but it can be solved from the compatibility conditions. One needs to have $p > 0$, which is always true when V is positive, but if V is negative then one must have $c < p/(\rho_0 V)$, so $V^2 < 2p_0 / (m-1)\rho_0)$.

The remainder of this chapter is primarily concerned with the solution of the present problem by means of series expansions in fractional powers of t.

§ 7. Notes on Chapter V. Since this chapter represents the extension of the one-dimensional analysis of the previous chapter to the more realistic case of three-dimensional gases, the fact that it occupies less space in the book is due to the fact that, in effect, all one must address is the fact that when one extends from a singular point moving on a line to a singular surface moving in space, one must mostly deal with the contributions that the transverse dimensions make to the same results.

First, one returns to the equations of motion in three-dimensional form [82]:

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = X_i - \frac{\partial^2 x_i}{\partial t^2}, \qquad i = 1, 2, 3.$$

As was shown in the previous chapter, any conflict between the Cauchy data and these equations can be resolved by the existence of a second-order discontinuity that propagates along characteristic curves. However, the extension of characteristic curves to three-dimensional characteristic hypersurfaces is not discussed until the final chapter of the book.

As usual, the gas is assumed to be barotropic, so $p = p(\rho)$. The equations of motion then take the form:

$$\frac{dp}{d\rho}\frac{\partial \ln \rho}{\partial x_i} = X_i - \frac{\partial^2 x_i}{\partial t^2}.$$

Now assume that one has a singular hypersurface S, over which $\partial^2 x_i / \partial t^2$ and $\partial \rho / \partial x_i$ have finite jump discontinuities. Assuming that the X_i and $dp / d\rho$ are continuous across S, the equations of motion then give the compatibility conditions:

[82] One notices that a recurring drawback to the mathematical literature of the era in which this book was written was the fact that the symbolic representation of systems of differential equations had yet to benefit from the introduction of a few well-chosen dimensional indices, so one generally had to deal with notation that appeared rather redundant by modern standards.

$$\frac{dp}{d\rho}\left[\frac{\partial \ln(1/\rho)}{\partial x_i}\right] = \left[\frac{\partial^2 x_i}{\partial t^2}\right].$$

Let λ_i , i = 1, 2, 3 represent the components of the discontinuity in the acceleration in the form:

$$\left[\frac{\partial^2 x_i}{\partial t^2}\right] = c^2 \lambda_i \,.$$

If n_i , i = 1, 2, 3 are the components of the unit normal to S then the compatibility conditions give:

$$\frac{dp}{d\rho}(\lambda_j\, n_j)\, n_i = c^2 \lambda_i \,.$$

Presumably, the λ_j do not vanish identically, so if the speed of propagation does not vanish either then one finds that the latter formula establishes the fact that a propagating discontinuity in a gas must be longitudinal with a speed that satisfies $c^2 = dp / d\rho$. On the other hand, if $c = 0$ then one must have that $\lambda_j\, n_j = 0$, which says that a stationary discontinuity must be transversal.

Hadamard then shows that the same thing is true for discontinuities of higher order, as well. That is, the propagating ones are longitudinal and have a speed given by the second order expression, while the stationary ones are transverse.

Furthermore, he points out that one can consider more general equations of state, such as ones where the pressure is inhomogeneous in space. The main alteration to the analysis is that one must replace the total derivative of pressure with respect to density with a partial derivative.

Now, since the expression for c is quadratic, it really allows for the propagation of waves in two directions, not just the one. However, the idea behind kinematic compatibility was that if one had such compatibility then the actual motion of S would have to have one sign or the other. Conversely, in the absence of compatibility it would be possible for S to split into two surfaces moving in opposite directions with speed c.

In order to examine this possibility, Hadamard first considers the more restrictive case of liquids, which cannot have normal discontinuities without that affecting the derivatives of the density. Now, if there were normal acceleration waves then a sufficient condition for the existence of normal second order discontinuities would be the existence of an acceleration potential Φ, which then obeys the defining condition that $\partial\Phi/\partial x_i = \partial^2 x_i / \partial t^2$. Although it not necessary to assume compatibility to obtain this result, nonetheless, when there is compatibility, it gives a consistent result. In any event, if an acceleration wave exists then it must be normal. Furthermore, it persists for higher order discontinuities and when there is compatibility one finds that any discontinuities that are tangential to S must be stationary. Therefore, since normal discontinuities cannot exist in liquids, one ultimately concludes that the only kind that one can consider is the stationary transverse kind.

Now consider a singular surface S in a liquid, across which one is given $[\partial^2 x_i / \partial a_j^2]$ and $[\partial^2 x_i / \partial t\, \partial a_j]$, but one wishes to derive the values of $[\partial^2 x_i / \partial t^2]$. One does not assume compatibility, but one does still assume that there is no cavitation in the liquid, so the regions of the liquid do not separate if S splits into two propagating surfaces. One must conclude that $[\partial^2 x_i / \partial t^2]$ vanishes – i.e., the acceleration is continuous across S – since any separation of S would have to involve normal discontinuities, but the only discontinuities that can exist are transverse, which is a contradiction. One can again extend this result to higher-order time derivatives; viz., they must be continuous.

Hadamard points out that this conclusion remains true even when the first time derivative – i.e., the velocity – is discontinuous, which is the case for shock waves and vortex sheets, which are the stationary manifestation of such discontinuities.

Recall that the vorticity tensor for a flow velocity is the anti-symmetric part of the velocity gradient $\omega_{ij} = 1/2(v_{i,j} - v_{j,i})$, which is the time derivative of the infinitesimal rotation $\theta_{ij} = 1/2(u_{i,j} - u_{j,i})$ of the displacement vector field u_i . In the present case, one has $u_{i,j} = \partial x_i / \partial a_j$. Hence, vorticity is related to the second-order partial derivatives of the form $\partial^2 x_i / \partial t\, \partial a_j$. Hadamard then defines a *vorticial discontinuity* to be a transversal discontinuity of the form $[\partial^2 x_i / \partial t\, \partial a_j]$, which makes it a second-order discontinuity of index one.

Hadamard gives an example of a uniformly rotating disc, which has a jump discontinuity $[\partial x_i / \partial t]$ in the velocity vector field at its rim. Inside the disc, there is a velocity potential of the form $\phi = k\theta$, where k is a constant and $\tan \theta = y/x$.

From a previous result in no. **93**, it follows that a second-order discontinuity of order one should produce first order discontinuities of order zero. That is, a vorticial discontinuity should produce discontinuities in the deformation gradient.

Next, Hadamard returns to the case of a gaseous medium in which one has a second-order discontinuity across a hypersurface S, but one does not assume compatibility. He then shows that normal waves must be produced. First, he shows this in the case where the derivatives of index zero are normal to S, and then in the general case, by drawing upon the previous result concerning liquids that transverse discontinuities must be stationary.

Considering that the book was written in 1903, Hadamard observes that up to that point in the history of hydrodynamics, the most important mathematical advances seemed to be related to the conservation of vorticity and circulation along fluid flows that had been investigated by Helmholtz and Lord Kelvin. For instance, one might confer the definitive treatise of Poincaré [1893] on the subject of vortex theory in that era.

