THE GREAT PROBLEMS IN SCIENCE
WORKS COLLECTED BY M^{me}. P. FEVRIER

X.

THE RELATIVISTIC THEORY

OF SPINNING FLUIDS

AN INVESTIGATION INTO THE DYNAMICS OF RELATIVISTIC SPINNING PARTICLES AND THE RELATIVISTIC HYDRODYNAMICS OF FLUIDS THAT ARE ENDOWED WITH AN INTERNAL ANGULAR MOMENTUM DENSITY

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TO THE MEMORY OF MY FATHER

MAURICE HALBWACHS

AN EXEMPLARY SCHOLAR

“Discover now, and the most profound level, the internal cohesion of the universe, perceive all of the forces that are at work, reveal the very essence of things, and finally, cease to avail oneself of words.”

(Goethe, *Faust*)

“There is nothing in matter that can excite in us any sensation, except for its motion, its shape or position, and the size of its parts.”

(Descartes, *Principes*)

“The river and the *drops* in the river…the place of each *drop*, its relation to the other drops, its interdependence upon the other ones, the direction of its (rectilinear, curved, circular) motion towards the top, towards the bottom…The “ideas,” like the *comprehending* of the different aspects of motion, of different drops (or “things”), of different currents…Such is the dialectical picture of the universe.”

(Papers on the Hegelian dialectic)

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PREFACE

In the early days of wave mechanics, there was a viewpoint that endowed that theory with a hydrodynamical aspect, and Madelung was the first to point out its existence.

In that epoch, the author of this preface had endeavored to utilize that hydrodynamical aspect in order to obtain a concrete interpretation of wave mechanics that would uphold the notion of the localization of a particle in space (viz., the theory of the double solution). Together with the work of Bohm, that attempt was reprised at the Institut Henri Poincaré, notably, by Vigier and his collaborators, Hillion, Halbwachs, Lochak, *et al.*

On the one hand, the notion of the "spin" of a particle, which introduced for the electron in 1925 by Uhlenbeck and Goudsmit, was incorporated into wave mechanics several years later, thanks to the theory of the spinning electron that was due to Dirac. Dirac's theory itself also presented a hydrodynamical aspect to it, like the original wave mechanics, but led naturally to hydrodynamics with a relativistic character, since it had to be in accord with relativistic concepts. Nevertheless, although Dirac's equations have been subjected to innumerable analyses for thirty years, it was only in the last few years that we have discovered all of the richness in its resemblance to mechanics and hydrodynamics that is hidden behind its formalism. That discovery was due to the work of Takabayasi, as well as Bohm and Vigier. It has recently led to a general theory of particles that are imagined as being very small fluid droplets that move in a state of relativistic rotation, or rather – to employ an analogy that seems more exact – as a very small region of space that sits in the field of a wavelike vortex. This new theory of particles, which defines every kind of particle by a set of quantum numbers when they are subsequently interpreted in the standard modern formulation of Gell-Mann, provides a model of all particles that conforms to a general ambition to bring concepts that are more concrete back to microphysics.

The work of Halbwachs, which was the subject of his doctoral thesis and which will be discussed in the present book, constitutes a very important contribution to the deeper study of the mechanical and hydrodynamical concepts that carry over into Dirac's relativistic formalism. After recalling the theory that was proposed by Frenkel, Mathisson, and Weyssenhoff (which is already rather old, but nonetheless quite instructive to contemplate), Halbwachs subjects the study of the motion of relativistic spinning particles to a very detailed analysis. Defining all of the aspects of this extremely complex theory in a painstaking fashion, he shows how it is expedient to define the "center of mass" and the "center of matter," which were previously defined in the work of Bohm and Vigier, in a rigorous manner, and these centers will play a fundamental role in the new theory of particles. Halbwachs then shows that if one seeks to endow these remarkable points with the structure of a small, relativistic system that spins in an arbitrary Galilean reference frame then the definition that one obtains will not have the necessary invariant character. In order to obtain an invariant definition of those two centers, he defines them in a fundamental reference that both Weyssenhoff and

Halbwachs call the "reference frame of inertia." Both points consequently prove to be very neatly well-defined.

The author gives a number of applications of these fundamental definitions, and then arrives at the introduction of two very important quantities, one of which has the character of a quadri-vector, and the other of which has the character of a pseudo-quadrivector, that he calls the "wobble" and the "gyration," respectively. The detailed study of the role that these quantities play then leads to a link with the quadri-vector of "spin" that comes from Dirac's formalism by means of formulas in which an angle A appears that was previously introduced by Takabayasi in his own work, and is an angle whose significance (which has remained rather mysterious up to now), can thus be clarified. I will not expound any further upon the results in that part of Halbwachs's thesis, since he always subjects all of the questions under consideration to a very meticulous examination.

Having been given these notions as a basis, Halbwachs then embarks upon a general theory of hydrodynamical models in which he lets the work of Takabayasi serve as his guide, since that was where the main conclusions were deduced. He then develops the hydrodynamical viewpoint in the case of the non-relativistic Schrödinger equations, then in the case of the Klein-Gordon equation for relativistic wave functions, and finally, in the case of the Dirac wave equations. In all of these cases, he attentively studies the interpretations that have been proposed, as well as some others that one might envision.

The last chapter of this book is dedicated to spinning fluids with molecular structure. The author first considers the mean properties of a fluid that is composed of particles that are fluid droplets that possess the infinitesimal characteristics that were studied in the preceding chapters, and then he studies the mean fluid that is thus defined in the case where spin is not involved, and then in the case where it is. He studies a series of related questions in great detail that lead to various viewpoints regarding such a fluid.

We conclude by saying that this book by Halbwachs constitutes an extremely important document for the study of problems that undoubtedly deserve to play a major role in the neighboring development of quantum microphysics.

By examining the questions that relate to the hydrodynamics of a body in rotation in a very concise fashion when one takes the theory of relativity into account, as well as collecting and analyzing all of the results that were obtained before by the other researchers collectively, besides numerous improvement or corrections, and his own personal contributions, Halbwachs has arrived at a book that is, in itself, quite important, and which might also contribute, as I pointed out in the beginning of this preface, to the consolidation of the basis for the reinterpretation of wave mechanics with the attempts that serve to obtain a concrete model of particles whose various types will correspond to the quantum states of liquid droplets (or, in my more exact sense: wave fields) that are in a state of relativistic rotation. From this point of view, Halbwachs's thesis is a definitive statement of its era and might be of very great benefit.

Louis de Broglie

FOREWORD

Whenever it is essential, this book will remain in the context of special relativity. Unless stated to the contrary, one will consider a Minkowski spacetime with a Euclidian metric that is referred to a Galilean reference frame exclusively. Thus, one does not need to distinguish covariant components from contravariant ones, so all tensorial indices will be placed in the lower position. A summation will be assumed to be performed over any indices that are repeated twice.

Greek indices (μ, ν, α, β) will be used for the spacetime components ($\mu = 1, 2, 3, 4$) with the fourth component $x_4 = ict$.

Latin indices $i, j, k, \ldots$ will denote the space components ($i = 1, 2, 3$).

One occasionally has to specify the time component of a spacetime vector V_μ to be real. One will then denote it by:

$$V_\otimes = \frac{1}{ic} V_4 ,$$

while one will reserve the notation V_0 for an invariant (e.g., norm, proper mass). The symbols 0, 1 in the upper position will denote the components of a vector in the proper frame Σ_0 or in a frame Σ_1, respectively; e.g.:

$$V_k^0 , \; V_4^0 , \; V_k^1 , \; V_4^1 .$$

Derivatives with respect to coordinates will be denoted by:

$$\partial_\mu = \frac{\partial}{\partial x_\mu} .$$

Derivatives with respect *proper* time will be denoted by:

$$\frac{d}{d\tau} \qquad \text{or by a dot} \qquad \frac{d}{d\tau} G_\mu = \dot{G}_\mu .$$

One utilizes the Kronecker symbol:

$$\begin{array}{lll} \delta_{\mu\nu} = 1 & \quad \text{if} \quad & \mu = \nu, \\ \delta_{\mu\nu} = 0 & \quad \text{if} \quad & \mu \neq \nu, \end{array}$$

as well as the generalized symbols:

$$\delta_{\alpha\beta}^{\mu\nu} = \delta_{\mu\alpha}\, \delta_{\nu\beta} - \delta_{\mu\beta}\, \delta_{\nu\alpha} ,$$

$$\delta^{\mu\nu\lambda}_{\alpha\beta\gamma} = \delta_{\mu\alpha}\,\delta_{\nu\beta}\,\delta_{\lambda\gamma} + \delta_{\mu\beta}\,\delta_{\nu\gamma}\,\delta_{\lambda\alpha} + \delta_{\mu\gamma}\,\delta_{\nu\alpha}\,\delta_{\lambda\beta}$$
$$- \delta_{\mu\alpha}\,\delta_{\nu\gamma}\,\delta_{\lambda\beta} - \delta_{\mu\gamma}\,\delta_{\nu\beta}\,\delta_{\lambda\alpha} - \delta_{\mu\beta}\,\delta_{\nu\alpha}\,\delta_{\lambda\gamma}.$$

One will also use the completely-antisymmetric Levi-Civita symbols ε_{ijk} and $\varepsilon_{\mu\nu\alpha\beta}$, which are zero when two indices are equal, have the value + 1 when ijk or $\mu\nu\alpha\beta$ are obtained from an even number of transpositions of the sequence 123 or 1234, resp., and the value – 1 when that number is odd. One employs the well-known relations:

$$\varepsilon_{ijk}\,\varepsilon_{ijl} = 2\,\delta_{kl}\,,$$
$$\varepsilon_{ijk}\,\varepsilon_{ilm} = \delta^{jk}_{lm}\,,$$

$$\varepsilon_{\mu\nu\sigma\rho}\,\varepsilon_{\mu\nu\sigma\tau} = 3!\,\delta_{\rho\tau}\,,$$
$$\varepsilon_{\mu\nu\sigma\rho}\,\varepsilon_{\mu\nu\sigma\tau} = 2\,\delta^{\sigma\tau}_{\alpha\beta}\,,$$
$$\varepsilon_{\mu\nu\sigma\rho}\,\varepsilon_{\mu\alpha\beta\gamma} = \delta^{\nu\sigma\tau}_{\alpha\beta\gamma}\,.$$

One will use Takabayasi's projection operator:

$$\eta_{\mu\nu} = \delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\,,$$

as well, which will permit one to form a covariant expression for the projection of a vector A_μ onto the proper space hyperplane that is orthogonal to the unit-length vector u_μ (i.e., $u_\mu\,u_\mu = -c^2$) by way of:

$$A^{(u)}_\mu = \eta_{\mu\nu}\,A_\nu\,.$$

Obviously, one will have:

$$\eta_{\mu\nu}\,\eta_{\mu\lambda} = \eta_{\nu\lambda}\,, \qquad \eta_{\mu\nu}\,u_\nu = 0.$$

Finally, if the vector A_μ is itself contained in proper space (i.e., $A_\mu\,u_\mu = 0$) then one will have $\eta_{\mu\nu}\,A_\nu = A_\mu$.

One will denote an antisymmetric tensor by $S_{[\mu\nu]}$:

$$S_{[\mu\nu]} = -S_{[\nu\mu]}\,,$$

and the antisymmetric part of an arbitrary tensor $S_{\mu\nu}$ by:

$$S_{<\mu\nu>} = \tfrac{1}{2}(S_{\mu\nu} - S_{\nu\mu}).$$

If $S_{\mu\nu}$ is itself antisymmetric then one will have $S_{[\mu\nu]} = S_{<\mu\nu>}$.

Recall that the contracted product of an antisymmetric tensor $S_{[\mu\nu]}$ with a symmetric tensor $A_{\mu\nu} = A_{\nu\mu}$ is zero $S_{[\mu\nu]}\,A_{\mu\nu} = 0$.

We shall call the quantities:

$$*s_k \;\;= \tfrac{1}{2}\,\varepsilon_{ijk}\, s_{ij}\,,$$
$$*\sigma_{[ij]} = \;\;\varepsilon_{ijk}\;\sigma_k\,,$$

the *spatial duals* of a tensor s_{ij} or a vector σ_k, respectively.

If the tensor is antisymmetric then the components of $s_{[ij]}$ and its dual $*s_k$ will be equal, and the dual of the dual of $s_{[ij]}$ will be $s_{[ij]}$.

Similarly, one will define the *spacetime duals* of the quantities $f_{\mu\nu\lambda}$, $s_{\mu\nu}$, σ_μ by:

$$*f_\alpha \;\;= \frac{i}{3!}\,\varepsilon_{\alpha\mu\nu\lambda}\, f_{\mu\nu\lambda}\,,$$
$$*s_{[\mu\nu]} \;= \frac{i}{2}\;\varepsilon_{\alpha\beta\mu\nu}\, s_{\mu\nu}\,,$$
$$*\sigma_{[\alpha\beta\gamma]} = \;i\;\varepsilon_{\alpha\beta\gamma\mu}\;\sigma_\mu\,.$$

If the tensor $s_{\mu\nu}$ is antisymmetric then the components of $s_{[\mu\nu]}$ or its dual $*s_{[\mu\nu]}$ will be equal, and the dual of the dual of $s_{[\mu\nu]}$ will be $-\,s_{[\mu\nu]}$. Similarly, the components of a completely-antisymmetric tensor $f_{[\mu\nu\lambda]}$ will be equal to those of its dual $*f_\mu$, and the dual of the dual of $f_{[\mu\nu\lambda]}$ will be $f_{[\mu\nu\lambda]}$.

In Appendixes *A*, *B*, and *C*, we have separately discussed the questions (which are classical, moreover) that are not directly connected with the study of spinning particles or spinning fluids. Nevertheless, we will refer to them constantly, and an understanding of those issues will be indispensible to one's understanding of the entire book. All of the references will refer to the bibliography at the end of this volume.

Before commencing with my exposé, I would like to take this opportunity to mention all of the correspondences that I had with Louis de Broglie and André Lichnerowicz, who consented to direct my efforts, and who gave me inestimable assistance by way of their advice and the work that they did. I would similarly like to express my gratitude for the numerous clarifications that I received from my mentors David Bohm of Bristol and Takehiko Takabayasi of Nagoya, and to all of the group of researchers at the Institut Henri Poincaré, who constituted an admirable team of collective labor along the lines of the causal interpretation. I would also like to thank Casimir Jausserain of Marseilles, who allowed be to dedicate myself to my research and who provided me with ideal working conditions, as well as Georges Bodiou of Marseilles, who guided me in certain mathematical questions. But most of all, I am indebted to Jean-Pierre Vigier, who has followed all of my efforts, step-by-step, and without whom I could not possibly imagines how this book could have managed to continue up to the end. To them, as my friends and masters, I extend my deepest gratitude.

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INTRODUCTION

§ 1. Madelung's hydrodynamical representation. By setting:

$$\varphi = R\, e^{iS/\hbar},$$

one will find that the Schrödinger equation in an external field that is derivable from a scalar potential $V(\mathbf{x})$, namely:

$$\frac{\hbar}{i}\frac{\partial \varphi}{\partial t} = \frac{\hbar^2}{2m}\Delta\varphi - V\varphi,$$

can be split into two real equations:

$$\text{(J)} \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0,$$

$$\text{(C)} \qquad \frac{\partial (R^2)}{\partial t} + \nabla\left(R^2\frac{\nabla S}{m}\right) = 0.$$

If one suppresses the $\hbar^2$ term then the first equation will represent the Hamilton-Jacobi equation, and will define the basis for the conceptualization of the causal interpretation of the theory [**16, 28, 29**]. In accord with the formalism of classical analytical dynamics, one regards the action functional S as a potential whose gradient represents a momentum vector:

$$m\mathbf{v} = \nabla S,$$

and in order to recover the exact expression for the Hamilton-Jacobi , one introduces a "quantum potential":

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R},$$

which is combined with the potential V of the force. One then has:

$$\text{(J)} \qquad \frac{\partial S}{\partial t} + \tfrac{1}{2}mv^2 + (V + Q) = 0,$$

$$\text{(C)} \qquad \frac{\partial (R^2)}{\partial t} + \nabla\cdot\left(R^2\mathbf{v}\right) = 0.$$

Equation (J) shows that the quantity $-\partial S/\partial t$ represents the classical *energy* of a particle of mass m and velocity $\mathbf{v}$ when it is placed in a field with the potential $V + Q$. If one takes the gradients of the two sides of the equation then one will see that this particle executes a classical motion when one takes the external force $-\nabla(V + Q)$ into account. In regard to the non-quantum motion, which satisfies the Hamilton-Jacobi equation, properly speaking, viz.:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = 0,$$

one sees that the quantum effects are introduced solely by the presence of the potential:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R},$$

which is determined entirely by the function R, which obeys the equation of analytic continuity (C) in its own right.

Given this, one may propose an interpretation, which we call "statistical" [**40**], for the function R: If one considers an ensemble of a large number of identical particles that are distributed with a density $\rho = R^2$ then equation (C) will express simply the conservation of the number of particles in the course of their motion [**16**]. This concept of ensembles – or clouds – of particles assumes a well-defined initial distribution, and in so doing raises some difficulties that could not be resolved until recently, by introducing the hypothesis of stochastic fluctuations of a wave around a mean state that is expressed by equations (C) and (J) (see above).

However, it is possible to mathematically construct a model of *representative fluids* solely on the basis of these equations by introducing a matter density $\rho = R^2$ (or a mass density $\mu = \rho m$) and a flow velocity $\mathbf{v} = (\nabla S) / m$ at each point.

Such a fluid simultaneously provides a hydrodynamical representation for the Schrödinger wave function and a continuous representation of the mean of the ensemble of particle in the preceding theory. The quasi-classical dynamical laws that govern the particle of the causal theory correspond to the laws of hydrodynamics here, which are likewise quasi-classical. This model was proposed by Madelung [**41**] in 1927, which was the same year that de Broglie set down the foundations of the statistical formulation.

As we have previously discussed, equation (C) insures the conservative character of the fluid. One remarks that by attempting to derive the velocity field from a potential S/m one will be limited to the case of irrotational flow, since the curl of a gradient is zero. Consider equation (J), when expressed in tensor notation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\partial_j S\,\partial_j S + V - \frac{\hbar^2}{2m}\frac{\partial_j\partial_j R}{R} = 0.$$

Taking its gradient, replacing $\partial_j S$ with mv_j, and multiplying by $\rho = R^2$ gives:

$$\rho\frac{\partial(mv_k)}{\partial t} + \rho v_j \partial_j(mv_k) = -\rho\,\partial_k V + \rho\left(\frac{\hbar^2}{2m}\frac{\partial_j\partial_j R}{R}\right).$$

If one is given equation (C):

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho v_j) = 0$$

then the left-hand side can be better written in the form:

$$\frac{\partial}{\partial t}(\rho m v_k) + \partial_j(v_j\, \rho m v_k),$$

in which we recognize the derivative d / dt of the density $\rho m v_k$ along the streamlines (see Appendix *A*). If one recalls that V refers to the potential of the force $F_k = -\,\partial_k V$ that acts on a *particle* then the other term $-\,\rho\,\partial_k V$ will exhibit the *external force density* f_k . One will then finally have:

$$\frac{d}{dt}(\rho m v_k) = f_k + \frac{\hbar^2}{2m}\rho\,\partial_k\left(\frac{\partial_j\partial_j R}{R}\right).$$

One recognizes Euler's equation from classical hydrodynamics, along with an extra external force density f_k and another quantum force density that is expressed as a function of the quantum potential in the causal interpretation.

This quantum force does not have the form of a scalar gradient. It is easy to show that it is a manifestation of the existence of *quantum stresses* in the fluid interior. Indeed, one may write:

$$\varphi_k = \frac{\hbar^2}{2m}R^2\,\partial_k\left(\frac{\partial_j\partial_j R}{R}\right) = \frac{\hbar^2}{2m}(R\,\partial_k\,\partial_j\partial_j R - \partial_k R\;\partial_j\partial_j R),$$

or, upon integrating the two terms by parts:

$$\varphi_k = \frac{\hbar^2}{2m}[\partial_j(R\,\partial_j\partial_k R) - \partial_j(\partial_j R\;\partial_k R)].$$

In this form, one sees that φ_k is the divergence of a symmetric tensor θ_{jk} that may be regarded as an *internal stress potential* with respect to a unit volume:

$$\varphi_k = -\,\partial_j\,\theta_{jk}\,,$$

with

$$\boxed{\theta_{kj} = \frac{\hbar^2}{2m}(\partial_k R\,\partial_j R - R\,\partial_k\partial_j R),}$$

which one may also write, upon introducing the density $\rho = -\,R^2$:

$$\theta_{kj} = -\frac{\hbar^2}{2m}R^2\,\partial_k\left(\frac{\partial_j R}{R}\right) = -\frac{\hbar^2}{2m}R^2\,\partial_k\partial_j\log R = -\frac{\hbar^2}{2m}\rho\,\partial_k\partial_j\log\sqrt{\rho}\;;$$

i.e.:

$$\boxed{\theta_{kj} = -\frac{\hbar^2}{4m}\rho\,\partial_k\partial_j \log\rho.}$$

One sees that the fluid is not perfect, since the off-diagonal components of the tensor are not zero, in general. Nevertheless, upon considering the diagonal components, one may define an internal pressure p (see Appendix *B*), which does not, in general, suffice to completely characterize the totality of the stress forces:

$$p = \tfrac{1}{3}\,\theta_{jj} = \frac{\hbar^2}{6m}[(\nabla R)^2 - R\,\Delta R],$$

or furthermore:

$$p = -\frac{\hbar^2}{12m}\rho\,\Delta(\log\rho).$$

This hydrodynamical representation renders a very great service that has the support of intuitive reasoning to one who is determined – as is the case in the research that is carried our in the context of the causal interpretation – to represent physical quantities by classical variables; i.e., to interpret the "observables" of quantum mechanics by means of "hidden variables" that possess a tensorial character and satisfy differential equations that reduce to deterministic laws, in principle. We will now find it expedient to rapidly show, by way of example, how one uses the Madelung fluid to prove the two fundamental theorems of the causal interpretation in the case of the Schrödinger equation.

§ 2. The guidance law of the causal interpretation. One knows that in the theory that is called the "double solution" [**16, 30, 65**], the particle aspect of a micro-object is represented by a singularity of the amplitude of a wave function. This wave function:

$$U = f\,e^{iS/\hbar}$$

obeys the customary linear wave equation at every point, with the exception of a small region of radius r_0 whose order of magnitude is that of the classical dimensions of a particle ($\sim 10^{-13}$ cm.). The function U may be decomposed into two functions $U = U_0 + \varphi$. The latter:

$$\varphi = R\,e^{iS'/\hbar}$$

is a *regular* function that, when regarded by itself, is a solution of the linear wave equation at each point. It may be represented by a regular fluid of density $\rho' = R^2$ and velocity:

$$\mathbf{v} = \frac{\nabla S'}{m}.$$

One assumes that the total function U does not begin to differ appreciably from the regular function φ, except in the interior of a sphere of radius r_1 ($r_1 \gg r_0$). As far as the real representation of the complete wave function is concerned, a *singularity* of the density occurs in this sphere; i.e., an extremely strong concentration of fluid. One assumes that on the scale of r_1 the function S', which is related to φ and its gradient, is uniform, as well as the external field.

