APPENDIX A

§ 1. Variances and total hydrodynamics derivatives. – The fundamental concepts that are at the basis of relativistic hydrodynamics are *space-time densities*, which have, by definition, the variance of tensor densities. The most important ones that we shall encounter are the matter density ρ, which is a scalar density, the current density j_μ and the momentum density g_μ, which are both vector densities, the spin density s_μ, which also a vector density, the internal angular momentum density $s_{[\mu\nu]}$, and the energy-momentum tensor density $t_{\mu\nu}$. Since we shall use only the transformations of the *proper* Lorentz group, and need not concern ourselves with reflections and time reversals, we shall call the magnitudes above *tensors* by abstracting from the density character of some of them, in order to simplify the language.

If one starts with these space-time tensor densities then one will obtain global quantities that have a simple tensorial variance by integrating them over a given *space-time* domain. Starting with the densities f, f_μ, $f_{\mu\nu}$, one will therefore obtain the tensors $\mathfrak{F}$, $\mathfrak{F}_\mu$, $\mathfrak{F}_{\mu\nu}$, but they will not be tensors that have any physical significance. The physical magnitudes will be the integrals of *space-time* densities that are taken over a certain *spatial* volume that is occupied by a certain material body at a given instant, which will be integrals that possess a simple relativistic variance only in very exceptional cases. As Louis de Broglie once remarked, "This is an example of the rather frequent contrasts that are presented between quantum ideas and relativistic ideas" [**1**]. In order to construct a physical model for the causal theory or to construct the operators of the probabilistic one, one must make other global magnitudes correspond to the tensor densities f, f_μ, $f_{\mu\nu}$, which will be spatial integrals that we denote by F, F_μ, $F_{\mu\nu}$.

It will sometimes be convenient for the expression of the densities that characterize a fluid to consider them to be constituted from a great number of point-like particles whose properties can be expressed by the *particle* magnitudes F, F_μ, $F_{\mu\nu}$, which will be tensorial, by definition. One must then assume that, on the one hand, each of these magnitudes varies in a continuous fashion when one considers a particle neighborhood, and on the other hand, that these particles are distributed with a mean density that also varies in a continuous fashion. At a point P in the fluid, one considers the *proper reference frame* Σ_0 that is attached to the particle that is found at P. One considers a (proper) volume element $d_3 \upsilon_0$ in the space of this reference frame that contains $d_3 n$ particles, and one defines the *matter density* by $\rho = d_3 n \,/\, d_3 \upsilon_0$.

One has $d_3 n = \rho \, d_3 \upsilon_0 \, d\tau \,/\, d\tau$, in which $d\tau$ is proper time. However, $d_3 \upsilon_0 \, d\tau$ is the space-time volume element $d_4 \omega$, which is invariant under a Lorentz transformation, since the contraction of the volume compensates precisely for the dilatation of time. Since $d_3 n$ represents a given number of particles, it will likewise be invariant, so one sees that $\rho \,/\, d\tau$ – and, as a consequence, ρ – will be a space-time invariant. The fluid will then be characterized by the densities ρF, ρF_μ, $\rho F_{\mu\nu}$.

Before studying dynamics, we must express the pure and simple state of motion for the fluid by defining the *velocity* of matter at each point. In order to define the relativistic unit-speed velocity, we shall consider the velocity **v** in an arbitrary Lorentz frame in which it possesses a material particle that is found at the point P, which is considered at the instant in question, and we the construct the four components:

$$u_i = \alpha\, v_i, \qquad u_4 = \alpha\, ic \qquad \text{with} \quad \alpha = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

One knows that, under these conditions, the u_μ will transform like the components of a space-time vector under a Lorentz transformation, and that they will represent the derivatives $u_\mu = dx_\mu / d\tau$ of the relativistic coordinates of the particles with respect to the local proper time τ. One also knows that the norm of any unit-speed velocity is $u_\mu\, u_\mu = -c^2$.

If we are to concern ourselves with the mean local properties or the properties that are attached to a well-defined quantity of the fluid then the unit-speed must be included in the unit-speed velocity. Indeed, any objective physical property is related to the matter itself, and to each of its distinct parts, which must be followed and identified in the course of the motion of the matter. This has a physical meaning that is not pure and simple. For example, it is the momentum at the point P at the instant t, but more precisely, it is the momentum of the material particle that is found at the point P at the instant t and possesses a well-defined velocity by which it will be led to a well-defined point P' at an instant t'. One must then necessarily complete the representation of the fluid properties by field magnitudes with some kinematic corrections.

Therefore, it is mathematically useful to consider the variations of a magnitude $f_{(xyzt)}$ at a given instant in a given reference frame, or its variation in time at a given point, which are variations that can be expressed by the usual partial derivatives of tensor analysis in their covariant combinations: gradient, divergence, rotation, d'Alembertian, etc., but it is obvious that one will arrive at physical realities only by seeking the variations that the magnitude f that is attached to a certain particle of the fluid will be subjected to when one follows the particle in its motion. One is therefore led to attribute a much larger importance to process of *total differentiation along a world-line that is followed by a particle.*

With Weyssenhoff [**2**], one can then define two kinds of derivatives:

On the one hand, for a global relativistic magnitude $\mathfrak{F}$ that expresses the properties of a particle or a quadri-dimensional domain that is cut out of the fluid, the total derivative along the streamline will be:

$$\frac{d\mathfrak{F}}{d\tau} = u_\mu\, \partial_\mu \mathfrak{F} = \frac{\partial \mathfrak{F}}{\partial x^\mu}\frac{dx^\mu}{d\tau}.$$

This expression is a generalization of the Lagrangian derivative of non-relativistic dynamics, namely:

$$\frac{d\mathfrak{F}}{d\tau} = v_i \frac{\partial \mathfrak{F}}{\partial x^i} + \frac{\partial \mathfrak{F}}{\partial t}.$$

Indeed, one has $\frac{\partial \mathfrak{F}}{\partial t} = ic\,\frac{\partial \mathfrak{F}}{\partial x^4}$, such that, upon setting $\alpha = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$, one will have:

$$\frac{d\mathfrak{F}}{d\tau} = \alpha\,\frac{d\mathfrak{F}}{dt} = \alpha\,(v_i\,\partial_i\,\mathfrak{F} + ic\,\partial_4\mathfrak{F}) = \alpha\,v_\mu\,\partial_\mu\,\mathfrak{F} = u_\mu\,\partial_\mu\,\mathfrak{F}\,.$$

In what follows, we shall denote this derivative by a dot:

$$\boxed{\frac{d\mathfrak{F}}{d\tau} = u_\mu \partial_\mu \mathfrak{F} = \dot{\mathfrak{F}}.}$$

In that case, we shall always draw attention to the fact that $\mathfrak{F}$ must necessarily represent a magnitude that is integrated over an *infinitesimal* domain, which is a necessary condition if we are not to equivocate over which point one should follow during the motion, and thus, which velocity u_μ that one must consider. When we are led to study the global magnitudes that are integrated over a finite volume, it will become indispensible to specify to which point of the volume in question one must attach this magnitude and to take the unit-speed velocity *at that point* to be the coefficient u_μ. Moreover, in that case, the point chosen will not coincide with the same particle of fluid in the course of its motion, in general.

One the other hand, if one is dealing with a space-time *density* f then one will consider four non-collinear infinitesimal vectors δx^α, δx^β, δx^γ, δx^λ at a point P, and construct an infinitesimal space-time domain $\delta\omega$ by means of their antisymmetric product. If one considers the fact that the extremities of the four vectors are material points, just like P, that displace with the fluid then the element $\delta\omega$ can be followed in the course of its motion, and the product $f\,\delta\omega$ will represent a well-defined quantity of the same nature as the magnitude $\mathfrak{F}$ that was studied above, namely, $f\,\delta\omega = \delta\mathfrak{F}$.

At the end of a time interval $d\tau$, this quantity $\delta\mathfrak{F}$ will be subjected to a variation $d(\delta\mathfrak{F})$ that we shall express in the sequel by:

$$d(\delta\mathfrak{F}) = u_\mu\,\partial_\mu\,(\delta\mathfrak{F})\,d\tau\,.$$

On the other hand, it follows that:

$$d(\delta\mathfrak{F}) = u_\mu\,\partial_\mu\,f\,d\tau\,\delta\omega + f\,d(\delta\omega)\,.$$

The variation $d(\delta\omega)$ of the space-time element in the proper time interval $d\tau$ will be determined by the infinitesimal vectors that constitute it. For example, the vector δx^1 whose two extremities are the unit-speed velocities u_μ and $u_\mu + \frac{\partial u_\mu}{\partial x^1}\,\delta x^1$ is subjected to a variation $\frac{\partial u_\mu}{\partial x^1}\delta x^1\, d\tau$, and it easy to see that the antisymmetric product $\delta\omega$ will be subjected to a variation $\frac{\partial u_\mu}{\partial x^\mu}\delta\omega d\tau$, (with summation over μ). In summation, one has:

$$u_\mu\,\partial_\mu\,(\delta\mathfrak{F})\,dt = u_\mu\,\partial_\mu\,f\,\delta\omega\,d\tau + f\,\partial_\mu\,u_\mu\,\delta\omega\,d\tau$$

or

$$\frac{d}{d\tau}(\delta\mathfrak{F}) = (u_\mu\,\partial_\mu\,f + f\,\partial_\mu\,u_\mu)\;\delta\omega = \partial_\mu\,(u_\mu\,f)\;\delta\omega.$$

One can therefore define a "density derivative" along a streamline for the density f by setting:

$$\frac{d(f\,\delta\omega)}{d\tau} = \frac{df}{d\tau}\,\delta\omega;$$

i.e.:

$$\boxed{\frac{df}{d\tau} = \partial_\mu(u_\mu\,f).}$$

We shall likewise denote this derivative with a dot:

$$\frac{df}{d\tau} = \partial_\mu\,(u_\mu\,f) = \dot{f}\,,$$

and we remark that the dot does not have the same significance for a global magnitude of type $\mathfrak{F}$ as it does for a density of type f.

Of course, nothing will change in the preceding definition if $\mathfrak{F}$ or f are vectors $\mathfrak{F}_\mu$ or f_μ, resp., or tensors.

§ 2. Conservation laws. – The fundamental physical significance of these derivatives along the streamline will appear when one wants to express the physical permanence – or *conservation* – of a magnitude that is attached to a fragment of matter, such as an invariant density f. In order to define the global magnitude that corresponds to a small material fluid element, we must integrate f over a spatial volume element, not over a space-time elements, as we just did, because matter is a continuum of the spatial kind, and only a quantity of matter in motion that is delimited by a certain spatial volume and the physical properties of that quantity of matter can have any physical significance. In the formal mathematical representation of matter in a four-dimensional universe, one must then return to the consideration of three-dimensional domains of the spatial kind that subjected to relativistic variations under a change of reference frame.

Therefore, we unambiguously define the physical magnitude F that is attached to an infinitesimal droplet of matter by multiplying the *space-time* density f with the proper volume element dV_0 and integrating over the *proper volume* V_0 of the droplet.

$$F = \int_{V_0} f\, d\upsilon_0 \,.$$

One then sees that the conservation of the quantity F by the droplet in the course of its motion can be expressed by the condition:

$$\dot{f} = 0.$$

Indeed, consider the expression for that derivative:

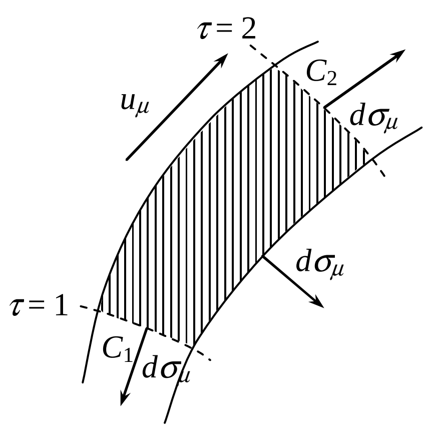
$$\dot{f} = \partial_\mu (f\, u_\mu) \,.$$

We multiply this by the space-time element $d\omega$ and integrate an element of the current tube that is generated by the streamlines that forms a "hyper-boundary" and is bounded by two "hyper-endcaps" C_1 and C_2 that are orthogonal to the current, and therefore constitute two proper space cuts, or – if one prefers – two instantaneous representations of the droplet at the instants 1 and 2.

One can then apply Gauss's theorem:

$$\int_\Omega \partial_\mu (f\, u_\mu)\, d\omega = \int_\Sigma f\, u_\mu\, d\sigma_\mu \,,$$

where Σ denotes the closed hypersurface that delimits the tube element, and $d\sigma_\mu$ denotes the area element of that hypersurface – in other words, the infinitesimal vector that is normal to the hypersurface.



One can decompose the integral into three parts:

1) An integral over the hyper-boundary that is generated by the streamlines, so $d\sigma_\mu$ will be orthogonal to u_μ – i.e., $d\sigma_\mu\, u_\mu = 0$ at every point – and its contribution to the hyper-boundary integral will be zero.

2) An integral over the hyper-endcap C_2, so $d\sigma_\mu$ will be collinear with u_μ at every point, and in the same sense, and one will have $d\sigma_\mu = u_\mu\, dV_0$, where dV_0 is the proper volume element. Thus, it will make a contribution:

$$\int_{V_0(2)} f\, u_\mu u_\nu\, dV_0 = -c^2 \int_{V_0(2)} f\, dV_0 .$$

3) An integral over the hyper-endcap C_1, so one will get the same result, up to the fact that $d\sigma_\mu$ is not oriented in the opposite sense – i.e., $d\sigma_\mu = -u_\mu\, dV_0$ – and one will get a contribution:

$$+c^2 \int_{V_0(1)} f\, dV_0 .$$

The space-time integral over hyper-boundary then reduces to:

$$\int_\Omega \partial_\mu(f\, u_\mu)\, d\omega = c^2 \int_{V_0(1)} f\, dV_0 - c^2 \int_{V_0(2)} f\, dV_0 ;$$

i.e., the variation of the quantity:

$$F = \int_{V_0} f\, dV_0$$

when the droplet passes from the instant 1 to the instant 2, namely:

$$\int_\Omega \dot{f}\, d\omega = c^2 (F_1 - F_2) .$$

If the two endcaps C_1 and C_2 correspond to an infinitesimally small time interval then this equality can be written as:

$$-c^2 \left(\int_{V_0} \dot{f}\, dV_0\right) d\tau = -c^2 \frac{dF}{d\tau}\, d\tau, \qquad \text{since} \quad d\omega = -c^2\, dV_0\, d\tau,$$

so one will finally have:

$$\boxed{\int_{V_0} \dot{f}\, dV_0 = \frac{dF}{d\tau}.}$$

If $\dot{f} = 0$ then the integral will be zero, and the global quantity F that is attached to the droplet will be conserved in the course of motion.

This argument, to which we shall frequently refer, is just as valid when f is a vector f_μ or a tensor $f_{\mu\nu}$.