By definition, if γ is a loop in a moving fluid with a flow velocity vector field [83] $v_i(x_j)$ then the *circulation* of v_i around γ is defined to be the loop integral:

$$\Gamma[\gamma] = \int_\gamma v_i dx^i .$$

[83] Actually, the most mathematically precise way of describing the v_i is to call them the components of the covelocity 1-form $v = v_i\, dx^i$, which is then a covector field, not a vector field. This is because the best way of defining integrals over curvilinear regions in space is in terms of exterior differential forms. However, the distinction between vector fields and covector fields only becomes unavoidable in relativistic continuum mechanics, where one raises and lowers indices with a less trivial metric tensor field than the Euclidian one that non-relativistic continuum mechanics assumes. Hence, as mentioned above, we shall not give into the temptation to drift too far from the subject of the treatise in question by discussing those modern aspects of the theory, except casually.

The Kelvin circulation theorem (see, e.g., Saffman [1992]) says that this circulation is conserved along the flow.

Several questions then arise concerning the effect of non-vanishing vorticity on the validity of the results above. One finds that the mere existence of non-vanishing vorticity does not affect their validity since propagating hydrodynamical discontinuities are normal, not transverse. Similarly, the conservation of circulation is not affected by the presence of a singular surface for a wave since the circulation integral involves only first-order derivatives. One can conclude from this that acceleration waves cannot produce vortices, which is an important consequence in the eyes of hydrodynamics.

Hadamard then considers the way that the earlier discussion of shock waves needs to be modified in order to account for the extra spatial dimensions. One finds that the one-dimensional compatibility conditions $[p] = -\rho_0\, c^2[\omega]$, where $\rho_0(a)$ is the initial density and $\omega = \partial x / \partial a$ now take the form:

$$[p]\, n_i = -\rho_1\, c^2\, \lambda_i\,,$$

in which ρ_1 is the density in the region that precedes the singular surface; i.e., the undisturbed region. The difference in form is due to the fact that in the one-dimensional case one was assuming that the initial state of the gas was its present state, while in the three-dimensional case this was no longer assumed.

Nevertheless, the Hugoniot adiabatic remains unchanged by the expansion of dimension. Consequently, contrary to the previous non-existence result for vortices in the context of acceleration waves, one finds that shock waves are indeed capable of producing vortices. This is another deep consequence of Hadamard's analysis in the eyes of hydrodynamics, and he defers its actual proof to an appendix.

§ 8. Notes on Chapter VI. The primary objective of this chapter is to apply the methods that were previously defined in general to waves that propagate in elastic media. Here, there is a considerable difference in the treatment of the subject depending upon whether one is concerned with infinitesimal deformations or finite ones. This essentially amounts to the statement that if a medium that supports the propagation of waves is regarded as a spatial distribution of oscillators then there is a considerable difference between the behavior of linear oscillators, which generally appear when one makes small-amplitude approximations, and nonlinear oscillators, whose behavior is already quite complicated.

Elastic media, which include some solid media as well as most compressible fluids, are considerably more involved in terms of their mechanical properties than the fluid media that were treated up to this point in the book. For one thing, there is generally a distinguished state of the medium that is defined by the equilibrium state of the body in the absence of applied loads. Hence, since the spirit of elastic deformation is related to the idea that the work done deforming an elastic body is completely reversible and path-independent, one can introduce the potential energy of the body and characterize the equilibrium state as a state of minimum energy.

Interestingly, as Volterra showed in 1907, the equilibrium state does generally not have to be a state of vanishing strain or stress, although that is true when the body is simply connected. For instance, if you bend a cylindrical rod into a torus and fuse the end faces together then when you remove all external loads the resulting body will be in equilibrium, even though it has it is in a state of non-vanishing strain. However, not all non-simply connected bodies must be in a state of non-vanishing strain, as the example of an initially unstrained elastic sheet with a hole punched out of it shows, if one ignores the strain that is introduced by the act of punching the hole..

It is customary to use the equilibrium state – or *natural* state – of the body as a reference configuration for the sake of defining coordinate systems, rather than the present state. Hence, one generally addresses the deformation of elastic bodies in terms of the Lagrangian formalism. One can also attribute this to the fact that the fundamental object in elasticity is the displacement vector field **u** that is defined by a deformation, rather than the velocity vector field **v** of a fluid flow. Another consequence of using the Lagrangian viewpoint is that one can use partial time derivatives instead of the convected ones.

In the approximation of infinitesimal deformations the distinction between initial and present state is moot and the Eulerian viewpoint agrees with the Lagrangian one. Furthermore, one usually derives linear systems of partial differential equations in that approximation, so the analysis is generally simpler. Hence, most of the empirical data that is catalogued for elastic materials in handbooks is oriented towards their properties under small deformations. The properties of elastic materials as one goes beyond the linear limit become increasingly complex and phenomenological in character. Generally, linear elasticity first turns into nonlinear elasticity until one reaches the yield point of the material, after which elastic deformation turns into plastic deformation, and eventually concludes with fracture. However, no general expression for this behavior seems even possible, and one sometimes approximates it with idealized models of elastic-plastic behavior, such as piecewise linear functions or polynomials.

When an infinitesimal strain e_{ij} on a body $B \subset \mathbb{R}^3$ produces an infinitesimal stress σ_{ij} , one calls the function on the body:

$$W(x_i, e_{ij}) = \sigma^{ij} e_{ij}$$

the *deformation energy density*. Its integral over B gives the total work done deforming the body.

A functional relationship $\sigma^{ij} = \sigma^{ij}(x^i, e_{ij})$ is called a *constitutive law* or *response function* for the material that B is composed of. If one has W to begin with then one can also obtain this relationship from the definition:

$$\sigma^{ij} = \frac{\partial W}{\partial e_{ij}}.$$

However, nowadays this possibility is not regarded as the most general response function and when this is the case, one calls the medium *hyperelastic* (see Truesdell [1961]). Hence, one must keep in mind that the media that Hadamard treated fell into this category.

Most commonly in practice, one considers linear constitutive laws of the form:

$$\sigma_{ij} = C_{ijkl}(x)\, e_{kl}\,.$$

When the functions $C_{ijkl}(x)$ are constants, one calls the material *homogeneous*, as well as linear. The deformation energy density then becomes a quadratic form that is defined by the part of the tensor $C = C_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ that is symmetric in the pair ij, symmetric in the pair kl, and symmetric under the exchange of these two index pairs. Such a deformation energy density W must necessarily be homogeneous of degree two in the infinitesimal strain. By Euler's theorem on homogeneous functions, one then must have:

$$W = \frac{1}{2}\frac{\partial W}{\partial e_{ij}} e_{ij} = \frac{1}{2}\,\sigma_{ij}\, e_{ij} = \frac{1}{2}\; C_{ijkl}\, e_{ij}\, e_{kl}\,.$$

A particular type of material that gets a lot of attention is that of an *isotropic* material, for which the components $C_{ijkl}(x)$ are invariant under the transformations that arise from rotations of the ambient space. In such a case, one has a constitutive law of the form:

$$\sigma_{ij} = \mu\, e_{ij} + \lambda\, e_{kk}\, \delta_{ij}\,,$$

in which the functions λ, μ, when they are constant, are called the *Lamé constants* of the material and can be related to tabulated data, such as the Young modulus and the shear modulus for the material.