From a wave-like viewpoint, this hypothesis signifies that the singularity is much smaller than the wavelength of the associated regular wave. The variations of the external field might not be negligible on the scale of the wavelength. This is precisely what characterizes the "quantum domain" and, in particular, the atomic fields. Our proof will remain valid, provided that the variations deviate from the scale of the singularity only negligibly; i.e., from the dimensions of the particle itself.

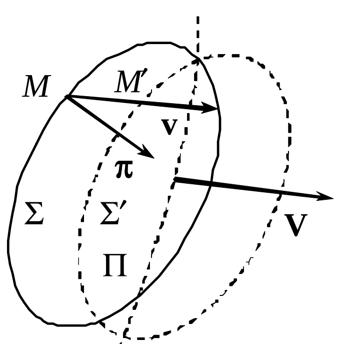
We have proved the Hamilton-Jacobi guiding relation:

$$m\mathbf{v} = \nabla S'$$

that relates the velocity $\mathbf{v}$ of the singularity to the flow velocity $\mathbf{V} = \nabla S' / m$ of a regular fluid, but under these conditions we will also show that this *classical* relation is valid at the *quantum* level.

One characterizes the singularity by the fact that in the interior of a sphere of radius r_1 the density $\rho = f^2$ increases very rapidly with $1 / r$, such that $\partial\rho / \partial r$ is much larger than ρ, and one must consider the ratio $\rho / (\partial\rho / \partial r)$ to be zero. One supposes from the outset that on the scale of r_1 the displacement of the singularity of density in the course of time with velocity $\mathbf{v}$ is effected without any deformation of the distribution of ρ inside of the singularity. The values of ρ are displaced in the course of time with a velocity that has the same magnitude and direction for all of the points of the singularity.

Having said this, the entire proof rests upon the following hypothesis, which is called "phase matching": In the domain of the singularity that is defined between the radii r_0 and r_1 there exists a closed surface Σ, upon which the density of the fluid ρ is the same at every point, and where the phase S of the complete wave is and remains constantly equal to the (uniform) phase S' of the regular wave φ, along with its first derivatives. This hypothesis does not have any hydrodynamical significance. It refers explicitly to the wave-like nature of the functions U and ρ, for which it represents a sort of resonance condition. It does not result from the particular form of the singular function.



Now, consider the surface Σ and its image Σ' when it is transported as a whole for a time δt such that each point M passes to M' with the same velocity $\mathbf{v}$ as that of the singularity. One represents the normal to the surface Σ at M by the vector $\mathbf{n}$, which is, consequently, collinear with $\nabla\rho$. One also represents a wave surface p of the regular wave, to which the flow velocity $\mathbf{V}$ of the fluid corresponds, which is uniform on the scale of r_1 and collinear with $\nabla S'$.

If the displacement of M is $\mathbf{v}\ \delta t$ then the density of the fluid at M at the instant $t + \delta t$ will be:

$$\rho_{M'} = \rho_M + \frac{\partial \rho}{\partial t}\,\delta t + \partial_k \rho\, v_k\ \delta t.$$

However, like the surface Σ, it will be displaced as a whole with the same value of ρ:

$$\rho_{M'} = \rho_M\,,$$

or

$$\partial_k \rho\, v_k\ \delta t + \frac{\partial \rho}{\partial t}\,\delta t = 0,$$

so

$$\partial_k \rho\, v_k = -\frac{\partial \rho}{\partial t}\,,$$

and the scalar product $\partial_k \rho\, v_k$ can be written:

$$\frac{\partial \rho}{\partial t}\, v \cos(\mathbf{v}, \mathbf{n}),$$

so, the norm of $\mathbf{v}$ will be:

$$v \cos(\mathbf{n}, \mathbf{v}) = -\,\frac{\partial \rho / \partial t}{\partial \rho / \partial n}\,.$$

Applying the continuity equation to the Madelung fluid gives:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \text{with} \quad \mathbf{v} = \frac{\nabla S}{m}\,.$$

The hypothesis of phase matching permits us to identify the gradient $\nabla S'$ of the phase of the regular wave with the uniform velocity $\mathbf{V}$ of a regular fluid. One then has:

$$m\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{V} + \rho\, \Delta S' = 0,$$

or furthermore:

$$m\frac{\partial \rho}{\partial t} + m\frac{\partial \rho}{\partial n}\, V \cos(\mathbf{n}, \mathbf{v}) + \rho\, \Delta S' = 0.$$

Dividing this by $\partial \rho / \partial n$ gives:

$$-\, mv \cos(\mathbf{n}, \mathbf{v}).$$

Therefore:

$$mv \cos(\mathbf{n}, \mathbf{v}) = mV \cos(\mathbf{n}, \mathbf{V}).$$

The values and direction of $\mathbf{v}$ and $\mathbf{V}$ are uniform on all of the surface Σ. On the contrary, $\mathbf{n}$ will vary and may point in any direction in space. That is why the condition above might not insure that the direction **v** and **V** are consistent at each point; that would entail the equality of the two velocities, as well. Hence, the singularities will be carried along and guided by the current of the regular fluid. This is the *guidance law* of quantum mechanics.

§ 3. Bohm and Vigier's statistical theorem. It is likewise by using the concept of the Madelung fluid that Bohm and Vigier [**31-33**] proved that the distribution $P = R^2$ of an ensemble of a large number of particles that have the same wave function does not represent a particular case that corresponds to a special choice of initial conditions, but, on the contrary, represents the limiting distribution towards which such a cloud of particles necessarily converges for any given initial distribution. We rapidly summarize this fundamental proof: The basic hypothesis is that the Madelung fluid represents the *mean state* of a real fluid that is subjected to disordered fluctuations, a fluid that always obeys the conservation law, but whose velocity is no longer the gradient of a Hamilton-Jacobi function. The fluctuations of the fluid reflect the fluctuations of the wave function that are due to either the continual variation of the limiting conditions (thermal agitation of the molecules from the mechanism of interference, for example) or to the influence of sub-quantum phenomena that are incoherent at the quantum level.

One assumes that this chaotic fluid involves a great number of particles that are distributed with a mean density $P(x, t)$ and obey the guidance law. Since the latter is a consequence of the continuity equation alone, which is an equation that is again true for chaotic fluids, it will be correct to think that the particles displace with the velocity **v** of the fluid and accompany it in its fluctuations. One supposes that the fluctuations are such that an element of fluid that is situated in an element of volume that is defined by the region in which the density ρ_0 of the mean fluid is non-zero will have a non-zero probability of going to any other element of the same region.

In order to describe these fluid displacements (which may dilate or contract), one uses a conformal representation of the space of x in a space of ξ that is constrained to be filled with a *mean density* that is uniform and constant. The volume element $d\mathbf{x}$ ($dx_1\, dx_2\, dx_3$) corresponds to the element $d\boldsymbol{\xi}$ ($d\xi_1\, d\xi_2\, d\xi_3$) by means of the Jacobian J of the transformation:

(J-1) $$d\mathbf{x} = J\left(\frac{\partial x_j}{\partial \xi_k}\right) d\boldsymbol{\xi}\,.$$

The quantity of matter that is contained in $d\mathbf{x}$ for a (mean) Madelung fluid is:

$$dQ = \rho_0(x, t)\, d\mathbf{x} = \rho_0(x, t)\; J\left(\frac{\partial x_j}{\partial \xi_k}\right) d\boldsymbol{\xi}\,.$$

In the space of ξ, this quantity must be proportional to the volume $d\boldsymbol{\xi}$:

$$\rho_0(x, t)\; J\left(\frac{\partial x_j}{\partial \xi_k}\right) = C,$$

or furthermore:

$$J\left(\frac{\partial x_j}{\partial \xi_k}\right) = \frac{1}{C}\ \rho_0(x, t).$$

This is only one condition for the determination of three functions $\xi_1(x, t)$, $\xi_2(x, t)$, $\xi_3(x, t)$. One thus has considerable indeterminacy. One may remark that the unbounded space of x corresponds to bounded values of ξ. Equal volumes of the space of ξ contain equal quantities of matter and correspond to volumes of the space of x that are much larger and whose density is sparser. Incidentally, since equation (1) depends upon time, the fabric (*les mailles*) of the space of ξ will deforms into the space of x in the course of time.

As far as the particles are concerned, if their density in the space of x is $P(x, t)$ then that density will transform into the space of ξ as:

$$F(\xi, t) = \frac{P(x,t)}{J\left(\dfrac{\partial \xi_k}{\partial x_j}\right)} = \frac{P(x,t)}{\rho_0(x,t)}.$$

One sees that if the density P of the particles is proportional to the density ρ_0 of the Madelung fluid then the function $F(\xi, t)$ will be uniform and constant in the space of ξ. Consider a volume element $\delta\xi$ that is centered on ξ at time t. At a *previous* instant t', the fluid in $\delta\xi$ was contained in another (equal) element that was centered on another point ξ_1. One may evaluate the probability that this other point ξ_1 was contained in an element $d\xi'$ that was centered at ξ':

$$d\Pi = K(\xi, \xi', t, t')\, d\xi'.$$

Since ξ_1 was certainly one of the points in the domain Δ of ξ (which represents the domain of x where ρ_0 is non-zero), the integral of this probability over all of the domain Δ will be equal to 1:

$$\int_\Delta K(\xi, \xi', t, t')\, d\xi' = 1$$

for a given ξ, t, and t'.

Since the particles follow the fluid in its fluctuations, one will have:

$$F(\xi, t) = \int_\Delta K(\xi, \xi', t, t')\, F(\xi', t')\, d\xi'. \tag{J-2}$$

At each instant, there exists a point $\xi_M(t)$ in the aggregate of this particle distribution where the density F is a maximum and equal to $M(t)$ and another point $\xi_m(t)$ where it is a minimum and equal to $m(t)$. The density at the point $\xi_M(t)$ at the instant t is well-defined and related to equation (2) by way of:

$$M(t) = \int_\Delta K[\xi_M(t), \xi', t, t']\, F(\xi', t')\, d\xi'.$$

If one replaces the function $F(\xi', t')$ under the integral sign with its maximum value $M(t')$ at the instant t' then one will obviously obtain a value that is greater than that of the integral above:

$$\int_\Delta K[\xi_M(t), \xi', t, t']\, M(t')\, d\xi' \geq M(t).$$

However, one may remove $M(t')$ from under the integral sign and the integral that remains, namely:

$$\int_\Delta K[\xi_M(t), \xi', t, t']\, d\xi',$$

will be equal to 1. One will then have:

$$M(t') \geq m(t).$$

When an interval of time ($t' < t$) elapses, the maximum density will diminish and the minimum density will increas. Therefore, the fluid will tend towards a state in which the two values are equal, and where the value of $F(\xi, t)$ will then be uniformly equal to its common limit in the entire domain. From that point onward, the situation where:

$$m(t) = m(t') = M(t) = M(t')$$

will be found to be the case, and will persist indefinitely. This implies that upon returning to ordinary space the particle density $P(x, t)$ will be constant and proportional to the density $\rho_0(x, t)$ of the mean Madelung fluid at each point.

The preceding proofs rest upon very general theorems, namely, the conservation law and the laws of statistics for Markov processes. Thus, they may be easily extended to other wave equations and to other types of micro-objects for which one can construct corresponding hydrodynamical interpretations. In each case, they permit us to account for a model – which is, in principle, deterministic – for the observable statistical distributions by taking the "hidden variables" to be the physical magnitudes, which are completely foreign to the statistical interpretation of quantum mechanics. One sees the importance with which this perspective is endowed: the study of the hydrodynamical interpretations for the various wave equations that are employed by quantum mechanics.

§ 4. A hydrodynamical model for the vacuum. One knows, moreover, that an essential characteristic of the causal interpretation is that the linear formalism of ordinary quantum mechanics is considered to be an *approximation* in it that is valid for a certain scale of reality, while the exact wave equations are *nonlinear*. The nonlinear terms relate appreciably to just the immediate neighborhood of the center of the singularity (where the distance r_0 is considered to be much larger), but do not modify the results that are given by the linear formalism at the atomic level. However, the hypothesis of their existence is essential to the theory at the quantum level. As we have seen, they are what is subordinate to the hypothesis of phase matching between a regular wave and a singularity, and consequently they will be responsible for the guidance law for the particles along the waves. They are what prevent the singularities from extending indefinitely by making them vanish at a distance and insure that the expansion of the wave amplitude diminishes. Finally, the perspective that the theory hopes to define is a satisfactory interpretation of the nuclear forces that are linked to ideas at the nuclear scale; they are the nonlinear terms that deviate considerably.

Now, it is very difficult to progress in the study of these nonlinear terms, partly because of the extremely arduous calculations that characterize the treatment of nonlinear equations, but also because of the lack of a directing hypothesis that would guide our research in a medium with an infinite variety of possible forms. It might be that the hydrodynamical scenario provides such a hypothesis and introduces a fruitful path to follow. Indeed, it is well-known that the study of the propagation of waves in material fluids in classical hydrodynamics uses linear equations that are valid only for regular waves of small amplitudes in a first approximation. However, one knows full well that it is only an approximation, and that the complete nonlinear equations that are provided by the properties of fluids in the presence of important constraints are frequently used for the study of the propagation of "shock waves," which are waves of very large amplitudes or ones that involve singularities.

Reasoning by analogy, one might hope to represent the linear waves of quantum mechanics as waves of small amplitude that propagate through a given fluid, and after that they will be guided by nonlinear equations that represent the propagation of shock waves through the same fluid, and thus arrive at the determination of the types of nonlinear terms that one must add to the quantum equation.

The hypotheses of that research introduce the hydrodynamical formalism into quantum mechanics on a plane that differs completely from the method of "hydrodynamical representations" that was described above. Here, the fluid is assumed to possess an objective reality and embody the essential physical properties of a "material medium" that are the true basis for wave-like phenomena. This new concept is introduced, instead of the usual concept of "field," or similarly that of "vacuum," in order that its properties might be specified by expressing them in hydrodynamical form. The present work, which has the fundamental hydrodynamical laws of certain more general types of fluid for its object of study, might serve as one such enterprise, and, more especially, might deduce the consequences of accepted general principles that show that the wave functions of particles that are given spin relative to the fluid possess an *internal angular momentum density*. One does not reach the following stage simply in order to study the propagation of vibratory motions in such fluids and to compare the equations of propagation thus obtained with the wave equations of quantum mechanics.

§ 5. Spinning fluids and spinning particles. It is remarkable that the various authors who originated the theory of fluids that possess an internal angular momentum density (i.e., "spinning fluids") – a theory in which it is well-advised to avoid confusing that concept with the vorticity of classical fluids [**46, 67**] – have introduced this new density from axiomatic viewpoint. They introduced it either, like Weyssenhoff [**2**], in the form of an antisymmetric, second-order tensor of "internal angular momentum density," or, like Costa de Beauregard [**68**], in the form of a vector that is dual to a completely antisymmetric, third-order tensor that represents a "spin density," or, like Pham Mau Quan [**69**], in the form of an electromagnetic field quantity, which implies that we are assuming that the medium is polarized and endowed with an electromagnetic moment density. However, it does not appear that any of the above authors arrive at a clear explanation for the internal angular momentum as a *kinetic moment*, properly speaking; i.e., one that relates to the *motion* of the matter. That seems to be a failure to see the consequences of a fact that was pointed out by Costa de Beauregard [**68**]: If one considers a small volume dv of fluid then its *proper* kinetic moment will be the integral of the moment of the quantity of motion with respect to a point *P that is contained in the volume dv*. It will obviously be of fifth order with respect to the linear dimensions of dv (as a moment of inertia). If one divides by the volume in order to obtain a mean density then one will get a quantity that will further be of second order, and which will, consequently, vanish as dv tends to zero. It appears from this that it is impossible to define an internal angular momentum density. In this fashion, Costa de Beauregard concluded, and rather inconsistently, the necessity of defining such a density by an axiomatic method without referring to the rotational motion or either of the two considerations of invariance and antisymmetry. However, it is also possible to draw an entirely different conclusion, namely, that passing to the limit is not legitimate, and that any fluid that is continuous at one scale might be resolved to a much smaller scale where it has a discontinuous structure. From this, one might consider a spinning fluid to be comprised of a finite number of tiny particles, each of which is in rotation about itself, and thus possesses a finite proper kinetic moment. One considers a volume δv that is non-vanishing, but large enough to contain a great number of particles, and one then defines the internal angular momentum density by dividing the sum of the proper kinetic moments of the particles by the volume δv. Under these conditions, the paradox of Costa de Beauregard, which, from a distance, proves the contradictory character of the concept of a spinning fluid, appears to simply support the well-known dialectical principle of the reciprocal relationship between the concepts of continuity and discontinuity.

It is this viewpoint that will be adopted in the present work, where one essentially proposes to illuminate a theory of spinning fluids that are imagined to be composed of discontinuous collections of tiny structures in rotation about themselves and are clearly understood to be coupled by interactions. It is this concept of "fluid tops," which was developed recently in terms of spinors by Bohm, Tiomno, and Schiller [**70**] in non-relativistic form, and then by Bohm, Vigier, Lochak, and myself [**17**] in relativistic form, that led, in this form, to a fluid expression for only "pure matter;" i.e., matter that is devoid of any interaction. In this work, we will avoid the use of the "top," which imprecisely evokes the image of a *solid* spinning particle.

Having constituted spinning fluids from such particles, we are then led to commence by studying the problems that are posed by the relativistic theory of spinning particles. We shall encounter it again, but in a form that is inverse to the dialectic of continuity of discontinuity. Indeed, the classical works on relativistic spinning particles (which are mostly old works, and which we shall recall in our first chapter) amount, in reality, to endowing the particle – *which is considered to be a point* – with a tensor that is christened the "angular momentum," but which is not related to the rotational motion of the particle for the excellent reason that a point cannot actually rotate about itself, since its mathematical expression does not contain the necessary parameters for describing such a rotation. In order for such a body to be called "rotating," in a strict sense, it must be decomposed into distinct parts that are capable of rotating about each other. The particle would spin in a precise, rigorous sense, thus contradicting the strict continuity of the spinning fluid. In other words – at least as far as proper rotations are concerned – the purely discontinuous conception has less physical meaning than the purely continuous one. One is in the presence of two opposing, but at the same time complementary, concepts, so it is impossible to conceive of one without the other. That is why the classical dynamics of relativistic spinning particles that was studied by Frenkel [**4**], Mathisson [**5**], and Weyssenhoff [**2**] represents only a special case of the various dynamical possibilities.

It is for this reason that in Chapter II we shall study and develop the general relativistic theory of spinning particles that is due to Møller [**12**], Pryce **13**], Bohm and Vigier [**14**], which describes the *global* motion of a classical physical fluid that is in a state of rotation, which is a theory for which we shall study several applications in detail in Chapter III. We can then proceed with the study of hydrodynamics, properly speaking, in our last two chapters. In Chapter IV, we will present the general axiomatic theory of spinning fluids, which is a theory that was established simultaneously and independently by Vigier, Lochak, and myself [**25**], and we will apply the hydrodynamical representation to various wave equations of quantum mechanics. Finally, in Chapter V, we will restrict and adapt the preceding theory to the case of fluids that are composed of relativistic spinning particles that are coupled by interactions, and we will show several applications of the formalism that was thus established. Although, as we said, at the start of this book we will seek to address the dynamics of spinning particles in a case that is more general than the one that is described by the classical Frenkel-Weyssenhoff theory, in the last chapter we will take the form of the fluid to be based in the specific particles that were studied in that classical theory. It is hardly claimed that we are proposing a complete theory of fluids that are endowed with an internal rotation. Furthermore, we promise to return in a later book to some cases that we have been obliged to set aside in order to abbreviate a treatise that is already lengthy.

§ 6. The structure of the vacuum in quantum mechanics. Despite the references that we will make to quantum wave functions at the end of Chapter V in the name of examples, it is important that treatise should be completely independent of quantum theory, and that it might be regarded as a contribution to several chapters of dynamics and classical relativistic hydrodynamics. It this context, it seems to blaze a path for the edification of a whole new series of topics in dynamics and hydrodynamics. However, we do not disguise the fact that we especially hope that it might be used fruitfully for research into quantum mechanics. Furthermore, we shall terminate this introduction with a rapid sketch of presently open issues in the case of the causal interpretation.

The essential point of departure seems to be currently provided by Bohm and Vigier's drop theory. The latter aides to Hillion and Lochak bring into consideration the possible existence of a limited number of excitation states of the drop, each of which characterizes, and in a global fashion, the existence of a certain periodicity in the evolution of all the parameters that define the internal motions of the drop. The classification of the stable states of the drop rejoins the customary classification of elementary particles [**51**] with a surprising degree of harmony between them. Naturally, the result should not be interpreted, as was believed thirty years ago, as showing that an elementary particle might be represented by a Bohm-Vigier drop in a given excited state.

In the case of the causal interpretation, one can regard the drops as representing various types of *sub-quantum* structures. One then supposes that the drops are fairly tiny in order for a large number of them to exist in a region of space on the order of 10^{-13} cm. These drops happen to be in a very violent, chaotic state, and are coupled by given interactions so as to constitute a material medium of the same type as the "Dirac ether," in which waves with singularities at the quantum level propagate, and by which the causal interpretation describes the elementary particles. From this perspective, the various types of drops can be regarded as the elementary sub-quantum constituents of a material medium that is continuous at the quantum level and which fills and constitutes the entire universe, in which it serves to support wave-like phenomena, and the fact that it might appear to be the different stable states of the same drop (which are, in principle, capable of transforming from one into the other) can be connected with the old-fashioned idea of the unity of matter, which seems to have severely comprised in recent years by the proliferation of elementary quantum particles.

Naturally, scientific prudence obliges us to consider the ideas that were enunciated here as being only quite vague and entirely hypothetical, as of yet. Their principal interest is in that they provide a skeleton for research and re-direct us towards a fairly neatly well-defined direction. Its essential points are certainly appealing to stipulate, that one must, perhaps, modify the precise mathematical edifice that gave birth to the theory, which could cause new problems and difficulties to appear. Nevertheless, we think that is not forbidden for us to place a few provisional landmarks along this path.

The first stage of this theory should necessarily consist of considering each of the stable states of the Bohm-Vigier drop separately, and studying their dynamics, first in the absence and then in the presence of external forces, and then constituting a relativistic fluid from each of these types of particle by introducing, on the one hand, an appropriate interaction, and on the other hand, a chaotic motion that endows each particle with a speed that is close to that of light, in order to realize an approximate isotropy that conforms to the recent ideas of Dirac.

The second stage will be to impose waves on this fluid that consist of *organized* motions of weak amplitude that are superimposed over *chaotic* motions of large amplitudes, in the manner by which resonance waves appear amongst molecules in chaotic motion in the kinetic theory of gases. This will compel us to direct our calculations, and in particular, to choose the internal stresses that we introduce in a fashion that will recover the equations of quantum mechanics as the equations of propagation.

The third stage consists of developing a theory of the "vacuum," which is regarded as being composed of a mixture of different particles that one studies separately and with specified interaction of each type, and introducing new interactions between the particles of different types that are capable of being made to appear between the various waves like the coupling terms that are customarily employed by quantum mechanics in order to treat (quantum) particles in interaction. At this stage, it seems that one might judge the value of the theory by essentially its capacity to completely re-interpret the collection of results that come from all of the linear formalism of first quantization.