We will never employ this argument without certain precautions. Indeed, it is convenient to remark that we have attributed a well-defined tensorial variance to the quantities:

$$f \qquad \text{and} \qquad \mathfrak{F} = f\, d\omega, \qquad \text{which are scalars,}$$

$$f_\mu \qquad \text{and} \qquad \mathfrak{F}_\mu = f_\mu\, d\omega, \qquad \text{which are vectors,}$$

$$f_{\mu\nu} \qquad \text{and} \qquad \mathfrak{F}_{\mu\nu} = f_{\mu\nu}\, d\omega, \qquad \text{which are tensors.}$$

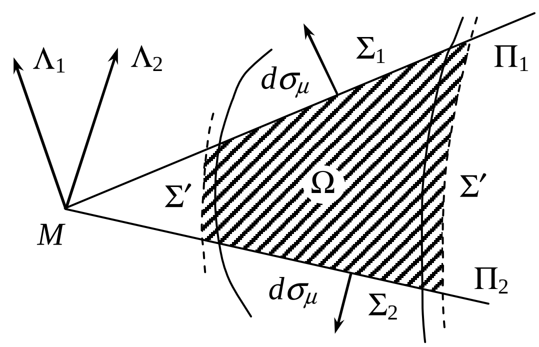
However, as we have remarked, it is difficult to attribute a clear physical significance to these magnitudes. They represent the usual physical properties of matter, which are the *volume* integrals of f, f_μ, $f_{\mu\nu}$ – namely, F, F_μ, $F_{\mu\nu}$ – and in fact, it is only by way of these magnitudes that one can attribute a physical meaning to the densities f, f_μ, $f_{\mu\nu}$ themselves, which are regarded as the *volume* densities of the *physical* magnitudes F, F_μ, $F_{\mu\nu}$ from this viewpoint, while we have attributed to the *variance* of *space-time* densities, by definition.

In general, the volume integral of a space-time tensor density will not be a tensor. Indeed, if we carelessly apply the usual procedure from non-relativistic physics in order to define a global quantity, such as F, that characterizes a certain fluid mass then we will place ourselves in a given reference frame, and integrate the density f over a fluid volume that is considered at a given instant. From the relativistic viewpoint, this will amount to making a cut of the spatial kind that is special to our reference frame in the space-time hyper-tube that is swept out by the fluid mass and defined by a constant value at our particular time. The spatial hyperplane Π that is orthogonal to the time axis Λ will then define an integration hypersurface Σ – in other words, an instantaneous volume of the fluid mass over which one can perform the integration. Since the integral that is obtained will be a sum of infinitesimal (pseudo-) vectors, it seems that one must get a vector. However, if we change the reference frame then it will result in not only an ordinary Lorentz transformation of the differential element, but in a change of the spatial hyperplane Π that is used to make the cut of the spatial kind that we have been led to make on the hyper-tube, so it will no longer be the same cut as before, since the domain of integration will change, and the new quantity will not be related to the old one by the Lorentz relations – i.e., it will not have the variance of a vector. Then again, if one follows de Broglie's expression relative to each reference frame then one will have to consider a different vector [**1**].

One obvious way to avoid this difficulty consists of defining the cut that one makes on the hyper-tube *in a covariant fashion*, once and for all. Since this cut is an intrinsic hypersurface that is defined independently of the reference frame, the tensors that are obtained by integrating over it will also be defined independently of the reference frame. We just recently did this by considering the *proper-space* that is related to the motion of the fluid and integrating over the proper volume of the droplet at each instant. This process seems to be applicable only in the case of an *infinitesimal* fluid mass because the same hyperplane Π might not be orthogonal to the current at each point of a finite fluid mass, and it does not seem possible to generalize this. In the case of a macroscopic hyper-tube, one can think to consider a non-planar hyper-endcap that will be tangent to the *local* proper space at each point, but the integrals that are obtained will not have a clear physical significance, because to quote an expression of Louis de Broglie, "it is physically natural, and likewise almost necessary, to define the tensors in each Galilean system with the aid of an endcap whose points are simultaneous in the same reference system" [**1**]. Furthermore, in the case of vortex motions, which is rightfully that is spinning fluids, surfaces that are orthogonal to the sheaf of streamlines will not be determined uniquely, and the integration will be impossible.

§ 3. Covariant volume integrals. Møller's theorem. – Meanwhile, in the case of macroscopic fluid masses, there exist certain volume integrals that are tensorially independent of the spatial hyperplane over which they are integrated. This is a consequence of an important theorem of Møller [**3**]. First, consider the most simple case, which is that of a vector density j_μ that expresses a certain property of matter, and assume that this vector is conservative $\partial_\mu j_\mu = 0$, by virtue of a kinematical or dynamical law.

The integral of the time component j_4 over a spatial hyper-endcap Σ will then be *invariant* under a Lorentz transformation, which will naturally accompany any modification of the hyper-endcap, as we have said.



In order to prove this, place the collective origin of the reference frames at a space-time point M, which we choose, for example, to be outside of the hyper-tube, and consider spatial hyperplanes Π_1 and Π_2 that cut the hyper-tube in two hyper-endcaps Σ_1 and Σ_2 that follow each other in time and define two Lorentz frames $\Pi_1\Lambda_1$ and $\Pi_2\Lambda_2$. The endcaps Σ_1 and Σ_2 delimit a certain frustrum of the hyper-tube, and we form a closed surface by completing the two endcaps with a hyper-boundary Σ' that is situated outside of the matter in its entirety, and over which, as a consequence, the vector j_μ will be zero. The set will enclose a space-time domain Ω over which we shall integrate.

Consider the conservation relation $\partial_\mu j_\mu = 0$, multiply the left-hand side by the space-time element $d\omega$, and integrate over the domain Ω:

$$\int_\Omega \partial_\mu j_\mu \, d\omega = 0.$$

One can apply Gauss's theorem by decomposing the hypersurface that bounds Ω into three parts Σ_1, Σ_2, and Σ':

$$\int_{\Sigma_1} j_\mu \, d\sigma_\mu + \int_{\Sigma_2} j_\mu \, d\sigma_\mu + \int_{\Sigma'} j_\mu \, d\sigma_\mu = 0.$$

The last term (viz., the hyper-boundary term) is zero, since Σ' is contained entirely *in vacuo*. Incidentally, since every element $j_\mu \, d\sigma_\mu$ is a scalar, one can evaluate it in any reference frame. We refer the vectors in the first integral to the reference frame $\Pi_1\Lambda_1$. One then sees that $d\sigma_\mu$ is parallel to the time axis Λ_1, and has components:

$$\left(0, 0, 0, d\sigma_4^{(1)}\right) = ic\, dV_1,$$

in which, dV_1 is the volume element for the hyperplane Π_1. One will then have:

$$(j_\mu\, d\sigma_\mu)^{(1)} = ic\ \ j_4^{(1)}\, dV_1\ .$$

Similarly, we use the reference $\Pi_2\Lambda_2$ for the second integral, and we will get a corresponding result, up to the fact that $d\sigma_\mu$ will be directed in the opposite sense, so one will have:

$$d\sigma_4^{(2)} = -\, ic\, dV_2\ .$$

Finally, one will then have:

$$ic \int_{\Sigma_1} j_4^{(1)}\, dV_1 - ic \int_{\Sigma_2} j_4^{(2)}\, dV_1 = 0.$$

In other words, the quantity:

$$\int_\Sigma j_4\, dV = ic\, J$$

will be an invariant, independently of the reference and the hyper-endcap that corresponds to it. Moreover, this result will not be modified if one chooses the origin M to be in the tube's interior.

One can easily show that the global quantity J will likewise be *constant in time.* Consider an arbitrary reference frame and its associated time coordinate $ict = x_4$. One has:

$$\frac{dJ}{dt} = ic \frac{dJ}{dx_4} = \frac{d}{dx_4} \int_\Sigma j_4\, dV = \frac{d}{dx_4} \int_\infty j_4\, dV\,,$$

because one can integrate over all space if one desires. Thus:

$$\frac{dJ}{dt} = \int_\infty \partial_4 j_4\, dV\ .$$

The conservation relation $\partial_\mu\, j_\mu = 0$ will permit us to replace $\partial_4\, j_4$ with $-\,\partial_k\, j_k$, and one will then have a *volume* integral:

$$\frac{dJ}{dt} = -\int_\infty \partial_k\, j_k\, dV\,,$$

which one can, by virtue of Gauss's theorem, transform into a surface integral:

$$\int_S j_k ds_k\ .$$

Now, the surface S will go to infinity on the spatial hyperplane Π, which is found entirely *in vacuo*, so the integral will be zero. One will then have:

$$\frac{dJ}{dt} = 0$$

precisely for any time that is used.

One then has a main theorem: The volume integral:

$$\int_\Sigma j_4 \, dV$$

of the time component of a *conservative* vector is a *scalar* that is constant in time.

It is easy to generalize these properties to the case of a tensor. For example, consider the conservative second-order tensor $t_{\mu\nu}$, for which one has $\partial_\mu t_{\mu\nu} = 0$.

We introduce an *arbitrary* auxiliary vector field k_μ that we assume to be *uniform*, and we form the vector:

$$f_\nu = k_\mu \, t_{\mu\nu}$$

at each point. It is conservative because:

$$\partial_\nu f_\nu = k_\mu \, \partial_\nu t_{\mu\nu} + t_{\mu\nu} \, \partial_\nu k_\mu \, .$$

The first term is zero, since $\partial_\nu t_{\mu\nu} = 0$, and the second is zero, as well, since k_μ is uniform, and one will then have:

$$ic\, F = k_\mu \int_\Sigma t_{\mu 4} \, dV \, ,$$

precisely, or upon setting:

$$ic\, G_\mu = \int_\Sigma t_{\mu 4} \, dV \, ,$$

we will get:

$$F = k_\mu \, G_\mu = \text{scalar.}$$

However, since k_μ is an *arbitrary* vector, but with the condition that it must be uniform, one will see from the tensoriality condition that the relation that we want to establish will show that k_μ is a *vector* that relates to a Lorentz transformation. Furthermore, by virtue of the preceding theorem:

$$\frac{dF}{dt} = 0,$$

and due to the fact that since k_μ is uniform it must obviously be constant in time, as well, from the relativistic viewpoint, one will likewise have:

$$\frac{dG_\mu}{dt} = 0.$$

Finally, the proof is exactly the same in the case of a conservative third-order tensor, such as $m_{\mu\nu\lambda}$, with $\partial_\lambda m_{\mu\nu\lambda} = 0$. It will suffice to multiply it by an arbitrary, uniform, second-order *tensor* $k_{\mu\nu}$, and one will immediately desire that the integral:

$$ic\, M_{\mu\nu} = \int_\Sigma m_{\mu\nu 4}\, dV$$

should be a second-order tensor that is constant in time.

One thus has the general theorem: The spatial integral of the fourth component of a *conservative* tensor is a *tensor* whose order is less than one that is constant in time.

In this form, Møller's proof does not seem irreproachable to us, because one cannot carelessly apply Gauss's theorem to the integral of a magnitude that is subject to a discontinuity on a surface of integration that is inside the domain of integration, as is the case for the quantities j_μ, $t_{\mu\nu}$, and $m_{\mu\nu\lambda}$, over the hypersurface of the tube that is swept out by the matter. One can naturally assume that the matter is not separated from the vacuum by a sharp discontinuity, but by a "transition layer," where the material magnitudes vanish gradually. However, in order for conservation equations such as:

$$\partial_\mu j_\mu = 0, \qquad \partial_\nu t_{\mu\nu} = 0, \qquad \partial_\lambda\ m_{\mu\nu\lambda} = 0$$

to remain valid in the transition layer (which is essential for the proof), the special conditions that the quantities j_μ, $t_{\mu\nu}$, and $m_{\mu\nu\lambda}$ obey in the transition layer must be realized, and if the thickness of the transition layer tends to zero then they must translate into the proper limits on the surface of the drop under these conditions. It then seems more suggestive to us to effectively consider a discontinuity surface that is characterized by special physical properties and apply Møller's argument to the hyper-tube that is swept out by the matter, while the boundary Σ' is not only *in vacuo*, but is also *generated by the same surface of the drop*. The application of Gauss's theorem will become legitimate under these conditions. By contrast, the boundary integrals:

$$\int_\Sigma j_\mu d\sigma_\mu\,, \qquad \int_\Sigma k_\mu t_{\mu\nu} d\sigma_\mu\,, \quad \int_\Sigma k_{\mu\nu} m_{\mu\nu\lambda} d\sigma_\lambda$$

are not automatically zero, and Møller's theorem can be applied only in the case where the boundary integrals are found to be zero by reason of the surface conditions that are determined by the physical phenomena. We have to examine that aspect of the problem whenever we appeal Møller's theorem.

Similarly, the proof that quantities such as J or $k_\mu\, G_\mu$ are constant in time must be repeated, since the integrals are now taken over domains that vary in time. Therefore, from a classical formula:

$$\frac{dJ}{dt} = ic\ \frac{d}{dx^4} \int_\Sigma j_4 dV$$

will give:

$$ic\ \int_\Sigma \partial_4 j_4 dV + \int_S j_4 v_k ds_k\,,$$

in which, v_k is the velocity of the matter through the surface S of the drop and in the reference $\Pi\Lambda$.

The equation $\partial_\mu j_\mu = 0$ will then permit us to write:

$$\int_\Sigma \partial_4 j_4 dV = -\int_\Sigma \partial_k j_k dV ,$$

which will transform into a surface integral $-\int_S j_k ds_k$, and one will finally have:

$$\frac{dJ}{dt} = \int_S (j_4 v_k - ic\, k_k)\, ds_k .$$

Whether or not this integral is zero must be examined in each case and in light of physical considerations.

APPENDIX B

§ 1. The relativistic hydrodynamics of classical fluids. It will prove useful for us to recall the basics of the relativistic equations of hydrodynamics in this chapter. One knows that the arbitrary fluids of non-relativistic hydrodynamics obey, on the one hand, the conservation equation:

$$\frac{\partial}{\partial t}(\rho m) + \partial_i(\rho\, m\, v_j) = 0,$$

and, on the other hand, the three Euler equations:

$$\frac{\partial}{\partial t}(\rho m\, v_i) + \partial_j(\rho m\, v_i\, v_j + \tau_{ij}) = F_i\,,$$

in which F_i is the force density or the external force per unit volume. The fluid is the site of internal stresses that are represented by the tensor τ_{ij}, which is symmetric in i and j: τ_{ij} is the component of the stress force along the x_i axis that is exerted on a section of unit area that is orthogonal to the x_i axis. One expresses this tensorially by saying that a stress force:

$$T_j\, dS = \tau_{kj}\, d\sigma_k = \tau_{jk}\, d\sigma_k$$

is exerted over the area element of measure dS that is represented by the infinitesimal ds_k.

One likewise knows that the stresses produce a volume force that is equal to:

$$\varphi_j\, dv = -\,\partial_k\, \tau_{jk}\, dv$$

over a small volume element dv. Hence, the tensor τ_{ij} will likewise play the role of a potential for the stress forces per unit volume. We finally remark that we have written the mass density $\rho\, m$ in the form of a product of a "matter density" ρ – viz., the density of a number of identical particles that the fluid is assumed to be comprised of – with the mass m of these particles.