A fluid medium is characterized by the fact that it is isotropic and does not support strains or shearing stress. Hence, in the absence of viscosity the stress tensor reduces to the pressure times the identity matrix:

$$\sigma_{ij} = p\,\delta_{ij}\,.$$

However, viscosity couples a shearing stress to the rate of deformation $\partial e_{ij} / \partial t$, which is also the infinitesimal strain in the flow velocity vector field.

In the isotropic case, the quadratic form defined by C_{ijkl} takes the form:

$$W[e_{ij}] = \mu\, e^{ij}\, e_{ij} + \lambda\,(e_{kk}\,)^2.$$

The equations of motion for a time-parameterized family of deformations, which is then described by a time-varying displacement vector field $u_i(t, x_i)$, are derived from the balance law for linear momentum – i.e., Newton's second law of motion – and take the form:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sigma^{ij}{}_{,j} + f_i\,,$$

in which f_i represents the external forces acting on the points of B, while the divergence of the stress tensor gives the force that acts on its boundary surface.

In the linear, isotropic, homogeneous case this takes the form:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \mu\Delta u_i + (\lambda + \mu)\frac{\partial^2 u_j}{\partial x^i \partial x^j} + f_i\,,$$

Customarily (see, e.g., Landau and Lifschitz [1959]), one decouples this equation in the unforced case into a pair of linear wave equations that describe longitudinal and transverse waves:

$$\frac{\partial^2 u_i}{\partial t^2} - c_l^2 \Delta u_i = 0, \qquad \frac{\partial^2 u_i}{\partial t^2} - c_t^2 \Delta u_i = 0,$$

in which the speeds of propagation are defined by:

$$c_l = \sqrt{\frac{2\mu + \lambda}{\rho}}\,, \qquad c_t = \sqrt{\frac{\mu}{\rho}}\,.$$

The longitudinal wave is coupled to the infinitesimal dilatation by way of λ, so it is a compression wave. However, the transverse wave is coupled to only the shear part of the infinitesimal strain, so it is a shear wave.

Since none of the material parameters ρ, λ, or μ are negative, both of the propagation speeds are real numbers.

A complete statement of a problem regarding the motion of B generally involves specifying not only the Cauchy data at some initial time point, but also the boundary data for ∂B. For instance, one might specify a particular surface motion – independent of the motion that it gets from solving the Cauchy problem – or perhaps a given external pressure that acts on the boundary surface.

Of course, this implies the possibility that the specified boundary data might be inconsistent with the data that one derives from the equations of motion, such as a disagreement between the values of the surface acceleration. However, this is exactly the sort of situation that Hadamard has been addressing all along, so in such a case one regards ∂B as a singular surface.

Let $\lambda_i = [\partial^2 u_i \,/\, \partial t^2]$ be the components of a jump discontinuity in the acceleration across ∂B, while n_i are the components of the unit normal to that surface, and c is the speed of propagation.

From the equations of motion, one derives a compatibility equation for this jump:

$$(\rho c^2 - \mu)\lambda_i = (\lambda + \mu)(\lambda_j\, n_j)\, n_i\,.$$

One can then distinguish two types of solutions to these equations: the longitudinal ones, for which the jump in the acceleration is normal ($\lambda_i = \alpha n_i$), and for which the speed of propagation is c_l, as above, and the transverse solutions, for which the jump is tangential to ∂B ($\lambda_j\, n_j = 0$) and makes c_t take on the previously-described value. A more general discontinuity can be decomposed into a normal and transverse part, which then produce longitudinal and transverse waves independently, due to the linearity of the equations of motion.

Hence, the results of compatibility considerations for jump discontinuities in the acceleration on the boundary surface are entirely consistent with the more general discussion that pertained to simply the form of the equations of motion.

One finds that the longitudinal waves have constant infinitesimal rotation – i.e., vanishing vorticity – while the transverse waves have constant density gradient.

If the singular surface is internal to B then, as Hadamard asserts, there will be one longitudinal wave and three transverse ones.

Previously, Christoffel showed that a discontinuity in the stress across the boundary surface would not produce an acceleration wave, but a shock wave.

Hadamard does not treat the case of anisotropic elastic media in the infinitesimal case, but returns to it later in the context of finite deformations. The usual modern treatment (Landau and Lifschitz [1959] or Brekhovskikh and Goncharov [1994]) of waves in anisotropic media in the regime of small amplitudes – i.e., infinitesimal deformations – is entirely analogous to the Fresnel analysis of the dispersion law for the propagation of electromagnetic waves in crystal optics (see Landau, et al. [1984] or Born and Wolf [1980]), except that electromagnetic waves do not have longitudinal modes; we shall return to this subject shortly.

If one wishes to pose the problem of determining the motion of an anisotropic deformable body B when one considers the deformations to be finite then the first thing that one must address is the choice of formulation, viz., Lagrangian or Eulerian. Customarily, in elasticity one chooses the Lagrangian viewpoint by regarding the natural state as the reference configuration for B and describing its points by means of the coordinates a_i , $i = 1, 2, 3$. The points of B as it moves through space are then defined by spatial coordinates $x_i(t, a_j)$, such that for each value of t the correspondence between each a_i and $x_i(t, a_j)$ is invertible, as well as the Jacobian matrix $\partial x_i / \partial a_j$ of the transformation. Indeed, one deals primarily with $a_i(t, x_j)$ and $a_{ij} = \partial a_i / \partial x_j$.

In order to obtain the equations of motion for finite deformation, Hadamard chooses to employ a variational formulation of those equations as the Euler-Lagrange equations for an appropriate action functional. First, he derives the equations of static equilibrium from using the deformation energy density $W(x_i, a_i, a_{ij})$ as the Lagrangian density for the *action functional*:

$$S[a_i] = \int_B W(x_i, a_i, a_{ij})\, dxdydz\,,$$

so the action that is associated with the deformation is the total work done by the deformation. Clearly, this assumes that the material that B is compressed of is hyperplastic.

The *first variation functional* $\delta S[.]$ for this action functional takes a variation δa_i of the natural state, which is best regarded as a vector field on B to the number:

$$\delta S[\delta a_i] = \int_B \left(\frac{\delta W}{\delta a_i} \delta a_i \right) dxdydz + \int_{\partial B} \left(\frac{\partial W}{\partial a_{ij}} \delta a_j \right) dS_i\,,$$

in which:

$$\frac{\delta W}{\delta a_i} = \frac{\partial W}{\partial a_i} - \frac{\partial}{\partial a_j} \frac{\partial W}{\partial a_{ij}}$$

is the *variational derivative* of W with respect to a_i .