The fourth stage consists of attacking the nonlinear domain – i.e., considering waves with singularities in the previously-constituted fluid and causing nonlinear terms to appear in the singular region, which will thus be determined, to a large extent, by the hydrodynamical properties of the fluid like the ones that occur for shock waves in the dynamics of material fluids. One will probably encounter considerable computational difficulties at this stage, but it might be that they are compensated for by the access that it gives to an interpretation of the nuclear forces and structures.

Finally, the last stage will be to try to comprehend how to effect the transition from one of the stable excitation states of the drop to another, how such a transition might be interpreted in terms of waves, and how it might account for the mutual transformations of elementary quantum particles.

One sees that we are concerned with a long-winded enterprise that would obviously necessitate the help of a good number of researchers. In the present book, we would like to give only a preliminary sketch of a mathematical tool that might, we hope, be of some utility along that line of research.

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CHAPTER ONE

THE CLASSICAL THEORY OF RELATIVISTIC SPINNING PARTICLES

§ 1. The Frenkel Lagrangian. The first formulation of the relativistic dynamics of spinning particles was given by the Soviet physicist Frenkel as a theory of the electron that was endowed with spin in an electromagnetic field [**4**]. Frenkel considered a particle that possessed a magnetic moment and an electric moment, and proposed the two simple hypotheses:

1) The magnetic moment **m** and the electric moment **q** of non-relativistic physics are, in reality, the various components of a single relativistic physical quantity: the *electromagnetic moment* $\mu_{\alpha\beta}$, which is a second-order antisymmetric tensor. If we place ourselves in an arbitrary reference frame then **m** will be, as one says, an *axial* vector, which can be taken to be the spatial dual of the tensor that is formed from the spatial components of $\mu_{\alpha\beta}$:

$$m_1 = \mu_{23}, \qquad m_2 = \mu_{31}, \qquad m_3 = \mu_{12},$$

or, by using the third-order, completely antisymmetric symbol:

$$m_k = \tfrac{1}{2}\,\varepsilon_{ijk}\,\mu_{ij}\ .$$

One then considers the polar vector **q** to be formed from the other three components of the same tensor $\mu_{\alpha\beta}$:

$$iq_k = \mu_{k4}\ .$$

Finally, conforming to the hypothesis of Uhlenbeck and Goudsmit, one supposes that the moment is purely magnetic (**q** = 0) *in the proper system of the electron*, which may be written in the covariant fashion:

(I.1) $$\mu_{\alpha\beta}\,u_\beta = 0,$$

as one may easily verify. If one considers an arbitrary reference frame in which the velocity of the electron is **v** then one will get a unit-speed velocity with components:

$$u_i = \alpha v_i \qquad \text{and} \qquad u_4 = \alpha ic, \qquad \text{with} \qquad \alpha = (1 - v^2/c^2)^{1/2},$$

and relation (1) can be written:

(I.2) $$\mu_{ji}\alpha v_i + \mu_{j4}\,\alpha ic = 0 \qquad \text{or} \qquad \mu_{ji}\,v_i + ic\mu_{j4} = 0.$$

Now, one has $\mu_{j4} = iq_j$, and on the other hand, $m_k = 1/2\ \varepsilon_{ijk}\ \mu_{ij}$, which will give $\mu_{im} = \varepsilon_{imk}\ m_k$ when it is contracted with ε_{imk}.

If one takes these expressions into account then relation (2) will change into:

$$-\varepsilon_{ijk}\ m_k\ v_i - cq_j = 0,$$

so that

$$\varepsilon_{ijk}\ v_i\ m_k = cq_j\ ,$$

which may be written in vectorial notation:

$$c\mathbf{q} = \mathbf{v} \times \mathbf{m},$$

and will obviously imply that:

$$\mathbf{q} \cdot \mathbf{m} = 0.$$

Thus, in an arbitrary reference frame, the electric moment of the particle is the vector product of the velocity with its magnetic moment.

2) The existence of an electromagnetic moment is connected with the existence of an internal angular momentum for the particle – i.e., with its rotation about itself, a rotation whose variation in the course of time is involved with dynamics. One supposes that this moment is expressed by an antisymmetric tensor $S_{\alpha\beta}$ that is related to the electromagnetic moment by the classical relation:

$$\mu_{\alpha\beta} = \frac{e}{m_0 c} S_{\alpha\beta}\ .$$

One will then get *Frenkel's auxiliary condition* for the internal angular momentum:

$$S_{\alpha\beta}\ u_\beta = 0. \tag{I.3}$$

Like $\mu_{\alpha\beta}$, the internal angular momentum can be decomposed in any reference frame into two spatial vectors:

$$s_k = \tfrac{1}{2}\,\varepsilon_{ijk}\ S_{ij} \qquad = \frac{m_0 c}{e} m_k\ ,$$

$$t_k = iS_{4k} \qquad = \frac{m_0 c}{e} q_k\ ,$$

and one will have:

$$c\mathbf{t} = \mathbf{v} \times \mathbf{s} \qquad \text{and} \qquad \mathbf{t} \cdot \mathbf{s} = 0.$$

If one puts the particle into an electromagnetic field that is represented in relativistic form by the Maxwell tensor $F_{\alpha\beta}$ then it will be subjected to a torque whose relativistic expression is obviously obtained by generalizing the classical expression $\boldsymbol{\gamma} = \mathbf{m} \times \mathbf{H}$ to:

$$\gamma_{\alpha\beta} = \mu_{\alpha\lambda}\ F_{\beta\lambda} - \mu_{\beta\lambda}\ F_{\alpha\lambda}\ .$$

With this, the relativistic generalization of the classical formula for the dynamics of the kinetic moment will be given by:

$$\dot{S}_{\alpha\beta} = \mu_{\alpha\lambda}\,F_{\beta\lambda} - \mu_{\beta\lambda}\,F_{\alpha\lambda}\;.$$

(The dot indicates the derivative of the particle with respect to proper time, or furthermore, the derivative along the world line that is described by the particle.) This can be interpreted in any reference frame as the two three-dimensional relations:

$$\frac{d}{dt}\left(\frac{m_0 c}{e}\mathbf{m}\right) = \mathbf{m}\times\mathbf{H} + \mathbf{q}\times\mathbf{E},$$

$$\frac{d}{dt}\left(\frac{m_0 c}{e}\mathbf{q}\right) = \mathbf{q}\times\mathbf{H} - \mathbf{m}\times\mathbf{E}.$$

In principle, these two reciprocal equations will permit us to determine the entire evolution of the vectors **m** and **q** when we know the distribution of the fields **E** and **H** and the initial conditions.

However, one sees that these equations do not imply the condition that $c\mathbf{q} = \mathbf{v}\times\mathbf{m}$. It can be assumed to be satisfied at the initial instant, but that does not continue to be the case for the rest of the motion, in general. It results that if one wants to adopt Frenkel's hypotheses then a simple relativistic generalization of the laws of dynamics for kinetic moments will not suffice. The relativistic internal angular momentum appears to express a reality that is more complex than that of the axial vector of classical dynamics.

Frenkel attacked the problem by means of a variational principle that was more general than the dynamical relation. Using the classical expressions for the energy of the electron and the two auxiliary conditions:

$$u_\alpha\,u_\alpha = -\,c^2 \qquad \text{and} \qquad S_{\alpha\beta}\,u_\beta = 0,$$

he formed the Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}M(u_\alpha\,u_\alpha + c^2) + a_\alpha\,S_{\alpha\beta}\,u_\beta + \frac{e}{c}\,\varphi_\alpha\,u_\alpha + \frac{1}{2}\frac{e}{m_0 c}\;S_{\alpha\beta}\,F_{\alpha\beta} + T^*.$$

The two coefficients M and a_α are Lagrange multipliers whose significance remains to be seen. Incidentally, one can recover the classical expression for the energy of charged particle and a dipole in an electromagnetic field that is represented by the Maxwell tensor $F_{\alpha\beta}$ or the spacetime potential φ_α (so one has: $F_{\alpha\beta} = \partial_\alpha\,\varphi_\beta - \partial_\beta\,\varphi_\alpha$).

Finally, the term T^* represents the kinetic energy of proper rotation, whose form remains to be seen. In addition, the two multipliers M and a_α of the system depend upon two groups of configuration variables: the coordinates x_α of the particle, along with their proper time derivatives:

$$\dot{x}_\alpha = a_\alpha,$$

and a group of angular variables that Frenkel represented by an antisymmetric tensor $\Omega_{[\alpha\beta]}$, along with its derivatives:

$$\omega_{[\alpha\beta]} = \dot{\Omega}_{[\alpha\beta]},$$

which generalized the classical notion of angular velocity.

The equation of the relativistic representation of the orientation of a body with respect to the four coordinate axes of a Galilean reference frame poses a series of problems that have been the object of numerous recent papers [**47-50**, **52**]. However, for the moment, and from Frenkel's purely formal standpoint, one does not have to use the variables $\Omega_{[\alpha\beta]}$ themselves (which define an anholonomic coordinate system), but only their variations $\delta\Omega_{[\alpha\beta]}$. This tensor differentiates to the tensor of relativistic angular momentum:

$$\delta\Omega_{\alpha\beta} = \omega_{\alpha\beta}\,\delta\tau.$$

As for the latter, one constructs everything naturally by considering, in a given reference frame, the classical axial vector **w** of "angular momentum," which one regards as the dual of the spatial tensor ω_{ij} ($\omega_{ij} = \varepsilon_{ijk}\, w_k$), and on the other hand, the acceleration γ of the particle, which corresponds to the temporal components of $\omega_{\alpha\beta}$ ($\omega_{k4} = i\gamma_k$).

The difficulty that was pointed out above stems from the fact that when one is given a tensor $\omega_{\alpha\beta}$ that is obtained in this fashion and its time evolution then the quantities:

$$d\Omega_{\alpha\beta} = \omega_{\alpha\beta}\, d\tau$$

will not be total differentials, in general. Thus, it is impossible to determine a system of holonomic, relativistic, angular variables by this method that would characterize the orientation of a particle relative to a given Galilean reference frame.

By generalizing a non-relativistic formula, one may then determine the variation of the internal angular momentum $S_{\alpha\beta}$ as a function of the infinitesimal rotation:

$$\delta\Omega_{\alpha\beta} = \omega_{\alpha\beta}\,\delta\tau.$$

Consider the rotational velocity vector **w** of classical mechanics and the infinitesimal rotation vector:

$$\delta\mathbf{W} = \mathbf{w}\,\delta t.$$

A magnetic dipole of magnetic moment **m** that is placed in the magnetic field **H** is subjected to a torque $\mathbf{m} \times \mathbf{H}$, and in the course of rotation $\delta\mathbf{W}$ will an amount of work $\delta\mathbf{W}(\mathbf{m} \times \mathbf{H})$. On the other hand, it possesses a magnetic energy $-\,\mathbf{m} \cdot \mathbf{H}$ that will undergo a variation $\delta(-\,\mathbf{m} \cdot \mathbf{H}) = -\,\delta\mathbf{m} \cdot \mathbf{H}$.

One will then have $\delta\mathbf{W}(\mathbf{m} \times \mathbf{H}) = -\,(-\delta\mathbf{m} \cdot \mathbf{H})$, or:

$$(\delta\mathbf{W} \times \mathbf{m}) \cdot \mathbf{H} = \delta\mathbf{m} \cdot \mathbf{H}.$$

The relation between $\delta\mathbf{m}$ and $\delta\mathbf{W}$ is independent of the field so this equation can be verified for any such **H**. One thus has:

$$\delta\mathbf{m} = \delta\mathbf{W} \times \mathbf{m}.$$

The relativistic generalization of this formula is immediate. The variation of the electromagnetic moment $\mu_{\alpha\beta}$ that results from an infinitesimal rotation $\delta\Omega_{\alpha\beta}$ can be written:

$$\delta\mu_{\alpha\beta} = \delta\Omega_{\alpha\gamma}\,\mu_{\beta\gamma} - \delta\Omega_{\beta\gamma}\,\mu_{\alpha\gamma},$$

and one will similarly have:

$$\delta S_{\alpha\beta} = \delta\Omega_{\alpha\gamma}\,S_{\beta\gamma} - \delta\Omega_{\beta\gamma}\,S_{\alpha\gamma}.$$

It still remains for us to examine the kinetic energy of proper rotation T^*. Frenkel proposed axiomatically, by analogy with classical dynamics, that the variation δT^* should be equal to:

$$\tfrac{1}{2}\,S_{\alpha\beta}\,\delta\omega_{\alpha\beta}.$$

This is a very debatable point. Indeed, since $S_{\alpha\beta}$ varies like an infinitesimal rotation, one cannot see why δT^* does not contain any terms in $\delta\Omega_{\alpha\gamma}$. In fact, the chosen expression for δT^* is not an exact total differential, and it is impossible to write down the expression for the Lagrangian completely.

§ 2. The Frenkel equations. At the very most, we may describe the Euler-Lagrange equations. By varying with respect to M and a_α, one will obviously obtain the two auxiliary equations:

$$u_\alpha\,u_\alpha = -c^2 \qquad \text{and} \qquad S_{\alpha\beta}\,u_\beta = 0.$$

Varying with respect to x_α and u_α gives:

$$\frac{\partial\mathcal{L}}{\partial x_\lambda} = \frac{e}{c}u_\alpha\partial_\lambda\varphi_\alpha + \frac{1}{2}\frac{e}{m_0 c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta},$$

$$\frac{\partial\mathcal{L}}{\partial u_\lambda} = \frac{e}{c}\varphi_\lambda + Mu_\lambda + a_\alpha S_{\alpha\beta}.$$

One will thus have the equation:

$$\begin{aligned}\frac{e}{c}u_\alpha\partial_\lambda\varphi_\alpha + \frac{1}{2}\frac{e}{m_0 c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta} &= \frac{d}{d\tau}\left(\frac{e}{c}\varphi_\lambda + Mu_\lambda + a_\alpha S_{\alpha\lambda}\right)\\ &= \frac{e}{c}u_\alpha\partial_\lambda\varphi_\alpha + \frac{d}{d\tau}\left(Mu_\lambda + a_\alpha S_{\alpha\lambda}\right),\end{aligned}$$

in which the derivative with respect to proper time is expressed by the operator $u_\alpha\,\partial_\alpha$ (see Appendix *A*).

One may combine the first term of the left-hand side with the first term of the right-hand side, from which the rotation $\partial_\lambda\, \varphi_\alpha - \partial_\alpha\, \varphi_\lambda$ appears, which is equal to $F_{\alpha\lambda}$, and one will get:

$$\frac{d}{d\tau}(Mu_\lambda + a_\alpha\, S_{\alpha\beta}) = \frac{e}{c}u_\alpha F_{\alpha\beta} + \frac{1}{2}\frac{e}{m_0 c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta},$$

or

(I.4) $$\frac{d}{d\tau}(Mu_\lambda + a_\alpha\, S_{\alpha\beta}) = f_\lambda + \frac{1}{2}\frac{e}{m_0 c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta}.$$

The *Lorentz force* appears in this:

$$f_\lambda = \frac{e}{c}u_\alpha F_{\alpha\beta}.$$

For a particle without spin, for which the last terms of the two sides of equation are zero, one can recover the classical formula:

$$\frac{d}{d\tau}(m_0\, u_\lambda) = f_\lambda,$$

which shows that the Lagrange multiplier M represents a proper mass. We write:

$$M = m_0 + \mu,$$

because, as we will see, spin causes a supplementary mass to appear.

One obtains a final group of Euler-Lagrange equations by considering the variations $\delta\Omega_{\alpha\beta}$ and $\delta\omega_{\alpha\beta}$. The variations $\delta\Omega_{\alpha\beta}$ produce a variation:

$$\begin{aligned}\delta\mathcal{L} &= \left(\frac{1}{2}\frac{e}{m_0 c}F_{\gamma\lambda} + a_\gamma u_\lambda\right)\delta S_{\gamma\lambda}, \\ &= \left(\frac{1}{2}\frac{e}{m_0 c}F_{\gamma\lambda} + a_\gamma u_\lambda\right)(\delta\Omega_{\alpha\beta}\, S_{\lambda\alpha} - \delta\Omega_{\lambda\sigma}\, S_{\gamma\sigma}) \\ &= \left(\frac{1}{2}\frac{e}{m_0 c}F_{\gamma\lambda} + a_\gamma u_\lambda\right) S_{\lambda\beta}\, \delta\Omega_{\alpha\beta} - \left(\frac{1}{2}\frac{e}{m_0 c}F_{\lambda\alpha} + a_\lambda u_\alpha\right)\delta\Omega_{\alpha\beta}.\end{aligned}$$

From (3), the term in $u_\lambda\, S_{\lambda\beta}$ is zero. As for the term $-\, a_\lambda\, u_\alpha\, S_{\lambda\beta}\, \delta\Omega_{\alpha\beta}$, by reason of the antisymmetry of $\delta\Omega_{\alpha\beta}$, it may be written:

$$+\, a_\lambda\, u_\alpha\, S_{\lambda\beta}\, \delta\Omega_{\alpha\beta} \qquad \text{or} \qquad -\frac{1}{2}(a_\lambda\, u_\alpha\, S_{\lambda\beta} - a_\lambda\, u_\beta\, S_{\lambda\alpha})\, \delta\Omega_{\alpha\beta}.$$

For the same reason, the other two terms:

$$\frac{1}{2}\frac{e}{m_0 c}(F_{\alpha\lambda}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha})\;\delta\Omega_{\alpha\beta}$$

may be written:

$$-\frac{e}{m_0 c}\;F_{\lambda\alpha}\,S_{\lambda\beta}\;\delta\Omega_{\alpha\beta} = -\frac{1}{2}\frac{e}{m_0 c}(F_{\lambda\alpha}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha})\;\delta\Omega_{\alpha\beta}\,.$$

One will then finally have:

$$\frac{\partial\mathcal{L}}{\partial\Omega_{\alpha\beta}} = -\frac{1}{2}\frac{e}{m_0 c}(F_{\lambda\alpha}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha}) - \frac{1}{2}\,a_\lambda\,(F_{\lambda\alpha}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha}).$$

Finally, it remains that:

$$\frac{\partial\mathcal{L}}{\partial\omega_{\alpha\beta}} = \frac{\partial T^*}{\partial\omega_{\alpha\beta}} = \frac{1}{2}\,S_{\alpha\beta}\,.$$

One therefore has the equation:

(I.5) $$\dot{S}_{\alpha\beta} = -\frac{e}{m_0 c}(F_{\lambda\alpha}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha}) - a_\lambda\,(F_{\lambda\alpha}\,S_{\lambda\beta} - F_{\lambda\beta}\,S_{\lambda\alpha}).$$

In order to elucidate the significance of a_λ, one contracts this equation with u_β, which will make the terms in $S_{\lambda\beta}$ disappear:

$$\dot{S}_{\alpha\beta}\,u_\beta = \frac{e}{m_0 c}\;F_{\lambda\beta}\,u_\beta\,S_{\lambda\alpha} - c^2\,a_\lambda\,S_{\lambda\alpha}\,.$$

However, from (3), one has:

$$\dot{S}_{\alpha\beta}\,u_\beta = -\,S_{\alpha\beta}\;\dot{u}_\beta \;= S_{\beta\alpha}\,\dot{u}_\beta\,,$$

and upon introducing the Lorentz force:

$$f_\lambda = \frac{e}{c}F_{\lambda\beta}\,u_\beta$$

one will obtain:

$$S_{\lambda\alpha}\left(-\dot{u}_\lambda - c^2 a_\lambda + \frac{1}{m_0}\,f_\lambda\right) = 0.$$

This equation will be true for any $S_{\lambda\beta}$ if we set:

$$c^2\,a_\lambda = -\,\dot{u}_\lambda + \frac{1}{m_0}\,f_\lambda\,.$$

If the particle does not have spin then one will have the relation:

$$m_0\dot{u}_\lambda = f_\lambda \qquad \text{so that} \qquad a_\lambda = 0.$$

One sees that the presence of spin involves a reduction of the acceleration when compared to the case of the particle with spin. The expression $m_0c^2a_\lambda$ then represents the excess Lorentz force in the derivative of the momentum:

$$m_0\dot{u}_\lambda .$$

Concerning the expression:

$$c^2a_\lambda = -\,\dot{u}_\lambda + \frac{e}{m_0c}F_{\lambda\beta}u_\beta ,$$

one remarks that:

$$a_\lambda\, u_\lambda = 0.$$

Therefore, the vector a_λ is in proper space.

Transform the first group of equations by accounting for the expression for M:

$$(m_0+\mu)\dot{u}_\lambda + \dot{\mu}\,u_\lambda + a_\alpha\dot{S}_{\alpha\lambda} + \dot{a}_\alpha S_{\alpha\lambda} = f_\lambda + \frac{1}{2}\frac{e}{m_0c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta} .$$

However, one has:

$$f_\lambda = m_0c^2a_\lambda + m_0\dot{u}_\lambda ,$$

so:

$$\mu\,\dot{u}_\lambda + \dot{\mu}\,u_\lambda + a_\alpha\dot{S}_{\alpha\lambda} + \dot{a}_\alpha S_{\alpha\lambda} = m_0c^2a_\lambda + \frac{1}{2}\frac{e}{m_0c}S_{\alpha\beta}\partial_\lambda F_{\alpha\beta} .$$

Upon contracting this with u_λ, it will then result that:

$$-\dot{\mu}c^2 + a_\alpha u_\lambda\dot{S}_{\alpha\lambda} = \frac{1}{2}\frac{e}{m_0c}S_{\alpha\beta}u_\lambda\partial_\lambda F_{\alpha\beta} = \frac{1}{2}\frac{e}{m_0c}S_{\alpha\beta}\dot{F}_{\alpha\beta} ,$$

which one will easily transform into:

$$-\dot{\mu}c^2 = \frac{1}{2}\frac{e}{m_0c}(S_{\alpha\beta}F_{\alpha\beta}) - \frac{1}{2}\frac{e}{m_0c}\dot{S}_{\alpha\beta}F_{\alpha\beta} - a_\alpha u_\beta\dot{S}_{\alpha\beta} ,$$

or

(I.6)
$$-\dot{\mu}c^2 = \frac{1}{2}\frac{e}{m_0c}(S_{\alpha\beta}F_{\alpha\beta}) - \frac{1}{2}\dot{S}_{\alpha\beta}\left(\frac{e}{m_0c}F_{\alpha\beta} + a_\alpha u_\beta - a_\beta u_\alpha\right),$$

by reason of the antisymmetry of $d/d\tau\, S_{\alpha\beta}$.