In order to pass on to relativistic mechanics, we must bring two types of considerations into view. On the one hand, the classical momentum $\rho\, m\, v_i$ brings only the energy that is related to the mass of the particles into play. Now, the internal stresses give rise to a special potential energy that, from the relativistic viewpoint, possesses inertia, and must therefore contribute to the momentum. If we consider an element of the fluid where the velocity is v_j then we will know that the force that acts on a surface element by way of the stresses is $T_j\, dS = \tau_{kj}\, d\sigma_k$.

During a time dt, the surface element is displaced by $v_j\, dt$, and the force $T_j\, dS$ has therefore performed work that is equal to:

$$T_j\, v_j\, dS\, dt = \tau_{kj}\, v_j\, d\sigma_k\, dt.$$

This expression shows that the unit area that is normal to $d\sigma_k$ is traversed in a unit time by an energy flux that is due to internal stresses and is equal to $\tau_{kj}\, v_j$.

This flux corresponds to a relativistic momentum density $1/c^2\, \tau_{kj}\, v_j$ that must be added to the momentum $\rho\, m\, v_k$ in the relativistic transcription of the two classical equations:

$$\frac{\partial}{\partial t}(\rho m) + \partial_j\left(\rho m v_j + \frac{1}{c^2}\tau_{jk} v_k\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\rho m v_i + \frac{1}{c^2}\tau_{jk} v_k\right) + \partial_j\left(\rho m\, v_i v_j + \frac{1}{c^2}\tau_{ik} v_j v_k\right) = F_i\ .$$

On the other hand, one knows that the equations of classical hydrodynamics, which are not invariant under Lorentz transformations, are valid with full rigor only in a particular frame of reference. If we consider a point P of the fluid at an instant t and a Galilean reference frame Σ_0 that is attached to this point (viz., *the local proper reference frame*) then the equations above describe the motion of the fluid in the immediate neighborhood of the point P, and only during an infinitesimal time interval dt. The equations above are therefore rigorous insofar as they are the projections of relativistically covariant equations onto the proper axes at P. The velocity v_i is zero in the proper system (while its derivatives with respect to the coordinates and time are not). It results from this that, in particular, the terms $\partial_j(\rho m\, v_i\, v_j)$ and $\partial_j(\tau_{ik}\, v_i\, v_j)$ will be zero. Relative to the proper axes, one will therefore have:

$$\frac{\partial}{\partial t}(\rho m) + \partial_j\left(\rho m v_j + \frac{1}{c^2}\tau_{jk} v_k\right) = 0, \tag{B.1}$$

$$\frac{\partial}{\partial t}\left(\rho m v_i + \frac{1}{c^2}\tau_{jk} v_k\right) + \partial_j \tau_{ij} = F_i\ . \tag{B.2}$$

We now introduce the tensorial quantities that enter into the relativistic equation and study their projections onto the proper axes, along with those of their derivatives. The unit-speed velocity u_μ has the general expression:

$$u_i = \alpha v_i\ , \qquad u_4 = \alpha\, ic \qquad [\alpha = (1 - v^2/c^{2)-1/2}].$$

Its projections onto the proper system are 0, 0, 0, ic. The derivatives are:

$$\partial_k\alpha = \frac{v\,\partial_k v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2},$$

$$\partial_4\alpha = \frac{v\,\partial_4 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}.$$

One then sets $v = 0$, which makes:

$$(\alpha)^0 = 1, \qquad (\partial_k\alpha)^0 = (\partial_4\alpha)^0 = 0,$$

$$(\partial_j\, u_k)^0 = \partial_j\, v_k\,, \qquad (\partial_4\, u_k)^0 = \frac{1}{ic}\frac{\partial}{\partial t}v_k\,, \qquad (\partial_j\, u_4)^0 = (\partial_4\, u_4)^0 = 0.$$

Neither the density ρ – i.e., the quantity of matter per unit volume – nor the mass m are relativistic invariants. In the proper system, they take the values ρ_0 and m_0, which characterize the matter at rest. One may then *define* two invariants that have the values ρ_0 and m_0 in *any* reference frame: m_0 is the *proper mass* and ρ (we shall suppress the index 0) is the *invariant matter density*. The conservation of matter translates into the covariant relation:

$$\boxed{\partial_\mu(\rho u_\mu) = 0;}$$

namely (see Appendix *A*):

$$\dot{\rho} = 0.$$

The number of molecules per unit volume remains constant along the streamline.

One defines a spacetime vector – viz., the *relativistic external force density* f_μ – which has the components:

$$f_k^0 = F_k\,, \qquad f_4^0 = 0$$

in the proper system. One thus has:

$$f_\mu\, u_\mu = 0.$$

Similarly, one defines a symmetric tensor of *internal stresses* $\theta_{\mu\nu}$, which has the components:

$$\theta_{ij}^0 = \tau_{ij}, \qquad \theta_{i4}^0 = \theta_{44}^0 = 0$$

in the proper system. One therefore has:

$$\theta_{\mu\nu}\, u_\nu = \theta_{\nu\mu}\, u_\nu = 0.$$

In the proper system, one has:

$$[\partial_\lambda(\theta_{\mu j}\, u_j)]^0 = \theta_{\mu j}^0(\partial_\lambda u_j)^0 = \theta_{\mu j}^0\partial_\lambda v_j\,,$$

$$[\partial_\lambda(\theta_{\mu 4}\, u_4)]^0 = ic\,(\partial_\lambda\,\theta_{\mu 4})^0,$$

and since:

$$\theta_{\mu j}\, u_j = -\theta_{\mu 4}\, u_4\,,$$

one has:

$$ic\,(\partial_\lambda\,\theta_{\mu 4})^0 = -\,\theta_{\mu j}^0\partial_\lambda v_j\,;$$

i.e.:

$$(\partial_i \theta_{k4})^0 = \frac{i}{c}\,\tau_{kj}\,\partial_i v_j\,, \qquad (\partial_i \theta_{44})^0 = 0,$$

$$(\partial_4 \theta_{k4})^0 = \frac{1}{c^2}\,\tau_{kj}\,\frac{\partial v_j}{\partial t}\,, \qquad (\partial_4 \theta_{44})^0 = 0.$$

One can arrive at the relativistic tensor equation by subjecting equations (1) and (2) to an arbitrary Lorentz equation, and seek to make the components of the tensorial quantities that we just defined evident in the transformed equation [**3**]. It is simpler to consider, as Lichnerowicz did [**44**], the tensorial equation:

(B.3) $$\boxed{\partial_\nu(\rho m_0 u_\mu u_\nu + \theta_{\mu\nu}) = f_\mu,}$$

and to show that its projections onto the proper axes are identical to equations (1) and (2). Since equation (3) is tensorial, one is then assured that when a Lorentz transformation is performed on equations (1) and (2) this would give equations that are the projections of the covariant equations onto any arbitrary reference.

The projections of equation (3) onto the space and time axes give:

$$\partial_4(\rho m_0\, u_4\, u_4 + \theta_{44}) + \partial_j(\rho m_0\, u_4\, u_j + \theta_{4j}) = f_4\,,$$

$$\partial_4(\rho m_0\, u_i\, u_4 + \theta_{i4}) + \partial_j(\rho m_0\, u_i\, u_j + \theta_{ij}) = f_i\,.$$

In the proper system:

$$f_4^0 = 0, \qquad [\partial_j(\rho m_0\, u_i\, u_j)]^0 = 0.$$

Now apply the formulas that we established for the expression of the derivatives:

$$\frac{\partial}{\partial t}(\rho m_0\, ic) + \partial_j(\rho m_0\, v_j\, ic) + \frac{i}{c}\,\tau_{ij}\,\partial_j v_i = 0,$$

$$\frac{\partial}{\partial t}(\rho m_0\, v_i) + \frac{1}{c^2}\,\tau_{ik}\,\frac{\partial v_k}{\partial t} + \partial_j \tau_{ij} = F_i\,.$$

These are precisely the equations that were given above.

The covariant equation $\partial_\nu(\rho m_0\, u_\mu u_\nu + \theta_{\mu\nu}) = f_\mu$ brings into view the symmetric tensor:

$$\boxed{t_{\mu\nu} = \rho m_0 u_\mu u_\nu + \theta_{\mu\nu},}$$

which one calls the *energy-momentum tensor*. The structure of the energy-momentum tensor determines the fundamental properties of a fluid in a general fashion and permits us to classify the various types of fluids. The form for the tensor $t_{\mu\nu}$ that we will arrive at characterizes an entity that we call a *classical fluid*. In order for that to be the case, assume that the tensor $t_{\mu\nu}$ possesses the following two properties:

1) It is *symmetric*, which, as we know, expresses the idea that the fluid does not have any internal kinetic moment.

2) If one contracts it with u_ν then one obtains:

$$t_{\mu\nu}\, u_\nu = -\rho\, m_0\, c^2\, u_\mu\,,$$

since

$$\theta_{\mu\nu}\, u_\nu = 0.$$

In other words, the unit-speed velocity is an *eigenvector* for the matrix $t_{\mu\nu}$. We know that this property expresses the idea that the momentum density vector of the fluid is collinear with the current. In the majority of cases (e.g., the "normal" schemes of Lichnerowicz), one may determine four directions by means of an arbitrary given tensor $t_{\mu\nu}$ that are the eigen-directions of the matrix $t_{\mu\nu}$, and which are necessarily time-like [**45**]. If one chooses the normal vector, when it is conveniently oriented along the fourth eigen-axis, to be the unit-speed velocity vector then the indicated property will be realized *by definition*. However, the fluids that we shall encounter in the hydrodynamical part of the present book will generally have a unit-speed velocity that is determined by "kinematical" considerations, independently of the energy-momentum tensor; hence, the property in question is not found to be realized, in general.

Our work is principally dedicated to the study of non-classical fluids that have a momentum that is non-collinear with the velocity and an internal angular momentum density. However, in order to guide us in this domain, it is indispensible to recall the principal results that are acquired from the study of classical fluids [**46**]. We will need to consider the evolution of these fluids only in the absence of external forces. The tensor $t_{\mu\nu}$ will then be conservative:

$$\boxed{\partial_\nu t_{\mu\nu} = 0.}$$

The various cases that one might encounter are distinguished by the peculiarities of the relativistic stress tensor $\theta_{\mu\nu}$. In order to review the principal ones, we proceed from the particular to the general and let ourselves be guided by non-relativistic hydrodynamics.

The simplest case is the one in which there are no internal stresses. The tensor $\theta_{\mu\nu}$ is then identically zero, and the energy-momentum tensor reduces to the kinetic part: $\rho m_0\, u_\mu u_\nu$. One then says that one is dealing with a *pure matter* fluid.

§ 2. Perfect fluids. The second case is the relativistic generalization of the *perfect fluid*, or inviscid fluid. One knows that in non-relativistic hydrodynamics this case is characterized by the fact that the stress force at a point P has a magnitude that is independent of the direction of the cut and a direction that is perpendicular to the cut. The non-relativistic stress tensor is then:

$$\tau_{ij} = p\ \delta_{ij}\ ,$$

p being the *pressure*, which suffices to characterize the internal forces. If one defines the relativistic tensor $\theta_{\mu\nu}$ by the fact that one has:

$$\theta_{ij}^{0} = \tau_{ij}, \qquad \theta_{i4}^{0} = \theta_{4i}^{0} = \theta_{44}^{0} = 0$$

in the proper system then one will easily see that $\theta_{\mu\nu}$ has the form:

$$\theta_{\mu\nu} \equiv p\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) \equiv p\,\eta_{\mu\nu}\,.$$

The second term makes no contribution to the components of θ_{ij}^{0}, θ_{i4}^{0}, and θ_{4i}^{0} in the proper system, but it does serve to annul the component θ_{44}^{0}. The energy-momentum tensor $t_{\mu\nu} = \mu_0\, u_\mu\, u_\nu + p\,\eta_{\mu\nu}$ can also be written:

(B.4) $$\boxed{t_{\mu\nu} = \mu_0 u_\mu u_\nu + p\,\delta_{\mu\nu}\,,}$$

if one sets:

$$\mu_0 = \rho M_0 = \rho m_0 + \frac{p}{c^2}$$

and introduces the variable proper mass – or *pseudo-mass*:

$$M_0 = \rho m_0 + \frac{p}{c^2}\,,$$

which differs from m_0 by only a relativistic term.

This new decomposition is more suggestive that the preceding one. It adds – in covariant form, this time – the supplementary mass that relates to the stress energy to the proper mass of the particle, which erases the absolute disjunction between the kinetic energy-momentum and the stresses, in conformity with relativistic ideas. This causes a new tensor to appear now:

$$\pi_{\mu\nu} = p\ \delta_{\mu\nu}\,,$$

which we shall call the *pressure tensor*, and which is no longer in the proper space, because it possesses a non-zero temporal component $\pi_{44}^{0} = p$ in the proper system.

Upon taking the divergence of this tensor, one will define a force density, and upon dividing it by μ_0 one will introduce a pressure force that relates to the pseudo-mass scalar that has the dimensions of a weight field. We call it the *internal pressure field:*

$$K_\mu = -\,\partial_\nu\,\pi_{\mu\nu}$$

or

$$\boxed{K_\mu = -\frac{\partial_\mu p}{\rho m_0 + p / c^2}.}$$

The fundamental relation $\partial_\nu t_{\mu\nu} = 0$ gives us:

$$\partial_\nu(\mu_0 u_\mu u_\nu) = -\partial_\mu p = \mu_0 K_\mu ,$$

or

(B.5) $$u_\mu \partial_\nu(\mu_0 u_\nu) + \mu_0 u_\nu \partial_\nu u_\mu = \mu_0 K_\mu .$$

Contracting this with u_μ gives:

$$-c^2\,\partial_\nu(\mu_0\,u_\nu) = \mu_0\,K_\mu\,u_\mu$$

because $u_\mu\,\partial_\nu\,u_\mu = 0$. If one replaces μ_0 with $\rho\,m_0 + p / c^2$ then it follows that:

(B.6) $$(\rho\,m_0 + p / c^2)\,\partial_\nu\,u_\nu + u_\nu\,\partial_\nu(\rho\,m_0) + u_\nu\,\partial_\nu(p / c^2) = -\frac{\mu_0}{c^2}K_\nu\,u_\nu = -\frac{1}{c^2}\partial_\nu\,p + u_\nu .$$

What then remains is:

$$\boxed{\left(\rho m_0 + \frac{p}{c^2}\right)\partial_\nu u_\nu + u_\nu \partial_\nu(\rho m_0) = 0.}$$

This is the *conservation equation* for perfect fluids. It may also be written (see Appendix *A*):

$$\partial_\nu(\rho\,m_0\,u_\nu) \equiv \frac{d}{d\tau}(\rho\,m_0) = -\frac{1}{c^2}p\,\partial_\nu\,u_\nu ,$$

or, since $d\rho / d\tau = 0$:

$$\rho\frac{dm_0}{d\tau} = -\frac{1}{c^2}p\,\partial_\nu\,u_\nu .$$

One sees that the special proper mass does not remain constant. The scalar $\partial_\nu\,u_\nu$, which may be evaluated in the proper system where it becomes $\partial_j\,u_j$, represents the divergence of the velocity – i.e., the *dilatation* of the matter. The work that is done by this dilatation against the internal pressure produces a specific energy $-\,p\,\partial_\nu\,u_\nu$ whose mass $-\,c^2 p\,\partial_\nu\,u_\nu$ gets added to the proper mass density.