An extremal of the action functional is a static configuration $a_i(x_j)$ that makes the first variation functional vanish for any variation δa_i that satisfies some set of boundary conditions that usually have the effect of making the boundary integral vanish. The necessary and sufficient condition for a static configuration to be an extremal is then the vanishing of the variational derivative, which gives the *Euler-Lagrange equations:*

$$\frac{\delta W}{\delta a_i} = 0.$$

Hadamard restricts W to be spatially homogeneous, so $\partial W / \partial a_i$ vanishes, and the equations of static equilibrium become:

$$0 = \frac{\partial}{\partial a_j}\frac{\partial W}{\partial a_{ij}} = \frac{\partial \sigma_{ij}}{\partial a_j}.$$

In order to go from elastostatics to elastodynamics, he then uses d'Alembert's principle, which amounts to saying that dynamic extremals are static extremals in spacetime when one includes the "inertial forces" $F_i - \rho\,\partial^2 x_i / \partial t^2$, which gives the equations of motion:

$$\rho\,\frac{\partial^2 x_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial a_j} + F_i\,,$$

Hadamard then verifies that these equations produce the hydrodynamical equations of motions that he previously treated by using $W = W(\det a_{ij}) \equiv W(D)$ and setting the pressure p equal to $-\rho_0\, dW / dD$.

We shall summarize the general flow of ideas in the remainder of the chapter as they were explained later in Truesdell [1961].

Consider a singular surface S in a body, which can be a boundary or an internal surface, and whose unit normal vector field is $\mathbf{n}$. Assume that there is a second-order discontinuity in the kinematical state across S.

The jumps $[a^a{}_{,ij}]$ and $[\ddot{a}^a]$ must then satisfy kinematical compatibility conditions:

$$[a^a{}_{,ij}] = \alpha^a\, a^b{}_{,i}\, a^c{}_{,j}\, n_b\, n_c\,, \qquad [\ddot{a}^a] = c^2\alpha^a$$

for some vector α^a and scalar c. If one recognizes that the differential map $a^b{}_{,i}$ to the deformation can be used to pull back components of covectors from the initial (i.e., reference) state to the present state then one can define $n_i = a^b{}_{,i}\, n_b$ to be the components of the unit normal relative to the present state and the first condition takes the form:

$$[a^a{}_{,ij}] = \alpha^a\, n_i\, n_j\,.$$

One also assumes a dynamical compatibility condition for the jump in the Cauchy stress tensor t^{ab}:

$$[t^{ab}]n_b = 0.$$

The stress tensor that one uses in the present state is the two-point tensor:

$$T^{ia}(a^a, x^i(a)) = \rho_0 / \rho\, t^{ab}\, x^i_{,b},$$

that one calls the *Piola-Kirchhoff tensor* (in addition to the references by Truesdell, see also de Veubeke [1979]). In this definition ρ_0 is the mass density of the object in the reference state, while ρ is its mass density in the present state.

One then expresses the constitutive law in the form $T^{ia} = T^{ia}(a^a_{,j}, \mathbf{e}_A)$, in which $\mathbf{e}_A$, A = 1, 2, 3 is an orthonormal triad at each point of the deformed state. This puts Cauchy's law of motion (balance of momentum) into the form:

$$T^{ia}_{,i} + \rho_0 f^a = \rho_0 [\ddot{a}^a].$$

Define the fourth-rank tensor field:

$$A_{ab}{}^{ij}(x^i_{,a}, \mathbf{e}_A) \equiv \frac{\partial T^i_a}{\partial a^b_{,j}}.$$

The equations of motion then take the form:

$$A_{ab}{}^{(ij)} a^b_{,ij} + \frac{\partial T^a_i}{\partial e^b_A} e^b_{A,i} + \rho_0 f^a = \rho_0 [\ddot{a}^a].$$

If one assumes that the only variables that experience a jump discontinuity across S are the second-order kinematical ones above then when the compatibility conditions are applied to this equation, one ultimately deduces the following *propagation condition* for the vector α^a:

$$Q^a{}_b(\mathbf{n})\alpha^b = \rho_0 c^2 \alpha^a,$$

in which we have defined:

$$Q_{ab}(\mathbf{n}) = A_{ab}{}^{(ij)} a^p_{,i}\, a^q_{,j}\, n_p\, n_q .$$

If we understand that $A_{ab}{}^{(pq)} = A_{ab}{}^{(ij)} a^p_{,i}\, a^q_{,j}$ are the components $A_{ab}{}^{(ij)}$ pulled back to the initial state then we can say that:

$$Q_{ab}(\mathbf{n}) = A_{ab}{}^{(pq)}\, n_p\, n_q .$$

One refers to the tensor field $Q(\mathbf{n}) = Q_{ab}(\mathbf{n})\, da^a \otimes da^b$ as the *acoustic tensor field* for **n**.

The symmetric part of $Q(\mathbf{n})$, namely $Q_{(ab)}(\mathbf{n})\, da^a da^b$ defines a quadratic form on $\mathbb{R}^3$ and has an associated ellipsoid defined by:

$$Q_{(ab)}(\mathbf{n})\, l^a l^b = 1$$

that one calls the *polarization ellipsoid*, which is analogous to the one that one defines in electromagnetism as a consequence of Fresnel analysis.

In the case of hyperelastic materials, which are the ones that Hadamard is concerned with, the components $Q_{ab}(\mathbf{n})$ are automatically symmetric. Hence, the eigenvalues of $Q^a{}_b(\mathbf{n})$ are real and there exists at least one orthonormal triad of eigenvectors for $Q^a{}_b(\mathbf{n})$. If we return to the propagation condition above then we see that it simply says that **a** must be one of those eigenvectors and the corresponding eigenvalue is $\rho_0 c^2$.

Hadamard then asserts that is the desired extension of the result that was established for infinitesimal deformations of isotropic media, which Truesdell [1961] calls the *Fresnel-Hadamard theorem*: For each wave normal vector **n** that lies on the polarization ellipsoid there are three mutually orthogonal directions $\mathbf{l}_A$, A = 1, 2, 3 in which second order discontinuities can propagate, namely, the principal directions of that ellipsoid, and the eigenvalues that correspond to the eigenvectors are then proportional to the squares of the propagation speeds in those principal directions.

Hence, in order for these speeds to be real the eigenvalues must be positive; in other words, the quadratic form $Q(\mathbf{n})$ must be positive-definite. This condition has much deeper physical ramifications.

We point out that, in general, the principal axes for the acoustic tensor – i.e., the *acoustic axes* – are distinct from the principal axes for the strain or stress tensors. However, this is the case in an isotropic medium, and, as a result, one sees that a principal wave must be either transverse or longitudinal, but not a combination of both.

In the case of liquids, $c^2 = dp \,/\, d\rho$, which is positive iff the equilibrium state of the fluid is stable. Hence, one suspects that the issue associated with the reality of the polarization ellipsoid is the stability of the equilibrium state of the deformable body.