One can show that the second term is zero. In order to do this, we recall equation (5):

$$\dot{S}_{\alpha\beta} = S_{\lambda\alpha}\left(\frac{e}{m_0 c}F_{\lambda\beta} + a_\lambda u_\beta\right) - S_{\lambda\beta}\left(\frac{e}{m_0 c}F_{\lambda\alpha} + a_\lambda u_\alpha\right),$$

in which we have the two zero terms:

$$S_{\lambda\beta}\, a_\alpha\, u_\lambda - S_{\lambda\alpha}\, a_\beta\, u_\lambda .$$

Upon introducing the tensor:

$$F'_{\lambda\beta} = F_{\lambda\beta} + \frac{m_0 c}{e}(a_\lambda\, u_\beta - a_\beta\, u_\lambda),$$

which, like $F_{\lambda\beta}$, is antisymmetric, it will then follow that:

$$\dot{S}_{\alpha\beta} = \frac{e}{m_0 c}(S_{\lambda\alpha}F'_{\lambda\beta} - S_{\lambda\beta}F'_{\lambda\alpha}),$$

which is the condensed form of equation (5).

However, the second term of (6) will then take on the form:

$$-\frac{1}{2}\frac{e}{m_0 c}(S_{\lambda\alpha}F'_{\lambda\beta} - S_{\lambda\beta}F'_{\lambda\alpha})\frac{e}{m_0 c}F'_{\alpha\beta} = -\frac{1}{2}\left(\frac{e}{m_0 c}\right)^2(S_{\lambda\alpha}F'_{\lambda\beta} + S_{\lambda\alpha}F'_{\lambda\beta})F'_{\alpha\beta},$$

by virtue of the antisymmetry of $F'_{\alpha\beta}$.

Given this, we will then have:

$$-\left(\frac{e}{m_0 c}\right)^2 S_{\lambda\alpha}F'_{\lambda\beta}F'_{\alpha\beta} = 0,$$

by virtue of the antisymmetry of $S_{\alpha\beta}$.

Ultimately, one will then have:

$$\dot{\mu}c^2 = -\frac{1}{2}\frac{e}{m_0 c}\frac{d}{d\tau}(S_{\alpha\beta}F_{\alpha\beta}),$$

and since μ supplements the mass as a result of spin, which then goes to zero with $S_{\alpha\beta}$, one may set:

$$\mu c^2 = -\frac{1}{2}\frac{e}{m_0 c}S_{\alpha\beta}F_{\alpha\beta} = -\tfrac{1}{2}\mu_{\alpha\beta}\, F_{\alpha\beta},$$

which represents, as it usually does, the energy of a dipole in an electromagnetic field.

We have then given a form to the Frenkel equations to which we shall frequently refer in what follows. If one considers equation (4) to be a generalization of the classical theorem of momentum then the quantity $(m_0 + \mu)\, u_\lambda - S_{\lambda\alpha}\, a_\alpha$ will represent the generalized momentum G_λ of the particle. One sees that it is composed of two terms: When one takes into account the supplementary mass that is provided by the electromagnetic energy of the dipole, the first one is a classical momentum that is collinear with the unit-speed velocity vector. The second one, which we write as:

$$- S_{\lambda\alpha}\, a_\alpha = - P_\lambda ,$$

represents a supplementary momentum that we call the *transverse momentum* because it is orthogonal to the velocity:

$$P_\lambda\ u_\lambda\ = 0.$$

It thus happens that in the proper space of the particle the particle possesses a residual momentum in the system of axes, relative to which it is at rest. The transverse momentum is, in turn, composed of two terms that appear when one specifies a_a:

A purely mechanical term:

$$-\ \frac{1}{c^2} S_{\lambda\alpha} \dot{u}_\alpha$$

and a second term that is connected with the electromagnetic field:

$$\frac{1}{m_0 c^2} S_{\lambda\alpha} f_\alpha .$$

One then has:

$$P_\lambda = -\frac{1}{c^2} S_{\lambda\alpha} \dot{u}_\alpha + \frac{1}{m_0 c^2} S_{\lambda\alpha} f_\alpha , \qquad G_\lambda = (m_0 + \mu)\, u_\lambda - P_\lambda .$$

Equation (4) then becomes:

$$\dot{G}_\lambda = f_\lambda + \frac{1}{2} \frac{e}{m_0 c} S_{\alpha\beta} \partial_\lambda F_{\alpha\beta} .$$

Similarly, one sees the term:

$$- a_\lambda\, (u_\alpha\, S_{\lambda\beta} - u_\beta\, S_{\lambda\alpha})$$

appear in equation (5), which takes the form $u_\alpha P_\beta - u_\beta P_\alpha$, or again $G_\alpha u_\beta - G_\beta u_\alpha$, which is an expression in which the part of G_α that is collinear with the current disappears. One then has:

$$\dot{S}_{\alpha\beta} = G_\alpha\, u_\beta - G_\beta\, u_\alpha + \frac{e}{m_0 c} (S_{\alpha\lambda}\, F_{\beta\lambda} - S_{\beta\lambda}\, F_{\alpha\lambda}).$$

In particular, in the absence of an electromagnetic field these equations will become:

$$G_\lambda = m_0\, u_\lambda - P_\lambda , \qquad P_\lambda = -\frac{1}{c^2} S_{\lambda\alpha} \dot{u}_\alpha ,$$

$$\dot{G}_\lambda = 0, \qquad \dot{S}_{\alpha\beta} = G_\alpha u_\beta - G_\beta u_\alpha .$$

§ 3. Mathisson's theory: formulation. – The Polish mathematician Mathisson chanced upon the problem of the spinning electron in the course of his work, and that formed the object of a series of publications between 1930 and 1937 [**5**], in which he developed a point of view that would eventually prove to have great fecundity in the recent developments of general relativity [**6**, **63**, **34**] and the causal interpretation of quantum mechanics [**7**, **11**, **30**]. The presence of matter in an external field, by its very nature, will lead to a supplementary field that obeys the same equations as the field, but possesses singularities along a certain world-line L, when it is superposed with a regular field. The nonlinear character of these field equations (which will be those of general relativity) involves restrictive conditions on the singular solution and the regular solution if they are to be superposable. These conditions, which relate to the world-line L, will appear in the form of the laws of motion for the material particle.

A more interesting approach is to employ, as Mathisson did, a method of successive approximations that, in the first approximation, will reduce a small body of matter to a material point of Newtonian mechanics, but which will produce, as a second approximation, another type of singularity that is endowed with an internal angular momentum and a non-classical dynamic in which we will recover the laws that were given by Frenkel.

We shall develop this theory by omitting the external gravitational field, which greatly complicated the calculations in Mathisson's papers, and which we will have no interest in considering.

In the absence of matter, we will therefore have a vacuum universe that is given a Euclidian "background metric," which we will refer to Galilean axes, so the metric tensor will simply be $\delta_{\mu\nu}$ then. The presence of a droplet of matter that travels through a certain very slender world-tube will lead to the addition of a supplementary metric tensor $\gamma_{\mu\nu}$ in all of the universe that will be *external to the tube*, and which we will assume to be very small in comparison to $\delta_{\mu\nu}$, so we will ignore the powers that are greater than one. One will have that the total metric tensor $g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}$ obeys Einstein's equations [**44**] at each point of the vacuum:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -2\chi\, T_{\mu\nu}$$

($T_{\mu\nu}$ is the electromagnetic energy-momentum tensor). One will also have that this equation leads, in the first approximation, to the linear equations:

$$\text{(I.7)} \qquad \Box \psi_{\mu\nu} = -2\chi\, T_{\mu\nu} \qquad \text{with} \qquad \psi_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\,\delta_{\mu\nu}\,\gamma_\lambda^\lambda ,$$

by way of the auxiliary condition:

$$\text{(I.8)} \qquad \nabla_\nu \psi_\mu^\nu = 0.$$

The symbols $\Box$ and ∇, as well as the operations of raising and lowering of indices, will be expressed in the context of the Euclidian background metric. One will remain in

Galilean axes as a reference system and preserve the formalism of special relativity. One will consider the tensor $\gamma_{\mu\nu}$ directly, which one calls the *gravitational potential*. The problem will then be to find solutions of the system (7), (8) that involve singularities that be likely to completely characterize the material droplet, more or less.

In order to do that, one considers a world-line L that is inside the tube that is described by the drop, and one establishes a correspondence between the point A on L and the point O in the universe by means of the retarded potential.

Consider the light cone with its vertex at O. Its "past" nappe will necessarily contain L and a well-defined point A that one makes correspond to the point O. On the whole, that goes back to the fact that at each moment of its motion, the point A, which travels along the world-line L, emits an electromagnetic signal into space, and thus connects that instant to another well-defined instant at each point of space. If we choose a point P along L to be the initial instant then every arc $PA = s$ will define a proper time τ that might therefore affect any point O of the universe whose light-cone contains A. The determination of proper time along the line L will, at the same time, define a scalar field $\tau(O)$ at each point O. Likewise, the quantity $f(\tau)$ [or $f_\mu(\tau)$] that is attached to every A will permit us to define a field $f(O)$ [$f_\mu(O)$, resp.] at each point of the universe. One may therefore attach a well-defined vector $u_\mu(O)$ to each point O that is the unit-speed velocity vector $u_\mu(\tau)$ at the point A on L that is associated with O.

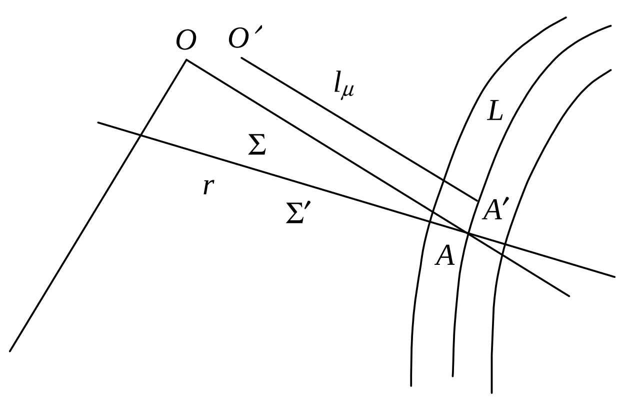
Likewise, one agrees to associate a vector l_μ to each point O that is simply a generator OA of a light-cone, and therefore an isotropic vector:

(I.9) $$l_\mu\, l_\mu = 0.$$

Finally, the two fields u_μ and l_μ permit us to define a scalar field:

(I.10) $$r = -\frac{1}{c}\, l_\mu\, u_\mu\,.$$

By projecting onto the proper axes at the point A and taking (9) into account, it is easy to see that r is simply the spatial distance between the points O and A in the proper system of A. One proceeds to introduce functions of r^{-1}, r^{-2}, … that then represent various classical types of singularities at A when one refers to the proper system.



It is useful and informative to calculate the gradient of the field that we just defined. The basis for that is a formula from the calculus of variations. Consider the vector l_μ that joins a given point $O(x_\mu)$ with the corresponding point $A(a_\mu)$. The points along the line OA will have $X_\mu = x_\mu + \alpha\, l_\mu$ for their coordinates, where α is a parameter.

One therefore has:

$$\frac{dX_\mu}{d\alpha} = l_\mu \,,$$

which is independent of α.

Form the integral:

$$I = \int_0^A l_\mu dX_\mu = \int_0^A l_\mu l_\mu \, d\alpha \;.$$

Consider a point O' that is very close to O then and a point A' that corresponds to it by the construction of the light-cone. The vector $O'A'$ is obtained from the vector OA by the introduction of a variation δX_μ at each point of OA – *including the extremities* – and one will have:

$$\delta I = 2 \int_0^A l_\mu \delta l_\mu \, d\alpha = 2 \left[l_\mu \delta X_\mu \right]_0^A - 2\int_0^A \frac{dl_\mu}{d\alpha} \delta X_\mu \, d\alpha \,.$$

The latter integral will be zero because l_μ is independent of α. On the other hand, one has $l_\mu \; l_\mu = 0$ along OA, as well as along $O'A'$, which are both isotropic vectors by construction. Thus, I and δI will both be zero. It will then result that:

$$\left[l_\mu \delta X_\mu \right]_0^A = 0 \text{ or } \qquad (l_\mu \, \delta X_\mu)_0 = (l_\mu \, \delta X_\mu)_A \,.$$

The point A moves along the line L, so one can write:

$$(l_\mu \, \delta X_\mu)_A = l_\mu \, u_\mu \, d\tau = - \, c\, \tau \, d\tau.$$

On the other hand:

$$(l_\mu \, \delta X_\mu)_0 = l_\mu \, dx_\mu \,.$$

A displacement dx_μ of the point O corresponds to a variation $d\tau$ of the scalar τ, and one will then have:

$$\boxed{\frac{d\tau}{dx_\mu} = \partial_\mu \tau = -\frac{l_\mu}{cr},}$$

which is the gradient that we seek for the scalar field τ. Similarly, for every quantity that is attached to the point A and defines a field $f(\tau)$ at the point O, one will have:

(I.11) $$\partial_\mu f(\tau) = - \, \dot{f} \, \frac{l_\mu}{cr} \qquad \text{with} \quad \dot{f} = \frac{df}{d\tau} \,.$$

For example, for u_μ, we will have:

$$\boxed{\partial_\mu u_\nu = -\dot{u}_\nu \frac{l_\mu}{cr}.}\,.$$

One easily obtains the gradient of the field l_μ by remarking that under the preceding variation, the variation δl_μ will be simply:

$$\delta \dot{l}_\mu = \delta a_\mu - \delta x_\mu = u_\mu \,\delta\tau - \delta x_\mu = u_\mu \left(-\frac{l_\nu}{cr}\right)\delta x_\nu - \delta x_\mu\,,$$

so one will get:

(I.12) $$\boxed{\partial_\nu l_\mu = -\delta_{\mu\nu} - \frac{l_\nu u_\mu}{cr}.}\,.$$

Finally, for the scalar:

$$r = -\frac{l_\mu u_\mu}{c}\,,$$

one will have that:

$$\partial_\mu r = (u_\nu \,\partial_\mu \,l_\nu + l_\nu \,\partial_\mu \,u_\nu),$$

which will make:

(I.13) $$\boxed{\partial_\mu r = \frac{1}{c}\left[u_\mu + \frac{l_\mu}{cr}(l_\nu \dot{u}_\nu - c^2)\right].}$$

§ 4. Mathisson's theory: monopole case. – Having said that, it will result from our hypothesis that any potential:

$$\psi(x_\mu) = \frac{f(\tau)}{r}$$

that is defined by an arbitrary function $f(t)$ will obey the Laplace equation:

$$\Box\psi = 0$$

at any point that is not situated on L. One easily proves this by expressing $\partial_\mu \partial_\mu \,\psi$ by means of formulas (11) and (13), while taking (9) and (10) into account. However, that will result directly from the fact that, by definition, ψ is a retarded potential, which will then propagate like an electromagnetic wave. Any function $f(t)$ that is attached to the point A will permit us to form a solution of the Laplace equation that possesses a first-order line singularity along L.

In particular, we can utilize such solutions to solve Einstein's equations in the absence of an electromagnetic field:

$$\boxed{\Box\psi_{\mu\nu} = 0.}$$

However, it happens that the solutions must also satisfy the auxiliary conditions:

$$\boxed{\partial_\mu \psi_{\mu\nu} = 0}$$

(when we revert to the notation of special relativity). One might remark that the approximating equation $\Box\psi_{\mu\nu} = 0$ is linear. However, the nonlinear character still leaves a trace, in the form of the condition that $\partial_\mu \psi_{\mu\nu} = 0$.

In the context of the ideas that were discussed above, it is fitting that this condition, which restricts the generality of the world-line L, should follow from the equations of motion. It is also fitting to recall that this condition will occur in reality unless we impose a restriction on the field $\psi_{\mu\nu}$ in order to determine a particular system of coordinates at each point in which the approximate equations will become linear.

We consider a symmetric tensor $m_{\mu\nu}(\tau)$ at A and form the potential:

$$\boxed{\psi_{\mu\nu}(x_\alpha) = \frac{m_{\mu\nu}(\tau)}{r}.}$$

We calculate the divergence $z_\nu = \partial_\mu \psi_{\mu\nu}$.

Upon employing the relations that we just established, it will follow that:

$$z_\mu = -\frac{1}{r}\frac{l_\mu}{r}\left(\frac{\dot{m}_{\mu\nu}}{c} + \frac{m_{\mu\nu}}{c^2}\frac{l_\lambda}{r}\dot{u}_\lambda\right) - \frac{1}{r^2}m_{\mu\nu}\left(\frac{u_\mu}{c} - \frac{l_\mu}{r}\right).$$

We have brought the ratio l_μ / r to prominence, which will remain finite in the limit as we approach L, and because of that fact, we have ordered the terms in powers of $1 / r$. Upon multiplying certain factors by $- l_\mu u_\mu / cr = 1$, we will obtain:

$$z_\mu = -\frac{1}{r}\frac{l_\mu}{r}\frac{l_\lambda}{r}\frac{1}{c^2}\left(m_{\mu\nu}\dot{u}_\lambda - \dot{m}_{\mu\nu}u_\lambda\right) + \frac{1}{r^2}\frac{l_\lambda}{r}\frac{1}{c^2}\left(m_{\mu\nu}u_\mu u_\lambda + c^2 m_{\lambda\nu}\right).$$

If one wants to have $z_\nu = 0$ for all values of $1 / r$ and l_μ / r then one must have:

(I.14) $$m_{\mu\nu}\, u_\mu\, u_\lambda + c^2\, m_{\lambda\nu} = 0,$$

(I.15) $$m_{\mu\nu}\dot{u}_\lambda - \dot{m}_{\mu\nu}u_\lambda = 0$$

separately. When equation (14) is projected onto the proper system (viz., $u_j = 0$), that will yield simply:

$$m_{jk} = 0 \qquad \text{and} \qquad m_{i4} = 0.$$

The tensor $m_{\mu\nu}$ will include only purely time-like components in the proper system. It is obligatory that it should then take the form:

$$\boxed{m_{\mu\nu} = \mathfrak{M}\, u_\mu u_\nu} \qquad (\mathfrak{M} \text{ is a scalar}).$$

When equation (15) is contracted with u_μ, that will give:

$$\dot{m}_{\mu\nu} = 0,$$

so one will get:

$$\dot{\mathfrak{M}} u_\mu u_\nu + \mathfrak{M} \dot{u}_\mu u_\nu + \mathfrak{M} u_\mu \dot{u}_\nu = 0.$$

Upon contracting with u_ν, it will result that:

$$\dot{\mathfrak{M}} u_\mu + \mathfrak{M} \dot{u}_\mu = 0. \tag{I.16}$$

Finally, upon contracting this with u_μ, one will get:

$$\dot{\mathfrak{M}} = 0, \qquad \text{so} \qquad \boxed{\mathfrak{M} = \text{constant.}}$$

Equation (16) will then reduce to:

$$\mathfrak{M}\,\dot{u}_\mu = 0, \qquad \text{so} \qquad \boxed{\dot{u}_\mu = 0.}$$

This is a condition that depends strongly upon the line L, and which imposes the constraint that the point A must describe a *uniform, rectilinear motion.*

In order to interpret this result physically, we refer to the method that was employed in the ordinary dualistic theory that considered matter to be a foreign body that evolved in space-time and created a gravitational field. One will then have that the gravitational potential at O can be computed by means of the retarded potential using the integral:

$$\Phi_{\mu\nu} = \frac{K}{2\pi} \int \frac{T_{\mu\nu}(t - r/c)}{r} dv .$$

The integral is taken over all of space, and t is the time at which one considers the point O. The fact that we use $T_{\mu\nu}$ $(t - r / c)$ signifies that the integral is taken over a three-dimensional multiplicity that is defined precisely by the "past" nappe of the light-cone whose vertex is at O. In the absence of an electromagnetic field, $T_{\mu\nu}$ will be zero at every point, except inside the world-tube that is swept out by the matter. With Lubanski [**8**], we may make the following hypotheses:

1. One deals with the case of "pure matter," in which the expression for the energy-momentum tensor inside of the tube will be:

$$T_{\mu\nu} = \rho\, m_0\, u_\mu\, u_\nu .$$

2. One ignores accelerations.

One can then replace the velocity at each material point P, which is considered to be on the light-cone Σ, with the velocity that it possesses as it traverses the hyperplane of proper space Σ' that relates to the point A that is the intersection of Σ and the world-line

L. One can then take the integral over the volume V_0 of the droplet relative to the proper system of the point A.

These hypotheses permit us to consider the tensor $T_{\mu\nu} = \rho\, m_0\, u_\mu\, u_\nu$ at the instant $t - r_0/c$ that characterizes the point A. Upon developing $1 / r$ in a series, we will get:

$$\frac{1}{r} T_{\mu\nu} = \rho\, m_0\, u_\mu\, u_\nu\, \frac{1}{r_0} + \rho\, m_0\, u_\mu\, u_\nu\, \partial_\lambda \left(\frac{1}{r}\right)_0 y_\lambda + \ldots,$$

in which y_λ are the coordinates of the point in question *with respective to the origin when it is defined to be the point A.* Now, finally, upon considering the drop to be an ensemble of discrete points of mass m_0 , we will have:

$$\Phi_{\mu\nu} = \frac{K}{2\pi}\frac{1}{r_0}\sum m_0 u_\mu u_\nu + \frac{K}{2\pi}\partial_\lambda\left(\frac{1}{r}\right)_0 \sum m_0 y_\lambda u_\mu u_\nu + \ldots$$

If we suppose that the point A that describes L is the *center of gravity of the drop* then we will have:

$$\sum m_0 y_\lambda = 0,$$

so in particular, the time-like component of $\Phi_{\mu\nu}$ will be:

$$\Phi_{44} = -\frac{K}{2\pi}\frac{1}{r_0}c^2\sum m_0 \;+ \text{(terms of order greater than 1 in } y_\lambda)$$

in the proper system of A. Obviously, one can identify this result with the one that we obtained just now:

$$\psi_{\mu\nu} = \frac{M u_\mu u_\nu}{r}, \qquad \text{given that} \qquad \psi_{44} = -\frac{Mc^2}{r},$$

which will permit us to compare the constant M with the equally-constant quantity $K / 2\pi \sum m_0$.

In conclusion, one can, as a first approximation, identify a material drop that moves in the absence of an electromagnetic field with a point-like singularity of the gravitational field that is localized to the center of gravity of the drop and animated with a uniform, rectilinear motion. That singularity will be of the form:

$$\psi_{\mu\nu} = \frac{M u_\mu u_\nu}{r},$$

in which the constant M will be the total mass of the drop, to a very good approximation.

§ 5. Mathisson's theory: dipole case. – We shall now pass on to a higher degree of approximation. We will obtain another solution of the Laplace equation by superposing a solution of the type:

$$\partial_\mu \left(\frac{m_{\mu\nu\lambda}}{r} \right)$$

with a solution of the preceding type, where $m_{\mu\nu\lambda}$ is a tensor that is attached to the point A and is symmetric in μ and ν. Obviously, the field that is defined in that way over all of space-time will obey the equation:

$$\Box\, \partial_\mu \left(\frac{m_{\mu\nu\lambda}}{r} \right) = \partial_\mu \Box \left(\frac{m_{\mu\nu\lambda}}{r} \right) = 0,$$

and for:

$$\psi_{\mu\nu} = \frac{m_{\mu\nu}}{r} + \partial_\mu \left(\frac{m_{\mu\nu\lambda}}{r} \right),$$

one will similarly have:

$$\Box \psi_{\mu\nu} = 0.$$

It will remain for us to see what restrictions will have the auxiliary $\partial_\mu \psi_{\mu\nu} = 0$ as a consequence.