Upon taking (6) into account, equation (5) will become:

$$-\frac{\mu_0}{c^2}K_\nu\,u_\nu\,u_\mu + \mu_0\,u_\nu\,\partial_\nu\,u_\mu = \mu_0 K_\mu ,$$

or, upon setting (see Appendix *A*):

$$u_\nu\,\partial_\nu\,u_\mu = \frac{d}{d\tau}u_\mu = \dot{u}_\mu ,$$

$$\dot{u}_\mu = \eta_{\mu\nu} K_\nu, \tag{B.7}$$

and if one replaces K_ν with its value then one will get:

$$\boxed{(\rho m_0 + p/c^2)\dot{u}_\mu = -\eta_{\mu\nu}\partial_\nu p,}$$

which are equations that, along with the relation:

$$\dot{x}_\mu = u_\mu,$$

constitute the *differential system of the streamlines* for the perfect fluid.

The most interesting case (which is the one that would be true for material fluids) is the one in which the fluid is constrained by an *equation of state* that makes the density – viz., the proper mass density, here – depend uniquely upon the pressure:

$$\rho\, m_0 = \Phi(p).$$

In this case, one may make the force field appear in the form of the gradient of a scalar potential. One sets:

$$K_\mu \equiv -\frac{\partial_\mu p}{\rho m_0 + p/c^2} = -c^2\,\partial_\mu \log F,$$

so the scalar potential is log F and the factor c^2 appears for reasons of homogeneity. Since $\rho\, m_0$, and as a consequence $\rho\, m_0 + p / c^2$, depend upon the coordinates only by the intermediary of p, one can integrate the last equation and obtain:

$$\boxed{\log F = \frac{1}{c^2}\int_{p_0}^{p}\frac{dp}{\rho m_0 + p/c^2}.}$$

log F is therefore a relativistic quantity in c^{-2}, and F is a pure number that is very close to unity, and can be reasonably approximated by:

$$F \approx 1 + \frac{1}{c^2}\int_{p_0}^{p}\frac{dp}{\rho m_0 + p/c^2}.$$

Lichnerowicz has proved [**44**] – and in the more general case of "holonomic fluids," which we shall define shortly – that the function F enjoys the following property: The fluid streamlines realize the extremum of the integral $\int F\,ds$, when it is taken between two fixed extremes. This analogy with the optics of Fermat-Malus leads us to give the name of *index* of the fluid to F.

One then introduces the vector:

$$\boxed{F u_\mu = C_\mu,}$$

which is collinear with the velocity and differs in magnitude by only a quantity in c^{-2}. We prefer to call this vector (which Lichnerowicz called the "current") the *weighted velocity*, while reserving the term "current" for the vector $j_\mu = \rho\, u_\mu$, which does not have the same dimensions, and which is intrinsically conservative (while C_μ is conservative only in a special case). The vector C_μ enjoys some important analytical properties: In particular, its circulation $\int C_\mu\, dx_\mu$ around a closed contour will not vary when one deforms the contour in such a manner that it remains on a current tube [**46**], and this will remain true in the general case of holonomic fluids.

The relativistic rotation of the weighted velocity vector:

$$\boxed{\Omega_{[\mu\nu]} = \partial_\mu C_\nu - \partial_\nu C_\mu}$$

is the *vorticity tensor*. If one specifies C_μ then:

$$\Omega_{\mu\nu} = \partial_\mu (F\, u_\nu) - \partial_\nu (F\, u_\mu) = F(\partial_\mu u_\nu - \partial_\nu u_\mu) + u_\nu\, \partial_\mu F - u_\mu\, \partial_\nu F$$

$$= F(\partial_\mu u_\nu - \partial_\nu u_\mu) - \frac{F}{c^2}(\, u_\nu K_\mu - u_\mu K_\nu).$$

The vorticity differs from a tensor that is collinear with (and almost equal to) the usual vorticity of the unit-speed velocity by a relativistic term that is the exterior product of the unit-speed velocity with the pressure force field. This being the case, it then results from the equations of motion that the vorticity tensor is orthogonal to the current, and consequently that all of its components are in proper space. Indeed:

$$\Omega_{\mu\nu}\, u_\nu = F(u_\nu \partial_\mu u_\nu - u_\nu \partial_\nu u_\mu) - \frac{F}{c^2}(-c^2 K_\mu - u_\mu K_\nu u_\nu).$$

The first term in the first parenthesis is zero, since $u_\nu\, u_\nu = -c^2$. From (7), the second parenthesis is equal to $-K_\nu\, \eta_{\mu\nu}$, so the second parenthesis will be annulled.

If one places oneself in proper space then one may form a proper space vector that is defined to be the spatial dual of the vorticity tensor, namely:

$$\theta_k^0 = \tfrac{1}{2}\varepsilon_{ijk}\Omega_{ij}^0 .$$

This vector, which is called the *vorticity vector*, is expressed in a covariant fashion by way of:

$$\boxed{\theta_\mu = \frac{i}{2c}\varepsilon_{\nu\alpha\beta\mu} u_\nu \Omega_{\alpha\beta},}$$

with $\theta_\mu\, u_\mu = 0$, as is easy to verify (Chap. III).

If one specifies $\Omega_{\alpha\beta}$ then one will see that:

$$\theta_\mu = \frac{i}{c}\,\varepsilon_{\nu\alpha\beta\mu}\,F\,u_\nu\,\partial_\alpha\,u_\beta + \frac{i}{c}\,\varepsilon_{\nu\alpha\beta\mu}\,u_\nu\,u_\beta\,\partial_\alpha\,F,$$

so

$$\boxed{\theta_\mu = \frac{i}{c}F\varepsilon_{\nu\alpha\beta\mu}u_\nu\partial_\alpha u_\beta,}$$

since the second term goes to zero, by antisymmetry.

One sees that the vorticity vector:

$$\theta_\mu = \frac{i}{c}\,\varepsilon_{\nu\alpha\beta\mu}\,u_\nu\,\partial_\alpha\,C_\beta$$

is collinear with the analogous vector that is formed from the rotation of the unit-speed velocity:

$$\frac{i}{c}\,\varepsilon_{\nu\alpha\beta\mu}\,u_\nu\,\partial_\alpha\,u_\beta,$$

but the latter does not correspond to an antisymmetric proper-space tensor that is analogous to $\Omega_{\mu\nu}$. One must guard against confusing the tensor $\Omega_{\mu\nu}$ and the vector θ_μ with the internal angular momentum tensor $S_{\mu\nu}$ and the spin σ_μ that we encountered all along in this book. Indeed, the latter quantities refer to a *proper* rotation that is characteristic of non-classical fluids and is independent of the vorticial or *orbital* motion that is described by $\Omega_{\mu\nu}$ and θ_μ. In the first place, $S_{\mu\nu}$ and σ_μ are *kinetic* quantities that involve the mass, while $\Omega_{\mu\nu}$ and θ_μ are *kinematical* quantities that involve only the velocity (weighted by the internal pressure potential).

A particular case of the perfect fluid serves to fix the role of the weighted velocity vector C_μ. Non-relativistic hydrodynamics studies the perfect *incompressible* fluid as a limiting case that is characterized by the state equation ρ = const.

The fluid density remains invariant for any magnitude of the pressure. On the other hand, since one has the conservation equation $\partial_\mu(\rho u_\mu) = 0$, this case will translate into the relativistic equation:

$$\partial_\mu\,u_\mu = 0.$$

However, a rigorously incompressible fluid will transmit deformations with an infinite velocity, which is contrary to the principles of relativity. In fact, one shows [**46**] that the velocity of a wave front in a perfect fluid that admits a state equation is:

$$v = \frac{1}{\sqrt{d(\rho m_0)/dp}}\,.$$

Since this velocity is necessarily less than c in relativity, one calls a bounded fluid in which the wave fronts are propagated with the velocity c an *incompressible fluid.* For the perfect, incompressible fluid, one will then have:

$$\frac{d(\rho m_0)}{dp} = \frac{1}{c^2},$$

which can be integrated immediately, and will give the equation of state:

$$\boxed{\rho m_0 - \frac{p}{c^2} = \text{const.}}$$

If we take the conservation equation into account, namely:

$$\left(\rho m_0 - \frac{p}{c^2}\right) \partial_\alpha u_\alpha + u_\alpha \partial_\alpha (\rho m_0) = 0,$$

then the equation of state will give:

$$u_\alpha \partial_\alpha \left(\rho m_0 - \frac{p}{c^2}\right) = 0,$$

such that:

$$u_\alpha \partial_\alpha (\rho m_0) = u_\alpha \partial_\alpha \left(\frac{p}{c^2}\right)$$

and

$$\left(\rho m_0 - \frac{p}{c^2}\right) \partial_\alpha u_\alpha + u_\alpha \partial_\alpha \left(\frac{p}{c^2}\right) = 0,$$

from which:

$$\partial_\alpha u_\alpha = - u_\alpha \frac{\partial_\alpha p / c^2}{\rho m_0 + p / c^2}.$$

This is the *relativistic incompressibility equation*, which replaces the equation:

$$\partial_\alpha u_\alpha = 0,$$

which is impossible to realize. However, the right-hand side is simply $u_\alpha K_\alpha / c^2$, and when one expresses K_α as a function of the index F, it will follow that:

$$\partial_\alpha u_\alpha = - u_\alpha \partial_\alpha \log F,$$

or

$$F \partial_\alpha u_\alpha = - u_\alpha \partial_\alpha F,$$

namely:

$$\partial_\alpha (F u_\alpha) = 0.$$

One sees that the relativistic incompressibility is expressed by:

$$\boxed{\partial_\alpha C_\alpha = 0.}$$

More generally, the divergence of the weighted velocity vector C_α characterizes the degree of incompressibility of the fluid. One may then give the name of *relativistic compressibility* to the scalar $\partial_\alpha C_\alpha$.

The notion of pressure that was introduced for perfect fluids and the decomposition (4) of the energy-momentum tensor can be generalized to an arbitrary classical fluid. We start with the decomposition:

$$t_{\mu\nu} = \rho m_0\, u_\mu\, u_\nu + \theta_{\mu\nu}, \qquad \theta_{\mu\nu}\, u_\nu = \theta_{\mu\nu}\, u_\mu = 0,$$

which characterizes the classical fluid.

If the fluid is perfect then:

$$\theta_{\mu\nu} = p\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right),$$

and the scalar that is obtained by contracting this will be:

$$\theta_{\mu\mu} = p\left(4 - \frac{c^2}{c^2}\right) = 3p.$$

One can always form the same scalar for an arbitrary fluid by setting:

$$\boxed{3p = \theta_{\mu\mu}.}$$

The stress tensor can then be decomposed into a term:

$$p\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) \equiv p\,\eta_{\mu\nu}$$

and a term $\Lambda_{\mu\nu}$ that disappears under contraction, so:

$$\Lambda_{\mu\mu} = 0,$$

$$\theta_{\mu\nu} = p\,\eta_{\mu\nu} + \Lambda_{\mu\nu}.$$

Since the first term, like $\theta_{\mu\nu}$, is situated in proper space and symmetric, the same thing will be true for $\Lambda_{\mu\nu}$:

$$\Lambda_{\mu\nu}\, u_\nu = \Lambda_{\mu\nu}\, u_\mu = 0 \qquad \text{and} \qquad \Lambda_{\mu\nu} = \Lambda_{\nu\mu}.$$

In what follows, we shall assume that the pressure *that is obtained by contracting the stress tensor* is physically meaningful, and as such, it will enter into an equation of state, from which it will follow that it determines only the density:

$$\rho \, m_0 = \Phi(p).$$

Moreover, as in the case of perfect fluids, it is possible to define an index F for the fluid by means of the expression:

$$\log F = \frac{1}{c^2}\int_{p_0}^{p} \frac{dp}{\rho m_0 + p / c^2} .$$

One likewise once more defines a weighted velocity:

$$C_\mu = F \, u_\mu ,$$

and one takes the divergence $\partial_\mu C_\mu$ to be a measure of the relativistic compressibility.

§ 3. Viscous fluids. The following level of discussion requires a relativistic generalization of the classical hydrodynamics of viscous fluids. One knows that the viscous stress tensor may be written:

$$\tau_{ij} = \pi \, \delta_{ij} + \tfrac{1}{2} \, \eta \, (\partial_i \, v_j + \partial_j \, v_i),$$

in which π is a non-relativistic pressure and η is the coefficient of viscosity. Lichnerowicz has shown [**46**] that in order to arrive at a relativistically correct formulation, one must introduce an index and a metric that is associated with this index. We suppose that the problem has been solved and that we are in possession of a relativistic viscous stress tensor $\theta_{\mu\nu}$. One may then determine a *relativistic pressure* p = 1/3 $\theta_{\mu\mu}$ (which is not be confused with π), from which we will get an index F and a weighted velocity C_μ. In addition, we shall define a Riemannian metric that we shall describe by means of the usual components that have been used up to now, and are referred to the Galilean reference frame of the Euclidian spacetime of special relativity. We choose a fundamental tensor:

$$g_{\mu\nu} = F^2 \, \delta_{\mu\nu} ,$$

so:

$$g^{\mu\nu} = F^{-2} \, \delta^{\mu\nu},$$

and we will suppose that the tensors that we have studied up to now are represented by tensors that have the same *covariant* components in terms of the new metric. Therefore, in terms of the new metric $g_{\mu\nu}$, the weighted velocity C_α gives us a vector whose covariant components will be C^α, and the contravariant components will be $C^\alpha = g^{\alpha\beta} \, C_\beta = F^{-2} \, \delta^{\alpha\beta} \, C_\beta$.