In the case of anisotropic solid media, the stability of equilibrium is more involved than it is for fluids. In particular, one needs to consider the second variation of the action functional, since equilibrium state itself is obtained from the first variation. One can think of this situation as a sort of infinite-dimensional analogue of the situation that one considers in the study of critical points of differential functions of a finite number of variables, although in practice that analogy is more heuristically probative than computationally useful.

When Duhem [1905] published his own work on elastic stability, he suspected that the discussion that Hadamard gave to elastic stability had flaws in the proof of his basic result. However, in 1946 Cattaneo gave a rigorous proof that confirmed it.

The Legendre sufficient condition for an extremal to be a (weak) local minimum of the action functional is that the quadratic form defined by the Hessian $\partial^2 W \,/\, \partial a_{ij}\, \partial a_{kl}$ be positive definite. However, the real issue is the positive definiteness of the second variation functional $\delta^2 S[\delta a_{ij}, \delta' a_{ij}]$, as a quadratic form on the infinite-dimensional vector space of variations, such as δa_{ij} and $\delta' a_{ij}$. Hence, one must regard this problem as defining a strong minimum, since the positive-definiteness of this functional does not have to imply the positive-definiteness of $\partial^2 W \,/\, \partial a_{ij}\, \partial a_{kl}$, although the converse is true.

One can easily convert the Hessian $\partial^2 W \,/\, \partial a_{ij}\, \partial a_{kl}$ to one of the form $\partial^2 W \,/\, \partial e_{ij}\, \partial e_{kl}$, which then gives a quadratic form of the form:

$$\mathfrak{S}[\delta a_{ij}, \delta a_{kl}] = \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \delta a_{ij}\, \delta a_{kl} + \Psi[\delta e_{ij}, \delta e_{kl}],$$

whose positive-definiteness gives the desired stability condition.

Truesdell asserts that this is equivalent to the condition:

$$A_{(ab)}^{(ij)}\, l^a\, l^b\, \lambda_i\, \lambda_j > 0 \qquad \text{for all } l^a, \lambda_i,$$

which he calls *strong ellipticity*, since it is intimately related to the symbol of the nonlinear differential operator that defines the equations of equilibrium for finite strains. This is the clarification of one his remarks in the second section of these notes that was previously promised.

Strong ellipticity can also be expressed in the form:

$$Q_{(ab)}(\mathbf{n})\, l^a\, l^b > 0 \qquad \text{for all } \mathbf{n}, \mathbf{l}.$$

Hence, it is equivalent to the positive-definiteness of the acoustic tensor for all $\mathbf{n}$. As one corollary, one sees that strong ellipticity is a sufficient condition for the reality of the speeds of propagation, and another says that this is always true in any isotropic material.

Hadamard shows that this is indeed consistent with the hydrodynamical condition when one substitutes the simplifying expressions that pertain to that case.

He then returns to an earlier question that was posed for the propagation of waves that involve finite deformations in isotropic media. He proceeds by analogy with the Fresnel analysis in crystal optics for anistropic media. However, when he reduces to the isotropic case, he obtains a negative result for the proposed extension. That is, unlike waves of infinitesimal deformations, waves of finite deformations in isotropic media are not solely longitudinal or transverse in general, but a combination of both.

§ 9. Notes on Chapter VII. This chapter is concerned with the theory of characteristics for hyperbolic second order partial differential equations. As Hadamard pointed out later in his Yale lectures [1922], the issues discussed in this chapter defined the starting point for that subsequent examination of the Cauchy problem in a purely mathematical context, rather than in the context of continuum mechanics that the present monograph centers around.

Since Hadamard had previously discussed the role of characteristic curves in the chapter on one-dimensional gas dynamics, the first section of this chapter represents the extension of that study to spaces of higher dimensions than just one, namely, dimension n.

The appearance of characteristic hypersurfaces is intimately related to the existence and uniqueness of solutions to the Cauchy problem, so we pose that problem in the present n-dimensional case for quasilinear second-order partial differential equations in scalar functions. However, many of the key notions can be generalized to higher-order partial differential equations and vector-valued functions (see, e.g., John [1982] or Folland [1976]).

A quasilinear second-order partial differential equation in a scalar function u on some region V in $\mathbb{R}^n$ takes the form:

$$(*) \qquad 0 = a^{ij}(x^k, u, u_k)\frac{\partial^2 u}{\partial x^i \partial x^j} + f(x^k, u, u_k),$$

in which the function $f(x^k, u, u_k)$ need not be linear in (u, u_k). Note that the symmetry of a^{ij} in its indices must follow naturally from the symmetry of mixed partial derivatives.

Suppose that one is given a hypersurface S in $\mathbb{R}^n$, which we express by making the coordinate system that we use be one that is adapted to the function that defines the hypersurface; i.e.:

$$x^n = x^n(x^1, \ldots, x^{n-1}).$$

Now, suppose that we are given a function u_0 on S. As long as one is assuming that u_0 is at least C^1 on S, one sees that the tangential derivatives u_{0i}, $i = 1, \ldots, n-1$ of u_0 are uniquely defined by a choice of u_0, so the only undetermined first derivative that can be defined arbitrarily must be the normal derivative u_{0n}. Hence, the Cauchy data for the Cauchy problem that is defined by our quasilinear partial differential equation above on the Cauchy hypersurface S – namely, $\{u, u_n\}$ – is the analogue of the corresponding initial-value problem for each point of S had we defined a system of ordinary differential equations, namely: Given u_0 and u_{0n} on S, find a u on $\mathbb{R}^n$ (or, at least, some neighborhood of S) that satisfies the partial differential equation in question and agrees with the Cauchy data on S.

When one goes to the next level of differentiation, one sees that if u_0 is assumed to be at least C^2 then, similarly, the tangential second derivatives u_{ij}, $i, j = 1, \ldots, n-1$ are all uniquely determined by the choice of u_0, and cannot be assigned arbitrarily. The question then arises whether the second partial derivatives u_{in} and u_{nn} can be determined arbitrarily when one also assumes that the differential equation (*) is in effect, along with the Cauchy data on S.

First, one sees that the functions $a^{ij}(x^k, u, u_k)$ and $f(x^k, u, u_k)$ are uniquely determined when one sets $x^k = x_0^k$, $u = u(x_0^k)$, $u_k = u_k(x_0^k)$. We expand the highest-order term in equation (*) to:

$$a^{nn} u_{nn} + 2a^{in} u_{in} + a^{ij} u_{ij}$$

Furthermore, on S, one also has that since x^n is a function of the remaining coordinates, one must have:

$$\frac{\partial^2 u_0}{\partial x^n \partial x^i} = \frac{\partial^2 u_0}{\partial x^n \partial x_0^i} + \frac{\partial x^n}{\partial x_0^i}\frac{\partial^2 u_0}{\partial x^n \partial x^n}, \qquad \frac{\partial^2 u_0}{\partial x^i \partial x^j} = \frac{\partial^2 u_0}{\partial x_0^i \partial x_0^j} + \frac{\partial x^n}{\partial x_0^i}\frac{\partial x^n}{\partial x_0^i}\frac{\partial^2 u_0}{\partial x^n \partial x^n}.$$

If we introduce the notation $K_i = \partial x^n / \partial x^i$, $i = 1, \ldots, n-1$ then the leading-order terms in (*) take the form:

$$(a^{nn} + 2a^{in} K_i + a^{ij} K_i K_j)\, u_{0nn} + 2a^{ni}\frac{\partial^2 u_0}{\partial x^n \partial x_0^i} + a^{ij}\frac{\partial^2 u_0}{\partial x_0^i \partial x_0^j}.$$

Hence, if u_0 is consistent with the differential equation then one can solve for u_{0nn} as long as one does not have:

$$0 = a^{nn} + 2a^{in} K_i + a^{ij} K_i K_j .$$

This can also be regarded as a partial differential equation for x^n; i.e., a compatibility condition on the Cauchy hypersurface S. When this condition is satisfied, one refers to the hypersurface S as a *characteristic hypersurface*.