We calculate $z_\nu = \partial_\mu \psi_{\mu\nu}$, while applying the Mathisson relations. That will give:

$$\partial_\alpha \left(\frac{m_{\mu\nu\alpha}}{r} \right) = -\frac{1}{cr^2} \left\{ \dot{m}_{\mu\nu\alpha} l_\alpha + m_{\mu\nu\alpha} \left[u_\alpha + \frac{l_\alpha}{cr} (l_\beta \dot{u}_\beta - c^2) \right] \right\},$$

so:

$$\begin{aligned}
&\partial_\mu \partial_\alpha \left(\frac{m_{\mu\nu\alpha}}{r} \right) \\
&\quad = -\frac{1}{cr^1} \left\{ -\ddot{m}_{\mu\nu\alpha} \frac{l_\mu l_\alpha}{cr} - \dot{m}_{\mu\nu\alpha} \left(\delta_{\mu\alpha} + \frac{l_\mu l_\alpha}{cr} \right) \right. \\
&\qquad - \dot{m}_{\mu\nu\alpha} \frac{l_\mu}{cr} \left[u_\alpha + \frac{l_\alpha}{cr} (l_\beta \dot{u}_\beta - c^2) \right] - m_{\mu\nu\alpha} \dot{u}_\alpha \frac{l_\mu}{cr} \\
&\qquad + m_{\mu\nu\alpha} \frac{1}{cr^2} \left[-\left(\delta_{\mu\alpha} + \frac{l_\mu u_\alpha}{cr} \right) (l_\beta \dot{u}_\beta - c^2) - l_\alpha \left(\dot{u}_\mu + \frac{l_\mu}{cr} l_\beta \ddot{u}_\beta \right) \right] \\
&\qquad \left. - m_{\mu\nu\alpha} \frac{1}{c^2 r^2} \left[u_\mu + \frac{l_\mu}{cr} (l_\lambda \dot{u}_\lambda - c^2) \right] l_\alpha (l_\beta \dot{u}_\beta - c^2) \right\} \\
&\qquad + \frac{2}{c^2 r^3} \left[u_\mu + \frac{l_\mu}{cr} (l_\lambda \dot{u}_\lambda - c^2) \right] \left\{ \dot{m}_{\mu\nu\alpha} l_\alpha + m_{\mu\nu\alpha} \left[u_\alpha + \frac{l_\alpha}{cr} (l_\beta \dot{u}_\beta - c^2) \right] \right\}.
\end{aligned}$$

Similarly, one has:

$$\partial_\mu\left(\frac{m_{\mu\nu}}{r}\right) = -\frac{1}{cr}\left\{\dot{m}_{\mu\nu} l_\mu + m_{\mu\nu}\left[u_\mu + \frac{l_\mu}{cr}(l_\alpha \dot{u}_\alpha - c^2)\right]\right\}.$$

As before, we order the subsequent powers of $1 / r$ after having divided each l_μ by r, in such a fashion as to obtain a vector $\lambda_\mu = l_\mu / r$ that remains finite when r goes to zero. When all calculations have been done, z will be composed of terms in r^{-3}, r^{-2}, r^{-1}. If one sees that the condition $z_\nu = 0$ is satisfied for any given r and λ_μ then the coefficients of the three terms must be annulled separately.

For the coefficient of the term in r^{-3}, one has:

$$A_\nu = -\frac{m_{\mu\nu\alpha}}{c^2}[-2\, u_\mu\, u_\alpha + 3c\,(\lambda_\mu\, u_\alpha + \lambda_\alpha\, u_\mu) + c^2\,(\delta_{\mu\alpha} - 3\lambda_\mu\ \lambda_\alpha)] = 0.$$

For the term in r^{-2}, one has:

$$B_\nu = -\frac{m_{\mu\nu\alpha}}{c}\left\{-\frac{\lambda_\beta \dot{u}_\beta}{c^2}[3(u_\mu\lambda_\alpha + u_\alpha\lambda_\mu) + c(\delta_{\mu\alpha} - 6\lambda_\mu\lambda_\alpha)] - \frac{1}{c}(\lambda_\mu \dot{u}_\alpha + \lambda_\alpha \dot{u}_\mu)\right\}$$
$$+\ \frac{\dot{m}_{\mu\nu\alpha}}{c}\left[\delta_{\mu\alpha} - 3\lambda_\mu\lambda_\alpha + \frac{2}{c}(\lambda_\mu u_\alpha + \lambda_\alpha \dot{u}_\mu)\right] + \frac{m_{\mu\nu}}{c^2}\lambda_\alpha(u_\mu u_\alpha + c^2\delta_{\mu\alpha}) = 0.$$

For the term in r^{-1}, one has:

$$C_\nu = \frac{\ddot{m}_{\mu\nu\alpha}}{c^2}\lambda_\mu\lambda_\alpha + 3\frac{\dot{m}_{\mu\nu\alpha}}{c^3}\lambda_\beta\dot{u}_\beta\lambda_\mu\lambda_\alpha + \frac{m_{\mu\nu\alpha}}{c^4}\lambda_\mu\lambda_\alpha[3(\lambda_\beta\dot{u}_\beta)^2 + c\lambda_\beta\ddot{u}_\beta]$$
$$-\ \frac{\dot{m}_{\mu\nu}}{c}\lambda_\mu - \frac{m_{\mu\nu}}{c^2}\lambda_\mu\lambda_\alpha\dot{u}_\alpha = 0.$$

The elaboration of these three conditions is rather complicated when compared to the case of the monopole singularity. Lubanski [**8**] decomposed the components of the tensor $m_{\mu\nu\alpha}$ along the proper axes. One can always write:

$$m_{\mu\nu\alpha} \equiv {}^*m_{\mu\nu\alpha} + S_{\alpha\nu}\, u_\nu + S_{\alpha\nu}\, u_\mu + q_{\mu\nu}\, u_\alpha + n_\alpha\, u_\mu\, u_\nu + w_\nu\, u_\mu\, u_\alpha + f\, u_\mu\, u_\nu\, u_\alpha\,.$$

The tensors that are introduced are all orthogonal to u_μ :

$$\begin{aligned}
&{}^*m_{\mu\nu\alpha}\, u_\mu &&= {}^*m_{\mu\nu\alpha}\, u_\nu &&= {}^*m_{\mu\nu\alpha}\, u_\alpha &&= 0,\\
&S_{\alpha\nu}\, u_\alpha &&= S_{\alpha\mu}\, u_\mu &&= 0, &&\\
&q_{\mu\nu}\, u_\mu &&= q_{\mu\nu}\, u_\nu &&= 0, &&\\
&n_\alpha\, u_\alpha &&= w_\alpha\, u_\alpha &&= 0, &&
\end{aligned}$$

Naturally, $^*m_{\mu\nu\alpha}$ and are symmetric in μ and ν.

Upon contracting several times by the unit-speed velocity, one will see that this decomposition is always possible and that it is unique. On the other hand, applying the Mathisson relations will permit us to prove an important identity that will simplify the decomposition: For any tensor in A that includes u_α as a factor, one will have:

$$\partial_\alpha\left(\frac{f\,u_\alpha}{r}\right) = \frac{1}{r^2}\left\{r\left(-\frac{\dot{f}\,l_\alpha}{cr}u_\alpha - f\,\frac{\dot{u}_\alpha\,l_\alpha}{cr}\right)-\frac{1}{c}\left[u_\alpha+\frac{l_\alpha}{cr}(l_\beta\dot{u}_\beta - c^2)\right]f\,u_\alpha\right\}$$
$$= \frac{1}{r^2}\left[r\,\dot{f}-\frac{f}{c}\dot{u}_\alpha l_\alpha + f\,c+\frac{f}{c}(l_\beta\dot{u}_\beta - c^2)\right],$$

or finally:

(I.17) $$\boxed{\partial_\mu\left(\frac{f\,u_\alpha}{r}\right) = \frac{\dot{f}}{r}.}$$

Therefore, in the expression for the "dipole" moment:

$$\partial_\alpha\left(\frac{m_{\mu\nu\alpha}}{r}\right),$$

one can apply this relation to the terms:

$$\partial_\mu\left(\frac{q_{\mu\nu}\,u_\alpha}{r}\right),\quad \partial_\alpha\left(\frac{w_\mu\,u_\alpha u_\alpha + w_\nu\,u_\alpha u_\mu}{r}\right),\qquad \partial_\alpha\left(\frac{f\,u_\mu u_\nu\,u_\alpha}{r}\right),$$

which will give:

$$\frac{\dot{q}_{\mu\nu}}{r},\qquad \frac{1}{r}\frac{d}{d\tau}(w_\mu\,u_\nu + w_\nu\,u_\mu),\qquad \frac{1}{r}\frac{d}{d\tau}(f\,u_\mu\,u_\nu),$$

respectively.

These terms are of the same form as the monopole term $m_{\mu\nu} / r$, and can be incorporated into it with no loss of generality. One then asserts that:

(I.18) $$m_{\mu\nu\alpha} \equiv {}^*m_{\mu\nu\alpha} + S_{\alpha\mu}\,u_\nu + S_{\alpha\nu}\,u_\mu + n_\alpha\,u_\mu\,m\,u_\nu\,.$$

Having said that, the first condition – viz., $A_\nu = 0$ – can be written as:

$$m_{\mu\nu\alpha}\,[-\,2u_\mu\,u_\alpha + 3c\,(\lambda_\mu\,u_\alpha + \lambda_\alpha\,u_\mu) + c^2\,(\delta_{\mu\alpha} - 3\lambda_\mu\,\lambda_\alpha)] = 0.$$

However, when one contracts this with u_μ, one will see that the quantity in brackets is zero identically. The terms in equation (18) that contain that factor will not be taken into account, and $m_{\mu\nu\alpha}$ can be replaced with $^*m_{\mu\nu\alpha} + S_{\alpha\mu}\,u_\nu$.

On the other hand, these latter two tensors are orthogonal to u_μ and u_α. Thus, one will likewise suppress the terms that include these vectors from the bracket. It will then result that:

$$(*m_{\mu\nu\alpha} + S_{\alpha\mu}\, u_\nu)\,(\delta_{\mu\alpha} - 3\lambda_\mu\, \lambda_\alpha) = 0.$$

If one projects this equation onto the proper axes, where all of the 4-components of $*m_{\mu\nu\alpha}$ and $S_{\alpha\mu}$ are zero, then it will decompose into two equations, provided that ν is an index of space or of time:

(I.19$_1$) $$*m_{ijk}\,(\delta_{ik} - 3\lambda_i\, \lambda_k) = 0,$$

(I.19$_2$) $$ic\, S_{ki}\,(\delta_{ik} - 3\lambda_i\, \lambda_k) = 0.$$

One can easily break these equations down with respect to the antisymmetric parts of $*m_{\mu\nu\alpha}$ and $S_{\alpha\mu}$. Indeed, one will then deduce that:

(I.19$_{1'}$) $$\tfrac{1}{2}(*m_{ijk} + *m_{kji})\,(\delta_{ik} - 3\lambda_i\, \lambda_k) = 0,$$

(I.19$_{2'}$) $$\tfrac{1}{2}(S_{ki} + S_{ik})\,(\delta_{ik} - 3\lambda_i\, \lambda_k) = 0.$$

If one remarks that in proper space the vector $\lambda_i = l_i / r$, where r represents the length of the space vector l_i, has unit norm (i.e., $\delta_{ik}\, \lambda_i\, \lambda_k = 1$) then one will see that $\delta_{ik}\,(\delta_{ik} - 3\lambda_i\, \lambda_k) = 0$.

The first symmetric factor in the left-hand sides of each of equations (1′) and (2′) then contains δ_{ik}, and the solutions will be of the form:

$$\tfrac{1}{2}(*m_{ijk} + *m_{kji}) = c^2\, m_j\, \delta_{ik}\,,$$
$$\tfrac{1}{2}(S_{ki} + S_{ik}) = c^2\, S\, \delta_{ik}\,,$$

in which $*m_j$ is a spatial vector and S is a scalar. Finally, upon taking all of the 4-components (which are zero) into account, one will get the general covariant solution:

$$\tfrac{1}{2}(*m_{\mu\nu\alpha} + *m_{\alpha\nu\mu}) = *m_\nu\,(c^2\, \delta_{\mu\nu} + u_\mu\, u_\alpha) \equiv c^2\, m_\nu\, \eta_{\mu\alpha}\,, \qquad \text{with} \qquad *m_\nu\, u_\nu = 0,$$

and

$$\tfrac{1}{2}(S_{\mu\alpha} + S_{\alpha\mu}) = S\,(c^2\, \delta_{\mu\alpha} + u_\mu\, u_\alpha) \equiv c^2\, S\, \eta_{\mu\alpha}.$$

One may express $S_{\mu\alpha}$ completely then. One has:

$$S_{\mu\alpha} = c^2\, S\, \eta_{\mu\alpha} + S_{<\mu\alpha>}\,.$$

As for $*m_{\mu\nu\alpha}$, which is symmetric in μ and ν, one will easily deduce the following expression:

$$*m_{\mu\nu\alpha} = c^2\, m_\nu\, \eta_{\mu\alpha} + c^2\, m_\mu\, \eta_{\nu\alpha} - c^2\, m_\alpha\, \eta_{\mu\nu}\,.$$

Now, considering the contribution that they make to the divergence z_ν, and as a result of the condition $z_\nu = 0$, the tensor $*m_{\mu\nu\alpha}$ and the symmetric part of $S_{\mu\alpha}$ will satisfy:

$$\partial_\alpha\left(\frac{*m_{\mu\nu\alpha} + S_{(\mu\alpha)}u_\nu + S_{(\nu\alpha)}u_\mu}{r}\right).$$

The first three terms that one gets are:

$$c^2\delta_{\mu\alpha}\partial_\alpha\left(\frac{*m_\nu + Su_\nu}{r}\right) + c^2\delta_{\nu\alpha}\partial_\alpha\left(\frac{*m_\mu + Su_\mu}{r}\right) - c^2\delta_{\mu\nu}\partial_\alpha\left(\frac{*m_\alpha + Su_\alpha}{r}\right).$$

One can simplify the notation by setting:

$$\Gamma_\nu = c^2\left(\frac{*m_\nu + Su_\nu}{r}\right),$$

and one will have that $\Box\Gamma_\nu = 0$, from a general theorem.

Thus, the three terms under consideration can be written $\partial_\mu\,\Gamma_\nu + \partial_\nu\,\Gamma_\mu - \delta_{\mu\nu}\,\partial_\alpha\Gamma_\alpha$, and the divergence of that will be:

$$\partial_\mu\,\partial_\nu\Gamma_\nu + \partial_\mu\,\partial_\nu\,\Gamma_\mu - \partial_\nu\,\partial_\alpha\Gamma_\alpha = \Box\Gamma_\nu + \partial_\nu\,\partial_\mu\,\Gamma_\mu - \partial_\nu\,\partial_\alpha\Gamma_\alpha\ .$$

This will be zero. Thus, the three terms under consideration do not contribute to the total divergence.

As for the other terms:

$$-\partial_\alpha\left(\frac{*m_\nu u_\mu u_\alpha}{r}\right) - \partial_\alpha\left(\frac{*m_\mu u_\nu u_\alpha}{r}\right) - \partial_\alpha\left(\frac{S\,u_\mu u_\mu u_\alpha}{r}\right) - \partial_\alpha\left(\frac{S\,u_\nu u_\alpha u_\nu}{r}\right),$$

one can transform them using formula (17) into:

$$-\frac{1}{r}\frac{d}{d\tau}(*m_\nu u_\mu + *m_\mu u_\nu + 2Su_\mu u_\nu)\,.$$

They will then be of the form $m_{\mu\nu} / r$, and can therefore be incorporated into the monopole term.

Therefore, as a consequence of the first relation, one will see that the only terms that remain in the development (18) of $m_{\mu\nu\alpha}$ will be the term in n_α and the antisymmetric part of the terms in $S_{\alpha\nu}$.

Henceforth, we shall write:

$$\boxed{m_{\mu\nu\alpha} = S_{[\alpha\mu]}u_\nu + S_{[\alpha\nu]}u_\mu + u_\alpha u_\mu u_\nu,}$$

where the tensor $S_{[\alpha\mu]}$ is *antisymmetric*.

One recalls that:

$$\boxed{S_{[\mu\nu]} = 0} \qquad \text{and} \qquad \boxed{n_\mu u_\mu = 0.}$$

It is possible to simplify this expression again when one seeks, as we have done for the monopole case, to give it a physical interpretation by comparing the potential that it provides us with, namely:

$$\psi_{\mu\nu} = \partial_\alpha \left(\frac{S_{\alpha\mu} u_\nu + S_{\alpha\nu} u_\mu + n_\alpha u_\mu u_\nu}{r} \right) + \frac{m_{\mu\nu}}{r},$$

with the one that the ordinary dualistic theory gives us:

$$\Phi_{\mu\nu} = \frac{1}{r_0} \sum m_0 u_\mu u_\nu + \partial_\alpha \left(\frac{1}{r} \right)_0 \sum m_0 y_\alpha u_\mu u_\nu + \ldots$$

If one compares the 44 components then one will get that:

$$\psi_{44} = -\, c^2 \partial_\kappa \left(\frac{n_\alpha}{r} \right) + \frac{m_{44}}{r} = -\, c^2 n_\alpha \partial_\kappa \left(\frac{1}{r} \right) + \frac{m_{44} - c^2 \partial_\kappa n_\alpha}{r}$$

and

$$\Phi_{44} = c^2 \sum m_0 y_\alpha \partial_\alpha \left(\frac{1}{r} \right)_0 - \frac{c^2}{r_0} \sum m_0,$$

upon neglecting the other terms.

One can then identify corresponding terms and set:

$$n_\alpha = \sum m_0 y_\alpha,$$

which is zero, since we have localized the point A, which we take to be the origin, at the center of gravity relative to the proper system.

One then finally has:

(I.20) $$\boxed{m_{\mu\nu\alpha} = S_{[\alpha\mu]} u_\nu + S_{[\alpha\nu]} u_\mu,}$$

and

$n_\alpha = 0$ implies that $\partial_\alpha n_\alpha = 0$ and $m_{44} = -\, c^2 \sum m_0$.

(We have suppressed the coefficient $-K / 2\pi$.)

It is possible to go further and identify the $4i$ components. One gets:

$$\psi_{4i} = -\partial_\kappa \left(\frac{ic\, S_{i\alpha}}{r} \right) + \frac{m_{4i}}{r} = -\frac{ic}{r} \left(\frac{m_{4i}}{ic} + \partial_\kappa S_{i\alpha} \right) + ic\, \partial_\alpha \left(\frac{1}{r} \right) S_{i\alpha},$$

and

$$\Phi_{4i} = \frac{ic}{r_0} \sum m_0 v_i + ic\, \partial_\kappa \left(\frac{1}{r} \right)_0 \sum m_0 v_i y_\alpha.$$

One then sees that this identification will affect the components that include a factor of v_i / c, as opposed to ψ_{44} and Φ_{44} (while the components ij include two). This then amounts to a second-order approximation, in comparison to the case of the monopole potential, where we were allowed to identify only the 44 components. It will then follow that:

$$\frac{m_{4i}}{ic} - \partial_\alpha S_{i\alpha} = \sum m_0 v_i$$

and

$$S_{i\alpha} = -\sum m_0 v_i y_\alpha .$$

Since we have agreed to retain only the antisymmetric part of $S_{\mu\nu}$, we will have, in reality:

$$S_{[i\alpha]} = \tfrac{1}{2}\sum m_0 (y_i v_\alpha - y_\alpha v_i) .$$

In other words: The space components of $S_{[\mu\nu]}$ in the proper system represent the *internal angular momentum* of the drop relative to the center of gravity, and one will indeed have:

(I.21) $$\boxed{S_{[\mu\nu]} u_\nu = 0,}$$

which is the Frenkel condition.

We now recall the second relation, which annuls the term in $1 / r^2$ in the divergence z_ν.

If we replace $m_{\mu\nu\alpha}$ by the expression $m_{\mu\nu\alpha} = S_{\alpha\mu}\, u_\nu + S_{\alpha\nu}\, u_\mu$ then we will have for the various terms:

$$m_{\mu\nu\alpha}(u_\mu\, \lambda_\alpha + u_\alpha\, \lambda_\mu) = S_{[\alpha\mu]}\,(u_\mu\, \lambda_\alpha + u_\alpha\, \lambda_\mu)\, u_\nu + S_{[\alpha\nu]}\, u_\mu\, u_\mu\, \lambda_\alpha + S_{[\alpha\nu]}\, u_\mu\, u_\mu\, \lambda_\alpha = -\, c^2\, \lambda_\alpha .$$

The first term goes to zero, by antisymmetry, while the last one will go to zero from (21):

$$m_{\mu\nu\alpha}(\delta_{\mu\alpha} - 6\lambda_\mu\, \lambda_\alpha) = \delta_{\mu\alpha}\, S_{\alpha\mu}\, u_\nu + S_{\alpha\mu}\, u_\mu - 6S_{\alpha\mu}\, \lambda_\alpha\, \lambda_\mu\, u_\nu - 6S_{\alpha\nu}\, \lambda_\alpha\, \lambda_\mu\, u_\mu = 6c\, S_{\alpha\nu}\, \lambda_\alpha ,$$

if one takes (10), (21), and the antisymmetry of $S_{\alpha\mu}$ into account.

$$m_{\mu\nu\alpha}(\lambda_\mu \dot{u}_\alpha + \lambda_\alpha \dot{u}_\mu) = S_{\alpha\mu}(\lambda_\mu \dot{u}_\alpha + \lambda_\alpha \dot{u}_\mu)u_\nu + S_{\alpha\nu}\dot{u}_\alpha u_\mu \lambda_\mu + S_{\alpha\nu}\lambda_\alpha u_\mu \dot{u}_\mu = -\, c^2\, S_{\alpha\nu}\, \dot{u}_\alpha ,$$

upon taking antisymmetry, (10), and the relation:

$$u_\mu \dot{u}_\mu = 0,$$

which follows from:

(I.22) $$u_\mu\, u_\mu = -\, c^2,$$

into account. On the other hand:

$$\dot{m}_{\mu\nu\alpha} = \dot{S}_{\alpha\mu} u_\nu + \dot{S}_{\alpha\nu} u_\mu + S_{\alpha\mu} \dot{u}_\nu + S_{\alpha\nu} \dot{u}_\mu ,$$

$$\dot m_{\mu\nu\alpha}\delta_{\mu\alpha} = \delta_{\mu\alpha}\dot S_{[\alpha\mu]}u_\nu + \dot S_{\mu\nu}u_\nu + \delta_{\mu\alpha}S_{[\alpha\mu]}\dot u_\nu = \dot S_{\mu\nu}u_\mu + S_{\mu\nu}\dot u_\mu = \frac{d}{d\tau}(S_{\mu\nu}u_\mu) = 0,$$

$$\begin{aligned}\dot m_{\mu\nu\alpha}\lambda_\mu\lambda_\alpha &= \dot S_{[\alpha\mu]}\lambda_\mu\lambda_\alpha u_\nu + \dot S_{[\alpha\nu]}\lambda_\alpha\lambda_\mu u_\mu + S_{[\alpha\mu]}\lambda_\alpha\lambda_\mu\dot u_\nu + S_{[\alpha o]}\lambda_\alpha\lambda_\nu\dot u_\mu \\ &= -c^2\dot S_{[\alpha\mu]}\lambda_\alpha + S_{[\alpha\mu]}\lambda_\alpha\dot u_\mu\lambda_\mu,\end{aligned}$$

$$\begin{aligned}\dot m_{\mu\nu\alpha}(\lambda_\mu u_\alpha + \lambda_\alpha u_\mu) &= (\dot S_{[\alpha\mu]}u_\nu + S_{[\alpha\mu]}\dot u_\nu)(\lambda_\mu u_\alpha + \lambda_\alpha u_\mu) \\ &\quad + \dot S_{[\alpha\nu]}u_\alpha\lambda_\mu u_\mu + \dot S_{[\alpha\nu]}\lambda_\alpha u_\mu u_\mu + S_{[\alpha\nu]}u_\alpha\lambda_\mu\dot u_\mu + S_{[\alpha\nu]}\lambda_\alpha u_\mu u_\mu \\ &= -c^2\dot S_{[\alpha\nu]}u_\alpha - c^2\dot S_{[\alpha\nu]}\lambda_\alpha,\end{aligned}$$

upon taking the antisymmetry of $S_{[\alpha\mu]}$ and equations (10), (21), and (22) into account.