It results from this that:

$$C_\alpha C^\alpha = C_\alpha C_\beta F^{-2}\ \delta^{\alpha\beta} = F^2\ u_\alpha u_\beta F^{-2}\ \delta^{\alpha\beta} = -c^2.$$

Given the metric $g_{\mu\nu}$, the weighted velocity vector will thus play the role of a *unit-speed* velocity. The ordinary derivatives must be replaced with the covariant derivatives. The connection will be given by the Christoffel symbols:

$$\begin{aligned}\begin{Bmatrix}\rho \\ \alpha\,\beta\end{Bmatrix} &= \tfrac{1}{2} g^{\rho\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \\ &= \tfrac{1}{2} F^{-2}\ \delta^{\rho\lambda}(\delta_{\beta\lambda}\,\partial_\alpha F^2 + \delta_{\alpha\lambda}\,\partial_\beta F^2 - \delta_{\alpha\beta}\,\partial_\lambda F^2) \\ &= F^{-1}(\delta^\rho_\beta\partial_\alpha F + \delta^\rho_\alpha\partial_\beta F - \delta_{\alpha\beta}\delta^{\rho\lambda}\partial_\lambda F)\,.\end{aligned}$$

If one introduces the internal pressure field:

$$K_\alpha = -c^2\,\partial_\alpha \log F$$

then it will follow that:

$$\begin{Bmatrix}\rho \\ \alpha\,\beta\end{Bmatrix} = -\frac{1}{c^2}(\delta^\rho_\beta K_\alpha + \delta^\rho_\alpha K_\beta - \delta_{\alpha\beta}\delta^{\rho\lambda}K_\lambda)\,,$$

such that for the covariant derivative of a vector V_β one will get:

$$\nabla_\alpha V_\beta = \partial_\alpha V_\beta + \frac{1}{c^2}(K_\alpha V_\beta + K_\beta V_\alpha - \delta_{\alpha\beta}\delta^{\rho\lambda}K_\lambda V_\rho)\,.$$

Having established this, we then consider the non-relativistic proper stress tensor to be something that corresponds to the relativistic tensor when it is referred to the spatial axes of the proper system *and the associated Riemannian metric*; i.e., we replace v_j with C^0_j, the ordinary derivatives with the covariant derivatives ∇_α, and the symbol δ_{ij} (which is the fundamental tensor of the Euclidian metric) with the tensor g^0_{ij}. One will then have:

$$\tau_{ij} = \theta^0_{ij} = \pi F^2\delta_{ij} + \tfrac{1}{2}\eta(\nabla_i C_j + \nabla_j C_i)^0,$$

which is why we also append the conditions $\theta^0_{i4} = \theta^0_{4i} = \theta^0_{44} = 0$.

We now pass on to the covariant formulation. One sees immediately that the first term is:

$$\pi\left(g_{\eta\nu} + \frac{C_\mu C_\nu}{c^2}\right) = \pi F^2\ \eta_{\mu\nu}\,.$$

Lichnerowicz gives the expression $\eta\ \gamma_{\mu\nu}$ to the second term, with:

$$2\gamma_{\mu\nu} = \nabla_\mu C_\nu + \nabla_\nu C_\mu + \frac{C^\lambda}{c^2}(\nabla_\lambda C_\mu \cdot C_\nu + \nabla_\lambda C_\nu \cdot C_\mu)\ .$$

Indeed, upon projecting onto the proper system, one will have:

$$2\gamma_{i4}^0 = (\nabla_i C_4)^0 + (\nabla_4 C_i)^0 + \frac{(C^4 C_4)^0}{c^2}(\nabla_4 C_i)^0.$$

Now, in the proper system, the relations $C^\mu C_\mu = -c^2$ and $C^\mu \nabla_\lambda C_\mu = 0$ give $(C^4 C_4)^0 = -c^2$, so $2\gamma_{i4}^0 = (\nabla_i C_4)^0$, and $(C^4 \nabla_\lambda C_4)^0 = 0$, namely, $(\nabla_\lambda C_4)^0 = 0$, so $2\gamma_{i4}^0 = 0$.

One likewise sees that $\gamma_{44}^0 = 0$.

We may revert to the usual Euclidian formalism completely by simply preserving the symbol ∇_α, which represents, by definition, the operation:

$$\boxed{\nabla_\alpha V_\beta = \partial_\alpha V_\beta + \frac{1}{c^2}(K_\alpha V_\beta + K_\beta V_\alpha - \delta_{\alpha\beta} K_\lambda V_\lambda).}$$

This being the case, the tensor:

$$\boxed{2\gamma_{\mu\nu} = \nabla_\mu C_\nu + \nabla_\nu C_\mu + \frac{u_\lambda}{c^2}(u_\mu \nabla_\lambda C_\nu + u_\nu \nabla_\lambda C_\mu)}$$

will be in proper space – i.e., $\gamma_{\mu\nu}\, u_\nu = 0$ – and one will have:

$$\boxed{\theta_{\mu\nu} = \pi F^2\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) + \eta\gamma_{\mu\nu}.}$$

If one contracts $\gamma_{\mu\nu}$ then one will obtain:

$$\gamma_{\mu\mu} = \nabla_\mu C_\mu = \partial_\mu C_\mu - \frac{2}{c^2} K_\mu C_\mu \equiv \kappa.$$

The scalar that we denote by κ differs from the relativistic compressibility $\partial_\mu C_\mu$ by only the relativistic term $2/c^2\, K_\mu C_\mu = 2u_\mu \partial_\mu F$.

This being the case, we can ultimately express the relativistic pressure p as $3p = \theta_{\mu\mu} = 3\pi F^2 + \eta\kappa$. Hence:

$$\boxed{\pi F^2 = p - \tfrac{1}{3}\eta\kappa,}$$

and the energy-momentum tensor will be written:

$$\boxed{t_{\mu\nu} = \left(\rho m_0 + \frac{p - \eta\kappa/3}{c^2}\right) u_\mu u_\nu + \delta_{\mu\nu}(p - \eta\kappa/3) + \eta\gamma_{\mu\nu},}$$

which is an expression that generalizes formula (4) for perfect fluids by introducing the pseudo-mass density:

$$\mu_0 = \rho\, m_0 + \frac{p - \eta\kappa/3}{c^2}$$

and the generalized pressure:

$$\pi F^2 = p - \eta\kappa / 3.$$

In order to establish the equations of motion, one must first express the divergence of $\gamma_{\mu\nu}$. One can first transform this tensor in such a fashion as to make the vorticity tensor $\Omega_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ appear, which, as one knows, is identified with the tensor $\nabla_\mu C_\nu - \nabla_\nu C_\mu$ for a Riemannian metric.

On the other hand, if one takes into account the relation $C_\lambda \nabla_\mu C_\lambda = 0$ then one will easily get:

$$2\gamma_{\mu\nu} = 2\nabla_\mu C_\nu + \Omega_{\mu\nu} + \frac{1}{c^2 F^2} C_\lambda [\Omega_{\lambda\mu}\, u_\nu + \Omega_{\lambda\nu}\, u_\mu] .$$

One can then set:

$$2\Theta_{\mu\nu} = \Omega_{\mu\nu} + \frac{u_\lambda}{c^2} [\Omega_{\lambda\mu}\, u_\nu + \Omega_{\lambda\nu}\, u_\mu] .$$

We stipulate that the tensor $\Omega_{\mu\nu}$ is no longer orthogonal to the current, as we shall verify shortly. The tensor $\Theta_{\mu\nu}$, which is derived from the vorticity, reduces to it only in the case of the perfect fluid, for which $\Omega_{\mu\lambda}\, u_\lambda = 0$.

Upon specifying the symbol ∇_μ, one will ultimately get:

$$\gamma_{\mu\nu} = \partial_\mu C_\nu + \frac{1}{c^2}(K_\mu C_\nu + K_\nu C_\mu - \delta_{\mu\nu} K_\lambda C_\lambda) + \Theta_{\mu\nu}.$$

If we differentiate this then that will give:

$$\partial_\nu \gamma_{\mu\nu} = \partial_\nu \Theta_{\mu\nu} + \partial_\nu \partial_\mu C_\nu + \frac{1}{c^2}[K_\mu \partial_\nu C_\nu + C_\mu \partial_\nu K_\nu + C_\nu \partial_\nu K_\mu + K_\mu \partial_\nu C_\nu - \partial_\mu (K_\lambda C_\lambda)].$$

By using the vorticity again, we transform the bracket into the form $\Omega_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$, and remark that $\partial_\nu K_\mu = \partial_\mu K_\nu$, since K_μ is gradient.

One has:

$$\frac{1}{c^2}[K_\nu \Omega_{\mu\nu} + K_\nu \partial_\mu C_\nu + C_\nu \partial_\mu K_\nu + C_\mu \partial_\nu K_\nu + K_\mu \partial_\nu C_\nu - \partial_\mu (K_\lambda C_\lambda)].$$

The second and third terms give $\partial_\mu (K_\nu C_\nu)$, which is annulled, along with the latter ones. One will then have:

$$\partial_\nu \gamma_{\mu\nu} = \partial_\nu \Theta_{\mu\nu} + \partial_\nu \partial_\mu C_\nu + \frac{1}{c^2}[K_\nu \Omega_{\mu\nu} + C_\mu \partial_\nu K_\nu + K_\mu \partial_\nu C_\nu],$$

which will give, upon replacing K_ν with $-c^2 \partial_\nu \log F$:

$$\partial_\nu \gamma_{\mu\nu} = \partial_\nu \Theta_{\mu\nu} - \Omega_{\mu\nu} \partial_\nu \log F + \partial_\nu \partial_\mu C_\nu - C_\mu \Box \log F + \partial_\nu C_\nu \partial_\mu \log \left(\frac{\partial_\lambda C_\lambda}{F}\right).$$

It is then easy to write down the equations of motion in the absence of external forces $\partial_\nu t_{\mu\nu} = 0$, namely:

(B-8) $$u_\mu \partial_\nu (\mu_0 u_\nu) + \mu_0 u_\nu \partial_\nu u_\mu + \partial_\mu (F^2 \pi) + \eta \partial_\nu \gamma_{\mu\nu} = 0.$$

Upon contracting this with u_μ, one will get the conservation equation:

$$\dot{\mu}_0 = \frac{1}{c^2} u_\lambda [\partial_\lambda (F^2 \pi) + \eta \partial_\nu \gamma_{\lambda\nu}],$$

which will give, when substituted in equation (8):

$$\frac{u_\mu}{c^2} u_\lambda [\partial_\lambda (F^2 \pi) + \eta \partial_\nu \gamma_{\lambda\nu}] + \mu_0 \dot{u}_\mu + \partial_\mu (F^2 \pi) + \eta \partial_\nu \gamma_{\mu\nu} = 0;$$

i.e.:

$$\mu_0 \dot{u}_\mu = -\left(\delta_{\mu\lambda} + \frac{u_\mu u_\lambda}{c^2}\right)[\partial_\lambda (F^2 \pi) + \eta \partial_\nu \gamma_{\lambda\nu}].$$

Namely, we will get this by specifying μ_0, $F^2\pi$, and $\partial_\nu \gamma_{\lambda\nu}$, and upon remarking that in the term in the latter expression that is collinear with the current, $C_\mu \Box \log F$ will disappear when one multiplies it by the projection $\delta_{\mu\nu} + u_\mu u_\nu / c^2$.

The conservation equation is:

$$\boxed{\dot{\mu}_0 = \frac{1}{c^2}\left[u_\lambda \partial_\lambda \left(p - \frac{\eta\kappa}{3}\right) + \eta \left\{u_\lambda \partial_\nu \Theta_{\lambda\nu} - u_\lambda \Omega_{\lambda\nu} \partial_\nu \log F + c^2 F \Box \log F + F \frac{d}{d\tau}\left(\frac{\partial_\alpha C_\alpha}{F}\right)\right\}\right]}$$

and the equations of the streamlines are:

$$\boxed{\left(\rho m_0 + \frac{p - \eta\kappa/3}{c^2}\right)\dot{u}_\mu = -\eta_{\mu\lambda}\left\{\partial_\lambda \left(p - \frac{\eta\kappa}{3}\right) + \eta \left[F \partial_\lambda \left(\frac{\partial_\alpha C_\alpha}{F}\right) + \Omega_{\alpha\lambda} \partial_\lambda \log F + \partial_\alpha \Theta_{\alpha\lambda}\right]\right\}.}$$

We shall use this equation in order to study the vorticity tensor. As in the case of the perfect fluid, one will find that the projection of $\Omega_{\mu\nu}$ onto the current is:

$$\Omega_{\mu\nu}\, u_\nu = F(u_\nu\,\partial_\mu\, u_\nu - u_\nu\,\partial_\nu\, u_\mu) - \frac{F}{c^2}(-\,c^2\, K_\mu - u_\mu\, K_\nu\, u_\nu)\ ;$$

i.e.:

$$\Omega_{\mu\nu}\, u_\nu = -\,F(\dot{u}_\mu - \,\eta_{\mu\nu}\, K_\nu\, u_\mu),$$

or, upon multiplying this by μ_0 and replacing K_ν as a function of $\partial_\nu p = -\left(\rho m_0 + \frac{p}{c^2}\right) K_\nu$, one will get

$$\mu_0\,\Omega_{\mu\nu}\, u_\nu = -\,F\left[\mu_0 \dot{u}_\mu + \frac{\mu_0}{\rho m_0 + p/c^2}\eta_{\mu\nu}\partial_\nu p\right],$$

so upon deducing $\mu_0\,\dot{u}_\mu$ from the equation of motion this will become:

$$\mu_0\,\Omega_{\mu\nu}\, u_\nu = F\eta_{\mu\nu}\left[\partial_\nu(F^2\pi) + \eta\partial_\lambda\gamma_{\nu\lambda} - \frac{\mu_0}{\rho m_0 + p/c^2}\partial_\nu p\right],$$

or, upon specifying μ_0 and $F^2\pi$:

$$\mu_0\,\Omega_{\mu\nu}\, u_\nu = \eta\, F\eta_{\mu\nu}\left(-\frac{1}{3}\partial_\nu\kappa + \partial_\lambda\gamma_{\nu\lambda} + \frac{\kappa}{3c^2}\frac{\partial_\nu p}{\rho m_0 + p/c^2}\right).$$

Upon re-introducing K_ν into the last term, the parentheses will give:

$$-\frac{1}{3}\partial_\nu\kappa - \frac{K_\nu}{3c^2} + \partial_\lambda\gamma_{\nu\lambda}\,,$$

which one easily transforms into $-\frac{1}{3}F\partial_\nu\left(\frac{\kappa}{F}\right) + \partial_\lambda\gamma_{\nu\lambda}$:

$$\boxed{\mu_0\Omega_{\mu\nu}u_\nu = \eta F\eta_{\mu\nu}\left[\partial_\lambda\gamma_{\nu\lambda} - \tfrac{1}{3}F\partial_\nu\left(\frac{\kappa}{F}\right)\right].}$$

Therefore, the tensor $\Omega_{\mu\nu}$ is no longer orthogonal to the current. One can further express this, as we shall do in the case of internal angular momentum, by means of the vector $\theta_\mu = i\,/\,c\ \varepsilon_{\mu\nu\alpha\beta}\, u_\nu\,\Omega_{\alpha\beta}$, but we must, in addition (see Chap. III), introduce a second vector $\lambda_\mu = 1/c\ \Omega_{\mu\nu}\, u_\nu$ that is orthogonal to the current, by reason of antisymmetry, and whose expression is:

$$\boxed{\lambda_\mu = \frac{1}{\mu_0 c}\eta F \eta_{\mu\nu}\left[\partial_\lambda \gamma_{\nu\lambda} - \tfrac{1}{3} F \partial_\nu\left(\frac{\kappa}{F}\right)\right],}$$

and we will have:

$$\Omega_{\mu\nu} = \frac{1}{c}[i\ \varepsilon_{\mu\nu\alpha\beta}\, u_\alpha\, \theta_\beta + (u_\mu\, \lambda_\nu - u_\nu\, \lambda_\mu)]\ .$$

The equations simplify in the case of irrotational motions, since the tensors $\Omega_{\mu\nu}$ and $\Theta_{\mu\nu}$ vanish, so the conservation equation will become:

$$\dot{\mu}_0 = \frac{1}{c^2} u_\lambda \left\{\partial_\lambda (F^2 \pi) + \eta F\left[c^2 \Box \log F + \frac{d}{d\tau}\left(\frac{\partial_\alpha C_\alpha}{F}\right)\right]\right\},$$

while the equation of the streamlines will become:

$$\mu_0 \dot{u}_\mu = -\ \eta_{\mu\lambda}\left[\partial_\lambda (F^2 \pi) + \eta F \partial_\lambda \left(\frac{\partial_\alpha C_\alpha}{F}\right)\right].$$

As far as that is concerned, we remark that the irrotationality condition can be written:

$$\nabla_\mu\, C_\nu - \nabla_\nu\, C_\mu = 0,$$

and one can then, upon considering this as being referred to the associated metric $g_{\mu\nu}$, contract it with C^μ:

$$C^\mu\, \nabla_\mu\, C_\nu - C^\mu\, \nabla_\nu\, C_\mu = 0.$$

The second term is zero, since $C^\mu\, C_\mu = -\, c^2$, and what will remain is the condition $C^\mu\, \nabla_\mu\, C_\nu = 0$, where, since C_μ has unit velocity for the metric $g_{\mu\nu}$, one recognizes that this is the *equation for a geodesic.* Therefore, the streamlines of an irrotational motion are geodesics of the associated Riemannian metric, which serves to show the importance of the introduction of that metric.