Now, one can regard the K_i as inhomogeneous coordinates for the projective space $\mathbb{R}\mathrm{P}^{n-1}$ and introduce the homogeneous coordinates by way of $K_i = k_i / k_n$, where $k_n \neq 0$ so the characteristic equation takes the form:

$$a^{\mu\nu} k_\mu k_\nu = 0, \qquad \mu, \nu = 1, \ldots, n.$$

The character of this hypersurface depends upon the character of the symmetric real matrix $a^{\mu\nu}$, which one calls the *principal symbol* of the second-order partial differential operator that we are considering. By Sylvester's principle of inertia, there is a matrix T^μ_ν that makes $T^\mu_\kappa T^\nu_\lambda a^{\kappa\lambda} = \mathrm{diag}[-1, \ldots, -1, +1, \ldots, +1]$ with p negative signs and q positive ones; one then calls the ordered pair (p, q) the *signature type* of $a^{\mu\nu}$, while the *signature* is defined to be $q - p$.

When $a^{\mu\nu}$ is either positive definite ($q = 0$) or negative definite ($p = 0$) the only real solution to the characteristic equation is 0. In such a case – namely, the *elliptic* case – one can always compute u_{0nn} from the given data.

When $a^{\mu\nu}$ has a *Lorentzian* signature type, so either $p = 1$, $q = n - 1$ or $p = n - 1$, $q = 1$, the hypersurface takes the form of a generalized cone in $\mathbb{R}^n$, since the line through any point on the hypersurface and the origin will be contained in that hypersurface. The shape of its spatial generator in $\mathbb{R}\mathrm{P}^{n-1}$ will take the form of an ellipsoid when $n = 4$. If one regards the homogeneous coordinates k_μ as describing points of the cotangent bundle T^*M in a local coordinate system (x^μ, k_μ) then when the signature type of $a^{\mu\nu}$ is Lorentzian the characteristic equation that it defines can either describe a hypersurface in T^*M when the functions $a^{\mu\nu}(x^\mu, k_\mu)$ are allowed to vary freely, and which Hadamard calls the *characteristic conoid*, or a hypersurface in each cotangent space when one fixes x^μ, and he calls this the *characteristic cone* (at x).

Hadamard singles out the case of a symbol with *multiple characteristics* as being associated with points where $a^{\mu\nu}(x^\mu, k_\mu)$ does not have maximal rank, which is then the *parabolic* case. For instance, each component of $a^{\mu\nu}(x^\mu, k_\mu)$ might be itself a quadratic form $a^{\mu\nu\kappa\lambda}(x)k_\kappa k_\lambda$ whose vanishing defines a conoid in T^*M, as well. The zero locus of $a^{\mu\nu}(x^\mu, k_\mu)$ could then consist of intersecting cones in each cotangent space or a more elaborate self-intersecting quartic, which is analogous to the situation one encounters in Fresnel analysis that leads to *birefringence* (double refraction) or *conical refraction*, respectively (see, e.g., Landau, et al, [1984] or Born and Wolf [1980]). However, in that analysis the components $a^{\mu\nu\kappa\lambda}(x)$ of the quartic form come about from a slightly different, but related, procedure to that of simply taking the principal symbol of the differential operator in question.

The symbol of the second-order quasilinear differential operator considered defines a generally nonlinear first-order partial differential equation:

$$g^{\mu\nu}(x)\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0.$$

In this formulation, we are assuming that our second-order differential operator is hyperbolic, so we have changed our notation for the coefficients of the leading term from a^{ij} to $g^{\mu\nu}$, and we are assuming that they do not depend explicitly upon u. The equation that we have defined on spacetime manifold M is a homogeneous form of the Hamilton-Jacobi equation, and when one restricts it to a spatial submanifold by considering only stationary wave functions one obtains the *eikonal* equation of geometrical optics.

The function $S(x^\mu)$ on M is called the *phase function* for the wave function $u(x^\mu)$ that solves our wave equation. This is because the way that one associates the latter first-order partial differential equation with the original second-order differential equation (*) is by way of the *geometrical optics* approximation, which we will discuss later in the context of linear wave equations.

We define the function $F(x^\kappa, k_\kappa)$ on T^*M by way of:

$$F(x^\kappa, k_\kappa) = \tfrac{1}{2}g^{\mu\nu}(x^\kappa, k_\kappa)k_\mu k_\nu .$$

The function F can be regarded as a Hamiltonian function on the phase space that is defined by T^*M, and its associated canonical equations are:

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu}k_\nu, \qquad \frac{dk_\mu}{d\tau} = -\tfrac{1}{2}\, g^{\kappa\lambda}{}_{,\mu}\, k_\kappa k_\lambda .$$

These equations are a specialized form of the characteristic equations for the first-order partial differential equation $F(x^\kappa, S_{,\kappa}) = 0$. For a more general equation, F is a function on $J^1(M, \mathbb{R})$ with the local form $F(x^\kappa, u, u_\kappa)$ and the resulting characteristic equations are:

$$\frac{dx^\mu}{d\tau} = \frac{\partial F}{\partial k_\mu}, \quad \frac{du}{d\tau} = \frac{\partial F}{\partial k_\mu}k_\mu, \quad \frac{dk_\mu}{d\tau} = -\left(\frac{\partial F}{\partial x^\mu} + \frac{\partial F}{\partial u}k_\mu\right).$$

As one sees, these equations reduce to the previous form when F does not depend upon u as long as:

$$0 = \frac{\partial F}{\partial k_\mu}k_\mu = \frac{dx^\mu}{d\tau}k_\mu .$$

In the case of F that we defined above, this expression takes the form $0 = g^{\mu\nu}k_\mu k_\nu$, which amounts to restricting k_μ to be a characteristic covector. This has the effect of saying that the velocity vector for the motion of the wave must be tangent to the characteristic hypersurface, which forces the characteristic curves to lie in it.

Since we are now using the word "characteristic" in two different senses, in order to avoid ambiguity, Hadamard suggests using the term *bicharacteristics* for the characteristic *curves* of the first-order partial differential equation that is defined by the characteristic *hypersurface* of the original second-order one. When space is one-dimensional, the situation is confused further by the fact that the hypersurfaces are also curves. In optics, the characteristic hypersurfaces represent elementary propagating wave fronts at each point of M, such as expanding spheres, while the bicharacteristic curves represent the light rays that come about in the geometrical optics approximation.