One thus gets:

$$\begin{aligned}&\frac{1}{c}S_{\alpha\nu}(3\lambda_\beta\dot u_\beta\lambda_\alpha - \dot u_\alpha + 3\lambda_\mu\dot u_\mu\lambda_\alpha) + \dot S_{\alpha\nu}(\lambda_\alpha - 2u_\alpha / c) \\ &= -\frac{1}{c}(S_{\alpha\nu}\dot u_\alpha + 2\dot S_{\alpha\nu}u_\alpha) + \dot S_{\alpha\nu}\lambda_\alpha = \dot S_{\alpha\nu}(\lambda_\alpha - u_\alpha / c),\end{aligned}$$

for the dipole terms of B_ν, since:

$$S_{\alpha\nu}\dot u_\alpha + \dot S_{\alpha\nu}u_\alpha = \frac{d}{d\tau}(S_{\alpha\nu}u_\alpha) = 0.$$

Finally, the monopole terms give:

$$m_{\mu\nu}(\lambda_\alpha - u_\alpha / c).$$

Thus, by definition:

(I.23) $$B_\nu = (\dot S_{[\alpha\nu]} + m_{\alpha\nu})(\lambda_\alpha - u_\alpha / c) = 0.$$

In order to understand this relation better, we decompose the tensor $m_{\alpha\nu}$ in the same way that we did with $m_{\mu\nu\alpha}$:

(I.24) $$m_{\alpha\nu} = {}^*m_{\alpha\nu} + P_\alpha u_\nu + P_\nu u_\alpha + q\,u_\alpha u_\nu,$$

with

$${}^*m_{\alpha\nu}u_\alpha = {}^*m_{\alpha\nu}u_\nu = P_\alpha u_\alpha = 0.$$

One then remarks that:

(I.25) $$u_\alpha(\lambda_\alpha - u_\alpha / c) = 0,$$

by virtue of (10). Thus, the terms in u_α in the development (24) will disappear identically in equation (23), which one can write as:

$$(\dot S_{\alpha\nu} + {}^*m_{\alpha\nu} + P_\alpha u_\nu)(\lambda_\alpha - u_\alpha / c) = 0.$$

If one contracts this with u_α then one will get the expression for the vector:

$$Q_\nu = -\frac{1}{c^2}\dot{S}_{\alpha\nu}u_\alpha = \frac{1}{c^2}S_{\alpha\nu}\dot{u}_\alpha ,$$

and one will that upon contracting this with u_ν, one will get:

$$Q_\nu\, u_\nu = \frac{1}{c^2}\dot{S}_{\alpha\nu}u_\alpha u_\nu = 0,$$

by antisymmetry.

If we then contract (26) with u_ν directly then it will happen that:

$$\dot{S}_{\alpha\nu}u_\nu - c^2 P_\alpha = 0,$$

so we will obtain the expression for P_α:

$$P_\alpha = \frac{1}{c^2}\dot{S}_{\alpha\nu}u_\nu ,$$

which we may substitute into equation (26), as well as that of Q_ν:

$$\dot{S}_{\alpha\nu} + {}^*m_{\alpha\nu} + \frac{1}{c^2}\dot{S}_{\alpha\beta}u_\beta u_\nu = \frac{1}{c^2}\dot{S}_{\nu\beta}u_\beta u_\alpha ,$$

$$\dot{S}_{\alpha\nu} + {}^*m_{\alpha\nu} = \frac{1}{c^2}u_\beta(\dot{S}_{\nu\beta}u_\alpha - \dot{S}_{\alpha\beta}u_\nu) .$$

One then sees that all of the terms are antisymmetric in α and ν, except for ${}^*m_{\alpha\nu}$, which is symmetric. Thus, ${}^*m_{\alpha\nu} = 0$, and what will remain is:

$$\dot{S}_{\alpha\nu} = \frac{1}{c^2}(\dot{S}_{\nu\beta}u_\beta u_\alpha - \dot{S}_{\alpha\beta}u_\beta u_\nu) . \tag{I.27}$$

We remark that the vector:

$$P_\alpha = \frac{1}{c^2}\dot{S}_{\alpha\nu}u_\nu$$

can also be written:

$$P_\alpha = -\frac{1}{c^2}S_{\alpha\nu}\dot{u}_\nu ,$$

since:

$$\dot{S}_{\alpha\nu}u_\nu + S_{\alpha\nu}\dot{u}_\nu = \frac{d}{d\tau}(S_{\alpha\nu}\ u_\nu) = 0.$$

Thus, the vector P_α can be identified with the transverse momentum of Frenkel's theory. Upon introducing it into equation (27), it will become:

$$\boxed{\dot{S}_{\alpha\nu} = P_\nu u_\alpha - P_\alpha u_\nu ,}$$

which is the second Frenkel equation, in the absence of an electromagnetic field.

It remains for us to annul the term in $1 / r$ in the divergence z_ν, namely, C_ν. If we leave out the expressions that go to zero by antisymmetry then, upon taking (20) into account, we will get:

$$\ddot{m}_{\mu\nu\alpha}\lambda_\mu\lambda_\alpha = -c^2\,\ddot{S}_{\alpha\nu}\lambda_\alpha + 2\ddot{S}_{\alpha\nu}\dot{u}_\mu\lambda_\mu\lambda_\alpha + S_{\alpha\nu}\ddot{u}_\mu\lambda_\mu\lambda_\alpha ,$$

$$\dot{m}_{\mu\nu\alpha}\lambda_\mu\lambda_\alpha = -c^2\,\dot{S}_{\alpha\nu}\lambda_\alpha + S_{\alpha\nu}u_\mu\lambda_\mu\lambda_\alpha ,$$

$$m_{\mu\nu\alpha}\,\lambda_\mu\,\lambda_\alpha = -c^2\,S_{\alpha\nu}\lambda_\alpha ,$$

That will give:

$$\frac{1}{c^2}[-c\ddot{S}_{\alpha\nu}\lambda_\alpha + 2\ddot{S}_{\alpha\nu}\dot{u}_\beta\lambda_\beta\lambda_\alpha + S_{\alpha\nu}\ddot{u}_\mu\lambda_\mu\lambda_\alpha] + 3\frac{\dot{u}_\beta\lambda_\beta}{c^3}(-c\,\dot{S}_{\alpha\nu}\lambda_\alpha + S_{\alpha\nu}\dot{u}_\beta\lambda_\beta\lambda_\alpha)$$
$$+\ 3\frac{(\dot{u}_\beta\lambda_\alpha)^2}{c^4}(-cS_{\alpha\nu}\lambda_\alpha) + \frac{\dot{u}_\beta\lambda_\beta}{c^3}(-cS_{\alpha\nu}\lambda_\alpha)$$
$$= -\frac{1}{c}\ddot{S}_{\alpha\nu}\lambda_\alpha - \frac{1}{c}\lambda_\beta\dot{u}_\beta\dot{S}_{\alpha\nu}\lambda_\alpha$$

for the dipole terms in C_ν.

Since the monopole terms are:

$$-\frac{1}{c}\dot{m}_{\alpha\nu}\lambda_\alpha - \frac{1}{c^2}\lambda_\beta\dot{u}_\beta m_\alpha\lambda_\alpha ,$$

one will see that one can set:

$$R_{\alpha\nu} = m_{\alpha\nu} + \dot{S}_{\alpha\nu} ,$$

and that one will have:

$$C_\nu = -\frac{1}{c}\dot{R}_{\alpha\nu}\lambda_\alpha - \frac{1}{c^2}\lambda_\beta\dot{u}_\beta R_{\alpha\nu}\lambda_\alpha .$$

Upon taking advantage of the facts that:

$$m_{\alpha\nu} = P_\alpha\,u_\nu + P_\nu\,u_\alpha + q\,u_\alpha\,u_\nu$$

and

$$\dot{S}_{\alpha\nu} = P_\alpha\,u_\nu - P_\nu\,u_\alpha ,$$

it will then happen that:

$$R_{\alpha\nu} = (2P_\nu + q\,u_\nu)\,u_\alpha ,$$

$$\dot{R}_{\alpha\nu} = (2\dot{P}_\nu + \dot{q}\,u_\nu + q\,\dot{u}_\nu)\,u_\alpha + (2P_\nu + q\,u_\nu)\,\dot{u}_\alpha\,.$$

Thus:

$$R_{\alpha\nu}\,\lambda_\alpha = -\,c\,(2P_\nu + q\,u_\nu)$$

from (10), so:

$$\dot{R}_{\alpha\nu}\lambda_\alpha = -\,c\;(2\dot{P}_\nu + \dot{q}\,u_\nu + q\,\dot{u}_\nu) + \lambda_\beta\dot{u}_\beta(2P_\nu + q\,u_\nu)\;,$$

or finally:

$$C_\nu = 2\dot{P}_\nu + \dot{q}\,u_\nu + q\,\dot{u}_\nu = 0.$$

In order to recover the Frenkel equations, one can set:

$$q = -\,2\mathfrak{M}_0\,, \qquad G_\mu = \mathfrak{M}_0\,u_\mu - P_\mu\,,$$

and one finally comes back to the first Frenkel equation:

$$\boxed{\dot{G}_\mu = 0.}$$

Thus, Mathisson's formalism permits us to show that the general principle of Einstein and Infeld that matter can be considered to be a singularity of the gravitational field can be applied to spinning particles, and not just classical material points, which will permit us to the describe the global motion of a material droplet more finely by the introduction of a supplementary quantity, namely, the internal angular momentum. As for the classical material point, one will recover the results that are obtained in the context of the usual dualistic conception of the dynamics of a spinning point.

§ 6. Weyssenhoff's theory. The dynamical equations of a spinning particle, for which we have indicated two different paths of approach, were deduced by Jan v. Weyssenhoff [**2**] by starting from a different viewpoint in some papers to which we will make frequent allusions in what follows, because he constructed his model in the context of a theory of a relativistic fluid that is given an internal rotation. The hydrodynamics of Weyssenhoff is constructed by axiomatic means. He considers a fluid that is defined at each point by the data of three relativistic tensor quantities: a *unit-length current velocity* u_μ, for which one has $u_\mu\,u_\mu = -\,c^2$, a *momentum density* vector g_μ, which we have not constrained to be collinear with the current, and finally, an antisymmetric *internal angular momentum density* tensor $s_{[\mu\nu]}$, for which one supposes that all of the components are in the local proper space, which is expressed by the "auxiliary kinematic condition" $s_{\mu\nu}\,u_\nu = 0$.

Weyssenhoff then made the fluid subject to two dynamical laws, which are expressed by the conditions of zero divergence for an *energy-momentum density* tensor and a *total moment of rotation density* tensor.

Weyssenhoff considered a fluid that was composed of "pure matter" and devoid of internal stresses. Now, in the case of such a fluid, and in the absence of internal rotation, one has that the momentum density will be collinear with the velocity: $g_\mu = \mu_0\,u_\mu$ (μ_0 being the invariant mass density) and the relativistic energy-momentum tensor will have:

$$t_{\mu\nu} = \mu_0\, u_\mu\, u_\nu = g_\mu\, u_\nu$$

for its expression. Weyssenhoff preserved the second expression, and further assumed that it was also valid in the case of a momentum that was not collinear with the velocity. It is essential that we remark that $t_{\mu\nu}$, which is symmetric in the classical case, actually becomes an asymmetric tensor. The first dynamical law will then become:

$$\partial_\nu\, t_{\mu\nu} = \partial_\nu\,(g_\mu\, u_\nu) = 0.$$

In order to write the second law, the path that Weyssenhoff followed amounted to adding a supplementary quantity that related to the internal angular momentum to the ordinary expression for the *"orbital" moment of rotation density* – which is $x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}$ – namely, the *"proper" moment of rotation density*, which one expresses by $s_{\mu\nu}\, u_\lambda$, which gives:

$$\partial_\lambda\,(x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}) + \partial_\lambda\,(s_{\mu\nu}\, u_\lambda) = 0$$

for the second dynamical equation. One may transform this expression by introducing the "derivatives along a streamline" (see Appendix *A*):

$$\dot{g}_\mu = \partial_\nu(u_\nu\, g_\mu), \qquad \dot{s}_{\mu\nu} = \partial_\lambda\,(u_\lambda\, s_{\mu\nu}),$$

and, on the other hand, by remarking that:

$$\partial_\lambda\,(x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}) = \delta_{\lambda\mu}\, t_{\nu\lambda} - \delta_{\lambda\nu}\, t_{\mu\lambda} + x_\mu\, \partial_\lambda\, t_{\nu\lambda} - x_\nu\, \partial_\lambda\, t_{\mu\lambda}\,.$$

The last two terms are zero by virtue of the first conservation law. The other two provide $t_{\nu\mu} - t_{\mu\nu}$, which is the antisymmetric part of the tensor $t_{\mu\nu}$. One will then have:

$$\partial_\lambda\,(x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}) = t_{\nu\mu} - t_{\mu\nu} \equiv g_\nu u_\mu - g_\mu\, u_\nu\,.$$

Together, the two equations imply that:

$$\dot{g}_\mu = 0 \qquad \text{and} \qquad \dot{s}_{\mu\nu} = g_\nu u_\mu - g_\mu\, u_\nu\,.$$

Up to now, our concerns have been those of hydrodynamics. Now, consider the *local* proper space at a point *P* of the fluid. One may cut out an infinitesimal droplet from this space that contains the point *P* – i.e., a droplet that sweeps out an infinitesimal world-tube in the course of its motions – and integrate the various densities in the proper system that we have considered over the proper volume V_0 of the droplet. One will then have the momentum of the droplet:

$$G_\mu = \int_{V_0} g_\mu dV_0$$

and its internal angular momentum:

$$S_{[\mu\nu]} = \int_{V_0} s_{\mu\nu} dV_0\,.$$

One can then integrate each side of the Weyssenhoff equations over the proper volume of the droplet, which will give:

$$\int_{V_0} \dot{g}_\mu dV_0 = 0, \qquad \int_{V_0} \dot{s}_{\mu\nu} dV_0 = \int_{V_0} g_\mu u_\nu dV_0 - \int_{V_0} g_\nu u_\mu dV_0 .$$

One finds that the type of derivation that is employed permits us to write:

$$\int_{V_0} \dot{g}_\mu dV_0 = \frac{d}{d\tau}\int_{V_0} g_\mu dV_0 = \dot{G}_\mu \qquad \text{(see Appendix } A\text{)},$$

and similarly:

$$\int_{V_0} \dot{s}_{\mu\nu} dV_0 = \dot{S}_{\mu\nu} .$$

On the other hand, the Weyssenhoff formalism amounts, on the one hand, to the separation of the true velocity at each point into two parts: a group velocity u_μ and a "pure" velocity, which pertains to only the unique characteristics of the fluid; namely, a momentum that is not collinear with the velocity and an internal angular momentum. One then has the right to assume that u_μ results from an average for the fluid that is estimated over a small domain of another point and varies relatively little, and in a continuous fashion. One neglects these variations on the scale of the droplet considered, which allows u_μ to emerge from the proper space integral. By definition, the local Weyssenhoff equations translate into the global equations:

$$\dot{G}_\mu = 0 \qquad \text{and} \qquad \dot{S}_{\mu\nu} = G_\mu u_\nu - G_\nu u_\mu$$

for the droplet, which, when combined with the condition $S_{\mu\nu} u_\mu = 0$ that one obtains in a similar fashion by integrating the hydrodynamical relation $s_{\mu\nu} u_\mu = 0$, will give us the system of Frenkel and Mathisson.

The extension of the Weyssenhoff theory to the case where there exists an external electromagnetic field results immediately in the hydrodynamical translation of the classical results on the action of a field $F_{\mu\nu}$ on an electric charge e and a dipole of electromagnetic moment $\mu_{\alpha\beta}$.

The charged point is subjected to a force:

$$F_\mu = \frac{e}{c} F_{\mu\nu} u_\mu \qquad \text{(viz., the Lorentz force).}$$

The dipole is subjected to a force:

$$\Phi_\mu = \tfrac{1}{2} \mu_{\alpha\beta}\, \partial_\mu F_{\alpha\beta} \qquad \text{(viz., the Stern-Gerlach force)}$$

and a torque:

$$N_{\mu\nu} = \mu_{\alpha\mu} F_{\alpha\nu} - \mu_{\alpha\nu} F_{\alpha\mu} .$$

The Weyssenhoff hypotheses lead us to assume that the fluid is given a *charge distribution* with a volumetric density of electricity ρe and an *internal electromagnetic moment distribution* of volume density $\rho\mu_{\alpha\beta}$. One will then have a *force density per unit volume:*

$$f_\mu = \frac{\rho e}{c} F_{\mu\nu} u_\nu + \frac{1}{2} \rho u_{\alpha\beta} \partial_\mu F_{\alpha\beta}$$

and a density of electromagnetic torque:

$$n_{\mu\nu} = \rho\, \mu_{\alpha\mu}\, F_{\alpha\nu} - \rho\, \mu_{\alpha\nu}\, F_{\alpha\mu}\ .$$

Moreover, Frenkel and Weyssenhoff suppose that the internal electromagnetic moment density $\rho\mu_{\alpha\beta}$ is proportional to and collinear with the internal angular momentum density $s_{\alpha\beta}$:

$$\rho\mu_{\alpha\beta} = \chi\, s_{\alpha\beta}\,,$$

in which χ is a constant. The classical laws of hydrodynamics then provide the equations:

$$\partial_\nu\, t_{\mu\nu} = \frac{\rho e}{c} F_{\mu\nu} u_\nu + \frac{1}{2} \rho u_{\alpha\beta} \partial_\mu F_{\alpha\beta}\,,$$

$$\partial_\lambda\, m_{\mu\nu\lambda} = \rho\mu_{\alpha\mu}\, F_{\alpha\nu} - \rho\mu_{\alpha\nu}\, F_{\alpha\mu}\,,$$

i.e.:

$$\dot{g}_\mu = \frac{\rho e}{c} F_{\mu\nu} u_\nu + \frac{\chi}{2} s_{\alpha\beta} \partial_\mu F_{\alpha\beta}\,,$$

$$g_\nu\, u_\mu - g_\mu\, u_\nu + \dot{s}_{\mu\nu} = \chi\, s_{\alpha\mu}\, F_{\alpha\nu} - \chi\, s_{\alpha\nu}\, F_{\alpha\mu}\ .$$

One integrates all of these equations over the proper volume of an infinitesimal fluid without difficulty by supposing that the variations of the field are negligible on the scale of the dimensions of the droplet and introducing the total charge:

$$Q = \int_{V_0} \rho e\, dV_0\ .$$

It happens that:

$$\dot{G}_\mu = \frac{Q}{c} F_{\mu\nu} u_\nu + \frac{\chi}{2} S_{\alpha\beta}\, \partial_\mu F_{\alpha\beta}\,,$$

$$\dot{S}_{\mu\nu} = G_\nu\, u_\mu - G_\mu\, u_\nu + \chi (S_{\alpha\mu}\, F_{\alpha\nu} - S_{\alpha\nu}\, F_{\alpha\mu}),$$

$$\dot{S}_{\mu\nu} u_\nu = 0.$$

These are the Frenkel formulas. One may show without difficulty that the momentum necessarily involves an electromagnetic term, and it is in the same fashion that we find the proper mass, which permits us to define the particle. The expressions for these terms will be the same as the ones in Frenkel's papers, and the identification will be complete.

It is interesting to remark that since one is concerned with fluids with no internal stresses the Weyssenhoff droplet will not be subjected to any action on the part of the fluid at rest, and one may consider it to be isolated, for that matter. The global equations above thus characterize the dynamics of a droplet that is as small as one pleases, and we therefore obtain the dynamics of an isolated, spinning, material particle. It is for this reason that the system of equations that we obtained is identical with the ones that were given by Frenkel and Mathisson in order to characterize an isolated spinning point.

Similarly, one may reverse Weyssenhoff's argument and, solely by reason of the absence of internal stresses, take as one's point of departure, not the continuous fluid that was defined axiomatically, but the Weyssenhoff droplet, which is identical with the Frenkel-Mathisson spinning particle, and is characterized by a momentum G_μ that is non-collinear with the velocity u_μ and an internal angular momentum $S_{\mu\nu}$.

If one considers a collection of such particles, with appropriate initial conditions, then that will constitute a fluid of "pure matter" that will not be different from the Weyssenhoff fluid. This viewpoint seems more productive to us, and we will adopt it in the sequel because it will permit us to construct other spinning fluids by introducing forces of interaction between the droplets that are expressed by the internal stresses in the fluid. Furthermore, if one demands that the fluid must propagate waves then consideration of internal stresses will be indispensible. The Weyssenhoff fluid will therefore appear to be a particular case of the fluids that one may construct from the Frenkel-Mathisson particles, which will be the case of a "pure matter" fluid.

CHAPTER II

THE GENERAL THEORY OF RELATIVISTIC SPINNING PARTICLES

§ 1. Principles. The works that we just discussed, most of which are already quite old, were inspired by the problem of the spinning electron, as it manifested itself in the study of spectral lines. Their objective was to permit a relativistic treatment of that problem in the spirit of the old quantum theory: Give a relativistic model of a *classical* spinning particle that yields the rules of quantization. However, quantization, which was attempted notably by Mathisson, did not give the correct results. Furthermore, one finds that the problem of the relativistic spinning particle has been resolved completely (at least, from the formal viewpoint) by Dirac's theory, and in the context of wave mechanics. However, it is not without interest, from our standpoint of non-quantum dynamics, to demand that the non-quantum spinning particle should lead to the Dirac formalism in such a way that the classical particle would be to the Dirac electron what the Newtonian particle is to the Schrödinger electron.