If the viscous fluid is incompressible then it is the term $F\ \partial_\lambda(\partial_\alpha\ C_\alpha\ /\ F)$ that disappears in the equations, and if the fluid is incompressible and restricted to an irrotational motion then it is everything in the parentheses that is a factor of η that vanishes, with the exception of the term in $\Box$ log F in the conservation equation:

$$\dot{\mu}_0 = \frac{1}{c^2} u_\lambda\, \partial_\lambda (F^2 \pi) + \eta\, F\, \Box \log F,$$

$$\mu_0\, \dot{u}_\mu = -\ \eta_{\mu\lambda}\, \partial_\lambda (F^2 \pi).$$

The second equation is analogous to that of a perfect fluid. Nevertheless, the viscosity again enters into the expression for $F^2 \pi = p - \eta\, \kappa/\ 3$, and $\mu_0 = \rho\, m_0\ +$

$\frac{p-\eta\kappa/3}{c^2}$, and κ reduces to $-2/c^2\, K_\mu\, C_\mu = 2\dot{F}$, since the compressibility term is zero. The complementary term is in $1/c^2$ in the pressure and in $1/c^4$ in the pseudo-mass.

One may thus say that an incompressible, viscous fluid admits an irrotational motion that will have the same streamlines as a perfect fluid that is subject to a pressure:

$$F^2\,\pi = p - \frac{2}{3}\eta\dot{F}\ .$$

Meanwhile, one may remark that the index F was defined by starting with the pressure p, not with the pressure $F^2\pi$. Although the differences are very small, the result, when taken in full rigor, is therefore less simple than the one that was given by Lichnerowicz – a possibility that he himself provided for in a footnote, moreover. This comes down to the particular choice that we made of a pressure that was equal to the contracted stress tensor.

§ 4. Holonomic fluids. One may generalize the hydrodynamics of perfect fluids in another manner. One may, in general, and in an infinitude of ways, decompose the energy-momentum tensor of an arbitrary fluid into a kinetic term that involves the pseudo-mass density and an internal pressure tensor that is not necessarily contained in proper space:

$$t_{\mu\nu} = \mu_0\, u_\mu\, u_\nu + \pi_{\mu\nu}\ .$$

For each of these decompositions, one can, upon taking the divergence of the pressure tensor and referring to it as a pseudo-mass, define an internal force field:

$$K_\mu = -\frac{1}{\mu_0}\,\partial_\nu\, \pi_{\mu\nu}\ .$$

Suppose that one wishes to find a particular decomposition such that K_μ is the gradient of a scalar function. One then says that one is dealing with a *holonomic fluid.* One may introduce an index F by setting:

$$K_\mu = -\, c^2\, \partial_\mu \log F.$$

Here, the pressure p, which is always possible to define, no longer plays the same simple role as it did in the preceding two cases. It no longer suffices to determine K_μ and F, which depend upon the set of internal actions.

One can, as we did in the other two cases, write down the equation:

$$\partial_\nu\, t_{\mu\nu} = u_\mu\, \partial_\nu\,(\mu_0\, u_\nu) + \mu_0\, u_\mu\, \partial_\nu\, u_\nu + \partial_\nu\, \pi_{\mu\nu} = 0,$$

namely:

$$\dot{\mu}_0 u_\mu + \mu_0 \dot{u}_\mu = \mu_0\, K_\mu = -\,\mu_0\, c^2\, \partial_\mu \log F.$$

One thus deduces a conservation equation:

$$\boxed{\dot{\mu}_0 = \mu_0 u_\mu \partial_\mu \log F}$$

and an equation for the streamlines:

$$\boxed{\dot{u}_\mu = -c^2 \eta_{\mu\nu} \partial_\nu \log F.}$$

The conservation equation can be written:

$$\partial_\nu u_\nu + u_\nu \partial_\nu \log \mu_0 = u_\nu \partial_\nu \log F.$$

If one introduces the weighted velocity $C_\mu = F\, u_\mu$ then the compressibility $\partial_\mu C_\mu$ can be written:

$$\begin{aligned} \partial_\mu C_\mu &= F\, \partial_\mu u_\mu + u_\mu \partial_\mu F \\ &= F(u_\nu \partial_\nu \log F - u_\nu \partial_\nu \log \mu_0) + F\, u_\mu \frac{\partial_\mu F}{F} \end{aligned}$$

i.e.:

$$\boxed{\partial_\mu C_\mu = F u_\nu \partial_\nu \log\left(\frac{F^2}{\mu_0}\right).}$$

If the fluid is incompressible then the ratio F^2 / μ_0 will remain constant along a streamline.

We shall give the proof of the fundamental property of the index F in this case:

Consider the function $1 / ic\, F \sqrt{u_\mu u_\mu}$, which is identical to F, and its circulation along an arbitrary world-line C that joins two points M_0 and M_1:

$$\mathcal{I} = \frac{1}{ic} \int_{M_0}^{M_1} F \sqrt{u_\mu u_\mu}\, ds .$$

Give an infinitely small variation δx_μ to each point x_μ of the line C, while now fixing the points M_0 and M_1 . One will then have:

$$\begin{aligned} \delta\mathcal{I} &= \frac{1}{ic} \int_{M_0}^{M_1} \delta\left(F \sqrt{u_\mu u_\mu}\right) ds \\ &= \frac{1}{ic} \int_{M_0}^{M_1} \left(\partial_\mu F \sqrt{u_\mu u_\mu}\, \delta x_\mu + \frac{F u_\mu}{\sqrt{u_\mu u_\mu}} \delta u_\mu \right) ds . \end{aligned}$$

One can replace $\sqrt{u_\mu u_\mu}$ with ic and integrate the second term by parts, where $\delta u_\mu = \delta(dx_\mu / d\tau)$ (the completely integrable part is zero at the limits):

$$\delta\mathcal{I} = \int_{M_0}^{M_1} \left[\partial_\mu F + \frac{1}{c^2} \frac{d}{d\tau}(F u_\mu) \right] \delta x_\mu \, ds .$$

The condition for the integral $\mathcal{I} = \int_{M_0}^{M_1} F\, ds$ to be stationary is then:

$$\partial_\mu F + \frac{1}{c^2} u_\mu u_\nu \partial_\nu F + \frac{1}{c^2} F\dot{u}_\mu \equiv \eta_{\mu\nu} \partial_\nu F + \frac{1}{c^2} F\dot{u}_\mu = 0,$$

namely:

$$c^2 \eta_{\mu\nu} \partial_\nu \log F + \dot{u}_\mu = 0,$$

which will be the equation for the streamlines.

Finally, we construct the vorticity tensor:

$$\Omega_{\mu\nu} = \partial_\mu (F u_\nu) - \partial_\nu (F u_\mu) = F(\partial_\mu \; u_\nu - \partial_\mu \; u_\nu) - \frac{F}{c^2} (K_\mu \; u_\nu - K_\mu \; u_\nu).$$

Contracting this with u_ν will give:

$$\begin{aligned} \Omega_{\mu\nu} u_\nu &= - F\dot{u}_\mu - \frac{F}{c^2} (- c^2 K_\mu - K_\nu u_\nu u_\mu) \\ &= - F[\dot{u}_\mu - K_\nu \eta_{\mu\nu}]. \end{aligned}$$

Upon replacing K_ν with $- c^2 \partial_\nu \log F$, one sees that the bracket is annulled by virtue of the equations of motion. Therefore, a characteristic property of holonomic fluids is that the vorticity tensor is orthogonal to the current. As in the case of perfect fluids, it may be replaced with a vorticity vector:

$$\theta_\mu = \frac{i}{c} \varepsilon_{\mu\nu\alpha\beta} \Omega_{\alpha\beta} = \frac{2iF}{c} \varepsilon_{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta .$$

____

APPENDIX C

§ 1. Tensorial quantities that are deduced from the general theory of fields. Here, we shall recall the principles of the classical theory of fields [**19**]. Let a wave function be defined at each spacetime point with n components ψ^r (r = 1, 2, 3, …, n), along with its derivatives $\psi^r_{,\mu} \equiv \partial_\mu \psi^r$. One gives it a Lagrangian that depends explicitly upon only the ψ^r and $\psi^r_{,\mu}$, namely, $\mathcal{L}(\psi^r, \psi^r_{,\mu})$.

This Lagrangian is an abstract mathematical "expression." By contrast, its integral over an arbitrary domain:

$$I = \int_\Omega \mathcal{L}\, d\omega$$

will be a "quantity;" i.e., it will possess the invariance properties that permit one to establish all of the formalism.

First of all, the integral I is subject to a condition of *stationary variation.* Suppose that the components of the wave function undergo arbitrary infinitesimal variations $\delta\psi^r$ at each point, which are continuous functions of the coordinates that go to zero on the boundary of the domain of integration. One then postulates that the integral I is stationary, so $\delta I = 0$ for such variations:

$$\delta I = \int_\Omega \delta\mathcal{L}\, d\omega = \int_\Omega \frac{\partial \mathcal{L}}{\partial \psi^r}\, \delta\psi^r\, d\omega + \int_\Omega \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}}\, \delta\psi^r_{,\mu}\, d\omega .$$

Since $\delta\psi^r_{,\mu} = \delta(\partial_\mu \psi^r) = \partial_\mu (\delta\psi^r)$, one can integrate the second integral by parts, and since the $\delta\psi^r$ go to zero on the boundary of the domain of integration, that part will integrate to zero:

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}}\, \delta\psi^r_{,\mu}\, d\omega = -\int_\Omega \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \right) \delta\psi^r\, d\omega ,$$

so

$$\delta I = \int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \psi^r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \right) \right] \delta\psi^r\, d\omega .$$

In order for this variation to be zero for an arbitrary $\delta\psi^r$, one must have:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi^r} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \right),}$$

which will give the general form for the *wave equation* that is associated with a given Lagrangian (more precisely, it is a *system* of n equations). We remark that since the ψ^r are complex, in general, one may consider their complex conjugates ψ^{*r} to be independent variables, which will give a second system of n equations:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi^{*r}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right).}$$

In the second place, one knows that the classical formalism expresses all of the observable quantities that relate to the particle by means of linear quantities that are constructed from the wave function, namely, $\psi^\dagger \Lambda \psi$, where Λ is a square matrix of rank n, ψ represents the set of all ψ^r, which are put into the form of a column matrix, and $\psi^\dagger$ represents the set of all ψ^{*r}, which are put into the form of a row matrix that is the adjoint of the former matrix. It will then result that the multiplication of all the components of the wave function by the same factor $e^{i\gamma}$ (γ = arbitrary real number) – which one calls a *gauge transformation of the first kind* – does not modify any of the observable quantities that relate to the particle. It is natural to insist that such a transformation should likewise leave the integral $I = \int_\Omega \mathcal{L}\, d\omega$ invariant, or what amounts to the same thing, the Lagrangian $\mathcal{L}$ itself. In particular, consider the case of an infinitesimal gauge transformation $\delta\gamma$:

$$\begin{aligned}\delta\psi^r &= \psi^r (e^{i\delta\gamma} - 1) = i\,\psi^r\, \delta\gamma, \\ \delta\psi^{*r} &= \psi^{*r} (e^{-i\delta\gamma} - 1) = -\, i\,\psi^{*r}\, \delta\gamma.\end{aligned}$$

It will then follow that:

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \psi^r} i\psi^r \delta\gamma - \frac{\partial \mathcal{L}}{\partial \psi^{*r}} i\psi^{*r} \delta\gamma + \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} i\psi^r_{,\mu} \delta\gamma - \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} i\psi^{*r}_{,\mu} \delta\gamma \\ &= i\delta\gamma \left(\frac{\partial \mathcal{L}}{\partial \psi^r} \psi^r - \frac{\partial \mathcal{L}}{\partial \psi^{*r}} \psi^{*r} + \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\mu} - \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \psi^{*r}_{,\mu} \right),\end{aligned}$$

or upon employing the wave equations:

$$\delta\mathcal{L} = i\delta\gamma \left[\psi^r \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \right) - \psi^{*r} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right) + \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \partial_\mu \psi^r - \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \partial_\mu \psi^{*r} \right],$$

namely:

$$\delta\mathcal{L} = i\delta\gamma\, \partial_\mu \left(\psi^r \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} - \psi^{*r} \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right).$$

Therefore, the variation of the integral I is:

$$\delta I = i\delta\gamma \int_\Omega \partial_\mu \left(\psi^r \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} - \psi^{*r} \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right) d\omega.$$

$$= \delta\gamma \int_\Sigma i \left(\psi^r \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} - \psi^{*r} \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right) d\sigma_\mu ,$$

where Σ represents the hypersurface that bounds the domain of integration and $d\sigma_\mu$ is the infinitesimal element of that hypersurface. One sees that the latter integral has the form of a flux for the quantity:

$$\boxed{j_\mu = -i \left(\psi^r \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} - \psi^{*r} \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \right).}$$

The variation δI represents the difference between the values of the integral I for two different values of γ. As for the latter, it must be a tensor invariant. Therefore, one must have that the quantity j_μ that represents the flux is a *vector*; one calls this vector the *current*. The flux must be zero for any $\delta\gamma$, so for any domain Ω, the current vector must be conservative:

$$\boxed{\partial_\mu j_\mu = 0.}$$

This fundamental *tensorial* relation expresses the invariance of the integral I under a gauge transformation.

We must now express the invariance of the integral I with respect to an arbitrary coordinate transformation. In particular, consider an infinitesimal transformation that transforms x_μ into $x'_\mu = x_\mu + \delta x_\mu$ and each wave function $\psi^r(x)$ into $\psi'^r(x) = \psi^r(x) + \delta\psi^r$.