Since the recurring theme of this book is that waves are best mathematically represented by propagating discontinuities, one finds that the relevance of characteristics and bicharacteristics to discontinuities is based in the fact that a second-order discontinuity in a wave function across a singular hypersurface represents a non-uniqueness in its second derivatives across that hypersurface. However, as we have seen, the only way that this is possible is when the hypersurface is characteristic. One then has the corollary that discontinuities must propagate along bicharacteristics. Hence, the association of propagating second-order discontinuities with waves seems quite natural. It is also a consequence of the kinematical compatibility conditions, which make the discontinuity transverse to the singular hypersurface and the velocity normal to it.

As Hadamard points out, an important class of transformations is defined by wave motion, namely, *contact transformations*. Primarily, they take solutions of the Hamilton-Jacobi equation to other solutions. If one thinks of the pair $(x, k) \in T^*M$ as representing a hyperplane in the tangent space at x – viz., the hyperplane $k(\mathbf{v}) = 0$ – then one sees that if this hyperplane is tangent to a wave hypersurface, which was once referred to as *first-order contact*, then one sees that contact transformations take hyperplanes tangent to the characteristic hypersurface to other such hyperplanes. Hence, they must preserve the characteristic hypersurface, as well as the bicharacteristic curves.

When Hadamard applies the method of characteristics to waves in gases and compressible fluids he deduces consistent conclusions, that if the discontinuity is normal to the (spatial) wave surface then the wave surface must be characteristic and the bicharacteristics must be normal to that wave surface (this assumes that one is using the Eulerian viewpoint for the dynamical model). In the case of three-dimensional elasticity, he finds that in an anisotropic medium, a given wave surface is compatible with three types of discontinuities: one normal and two transverse ones, while in an isotropic medium, there is one normal type and one transverse type.

In the second section of Chapter VII, Hadamard focuses on applying the methods of characteristics to the problem of proving the actual existence of solutions to the Cauchy problem. Prior to the publication of his book, existence proofs for restricted cases of the Cauchy problem had been known to Cauchy, Kowalevski [84], Goursat, and Beudon.

The Cauchy-Kowalevski theorem asserted the existence of a unique local solution about each point of a non-characteristic Cauchy hypersurface when the Cauchy data, as well as the coefficients of the differential equation, were analytic. What Goursat had shown was that when one is given an analytic partial differential equation for a scalar function $u(x, y)$ of two independent variables x, y:

[84] Since there are numerous alternate spellings of this last name, we defer to the argument of Fritz John [1982], who points out that the spelling that Sonja Kowalevski herself used in the papers that she submitted to the *Acta Mathematica* was the one we have chosen.

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

i.e., F is analytic, and two intersecting curves γ_1, γ_2 that are tangent to the characteristic curves through the point of intersection then if one is given analytic Cauchy data for u on *both* characteristics there will be a unique solution u to this variation of the Cauchy problem.

A corollary to this is the fact that if u is given on *only one* of the characteristic curves then there will be an infinitude of solutions, which suggests what breaks down when the Cauchy problem is defined upon a characteristic initial hypersurface.

What Beudon contributed was the extension of this result to the case of an arbitrary finite number of independent variables; i.e.:

$$F(x^i, u, u_i, u_{ij}) = 0.$$

One must then consider two intersecting initial hypersurfaces that are tangent to the characteristic hypersurfaces at the curve of intersection.

Hadamard then proposes to prove Beudon's result under the slightly weaker hypothesis that only one of the initial hypersurfaces needs to be characteristic. He deduces an analogous result that includes the corollary that when the Cauchy data is defined upon a characteristic initial hypersurface there will be an infinitude of solutions to the Cauchy problem.

He applies his result to various problems in hydrodynamics, such as the crossing of irrotational waves, which then admit velocity potentials. One can then obtain the resulting motion after the crossing in the case when the wave surfaces are analytic, along with the motion of the fluid.

Next, Hadamard generalizes his existence result to the case of systems of second-order partial differential equations for vector-valued wave functions $u^a(x^i)$:

$$F^b(x^i, u^a, u_i^a, u_{ij}^a) = 0, \qquad a, b = 1, 2, 3.$$

He then applies this to the initial-value problem in gas dynamics in which both the initial motion of the gas and the motion of the boundary that encloses it are given. One also assumes that the fluid and the wall are in constant contact and that the new motion of a wave agrees with the original one. As long as the intersection of the wave with the wall is not tangent to a bicharacteristic, one can determine the resulting wave uniquely. However, the method does not apply to the crossing of waves in this case.

In the final section of this chapter, Hadamard examines the form that the analysis takes when one restricts oneself to linear second-order partial differential operators, which then take the form:

$$L = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(x)\frac{\partial}{\partial x^i} + c(x).$$

For the case where the operator L is the d'Alembertian operator:

$$\square = \eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial t^2} - \frac{1}{c^2}\delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j},$$

solutions to the stationary initial-boundary-value problem had been found by Poisson and Kirchhoff. One seeks solutions of the form:

$$u(t, x^i) = T(t)\phi(x^i),$$

which converts the linear wave equation $\square\, u = 0$ into the pair of equations:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0, \qquad \Delta\phi + (\omega/c)^2\phi = 0,$$

when one introduces the separation constant ω, which then represents the frequency of the elementary oscillators that define the wave medium.

The first of these equations can be solved by the pair of sinusoidal functions:

$$T(t) = e^{\pm i\omega t}.$$

The second is a spatial second-order partial differential equation that one refers to as the *Helmholtz equation*. Its solution ϕ essentially defines the shape of the wave envelope for the motion. The operator that defines it is self-adjoint for the Euclidian metric on the spatial manifold; hence, it is elliptic. One can then solve the Dirichlet or Neumann problems for it by the method of Green or Neumann functions. However, one must observe that the only way that one can specify both the boundary values of the function ϕ and its normal derivative is when these two data are compatible, as one might obtain when they are derived from the original Cauchy data by separation of variables.

Kirchhoff's solution to the problem amounted to obtaining the Neumann function for the d'Alembertian operator in spherical coordinates in the form:

$$N(x^i, y^i) = \frac{e^{ik(r-ct)}}{r}, \qquad r = \| x - y \| = [(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2]^{1/2}.$$

This is where physics usually introduces the *geometrical optics approximation* as a way of simplifying the solution of the Cauchy problem for wave motion, while one generally refers to the explicit solution for the wave function u as *wave optics.*

One first looks for all separable wave functions:

$$u(t, x^i) = A(x)e^{iS(t,\, x)},$$

in which $A(x)$ is called the *amplitude function* for u and $S(t, x^i)$ is its *phase function.*

This converts the d'Alembertian of u into:

$$\square u = g^{\mu\nu}[A_{\mu\nu} - AS_\mu S_\nu + i(A_\mu S_\nu + A_\nu S_\mu + AS_{\mu\nu})]\, e^{iS}.$$

One further restricts the class of wave functions by the approximation that the amplitude must vary slowly in space compared to the rate at which S varies, which amounts to a *small-wavelength* (or high-frequency) approximation; in practice, one might simply assume that the amplitude is constant. This approximation has the effect of eliminating all derivatives of A. As long as one also assumes that:

$$0 = \Box S = k^{\mu}{}_{,\mu}$$

which is a sort of incompressibility condition on the covector field k_μ , one can replace the linear wave equation for u with the Hamilton-Jacobi equation for S:

$$0 = g^{\mu\nu} S_\mu S_\nu \;;$$

again, it would be sufficient to assume that the frequency and wave number of the wave are constant.