One finds that the operators that are formed from the Dirac matrices by means of the wave function permit one to express a series of tensor quantities that appear in quantum theory as representing "mean densities" for the observable properties of the electron. From the classical viewpoint, one can consider these tensors to be ones that expressed the physical properties of a classical particle directly. We review the "classical Dirac particle" in detail in the next chapter. For now, note only that Dirac's theory causes a second-order antisymmetric tensor to appear, among others:

$$\mu_{\alpha\beta} = \frac{i}{2} B\, \psi^{+}\, \gamma_4\, (\gamma_\alpha\, \gamma_\beta - \gamma_\beta\, \gamma_\alpha)\, \psi$$

(B is the Bohr magneton), which provides the mean density of *electromagnetic moment*, such that one can observe it when an electromagnetic field acts upon the electron. However, one recalls that Frenkel posed two hypotheses as the basis for his model of spinning particles:

1) The electromagnetic moment and the internal angular moment are proportional:

$$\mu_{\alpha\beta} = \frac{e}{m_0 c} S_{\alpha\beta}\,.$$

2) Both moments belong to proper space:

$$\mu_{\alpha\beta}\, u_\beta = S_{\alpha\beta}\, u_\beta = 0.$$

It seems reasonable to preserve the first hypothesis in the Dirac case, and to set:

$$S_{\alpha\beta} = \frac{i}{2}\,\frac{Bm_0c}{e}\,\psi^+\,\gamma_4\,(\gamma_\alpha\,\gamma_\beta - \gamma_\beta\,\gamma_\alpha)\,\psi\,.$$

However, one will then arrive at a tensor that does not belong to proper space:

$$S_{\alpha\beta}\,u_\beta \neq 0.$$

If one gives a physical interpretation to such a particle in the classical theory then one must arrive at a theory of the dynamics of a spinning particle that has the Frenkel-Weyssenhoff dynamics as a special case and which might be of great interest as far as the relativistic rotations, in their full generality, are concerned if one desires to build a model of fluids that are given internal angular momentum.

An attempt that is guided by these considerations will permit us to show that it is the analysis of the pure and simple problem of the relativistic rotation of a particle that will necessarily introduce the supplementary parameters that were not understood by the theoreticians whose work we discussed. Non-relativistic mechanics (and even relativistic mechanics, when one does not consider "internal rotations) introduces the notion of dimensionless *point matter* as the constitutive element of matter in a completely natural fashion, without having to account for the fact that this concept is defined in reality by considering a small, *solid* body of the kind that we presently experience, and then letting its dimensions tend to zero. If the particle spins about itself then the dimensionless material point will no longer suffice as a representation. It must be considered to be a differential edifice if it is small and simplified, because an internal motion will necessarily assume that there are distinct parts that are separated in space and move with respect to each other.

Naturally, one can proceed by analogy with Newtonian point matter, when it is regarded as the limit of a small, solid body, and envision a small, *rigid* body in motion; our particles will then be small "tops." This is really the hypothesis of Bohm, Vigier, and Lochak [**17**]. However, that is not the only hypothesis, and in fact, it will raise great difficulties, in principle. Indeed, the solid body that appeals to the intuition of common sense is impossible to conceive with full rigor in relativistic physics. The rigorous simultaneity of the displacement of all of its points must be realized in an arbitrary reference frame. One can also say that if it is perfectly rigid then it will transmit deformations with an infinite velocity, and these are things that are forbidden by relativity. These difficulties must therefore necessarily arise in a relativistic treatment when we endow the solid body with distinct parts that are as small as necessary and when we renounce the Newtonian point matter that is devoid of any parts.

When one renounces the constitution of particles by tiny, solid structures, one will attempt to consider the fluid (or "sub-fluid") to be composed of tiny, fluid droplets whose form is simply that of a *classical* fluid. One might rightfully reproach such a model as being arbitrarily "mechanistic." To be sure, the classical fluid is an "element" of our current intuition (in the sense of the four "elements" of physics in antiquity). That is true by virtue of our gross experience with matter at our level. However, it can be remarked that under these conditions, it is no more "mechanistic" than a classical solid.

Constituting a fluid (which is purely representative, moreover) by means of particles that are, in themselves, liquid droplets is certainly a debatable step, but no more so than the hypothesis that constitutes gases, liquids, and solids by means of particles that are themselves conceived to be small, solid grains. However, that is the path that was followed by all atomic physicists since Democritus; i.e., to employ the precise and deterministic form that Louis de Broglie called "the Cartesian representation of phenomena by figures and motions [**11**]." That path, which was refuted *a priori* a number of times by philosophers, is nonetheless one that has led to brilliant successes in numerous domains in the hands of physicists and chemists, and has given birth to such admirable edifices as the kinetic theory of gases, the molecular theory of crystals, the stereochemistry of carbon, etc. In any case where it fails (the mechanical theory of the ether, for example), this concept can then be accused of mechanism and rejected with no regrets. Any time that it does not succeed in rendering a correct account of phenomena, it is legitimate to adjust one's confidence in its practical efficacy to that fact, and it will be likewise legitimate to see the reflection of a more profound physical significance in that selfsame efficacy.

It is in that spirit, and in view of its practical applications, that we shall now being the study of the *classical* dynamics of relativistic fluid masses in rotation by adopting a point of view that will lead us to a new dynamics of corpuscles, and in turn, to a new hydrodynamics that is capable if representing the wave functions of quantum corpuscles that are endowed with spin.

§ 2. Kinematical quantities. – The great merit of the theory in the former work of Møller [**12**] and Pryce [**13**], which is due to essentially to D. Bohm and J. P. Vigier [**14**], is to introduce quantities that represent matter in motion about itself in addition to the quantities that represent the energy of matter, which will be quantities that are abstracted from its mass, energy, and the internal stresses that are necessary to maintain its cohesion. The classical tensor $t_{\mu\nu}$ – viz., the energy-momentum tensor – is involved at once, as well as, in principle, without specifying or differentiating them, all of the forms of energy that the "molecules" possess, such as proper energy, kinetic energy, or potential energy that is due to stresses. However, one can give a purely kinematical initial description of a fluid by means of the local *unit-speed* velocity u_μ ($u_\mu \, u_\mu = - c^2$) and the *invariant matter density* ρ, which involves only the number of "molecules" that the fluid is composed of (which we assume to be identical), and ignoring their mass or energy.

One therefore defines a vector: the spacetime current density $j_\mu = \rho \, u_\mu$.

If we refer this vector to the local proper system then we will have:

$$j_k^0 = 0, \qquad j_4^0 = \rho \, ic, \qquad \text{or} \qquad j_\otimes^0 = \rho.$$

The time component of j_μ in the local proper system will thus be simply the invariant matter density. In an arbitrary reference frame, the time component $j_\otimes$ will again be the spatial density of matter relative to the reference frame in question, while the spatial components will form a spatial vector – viz., the "matter current density" – that is $j_i = j_\otimes \, v_i$, each of which represents the matter flux (or "molecular" flux) through a unit area that is perpendicular to the corresponding axis. One can remark that if one assumes that the molecules each carry the same electric charge e then $j_\otimes\, e$ will represent the electric charge density, and $j_i\, e$ will represent the components of the electric current density, and that will be true in any reference frame, since e is a relativistic invariant. That remark will serve to make the physical significance of the quadri-vector j_μ more precise, which will be a significance that persists in the absence of charge, of course.

The current vector is conservative, independently of all dynamical laws. Indeed, the relation:

$$\boxed{\partial_\mu j_\mu = \partial_\mu (\rho u_\mu) = 0,}$$

expresses simply the idea that:

$$\dot{\rho} = 0\,,$$

i.e., that ρ is conserved along the streamline (see Appendix A), and we will also have that this signifies that the proper volume integral:

$$Q = \int_{V_0} \rho\, dV_0$$

that is attached to a given droplet (which is nothing but the quantity of matter that is contained in this droplet) is conserved in the course of its motion.

We shall now consider a macroscopic fluid mass, which we assume to be bounded in space and to remain connected in the course of its motion. (Poincaré has shown that the existence of internal stresses makes such a mass generally tend to a stable form, such as for example, a rotating torus [**15**].) We shall see the restrictions that the relation $\partial_\mu j_\mu = 0$ imposes on the possible motion of that mass upon directing our attention to the global properties of a fluid mass when they are taken over its entirety.

In order to do that, we shall utilize Møller's method (see Appendix A) and cut the spacetime hyper-tube that is swept out by the matter in its motion with a space-like hyperplane Π that is orthogonal to a time-like axis Λ. One then considers the integral while holding time constant and taking it over all of the fluid volume.

If we then "weight" each of the points of Σ with the matter density:

$$j_\otimes = \frac{1}{ic}\, j_4\,,$$

where j_4 is the projection of the current j_μ onto the Λ axis, then we can define two sorts of integrals:

1) The integral:

$$\boxed{J = \int_\Sigma j_\otimes\, d\upsilon,}$$

which we call the *quantity of matter*, because it represent simply the total quantity of matter (i.e., the total number of molecules) that are contained in the fluid mass. It is physically obvious that this quantity is invariant from any viewpoint – i.e., it is independent of the reference frame $\Pi\Lambda$ in question, and it is constant in the course of its motion, in such a way that during the time interval in which one described its motion:

$$\boxed{\frac{dJ}{dt} = 0.}$$

Upon remarking that:

$$J = \frac{1}{ic}\int_\Sigma j_4 \, d\upsilon ,$$

one will find general proof of the tensorial character and constancy in time of such integrals in Appendix *A*, which is a proof that is necessitated by the condition $\partial_\mu j_\mu = 0$.

One recalls that Møller's theorem is necessitated from the outset by the condition that the integral $\int j_\mu \, d\sigma_\mu$ should be zero on the boundary that is swept out by the surface of the drop. That will immediately imply the fact that $d\sigma_\mu$ is orthogonal to the space-time velocity at each point on the boundary; i.e., $j_\mu \, d\sigma_\mu = 0$.

One will likewise see that the derivative:

$$\frac{dJ}{dt} = \int_S (j_4 V_k - ic \, j_k) \, ds_k$$

is zero in an arbitrary reference frame. Indeed, if one evaluates the scalar differential element $(j_4 \, V_k - ic \, j_k) \, ds_k$ in the local proper reference frame of the area element ds_k (as one has every right to do) then one will see that:

$$V_k^0 = 0 \qquad \text{and} \qquad j_k^0 = 0,$$

and the term in parentheses will be zero.

2) The integrals:

$$\boxed{J \, \eta_k = \int_\Sigma j_\otimes x_k \, d\upsilon}$$

define the coordinates η_k of a point on the hyperplane Π and resemble the classical formulas that define the barycenter in non-relativistic physics. (If one stipulates that the spatial coordinates x_k of the present point must be assumed to be referred to a triad *that is situated in the hyperplane* Π then coordinates x_k will, of course, become uniform, since the integration is performed with constant time.) However, the present case is much more complicated than that of the barycenter, since the point η_k is not unique; it will change when we consider another reference frame $\Pi'\Lambda'$. For that reason, we shall call that point the *pseudo-center of matter relative to the hyperplane* Π, and we stress that one will not have the right to speak of a pseudo-center unless one has specified the hyperplane to which it is associated.

At first, we shall proceed physically with an example. Suppose that the fluid mass in question is a sphere of homogeneous composition that is centered at *O* at a given moment

t (when one follows a slice $x_4 = ic\, t$ that is parallel to the hyperplane Π) in the reference frame $\Pi\Lambda$ and relative to the simultaneity that is defined by this particular reference frame, and suppose that this sphere expands in the course of time. Of course, the pseudo-center of matter will be the point O in the reference $\Pi\Lambda$. However, if we consider another reference frame $\Pi'\Lambda'$ then the points that are considered to be simultaneous will longer be the same ones. For example, the points to the right of Π' will be considered to be an instant that is previous to t (i.e., since the sphere is expanding, they will be closer to O than in the preceding calculation), so the points to the left, which are considered to be an instant that is later than t, will be more distant from O. The pseudo-center of matter will no longer be O, but will be translated to the left.

In order to show the same thing mathematically, we first remark that the three integrals:

$$J\, j_k = \int_\Sigma j_\otimes x_k \, d\upsilon$$

are performed by setting each x_k equal to a constant, and that the point thus-obtained will obviously have $\eta_k = x_k$ for its fourth component. It will then result that if we define the fourth integral:

$$\int_\Sigma j_\otimes x_4 \, d\upsilon = \int_\Sigma j_\otimes x_4 \, d\upsilon$$

in an analogous fashion then we will find that:

$$\eta_4 \int_\Sigma j_\otimes x_4 \, d\upsilon = \eta_4\, J.$$

One can thus define the pseudo-center of matter relative to Π by its *space-time* coordinates:

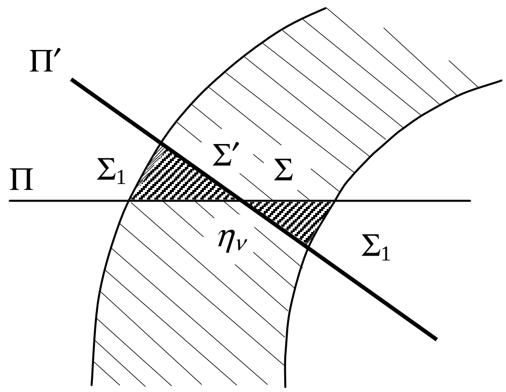
(II.1) $$J\, \eta_\mu = \int_\Sigma j_\otimes x_\mu \, d\upsilon .$$

η_μ will then represent the *space-time* vector that defines the pseudo-center of matter *that is attached to the hyperplane* Π in the course of its motion as a space-time point, and in any system of axes. By contrast, the right-hand side of the last equation does not represent a vector (we shall prove that much, at least), and the equality (1) will not be valid in the system $\Pi\Lambda$; we shall be inspired by Møller's method (see Appendix *A*).

Cut the tube with another hyperplane Π' that defines another Lorentz reference frame $\Pi\Lambda'$. In that new reference frame, the same conditions will define a space-time point η'_μ that will be the pseudo-center of matter *relative to the hyperplane* Π':

$$J\, \eta'_\mu = \int_\Sigma j_\otimes x_\mu \, d\upsilon .$$

We must consider the two points η_ν and η'_ν at the same instant t, which – if it will not confuse us – will translate into the condition $\eta_4 = \eta'_4$ *in an arbitrary reference frame*, which is a condition that restricts the choice of the hyperplane Π'. For example, if we know the point η_ν relative to the system $\Pi\Lambda$ then we will force the hyperplane Π' to pass through it. The hyperplanes Π and Π', along with the boundary Σ_1 that is swept out by the surface of the drop, will therefore delimit a certain hyper-volume that we shall call Ω.



Now, consider the tensor $j_\mu\, x_\nu$. We compute its divergence:

$$\partial_\mu\,(j_\mu\, x_\nu) = x_\nu\, \partial_\mu\, j_\mu + j_\mu\, \partial_\mu\, x_\nu\,.$$

The first term is zero, since $\partial_\mu\, j_\mu = 0$.

The second one will become $j_\mu\, \delta_{\mu\nu} = j_\nu$.

Now, consider a *uniform and arbitrary* vector field k_ν, and scalar multiply k_ν by the two derived expressions $k_\nu\, j_\nu = \partial_\mu\,(k_\mu\, j_\nu\, x_\nu)$, since k_ν is uniform.

Multiply this by the hyper-volume element $d\omega$ and integrate the result over the domain Ω that we just defined:

$$\text{(II.2)} \qquad \int_\Omega k_\nu\, j_\nu\, d\omega = \int_\Omega \partial_\mu (k_\nu\, j_\mu x_\nu)\, d\omega\,.$$

By applying Gauss's theorem, the second integral can be transformed into a hypersurface integral that we split into three parts that relate to the three hypersurfaces Σ, Σ, and Σ_1:

$$\int_\Omega k_\nu\, j_\nu\, d\omega = \int_\Sigma k_\nu\, j_\mu x_\nu\, d\sigma_\mu + \int_{\Sigma'} k_\nu\, j_\mu x_\nu\, d\sigma_\mu + \int_{\Sigma_1} k_\nu\, j_\mu x_\nu\, d\sigma_\mu\,.$$

The hypersurface integral on Σ_1 will be zero, since $j_\mu\, d\sigma_\mu = 0$. On the other hand, since the quantities k_ν, j_μ, x_ν, and $d\sigma_\mu$ are vectors, every differential element $k_\nu\, j_\mu\, x_\nu\, d\sigma_\mu$ will be a scalar that keeps the same value in any reference frame to which one refers the four vectors that comprise it.

Refer the vectors that are contained in the first surface integral to the reference frame $\Pi\Lambda$. The only non-zero component of $d\sigma_\mu$ will be the time component $d\sigma_4 = ic\; d\upsilon$, and one will have:

$$\left(\int_\Sigma k_\nu\, j_\mu x_\nu\, d\sigma_\mu\right)_{\Pi\Lambda} = \left(k_\nu \int_\Sigma j_4 x_\nu\, ic\, d\upsilon\right)_{\Pi\Lambda};$$

i.e.:

$$ic\left(k_\nu \int_\Sigma j_4 x_\nu \, d\upsilon\right)_{\Pi\Lambda} = ic\,(k_\nu J\,\eta_\nu)_{\Pi\Lambda}\,,$$

since one will find that it belongs to the reference frame $\Pi\Lambda$.

Similarly, if we evaluate the second surface integral in the reference frame $\Pi'\Lambda'$ then, upon remarking that in that case the only non-zero component of $d\sigma_\mu$ will be $d\sigma_4 = -\,ic\,d\upsilon$, the integral will become:

$$-\,ic\,\left(k_\nu \int_{\Sigma'} j_4 x_\nu \, d\upsilon\right)_{\Pi'\Lambda'} = -\,ic\,(k_\nu J\,\eta'_\mu)_{\Pi'\Lambda'}\,;$$

k_ν and η'_μ are, of course, evaluated in the reference $\Pi'\Lambda'$, this time.

The right-hand side of equation (2) will then reduce to the difference:

$$ic\,J\,[(k_\nu\,\eta_\nu)_{\Pi\Lambda} - (k_\nu\,\eta'_\mu)_{\Pi'\Lambda'}]\,.$$

If the point under scrutiny is independent of the chosen hyperplane then η_ν and η'_μ will represent the components of the *same* vector in the two references frames $\Pi\Lambda$ and $\Pi'\Lambda'$, and as k_μ is also a vector, the expression $k_\mu\,\eta_\mu$, which is the scalar product of two vectors, will be independent of the reference frame that is used for the evaluation:

$$(k_\nu\,\eta_\nu)_{\Pi\Lambda} = (k_\nu\,\eta'_\mu)_{\Pi'\Lambda'}\,.$$

The condition of the invariance of the point η_ν is therefore that the right-hand side of equation (2) must be zero for any k_ν :

$$\int_\Omega k_\nu\, j_\nu \, d\omega \;=\; k_\nu \int_\Omega j_\nu d\omega\,.$$

However, this time the integral will be a *vector*, and the condition that its scalar product with any *arbitrary* vector k_ν must be zero will become the condition that the vector itself must be zero:

$$\int_\Omega j_\nu d\omega \;=0.$$

This will produce four independent equations, while we have introduced two conditions:

$$\eta_4 = \eta'_4 \qquad\text{and}\qquad \partial_\mu\, j_\mu = 0.$$

Therefore, the requisite condition is satisfied, in general, and the point η_ν will depend upon the reference frame to which we have referred the fluid mass.

We shall now study the spatial velocity of an arbitrary pseudo-center of matter relative to its defining reference frame.

If $x_4 = ic\,t$ is the fourth coordinate, which is measured along the Λ axis, then the spatial velocity of the point whose spatial coordinates η_k in the hyperplane Π is:

$$V_k = \frac{d\eta_k}{dt} = ic\,\frac{d\eta_k}{dx_4}.$$

This gives:

$$J\,V_k = ic\,\frac{d}{dx_4}(J\eta_k), \quad \text{since} \quad \frac{dJ}{dt} = 0.$$

Upon taking the relation that defines η_k into account and recalling that the domain of integration varies in the course of time, it will result that:

$$J\,V_k = ic\frac{d}{dx_4}\left(\frac{1}{ic}\int_{\upsilon_0} j_4 x_k\, d\upsilon^0\right) = \int_\Sigma \partial_4 (j_4 x_k)\, d\upsilon + \int_S \frac{1}{ic}\, j_4 x_k\, \upsilon_i\, ds_i\,,$$

in which S is the surface of the drop in the reference frame Σ.

Since x_4 and x_k are independent variables, one will have $\partial_4\,(j_4\,x_k) = x_k\,\partial_4\,j_4$, and upon taking the conservation relation $\partial_\mu\,j_\mu = 0$ into account, one will get: $\partial_4\,j_4 = -\,\partial_i\,j_i$.

Finally, upon integrating by parts:

$$x_k\,\partial_4\,j_4 = -\,x_k\,\partial_i\,j_i = -\,(x_k\,j_i) + j_i\,\delta_{ik}\,,$$

so one will get:

$$J\,V_k = -\int_\Sigma \partial_i (x_k\, j_i)\, d\upsilon + \int_\Sigma j_k\, d\upsilon + \int_S j_\otimes\, x_k V_i\, ds_i\,.$$

The first term defines a surface integral that will give:

$$\int_S (j_\otimes V_i - j_i)\, x_k\, ds_i$$

when it is combined with the former one, which will be zero, since $j_i = j_\otimes\,V_i$. It will then result that:

$$J\,V_k = \int_\Sigma j_k\, d\upsilon\,.$$

Therefore, the velocity of the pseudo-center of matter in its defining reference frame will be:

$$\boxed{V_k = \frac{1}{J}\int_\Sigma j_k\, d\upsilon.}\,.$$

These three expressions, which behave like the components of a spatial vector relative to a rotation of the spatial axes of the hyperplane Π, will, on the contrary, depend upon the choice of hyperplane Π in a complicated manner, since under a change of hyperplane that corresponds to a Lorentz transformation every j_k will transform like the spatial components of a quadri-vector, but the integration will no longer be performed over the same domain, moreover. One will then be concerned with the j_k that are found at different points, and it will generally be impossible to specify the dependency of V_k upon the choice of hyperplane Π without knowing all of the details of the motion of the fluid.

§ 3. The energy-momentum density and the moment of rotation. – We shall now introduce some new quantities that relate to the *dynamical* laws that govern the motion of the fluid and succeed in determining it, as well.

One knows that relativistic hydrodynamics expresses the properties of a classical fluid by means of the tensor that represents the total energy-momentum density and is composed of two terms:

$$\boxed{t_{\mu\nu} = \rho m_0\, u_\mu u_\nu + \theta_{\mu\nu}} \qquad \text{(see Appendix } B\text{)}.$$

ρ and u_μ have the same meanings as before, m_0 is the individual proper mass of the molecules, which are assumed to be identical, and $\theta_{\mu\nu}$ is the internal stress tensor, which will be a proper space tensor (i.e., $\theta_{\mu\nu}\, u_\nu = \theta_{\mu\nu}\, u_\mu = 0$) and will play the role of a potential for the force density $f_\mu = -\partial_\nu\, \theta_{\mu\nu}$.