In order to establish the variations that result from this, we use a method that was proposed by E. Noether [**20**]. One has:

$$\delta I = \int \mathcal{L}[\psi'^r(x'), \psi'^r_{,\mu}(x')]\, d\omega' - \int \mathcal{L}[\psi^r(x), \psi^r_{,\mu}(x)]\, d\omega ,$$

in which the integrals are taken over corresponding domains in the two systems. One sees that the transformation δx_μ influences the integral in three ways: by the variation of the x_μ that represent a given point, and which depend upon the wave functions, by the variation of these functions themselves, which become other functions of x_μ (except for the case of a scalar wave function), and finally, by the variation of the infinitesimal domain $d\omega$.

In order for one to group the two differential elements together into the form of a difference under the same integral sign and then integrate them, one must first express them in terms of the same variables – namely, x_μ . We transform the first integral in this way:

$$\mathcal{L}[\psi'^r(x'),\psi'^r_{,\mu}(x')]\,d\omega' = \left\{\mathcal{L}[\psi'^r(x),\psi'^r_{,\mu}(x)]+\partial_\nu\mathcal{L}[\psi'^r(x),\psi'^r_{,\mu}(x)]\delta x_\nu\right\} J\left(\frac{\partial x'}{\partial x}\right) d\omega\,,$$

in which $J(\partial x' / \partial x)$ is the Jacobian of the transformation (which reduces to $1 + \partial_\nu\, \delta x_\nu$ for an infinitesimal transformation), and all of the functions are now expressed in terms of x_μ. It then follows, upon neglecting the second-order terms, that:

$$\begin{aligned}
&\mathcal{L}[\psi'^r(x'),\psi'^r_{,\mu}(x')]\,d\omega' \\
&\qquad = \left[\mathcal{L}(\psi'^r(x),\psi'^r_{,\mu}(x))+\partial_\nu\mathcal{L}(\psi'^r(x),\psi'^r_{,\mu}(x))\,\delta x_\nu+\mathcal{L}(\psi'^r,\psi'^r_{,\mu})\,\partial_\nu\delta x_\nu\right] d\omega \\
&\qquad = \left\{\mathcal{L}(\psi'^r,\psi'^r_{,\mu})+\partial_\nu[\mathcal{L}(\psi'^r,\psi'^r_{,\mu})\,\delta x_\nu]\right\} d\omega\,,
\end{aligned}$$

or, up to second order:

$$= \left\{\mathcal{L}(\psi'^r,\psi'^r_{,\mu})+\partial_\nu[\mathcal{L}(\psi^r,\psi^r_{,\mu})\,\delta x_\nu]\right\} d\omega\,.$$

One may then write, while expressing everything in terms of x_μ :

$$\delta I = \int\left\{\mathcal{L}(\psi'^r,\psi'^r_{,\mu})-\mathcal{L}(\psi^r,\psi^r_{,\mu})+\partial_\nu[\mathcal{L}(\psi^r,\psi^r_{,\mu})\,\delta x_\nu]\right\} d\omega\,.$$

The first two terms express the difference between the transformed Lagrangian and the untransformed one for the same values of x_μ – i.e., at *two different points* of spacetime that have the *same coordinates* in the two systems, respectively (viz., the "substantial" variation); we denote this by $\bar{\delta}$. This is expressed by means of:

$$\bar{\delta}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi^r}\bar{\delta}\psi^r+\frac{\partial\mathcal{L}}{\partial\psi^r_{,\mu}}\bar{\delta}\psi^r_{,\mu}\,,$$

or, upon integrating the second term by parts:

$$\bar{\delta}\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\psi^r}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\psi^r_{,\mu}}\right]\bar{\delta}\psi^r+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^r_{,\mu}}\bar{\delta}\psi^r\right).$$

The bracket is zero, by reason of the field equations. What will then remain is:

$$\bar{\delta}\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^r_{,\mu}}\bar{\delta}\psi^r\right).$$

It remains for us to express the substantial variation $\bar{\delta}\psi^r$ of the wave function itself. By its definition, it is clear that it is composed of two parts: One part is the variation that is produced by the changing of the axes for the function ψ^r *at a given point* of spacetime (so, as a consequence, the coordinates will change). This is the "local" variation, which is determined by the variance of the wave function. The other part is the variation that is due to considering two different points of spacetime that have the same coordinates in two respective coordinate systems. This variation will involve the gradient of the function ψ^r and a displacement that is $-\delta x_\mu$, since δx_μ represents the variation that coordinates of *the same spacetime point* undergo. Ultimately, one will thus have:

$$\bar{\delta}\psi^r = \delta\psi^r - \partial_\nu \psi^r,$$

which will give:

$$\bar{\delta}\mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} (\delta\psi^r - \psi^r_{,\nu}\delta x_\nu) \right].$$

One thus obtains:

$$\delta I \equiv \int \partial_\mu \left[\left(\delta_{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} \right) \delta x_\nu + \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \delta\psi^r \right] d\omega = 0,$$

and since the domain of integration is arbitrary, that will demand that one have the Noether formula:

$$\boxed{\partial_\mu \left[\left(\delta_{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} \right) \delta x_\nu + \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \delta\psi^r \right] = 0.}$$

Of course, one must give a definite form to δx_ν that depends upon the type of transformation that is envisioned, and the $\delta\psi^r$ are expressed as functions of the δx_ν according to expressions that depend upon the variance of the wave function.

First, let us apply the Noether formula to the case of a simple infinitesimal transformation of the axes. The $\delta\psi^r$ are zero, since the axes of projection remain invariant. The δx_μ are the same at every point. One will then have:

$$\delta x_\mu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} - \delta_{\mu\nu}\mathcal{L} \right) = 0,$$

which will demand that the divergence must be zero, since the δx_μ are arbitrary:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} - \delta_{\mu\nu}\mathcal{L} \right) = 0.$$

Finally, since the arbitrary δx_μ are the components of a vector, the criterion for tensoriality will teach us that the quantity above is likewise a vector, and ultimately that the expression:

$$\boxed{t_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} - \delta_{\mu\nu} \mathcal{L}}$$

is a tensor, which one calls the *canonical energy-momentum tensor.*

Of course, this relation involves not only the functions ψ^r, but also their complex conjugates ψ^{*r}, so the complete expression can be written:

$$t_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} + \frac{\partial \mathcal{L}}{\partial \psi^{*r}_{,\mu}} \psi^{*r}_{,\nu} - \delta_{\mu\nu} \mathcal{L}\,,$$

which also obviously indicates that:

$$t_{\mu\nu} = \tfrac{1}{2} \mathrm{Re}\left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\mu}} \psi^r_{,\nu} \right) - \delta_{\mu\nu} \mathcal{L}\,.$$

We shall consider the ψ^{*r} to be subordinate as independent variables.

The tensorial conservation relation:

$$\text{(C.1)} \qquad \boxed{\partial_\nu t_{\mu\nu} = 0}$$

then expresses the invariance of the integral I under an infinitesimal translation of the axes.

One will obtain a final tensorial equation for the field by applying Noether's formula to the case of an infinitesimal rotation – or, more precisely, a Lorentz transformation – that one can define by means of an antisymmetric tensor $\omega_{[\mu\nu]}$. One has $\delta x_\mu = \omega_{[\mu\nu]}\, x_\nu$, and the variation of the wave function is expressed by means of:

$$\delta\psi^r = \omega_{[\mu\nu]}\, \mathcal{Z}^{rs}_{[\mu\nu]}\, \psi^r.$$

$\mathcal{Z}^{rs}_{[\mu\nu]}$ is an operator that is antisymmetric in μ and ν whose form depends upon the nature of the wave function. Thus, for a vector field, one has:

$$\mathcal{Z}^{rs}_{[\mu\nu]} = \tfrac{1}{2} \delta^{rs}_{\mu\nu} = \tfrac{1}{2} (\delta_{r\mu}\, \delta_{s\nu} - \delta_{r\nu}\, \delta_{s\mu}).$$

Likewise, if the wave function is a 4-spinor that one can write in the form of a column matrix ψ then one has:

$$\delta\psi = \omega_{[\mu\nu]}\, \mathcal{Z}_{\mu\nu(\text{op.})}\, \psi$$

with

$$\mathcal{Z}_{\mu\nu(\text{op.})} = \tfrac{1}{8}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu),$$

in which the γ_μ are the four Von Neumann matrices [**19**].

Noether's formula then gives:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\nu}} \omega_{\mu\lambda} \mathcal{Z}^{rs}_{[\mu\lambda]} \psi^s - t_{\mu\nu} \omega_{\mu\lambda} x_\lambda \right) = 0.$$

Just as we (implicitly) compared the infinitesimal transformation δx_μ that results from a translation of the axes to a point-like transformation in a fixed system of axes – from which, its tensorial character is derived – similarly, we can also compare the coordinate transformation $\omega_{[\mu\nu]}$ to an infinitesimal rotation that is considered in a fixed system of axes, which is a rotation that, as one knows, has the variance of a second-order, antisymmetric tensor. Furthermore, since this tensor is independent of the coordinates, it may be taken out of the divergence like an arbitrary tensor, which gives:

$$\omega_{\mu\lambda} \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \psi^r_{,\nu}} \mathcal{Z}^{rs}_{[\mu\lambda]} \psi^s - t_{\mu\nu} x_\lambda \right) = 0.$$

An argument that is identical to the one in the preceding paragraph shows us that:

$$\boxed{f_{[\mu\nu]\lambda} = \frac{\partial \mathcal{L}}{\partial \psi^r_{,\nu}} \mathcal{Z}^{rs}_{[\mu\lambda]} \psi^s,}$$

which is antisymmetric in μ and ν, is a *tensor* [**21**, **22**]; we give it the name of *Belinfante tensor*. Upon taking into account the arbitrary and antisymmetric character of $\omega_{\mu\lambda}$, one will then have:

$$\tfrac{1}{2}\partial_\nu(x_\mu t_{\lambda\nu} - x_\lambda t_{\mu\nu}) + \partial_\nu f_{\mu\lambda\nu} = 0.$$

If one performs the first derivation, while taking into account the relation $\partial_\nu t_{\mu\nu} = 0$, then one will get:

(C.2) $$\boxed{\tfrac{1}{2}(t_{\mu\lambda} - t_{\lambda\mu}) = \partial_\nu f_{[\mu\lambda]\nu}.}$$

This equations is the usual form in which one expresses the invariance of the integral I under infinitesimal Lorentz transformations.

§ 2. Gauge transformations. Sometimes, one also preserves the condition that the divergence be zero:

$$\partial_\nu [\tfrac{1}{2}(x_\mu t_{\lambda\nu} - x_\lambda t_{\mu\nu}) + f_{\mu\lambda\nu}] = 0,$$

or furthermore:

$$\partial_\nu[(x_\mu\, t_{\lambda\nu} - x_\lambda\, t_{\mu\nu}) + (f_{\mu\lambda\nu} - f_{\lambda\mu\nu})] = 0.$$

The first set of parentheses represents the moment (with respect to the origin) of the canonical tensor. One can, by a simple transformation, likewise put the second set of parentheses into the form of a moment [**23**]. One can write:

$$\partial_\nu f_{\mu\lambda\nu} = \partial_\nu f_{\sigma\lambda\nu}\,\delta_{\sigma\mu} = \partial_\nu f_{\sigma\lambda\nu}\,\partial_\sigma x_\mu\,,$$

or, upon integrating by parts:

$$\partial_\nu f_{\mu\lambda\nu} = \partial_\sigma(x_\mu\,\partial_\nu f_{\sigma\lambda\nu}) - x_\mu\,\partial_\sigma\partial_\nu f_{\sigma\lambda\nu}\,,$$

One may further integrate the second term by parts and get:

$$\partial_\nu f_{\mu\lambda\nu} = \partial_\sigma(x_\mu\,\partial_\nu f_{\sigma\lambda\nu}) - \partial_\nu(x_\mu\,\partial_\sigma f_{\sigma\lambda\nu}) + \delta_{\nu\mu}\,\partial_\sigma f_{\sigma\lambda\nu} = \partial_\nu\,[x_\mu\,\partial_\sigma\,(f_{\lambda\nu\sigma} - f_{\sigma\lambda\nu})] + \partial_\nu f_{\nu\lambda\nu}\,,$$

One can apply the same process to the last term and get:

$$\partial_\nu f_{\mu\lambda\nu} = \partial_\nu f_{\mu\sigma\nu}\,\partial_\sigma x_\lambda = \partial_\sigma(x_\lambda\,\partial_\nu f_{\nu\sigma\mu}) - x_\lambda\,\partial_\sigma\partial_\nu f_{\nu\sigma\mu}\,.$$

The second term will be annulled, since $f_{\nu\sigma\mu}$ is antisymmetric in ν and σ. What will then remain is:

$$\partial_\nu f_{\mu\lambda\nu} = \partial_\nu\,[x_\lambda\,\partial_\sigma(f_{\nu\mu\sigma} - f_{\sigma\mu\nu}) - \partial_\nu\,(x_\mu\,\partial_\sigma f_{\sigma\nu\lambda}).$$

Upon subtracting these two inequalities, one will get:

$$\partial_\nu(f_{\lambda\mu\nu} - f_{\mu\lambda\nu}) = \partial_\nu\,[x_\lambda\,\partial_\sigma(f_{\nu\mu\sigma} - f_{\sigma\mu\nu} + f_{\nu\sigma\mu}) - x_\mu\,\partial_\sigma\,(f_{\nu\lambda\sigma} - f_{\sigma\lambda\nu} + f_{\nu\sigma\lambda})].$$

One therefore gives rise to a tensor:

$$\boxed{K_{\mu(\nu\sigma)} = -(f_{\mu\nu\sigma} - f_{\mu\sigma\nu} + f_{\nu\sigma\mu}),}$$

and it is easy to see that it is antisymmetric in μ and ν, due to the antisymmetry of f.

If one sets:

$$\boxed{\tau_{\mu\nu} = \partial_\sigma K_{\mu\nu\sigma}}$$

then our equation will become:

$$\partial_\nu\,[(x_\lambda\, t_{\mu\nu} - x_\mu\, t_{\lambda\nu}) + (x_\lambda \tau_{\mu\nu} - x_\mu \tau_{\lambda\nu})] = 0,$$

or furthermore, upon setting $t'_{\mu\nu} = t_{\mu\nu} + \tau_{\mu\nu}$, one will get:

$$\boxed{\partial_\nu\,(x_\lambda t'_{\mu\nu} - x_\mu t'_{\lambda\nu}) = 0.} \tag{C.3}$$

Therefore, the moment of the tensor $t'_{\mu\nu}$, which one calls the *total energy-momentum tensor*, is conservative. As for the tensor $\tau_{\mu\nu}$, which one calls the *complementary energy-momentum tensor*, it will then result from the antisymmetric character of $K_{\mu\nu\sigma}$ that $\partial_\nu \tau_{\mu\nu} = \partial_\nu \partial_\sigma K_{\mu[\nu\sigma]} = 0$.