The resulting bicharacteristic equations:

$$\frac{dx^\mu}{ds} = g^{\mu\nu}k_\nu = k^\mu, \qquad\qquad \frac{dk_\mu}{ds} = -\tfrac{1}{2}\, g^{\kappa\lambda}{}_{,\,\mu}\, k_\kappa\, k_\lambda \,,$$

can be combined into a single system of second-order ordinary differential equations:

$$\frac{d^2x^\mu}{ds^2} + \begin{Bmatrix} \mu \\ \kappa\lambda \end{Bmatrix} x^\kappa x^\lambda = 0,$$

in which we have introduced the *Christoffel* symbols:

$$\begin{Bmatrix} \mu \\ \kappa\lambda \end{Bmatrix} = \tfrac{1}{2}\, g^{\mu\alpha}(g_{\alpha\kappa,\lambda} + g_{\alpha\lambda,\kappa} - g_{\kappa\lambda,\alpha}).$$

At the present stage of history, the geometrical significance of these equations is well-known to differential geometry and physics alike: They are the equations for the geodesics that are associated with the Lorentzian metric $g^{\mu\nu}$ by way of the *Levi-Civita connection* that it defines. Geodesics can be regarded as the curved-space analogues of the straight lines that one finds in Euclidian spaces, for which the Christoffel symbols will vanish.

When one combines this fact with the fact that the velocity vector field k^μ will always have to be characteristic – or null – one sees that the bicharacteristic curves, as light rays, will represent null geodesics.

The reason that we are using the symbol s for our curve parameter instead of τ is simply that in the eyes of general relativity, the differential increment $d\tau$ of the proper time parameter τ must always be null – i.e., $g_{\mu\nu}k^\mu\, k^\nu$ – on light rays, so one must use some other parameter if one is to define non-degenerate light rays. The usual choice is to make s one of the class of *affine parameterizations*, which then differ by the replacement of s with $as + b$, with $a \neq 0$ and b constants. Affine parameters are characterized by the fact that they will put the geodesic equation into the form described above, while any other parameterization will introduce a non-zero contribution to the right-hand side of the geodesic equation.

The geometrical optics approximation is generally more than adequate for the most elementary problems of optics involving visible light wavelengths, such as describing reflection, refraction, and even dispersion in optical systems. However, it breaks down when one goes to radio frequencies, whose associated wavelengths can be in the meters and more, and – more to the point – when one considers the effects of diffraction on wave motion. These effects usually come about when waves of a certain wavelength pass through slits or edges whose characteristic dimensions are comparable to those wavelengths and amount to the appearance of non-zero light intensities inside the shadow of the obstacle. Traditionally, they are treated by asymptotic series expansions of the wave functions, in which the geometrical optics approximation represents the leading-order term and successive terms introduce the diffraction corrections. Since Hadamard did not mention this topic in his book, we shall, however, suspend our commentary with that brief observation.

§ 10. Notes on the appendixes. In general, the notes at the end of this book take up some of the open issues in the main body of the text. For instance, some of the conjectures in earlier chapters could not be rigorously proved until material in the later chapters had been introduced.

In Note I, Hadamard returns to a statement that was established in the final chapter on characteristics that if two integral surfaces to the same Monge-Ampère equation are tangent along a line then that line must be a characteristic line. He points out that the proof used breaks down when the intersection has higher-order contact than one.

This observation then gives way to a generalization of the Cauchy problem in which the Cauchy data include higher-order derivatives. The question to be resolved is: Under what conditions will this generalized Cauchy problem admit a unique solution assuming that the Cauchy data is not characteristic? This also suggests a generalization of the Cauchy-Kowalevski theorem must be posed.

About the same point in time when Hadamard first published this book, Holmgren [1904] established his uniqueness theorem (cf., e.g., John [1982], Folland [1976], or Hörmander [1969]), which stated, in effect, that when one is concerned a linear system of analytic first-order partial differential equations the only C^k, but not analytic, solution to the Cauchy problem with vanishing Cauchy data that is defined on some initial Cauchy hypersurface is the null solution.

Hence, the bulk of the material that is discussed in this Note is concerned with giving Hadamard's proof of Holmgren's theorem.

In Note II, Hadamard returns to the study of stationary discontinuities in fluids that he briefly introduced in Chapter V.

In order zero, one is concerned with discontinuities in some function, such as position. Hadamard refers to such position discontinuities as "slips" (*glissements*), although nowadays it is more common to call them *shears*. In order one, stationary discontinuities in the velocity vector field across a singular surface are referred to as *vortex sheets.*

Up to the point in question in Chapter V, nothing obstructed the persistence of fluid shears in the absence of viscosity, but their actual creation would have been impossible under the assumptions that had been made. Hence, Hadamard deferred the discussion of the assumptions under which fluid shears could be created to this appendix.

His basic observation is that nothing in the equations of motion of hydrodynamics forbids the existence of discontinuous solutions. Indeed, the cavitation that occurs near vortices, such as propellers, suggests that such solutions occur naturally.

The primary purpose of this Note is to establish that cavitation is a necessary condition for the creation of fluid shears. The proof is founded on the fact that at each instant in the motion of a fluid shear the jump in acceleration across a singular surface is normal.

In Note III, Hadamard returns to a previous observation (nos. **254-255**) that the existence of second-order discontinuities does not invalidate the classical theorems of vorticity, such as the conservation of circulation and velocity potential, to give a more rigorous proof of that assertion.

As a corollary, he shows that it is entirely possible for shock waves to produce vortices, at least when one assumes the Poisson adiabatic. Recall that this result was regarded as one of the fundamental lasting contributions from this book in the opinion of Truesdell.

When one uses the more physically realistic Hugoniot adiabatic, the quantity dp/ρ is not longer an exact differential, and the basic assumptions that the classical theory of vortices rests upon are no longer valid.

Finally, in Note IV the Hadamard returns to a question that had been left open in Chapter IV, namely: What happens to the initial/boundary-value problem when one includes both the initial motion of the gas in a pipe, as well as the initial motion of a piston, instead of only the initial motion of the gas itself?

Hadamard points out that the problem could be solved directly in at least one physically useful case, namely, the case in which the piston is assumed to exhibit uniform rectilinear motion, such as a fixed piston. The solution, which involved the use of results that were not obtained until Chapter VII, then included the possibility of reflection of the wave from the fixed piston.

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