"Bermerkungen zur Variationsrechnung," Math. Ann. **58** (1904), 148-168.

Remarks concerning the calculus of variations

By

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In volume 55 of Mathematische Annalen, **Kneser** published an article in which he returned to the already oft-treated question of whether a necessary condition for the occurrence of an extremum that would be analogous to the so-called Jacobi criterion also exists in the case of the simplest isoperimetric problems. Except for an exceptional case, that question is answered in the affirmative, while that exception case remains unresolved, as in all of the previous proofs that are known to me (*). However, since it is known that the isoperimetric problems can be reduced to the so-called Lagrange problem in a simple way, which was treated by **G. von Escherich** in a series of thorough investigations (**), that suggests that one might pose the question of whether the aforementioned question cannot be resolved completely on the grounds of the results of those investigations. To that end, it would be necessary to test the extent to which the assumptions that von Escherich expressly built his investigations upon (***) are fulfilled in the isoperimetric problem. If one overlooks the inevitable assumptions about certain continuity properties of the functions that appear, then they will be the following assumptions:

1. Any regular curve that produces the desired extremum, together with the multipliers, define a continuous solution of the Lagrange system of differential equations.

2. The equations of condition can be replaced with their equivalent linear equations.

For the isoperimetric problem, the question of the justification for those assumptions is easy to resolve. However, it will also show that results that are generally valid can also be achieved without having to restrict to the isoperimetric problems. The first three sections of the following work are devoted to their derivation.

(*) When I submitted the present article for editing, Bolza's treatise (Math. Ann. Bd. 57) "Zur zweiten Variation bei isoperimetrischen Problemen" had not yet appeared.

(**) "Die zweite Variation einfacher Integrale, Mitt. I, II, III," Wiener Ber. **107** (1898), *ibid.* Mitt. IV **108** (1899), *ibid.* Mitt. V **110** (1901). I have applied an analogous method to a somewhat-more-general problem: "Zur Theorie der zweiten Variation einfacher Integrale," Monatshefte für Math. und Phys., Bd. 14.

(***) Cf., *loc. cit.*, Mitt. II, **§ 2**, Mitt. V, **§ 15**.

As far as the first of those two assumptions are concerned, the known investigations of **A. Mayer** allow one to achieve that goal by means of some simple arguments. Matters are not so simple in regard to the second assumption. Like the method of Lagrange multipliers, it was adapted from the theory of constrained maxima and minima of functions of several variables with no further tests in the calculus of variations in its own right, but without entering into the details of the question of whether, and under what circumstances, that would be justified, at least, to my knowledge as of now. That is closely related to the question of the assumptions under which an arbitrarily-small extremal arc can be varied in an admissible way, which I shall try to answer in the second section. The last section will give a short summary and applications to the isoperimetric problem. The Jacobi condition will be itself expressed in the most-general form that is possible, such that, in particular, the case of analytic functions will be resolved completely, which was the only case that Kneser considered.

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