Like $t_{\mu\nu}$, $\tau_{\mu\nu}$ is also conservative; therefore, $t'_{\mu\nu}$ is likewise conservative. Finally, since $t'_{\mu\nu}$ is conservative and has a conservative moment, it will result immediately that it is symmetric, which is why one sometimes calls it the *symmetric energy-momentum tensor*. Moreover, this amounts to writing the fundamental equation (3) in the form:

$$\tfrac{1}{2}(t_{\mu\nu} - t_{\nu\mu}) + \tfrac{1}{2}(\tau_{\mu\nu} - \tau_{\nu\mu}) = 0,$$

and, in fact, one easily sees that:

$$\partial_\sigma f_{\mu\nu\sigma} = \tfrac{1}{2}(\tau_{\mu\nu} - \tau_{\nu\mu}).$$

Takabayasi has shown [**24**] that the symmetrization process above, which is due to Belinfante and Rosenfeld, is, in fact, only a particular consequence of a general property of gauge invariance that pertains to the two fundamental equations (1) and (2). Indeed, let $\varphi_{[\mu\nu]\rho}$ be an arbitrary tensor with three indices that is antisymmetric in μ and ν. One deduces another third-rank tensor from it, namely:

$$\Phi_{\mu[\nu\rho]} = \varphi_{\mu\nu\rho} - \varphi_{\mu\rho\nu} + \varphi_{\nu\rho\mu},$$

which is antisymmetric in ν and ρ, as one immediately sees. Now, perform the transformation:

$$t'_{\mu\nu} = t_{\mu\nu} - \partial_\rho \Phi_{\mu\nu\rho},$$
$$f'_{\mu\nu\rho} = f_{\mu\nu\rho} - \varphi_{\mu\nu\rho}.$$

One has:

$$\partial_\nu t'_{\mu\nu} = \partial_\nu t_{\mu\nu} - \partial_\nu \partial_\rho \Phi_{\mu[\nu\rho]}.$$

The second term is zero, by antisymmetry. The relation $\partial_\nu t_{\mu\nu} = 0$ is therefore likewise verified for the transformed tensor: $\partial_\nu t'_{\mu\nu} = 0$.

On the other hand:

$$2\partial_\rho f'_{\mu\nu\rho} = 2\,\partial_\rho f_{\mu\nu\rho} - \partial_\rho \varphi_{\mu\nu\rho},$$
$$t'_{\mu\nu} - t'_{\nu\mu} = t_{\mu\nu} - t_{\nu\mu} + \partial_\rho \Phi_{\mu\nu\rho} + \partial_\rho \Phi_{\nu\mu\rho}.$$

The last two terms are expressed by:

$$-\,\partial_\rho(\varphi_{\mu\nu\rho} - \varphi_{\mu\rho\nu} - \varphi_{\nu\rho\mu} - \varphi_{\nu\mu\rho} + \varphi_{\nu\rho\mu} - \varphi_{\mu\rho\nu}) = -\,2\partial_\rho \varphi_{\mu\nu\rho},$$

which vanishes, along with the similar term in $2\partial_\rho f'_{\mu\nu\rho}$.

One thus has:

$$\tfrac{1}{2}(t'_{\mu\nu} - t'_{\nu\mu}) = \partial_\rho f'_{\mu\nu\rho} .$$

The second fundamental relation is also verified with the new tensors $t'_{\mu\nu}$ and $f'_{\mu\nu\rho}$.

Finally, if the field is localized into a certain region of space then the wave functions and the fundamental tensors will be annulled at every point that is external to a certain world-hypertube $\mathcal{T}$, and it is possible to extend some of the considerations that we developed in Chapter II that related to a *classical* fluid mass to the present formalism. Therefore, one can define a *total field momentum:*

$$G_\mu = \int_\Pi t_{\mu 4} dv ,$$

in which the numeral 4 indicates the component along a temporal axis Λ that is orthogonal to a certain spatial hyperplane Π, and the integral is extended over the region of that plane where $t_{\mu\nu}$ is non-zero. One then knows that, by reason of the equation $\partial_\nu t_{\mu\nu} = 0$, G_μ will be a vector that is independent of the choice of the hyperplane Π (cf., Appendix *A*). Now, if one performs a gauge transformation then one will have:

$$G'_\mu = \int_\Pi t_{\mu 4} dv - \int_\Pi \partial_\rho \Phi_{\mu 4\rho} dv .$$

The second integral can be subdivided into $-\int_\Pi \partial_4 \Phi_{\mu 44} dv$, which is zero, due to the antisymmetry of $\Phi_{\mu[\nu\rho]}$, and $-\int_\Pi \partial_k \Phi_{\mu 4k} dv$, which gives $-\int_\Sigma \Phi_{\mu 4k} d\sigma_k$.

This integral is taken over the spatial surface Σ that bounds the volume considered. The integral will be zero, since that surface is situated completely inside the spatial region where the field tensors are zero (on the condition that one must assume that the gauge tensors are also zero in the same region, which is a hypothesis that one can always make without changing anything that happens inside the region where the field in non-zero). One will thus have:

$$G'_\mu = G_\mu ,$$

so the total momentum vector will be gauge invariant.

The same thing is true for the integrated quantities that one can define by means of the Belinfante tensor. If one considers the tensor $\tfrac{1}{2}(x_\mu\, t_{\lambda\nu} - x_\lambda\, t_{\mu\nu}) + f_{\mu\nu\lambda}$ (which, as we know, is conservative) then one can define a total angular momentum for the field:

$$\Gamma_{[\nu\lambda]} = \int_\Pi \left[\tfrac{1}{2}(x_\mu t_{\lambda 4} - x_\lambda t_{\mu 4}) + f_{\mu\lambda 4}\right] dv ,$$

which will be a tensor. A gauge transformation will then give:

$$\Gamma'_{\mu\lambda} = \int_\Pi \left[\tfrac{1}{2}(x_\mu t_{\lambda 4} - x_\lambda t_{\mu 4}) - \tfrac{1}{2}(x_\mu \partial_\rho \Phi_{\lambda 4\rho} - x_\lambda \partial_\rho \Phi_{\mu 4\rho})\right] dv + \int_\Pi (f_{\mu\lambda 4} - \varphi_{\mu\lambda 4})\, dv$$

$$= \Gamma_{\mu\lambda} - \tfrac{1}{2}\int_{\Pi}\Big[(x_\mu \partial_\rho \Phi_{\lambda 4\rho} - x_\lambda \partial_\rho \Phi_{\mu 4\rho})\Big]dv + \int_{\Pi} \varphi_{\mu\lambda 4}\, dv\,.$$

The first integral is integrated by parts, and the completely integrable part reverts to a surface integral over Σ, and is therefore annulled. What will remain is:

$$\Gamma'_{\mu\lambda} = \Gamma_{\mu\lambda} + \tfrac{1}{2}\int_{\Pi}\Big[(\Phi_{\lambda 4\rho}\partial_\rho x_\mu - \Phi_{\mu 4\rho}\partial_\rho x_\lambda)\Big]dv + \int_{\Pi} \varphi_{\mu\lambda 4}\, dv\,.$$

The first integrand becomes:

$$\tfrac{1}{2}(\Phi_{\lambda\, 4\mu} - \Phi_{\mu\, 4\lambda}),$$

namely:

$$\tfrac{1}{2}(\varphi_{\lambda\, 4\mu} - \varphi_{\lambda\mu\, 4} - \varphi_{\, 4\mu\lambda} - \varphi_{\mu\, 4\lambda} + \varphi_{\mu\,\lambda\, 4} + \varphi_{4\lambda\mu}) = \varphi_{\mu\, 4\lambda}\,,$$

which is annulled, along with the second integrand. One thus has:

$$\Gamma'_{\mu\lambda} = \Gamma_{\mu\lambda}\,,$$

so the total angular moment is also gauge invariant.

In summation, a system of wave functions whose evolution is determined by a Lagrangian formalism corresponds to a unique tensorial field that is determined by tensors $t_{\mu\nu}$ and $f_{\mu\nu\lambda}$ that obey two fundamental conservation relations. By contrast, these two equations determine the two tensors only up to a very large gauge indeterminacy. It is remarkable that the change of gauge leaves invariant the two tensors G_μ and $\Gamma_{\mu\nu}$ that result from integrating over the domain of the field.

It is also possible to say (and this will permit some very interesting developments) that a system of wave functions can provide the two quantities G_μ and $\Gamma_{\mu\nu}$ in a unique fashion, but they will correspond to an infinitude of systems of tensors $t_{\mu\nu}$ and $f_{\mu\nu\lambda}$ that obey the two fundamental conservation equations, but differ in their gauges, and it is permissible to choose a particular one of these systems by subjecting the choice of gauge to some suitable supplementary conditions. Therefore, if one chooses simply $\varphi_{\mu\,\nu\lambda} = f_{\mu\,\nu\lambda}$ then the new Belinfante tensor $f'_{\mu\nu\lambda}$ will be zero and the tensor $t'_{\mu\nu}$ will be symmetric, so it will not only be conservative, but so will its moment; as we have stated, it is simply the symmetric energy-momentum tensor, which may, if one desires, be chosen to represent the formalism collectively. As we know, some choices of gauge are more interesting than others.

BIBLIOGRAPHY

[1] L. DE BROGLIE. – *Théorie des particules de spin* 1/2, Paris, Gauthier-Villars, 1952, chap. IV.
[2] J. V. WEYSSENHOFF and RAABE. – Acta Phys. Pol. **9** (1947), 8.
[3] C. MØLLER. – *Theory of Relativity*, Oxford, Clarendon Press, 1955, chap. V.
[4] FRENKEL. – Zeit. Phys. **37** (1926), 243.
[5] MATHISSON, Zeit. Phys., **67** (1931), 270 and 826; Math. Ann. **107** (1933), 400; Acta Phys. Pol. **6** (1937), 163 and 218.
[6] EINSTEIN and GROMMER. – Sitz. Preuss. Akad. Wiss. (1927), 1.
[7] J. P. VIGIER. – *Structure des microobjets*, Paris, Gauthier-Villars, 1956, chap. III.
[8] LUBANSKI. – Acta Phys. Pol. **6** (1937), 356.
[9] TAKABAYASI. – Prog. Theor. Phys., Suppl. **4** (1957).
[10] EDDINGTON. – *Mathematical Theory of Relativity*, Oxford, Clarendon Press.
[11] L. DE BROGLIE. – *La physique quantique restera-t-elle indéterministe?*, Paris, Gauthier-Villars, 1953, exp. gén.
[12] MØLLER. – Ann. Inst. H. Poincaré **10** (1949), 5.
[13] PRYCE. – Proc. Roy. Soc. London A**195** (1948), 62.
[14] BOHM and VIGIER. – Phys. Rev. **109** (1958), 1882.
[15] H. POINCARÉ. – Acta Math. **7** (1885), 259.
[16] L. DE BROGLIE. – Journ. Phys. (7) **8** (1927), 225, § 111.
[17] BOHM, VIGIER, LOCHAK, HALBWACHS, Séminaire L. de Broglie, Institut H. Poincaré, no. 8, 1956.
[18] HALBWACHS. – C. R. **243** (1956), 1022 and 1098.
[19] CORSON. – *Introduction to Tensors, Spinors, and relativistic Wave Equations,* Blackie and Sons, London, 1953, chap. I.
[20] NOETHER. – Göttinger Nachrichten (1918), 235.
[21] ROSENFELD. – Acad. Roy. Belgique **18** (1940), 6.
[22] BELINFANTE. – Physica **7** (1939), 887.
[23] WINOGRADSKI. – Cahiers de Phys. 67-1956 and 16-1959.
[24] TAKABAYASI. – Séminaire L. de Broglie, Institut H. Poincaré, January 1957.
[25] HALBWACHS, LOCHAK, VIGIER. – C. R. **241** (1955), 692.
[26] HALBWACHS, LOCHAK, VIGIER. – C. R. **241** (1955), 744.
[27] HALBWACHS. – Journ. Phys. **17** (1956), 67.
[28] BOHM. – Phys. Rev. **85** (1952), 166 and 180.
[29] VIGIER. – *Structure des microobjets*, Paris, Gauthier-Villars, 1956, Introduction générale.
[30] L. DE BROGLIE. – *Formulation non linéaire et causale de la Mécanqiue Ondulatoire*, Paris, Gauthier-Villars, 1957.
[31] BOHM. – Phys. Rev. **89** (1953), 458.
[32] BOHM and VIGIER. – Phys. Rev. **96** (1954), 208.
[33] VIGIER. – *Structure des microobjets*, Paris, Gauthier-Villars, 1956, chap. IV.
[34] EINSTEIN, INFELD, HOFFMANN. – Ann. Math. **41** (1940), 455.

[35] INFELD and WALLACE. – Phys. Rev. **57** (1940), 797.

…(gap in bibliography numeration)

[40] TAKABAYASI. – Prog. Theor. Phys. **8** (1952), 143.
[41] MADELUNG. – Zeit. Phys. **40** (1927), 322.
[42] L. DE BROGLIE. – C. R. **185** (1927), 1118.
[43] TAKABAYASI. – Prog. Theor. Phys. **9** (1953), 187.
[44] LICHNEROWICZ. – *Éléments de calcul tensorial*, Paris, Colin, 1955, chap. VII.
[45] LICHNEROWICZ. – Ann. Scient. de l'E. N. S. **60** (1943), 247.
[46] LICHNEROWICZ. – Cours au Coll. de Fr. (mimeograph), 1952-1953.
[47] LOCHAK. – Séminaire L. de Broglie, Institut H. Poincaré, 1956., annex.
[48] JAKOBI and LOCHAK. – C. R. **243** (1956), 234 and 357.
[49] AYMARD. – C. R. **243** (1956), 885 and 1100; **244** (1957), 3133.
[50] TAKABAYASI and VIGIER. – Prog. Theor. Phys. **18** (1957).
[51] VIGIER, HILLION, and LOCHAK. – C. R. **246** (1958), 394, *et seq*.
[52] UNAL and VIGIER. – C. R. **245** (1957), 1787 and 1891.
[53] TAKABAYASI. – C. R. **246** (1958), 64.
[54] GÜRSEY. – Nuov. Cim. **5** (1957), 784.
[55] L. DE BROGLIE. – *Théorie des particules de spin* 1/2, Paris, Gauthier-Villars, 1952, chap. VI.
[56] PAULI. – Ann. Inst. H. Poincaré **6** (1936), 109.
[57] KOFFINCK. – Ann. d. Phys. **38** (1940), 421.
[58] PETIAU. – Journ. Math. pures et appl. **25** (1946), 235.
[59] KARPMANN and RAMAN. – C. R. **243** (1956), 1284.
[60] HALBWACHS. – C. R. **245** (1957), 1298.
[61] HALBWACHS. – C. R. **245** (1957), 1392.
[62] HALBWACHS. – C. R. **245** (1957), 1513.

…(gap in bibliography numeration)

[65] VIGIER. – *Structure des microobjets*, Paris, Gauthier-Villars, 1956, chap. II.
[66] JAKOBI and LOCHAK. – C. R. **244** (1957), 1014.
[67] SCHÖNBERG. – Nuov. Cim. **1** (1955), 543.
[68] COSTA DE BEAUREGARD. – Thesis, Paris, 1943.
[69] PHAM MAU QUAN. – Journ. of rat. Mech. and Anal. **5** (1956), 473.
[70] BOHM, TIOMNO, and SCHILLER. – Nuov. Cim., Supp. I (1955), 48.
[71] BHABHA. – Rev. Mod. Phys. **21** (1949), 451.
[72] YVON. – Journ. Phys. **1** (1940), 18.