When referred to the local proper system, the components of the energy-momentum tensor will be:

$$t_{ij}^0 = \theta_{ij}^0 ,$$

respectively (i.e., the internal stress tensor that is usually defined in non-relativistic dynamics):

$$t_{i\otimes}^0 = t_{\otimes i}^0 = 0 \qquad \text{and} \qquad t_{\otimes\otimes}^0 = \rho\, m_0$$

(i.e., the proper mass density). Thus, in relativistic form, the fundamental law of hydrodynamics is expressed by the *conservative* character of the energy-momentum tensor:

$$\boxed{\partial_\mu t_{\mu\nu} = 0,}$$

namely:

$$\partial_\nu\, (\rho\, m_0\, u_\mu\, u_\nu) = -\,\partial_\nu\, \theta_{\mu\nu} .$$

Indeed, if we remark that the product $\rho\, m_0\, u_\mu$ that appears in the left-hand side is nothing but the relativistic momentum density g_0, and if, on the other hand, we involve the force of stress per unit volume $f_\mu = -\,\partial_\nu\, \theta_{\mu\nu}$ then, conforming to the usual notions, it will happen that:

$$\partial_\nu\, (g_\mu\, u_\nu) = \dot{g}_\mu = f_\mu \qquad \text{(see Appendix } A\text{)}.$$

If we multiply this by the hyper-volume element $d\omega$ and integrate the result over the hyper-tube that is swept out by the infinitesimal droplet in the proper time interval $d\tau$ then we will get:

$$\int_\omega \dot{g}_\mu\, d\omega = \int_\omega f_\mu\, d\omega .$$

However, one has (Appendix A) that if one introduces the total momentum of the droplet:

$$G_\mu = \int_{V_0} g_\mu \, d\upsilon^0$$

then one can write:

$$\int_\omega \dot{g}_\mu \, d\omega = \left(\frac{d}{d\tau} G_\mu \right) d\tau .$$

On the other hand, if one makes the time interval sufficiently small then one will likewise have:

$$\int_\omega f_\mu \, d\omega = d\tau \int_{V_0} f_\mu \, d\upsilon^0 = d\tau F_\mu ,$$

if we let F_μ denote the relativistic force that acts upon the droplet due to the stresses in the surrounding fluid:

$$F_\mu = \int_{V_0} f_\mu \, d\upsilon^0 .$$

Moreover, if the integration is performed over the hyper-tube in question then the equation of conservation $\partial_\mu t_{\mu\nu} = 0$ will translate into simply the relation:

$$\boxed{\dot{G}_\mu = F_\mu ;}$$

i.e., the *relativistic theorem for the quantity of motion* of the droplet.

On the other hand, the stresses that are introduced by classical dynamics are assumed to be derived from a *symmetric* tensor $\theta_{\mu\nu}$, and since the term $\rho\, m_0\, u_\mu\, u_\nu$ is likewise symmetric, the classical theory considers only symmetric energy-momentum tensors. It is easy to see that this hypothesis will lead to the introduction of a new *conservative* tensor, namely, the *rotational moment density:*

$$\boxed{m_{[\mu\nu]\lambda} = x_\mu t_{\nu\lambda} - x_\nu t_{\mu\lambda} ,}$$

which will be of order three and antisymmetric in μ and ν. Indeed, its divergence is:

$$\partial_\lambda\, m_{\mu\nu\lambda} = x_\mu\, \partial_\lambda\, t_{\nu\lambda} - x_\nu\, \partial_\lambda\, t_{\nu\lambda} + \delta_{\mu\lambda}\, t_{\nu\lambda} - \delta_{\nu\lambda}\, t_{\mu\lambda} ,$$

The first two terms will be zero, since $t_{\mu\nu}$ is conservative. The last two terms simply express the antisymmetric part $t_{\nu\mu} - t_{\mu\nu}$ of the energy-momentum tensor. If it is symmetric then the angular momentum density will be conservative, and vice versa.

One can then exhibit the dynamical significance of the symmetric character of $t_{\mu\nu}$ by following the same path that we did just now. Upon replacing $t_{\mu\nu}$ with its expression, the relation:

$$\boxed{\partial_\lambda m_{\mu\nu\lambda} = 0}$$

will give:

$$\partial_\lambda (x_\mu g_\nu u_\lambda - x_\mu g_\nu u_\lambda) = - \partial_\lambda (x_\mu \theta_{\nu\lambda} - x_\nu \theta_{\mu\lambda}) .$$

The left-hand side of this is the derivative of the expression $n_{[\mu\nu]} = x_\mu g_\nu - x_\mu g_\nu$ along the streamline; i.e., the moment of momentum g_μ .

Upon taking into account the symmetric character of the stress tensor $\theta_{\mu\nu}$, the right-hand side will reduce to:

$$x_\mu \partial_\lambda (-\theta_{\nu\lambda}) - x_\nu \partial_\lambda (-\theta_{\mu\lambda});$$

i.e., to:

$$x_\mu f_\nu - x_\mu f_\nu = \gamma_{\mu\nu} ,$$

which is the internal stress dipole moment per unit volume.

One will then have:

$$\dot{n}_{\mu\nu} = \gamma_{\mu\nu} .$$

Finally, if one integrates over the hyper-tube element, as we did just recently, then if one sets:

$$\int_{V_0} n_{\mu\nu} d\upsilon_0 = M_{\mu\nu}$$

(viz., the internal angular momentum of the droplet) and:

$$\int_{V_0} \gamma_{\mu\nu} d\upsilon_0 = \Gamma_{\mu\nu}$$

(viz., the total dipole moment of torsion) then the condition $t_{\mu\nu} = t_{\nu\mu}$ (or $\partial_\lambda m_{\mu\nu\lambda} = 0$) will be equivalent to:

$$\boxed{\dot{M}_{\mu\nu} = \Gamma_{\mu\nu};}$$

i.e., to the *relativistic theorem of the kinetic moment*, as it is applies to the droplet.

In order to define the dynamical properties of the fictitious fluid that we are in the process of studying, the example of classical relativistic hydrodynamics will lead us to introduce an energy-momentum tensor $t_{\mu\nu}$ and its moment:

$$\boxed{m_{\mu\nu\lambda} = x_\mu t_{\nu\lambda} - x_\nu t_{\mu\lambda},}$$

but it is not useful to specify the complete expression for the tensor $t_{\mu\nu}$ as a function of the various kinds of energy that one deals with for material fluids.

We must be content to require (see Appendix *B*) that:

1) The tensor $t_{\mu\nu}$ is conservative:

$$\boxed{\partial_\nu t_{\mu\nu} = 0.}$$

2) It admits u_μ as a proper vector:

$$t_{\mu\nu} u_\nu = K u_\mu .$$

3) It is symmetric:

$$t_{\mu\nu} = t_{\nu\mu} ,$$

or – what amounts to the same thing – that the tensor $m_{\mu\nu\lambda}$ is also conservative:

$$\boxed{\partial_\lambda m_{\mu\nu\lambda} = 0.}$$

We remember that this hypothesis expresses (in a general form that is still classical) the two fundamental laws of dynamics – viz., the conservation of the quantity of motion and the conservation of the kinetic moment – which, with the hypothesis that was given to begin with that:

$$\boxed{\partial_\mu j_\mu = 0,}$$

which translates into the conservation law of matter, and we thus succeed in characterizing our fluid entirely, as well as characterizing it as a *classical* fluid.

§ 4. Total momentum. – We have seen how one can represent the quantity of motion of an *infinitesimal* droplet by the vector:

$$G_\mu = \int_{V_0} g_\mu \, d\upsilon_0 .$$

We shall seek to extend this concept to our *macroscopic* mass. First, we remark that one can always define G_μ by the integral:

$$G_\mu = -\frac{1}{c^2}\int_{V_0} t_{\mu\nu} u_\nu \, d\upsilon_0 ,$$

for that matter. Indeed, if we take the expression for $t_{\mu\nu}$ into account then we will get:

$$G_\mu = \int_{V_0} \left(-\frac{1}{c^2} g_\mu u_\nu u_\nu - \frac{1}{c^2} \theta_{\mu\nu} u_\nu \right) d\upsilon_0 .$$

The second term will be zero since $\theta_{\mu\nu}$ belongs to proper space, and since $u_\mu \, u_\mu = - c^2$, it will result that:

$$G_\mu = \int_{V_0} g_\mu \, d\upsilon_0 ,$$

which is precisely the expression that was given above. Since we shall no longer specify the expression for $t_{\mu\nu}$ from now on, we shall start with the form:

$$G_\mu = -\frac{1}{c^2}\int_{V_0} t_{\mu\nu} u_\nu \, d\upsilon_0 .$$

However, one will encounter difficulties when one tries to generalize this form to the case of a macroscopic fluid mass. Indeed, it is obtained by cutting the hyper-tube that is swept out by the droplet with a particular hyperplane that is orthogonal to the current in such a way that one integrates over the *proper* volume of the droplet. That will no longer be possible in the case of a macroscopic fluid mass. As we did in relation to kinematic magnitudes, one must cut the macroscopic hyper-tube with a hyperplane Π and integrate over the volume Σ that it determines, but that hyperplane might not be orthogonal to the vector u_μ at all points, since it varies from point to point. Thus, we will seek a different definition to associate with that of the infinitesimal droplet. In order to do that, we remark that in the infinitesimal case, if one calculates G_μ in the *proper* reference frame then the differential element will contain only the terms:

$$t^0_{\mu 4}\, u^0_4 = ic\; t^0_{\mu 4}\,,$$

so:

$$G^0_\mu = -\frac{ic}{c^2}\int_{\Sigma_0} t^0_{\mu 4}\, d\upsilon^0 = \int_{\Sigma_0} t^0_{\mu\otimes}\, d\upsilon^0 \;\;.$$

This expression can be generalized to the case of a macroscopic fluid mass under the condition that one must consider the intersection Σ of the hyper-tube with a hyperplane Π to have an *arbitrary* space-like nature. That hyperplane and the orthogonal axis Λ will define a Lorentz frame, and we can write the components of the total quantity of motion (or momentum) *relative to that same reference frame* as:

$$\boxed{G_\mu = \int_\Sigma t_{\mu\otimes}\, d\upsilon,}$$

in which $d\upsilon$ is the volume element of the hyperplane Π.

We see that G_μ takes the form of a volume integral of the fourth components of a conservative tensor. One can attempt to apply Møller's theorem to it. In order to do that, one must seek the conditions under which the hyper-boundary integral $k_\mu \int_\Sigma t_{\mu\nu} d\sigma_\nu$ will be annulled. We shall study the scalar differential element $d\alpha = k_\mu\, t_{\mu\nu}\, d\sigma_\nu$ over a small domain of the hyper-boundary, and we shall choose a particular system of axes, namely, we shall place ourselves in the local proper system that is defined by the element of the surface considered. If one is given that $d\sigma_\nu$ is orthogonal to the current then one will have that $d\sigma_4 = 0$ in the proper system, and one will have to consider only the proper space components of $t_{\mu\nu}$, which will be identical with those of the internal stress tensor:

$$d\alpha = k^0_i\, \theta^0_{ij}\, d\sigma^0_j\,.$$

We refer proper space to two axes $x_k^{(1)}$, $x_k^{(2)}$ that are tangent to the surface of the drop, while the third one $x_k^{(3)}$ is collinear with the proper surface element $ds_k^{(0)}$. One will then have:

$$d\sigma_k^{(0)} = \frac{i}{c}\varepsilon_{ij4k}\, dx_i^{(1)}\, dx_j^{(2)}\, ic\, d\tau$$
$$= \varepsilon_{ij4k}\, dx_i^{(1)}\, dx_j^{(2)}\, d\tau = ds_k^{(0)}\, d\tau,$$

so

$$d\alpha = k_i^0\, \theta_{ij}^0\, ds_j^0\, d\tau .$$

In order for that integral to be zero for any form of the drop and any choice of k_μ, one must have that $\theta_{ij}^0\, ds_j^0$ is zero at each point, or that it goes to zero on all of the surface due to some special force that manifests itself on just the surface. Now, in reality, there exist forces of *surface tension* for material fluids that are endowed with such forces, and it seems very physically reasonable to introduce a supplementary hypothesis that would endow the surface of our drop with the property that gives rise to the phenomenon of surface tension. One sees, moreover, that the quantity $\theta_{ij}\, ds_j$ (we shall drop the superscript 0 from now on) simply represents the force that is developed by the internal stresses over the surface element ds_j .

It can be shown that it is always possible to equilibrate that force at each point by means of suitable surface tensions. We will give the proof of this (*), which will necessitate using the formalism of Riemannian geometry. We keep the Latin indices i = 1, 2, 3 in order to denote the spatial coordinates, and use Greek indices α = 1, 2 in order to denote the coordinates on the surface S.

We consider a system of curvilinear coordinates $u^i = u^\alpha, u^3$, whose linear coordinates are mutually-orthogonal and tangent to the three local axes $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ that we just defined at each point. The surface S will be represented by an equation u^3 = const.

The linear element that is $ds^2 = \delta_{ij}\, dx^i\, dx^j$ (since the space is Euclidian) in an arbitrary system of Cartesian axes x^i can be written $ds^2 = g_{ij}\, dx^i\, dx^j$, and if one is given a choice of a coordinate system then the metric tensor will be reduced to its diagonal components g_{11}, g_{22}, g_{33} , while the contravariant components will be:

$$g^{11} = \frac{1}{g_{11}}, \qquad g^{22} = \frac{1}{g_{22}}, \qquad g^{33} = \frac{1}{g_{33}}.$$

If one restricts oneself to displacements that leave u^3 constant then one can define a metric on the surface S by $ds^2 = \gamma_{\alpha\beta}\, du^\alpha\, du^\beta$, and it will become obvious that the $\gamma_{\alpha\beta}$ are identical with the $g_{\alpha\beta}$. One can further define the components of a metric connection on the surface S by:

$$\begin{Bmatrix} \rho \\ \alpha\ \beta \end{Bmatrix} = \frac{1}{2} g^{\rho\lambda}\left[\frac{\partial g_{\alpha\lambda}}{\partial u^\beta} + \frac{\partial g_{\beta\lambda}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\lambda}\right].$$

These components are the same as the analogous components in space. Indeed, in the latter case, one will have to consider a term $g^{\rho 3}$, as well, but it will be zero, since $g^{\rho 3} = 0$.

(*) Whose essential details come from a suggestion of Francis Fer.

In order to simplify the notation, we have located quantities that are themselves spatial *vectors* in the Riemannian manifold *S*, and we shall preserve the vectorial notation for them. Therefore, if we connect the current spatial point *M* in space to an arbitrary origin *O* and set:

$$\mathbf{e}_i = \frac{\partial OM}{\partial u_i}$$

then we will define three orthogonal vectors in Euclidean space that are situated along the axes of the local coordinates (with $\mathbf{e}_3$ along the normal to the surface). One can choose the parameterization of the u_i in such a way that the vectors will have unit length.

Having said that, suppose that we make an incision du^1 into the surface. In order to prevent the lips from separating, we must act upon it with two equal and opposite forces, which will be represented by the vector $\mathbf{t}_1\, du^1$, with a suitable orientation. Similarly, for an incision du^2 one will have a force $\mathbf{t}_2\, du^2$, and for an arbitrary incision du^1, du^2, a force $\mathbf{t}_\beta\, du^\beta$ (which is summed over β). The set of to linear densities ($\mathbf{t}_1$, $\mathbf{t}_2$) will constitute a *surface tension*. On the other hand, consider the force:

$$\mathbf{F}\, ds\sqrt{\gamma}\,,$$

which is the vectorial representation of the force $\theta_{ij}\, d\sigma_k$ that acts upon the surface by way of internal tensions. (Since one is dealing with Riemannian geometry, one must introduce $\sqrt{\gamma}$, where $\gamma = \|\, \gamma_{\alpha\beta} \,\| = g_{11}\, g_{22}$.)

We must now show that it is possible, in general, to determine a surface tension that equilibrates the action of the internal stresses.

Consider a portion *S* of the surface that is bounded by a contour *C*. Equilibrium will be expressed by the two equations:

$$\int_C \mathbf{t}_\beta du^\beta + \int_S \mathbf{F}\sqrt{\gamma} ds = 0,$$

$$\int_C (\mathbf{OM}\times\mathbf{t}_\beta)\, du^\beta + \int_C (\mathbf{OM}\times\mathbf{F})\sqrt{\gamma}\, ds = 0;$$

i.e., by virtue of Stokes's theorem:

(II.3)
$$\frac{\partial \mathbf{t}_2}{\partial u^1} - \frac{\partial \mathbf{t}_1}{\partial u^2} + \mathbf{F}\sqrt{\gamma} = 0,$$

(II.4)
$$\frac{\partial \mathbf{OM}}{\partial u^1}\times\mathbf{t}_2 - \frac{\partial \mathbf{OM}}{\partial u^2}\times\mathbf{t}_1 \equiv \mathbf{e}_1\times\mathbf{t}_2 - \mathbf{e}_2\times\mathbf{t}_1 = 0.$$

We decompose each of the vectors $\mathbf{t}_\alpha$ along the three axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{t}_1 = n_1\, \mathbf{e}_1 + \tau_1\, \mathbf{e}_2 + \alpha_1\, \mathbf{e}_3,$$
$$\mathbf{t}_2 = \tau_2\, \mathbf{e}_1 + n_2\, \mathbf{e}_2 + \alpha_2\, \mathbf{e}_3\,.$$

Upon substituting these expressions into equation (4), it will follow that:

$$\mathbf{e}_1 \times n_2\, \mathbf{e}_2 + \mathbf{e}_1 \times \alpha_2\, \mathbf{e}_3 - \mathbf{e}_2 \times n_1\, \mathbf{e}_1 - \mathbf{e}_2 \times \alpha_1\, \mathbf{e}_3 = 0\,.$$

Since one has:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3\,, \qquad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1\,, \qquad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2\,,$$

it will remain that:

$$(n_1 + n_2)\, \mathbf{e}_3 - \alpha_2\, \mathbf{e}_2 - \alpha_1\, \mathbf{e}_1 = 0,$$

so:

$$\alpha_1 = \alpha_2 = 0, \qquad\qquad n_1 + n_2 = 0.$$

It then results from the torque equation that the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are in the plane that is tangent to the surface. One can express that by:

$$\mathbf{t}_\beta = t_\beta^\alpha\, \mathbf{e}_\alpha \qquad \text{with} \qquad t_1^1 = -\, t_2^2\,.$$

Equation (3) will then lead to:

$$\frac{\partial}{\partial u^1}(t_2^\alpha \mathbf{e}_\alpha) - \frac{\partial}{\partial u^2}(t_1^\alpha \mathbf{e}_\alpha) + \mathbf{F}\sqrt{\gamma} = 0.$$

Now, since the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ have unit length, their covariant derivatives (in the three-dimensional metric) will be zero:

$$\nabla_\beta\, \mathbf{e}_\alpha \equiv \partial_\beta\, \mathbf{e}_\alpha - \begin{Bmatrix} i \\ \alpha\,\beta \end{Bmatrix} \mathbf{e}_i = 0,$$

so:

$$\partial_\beta\, \mathbf{e}_\alpha = \begin{Bmatrix} i \\ \alpha\,\beta \end{Bmatrix} \mathbf{e}_i\,.$$

One will then have:

$$\left(\frac{\partial t_2^\alpha}{\partial u^1} - \frac{\partial t_1^\alpha}{\partial u^2}\right)\mathbf{e}_\alpha + t_2^\alpha \begin{Bmatrix} i \\ \alpha\,1 \end{Bmatrix}\mathbf{e}_i - t_1^\alpha \begin{Bmatrix} i \\ \alpha\,2 \end{Bmatrix}\mathbf{e}_i + F^i\sqrt{\gamma}\,\mathbf{e}_i = 0,$$

hence:

(II.5) $$\frac{\partial t_2^\alpha}{\partial u^1} - \frac{\partial t_1^\alpha}{\partial u^2} + t_2^\beta \begin{Bmatrix} \alpha\,1 \\ \beta \end{Bmatrix} - t_1^\beta \begin{Bmatrix} \alpha\,2 \\ \beta \end{Bmatrix} + F^\alpha\sqrt{\gamma} = 0$$

along the $\mathbf{e}_\alpha$ axes and:

$$t_2^\beta \begin{Bmatrix} 3 \\ \beta\,1 \end{Bmatrix} - t_1^\beta \begin{Bmatrix} 3 \\ \beta\,2 \end{Bmatrix} + F^3\sqrt{\gamma} = 0$$

along the $\mathbf{e}_3$ axis.

If one adds the quantity:

$$t_\lambda^\alpha \begin{Bmatrix} \lambda \\ 21 \end{Bmatrix} = t_\lambda^\alpha \begin{Bmatrix} \lambda \\ 12 \end{Bmatrix}$$

to equation (5) and simplifies then one will bring about the covariant derivative:

$$\nabla_1 t_2^\alpha = \frac{\partial t_2^\alpha}{\partial u^1} + t_2^\beta \begin{Bmatrix} \alpha \\ \beta\, 1 \end{Bmatrix} - t_\lambda^\alpha \begin{Bmatrix} \lambda \\ 2\,1 \end{Bmatrix}$$

that relates to the *two*-dimensional metric.

One will then have the system:

$$\boxed{\begin{array}{ll} \nabla_1 t_2^\alpha - \nabla_2 t_1^\alpha + F^\alpha \sqrt{\gamma} = 0, & \text{(2 equations)} \\ t_2^\alpha \begin{Bmatrix} 3 \\ \alpha\, 1 \end{Bmatrix} - t_1^\alpha \begin{Bmatrix} 3 \\ \alpha\, 2 \end{Bmatrix} + F^3 \sqrt{\gamma} = 0, & \text{(1 equation)} \end{array}}$$

which are then three equations that determine the three quantities:

$$t_1^1 = t_2^2,\ t_2^1,\ t_1^2,$$

when one knows the coefficients of the connection on the surface (which results from knowing the form that is assumed by the surface) and the components F^i of the force that is produced by the internal stresses in the surface.

It is then possible to pose the problem in full generality, and one can assume the existence of surfaces tensions on the surface of the drop that equilibrate the internal stresses at each point.

Naturally, one assumes that these surface tensions are included in the expression for the energy-momentum tensor, along with the internal stresses, and consequently, in the vector G_μ. It results from their presence that the integral $\int k_\mu t_{\mu\nu} d\sigma_\nu$ will be zero when it is taken over the hyper-boundary, and consequently, $k_\mu G_\mu$ will be invariant.

Likewise, the same hypotheses will permit us to prove that G_μ is constant in time. One knows (Appendix *A*) that one has:

$$\frac{d}{d\tau}(k_\mu G_\mu) = \int_S k_\mu (t_{\mu 4} v_k - ic\, t_{\mu k})\, d\sigma_k\,.$$

Since the differential element is a scalar, we may express this in the proper system. The first term is zero, since $v_k^0 = 0$.

It remains that:

$$-\,ic\ t_{\mu k}^0\ d\sigma_k^0 = -\,ic\ \theta_{\mu k}^0\ d\sigma_k^0\,,$$

and one will have to consider only the spatial terms:

$$\theta_{jk}^{0}\, d\sigma_{k}^{0}\ .$$

Now, one knows that this quantity is zero at each point of the surface when one takes surface tension into account. Thus, one can apply Møller's theory, which will show that, on the one hand, the total momentum G_μ transforms as a *vector* under a Lorentz transformation that simultaneously changes the coordinates axes and the hyperplane Π that defines the domain of integration, and on the other hand, that the vector G_μ is *constant* in time along any axis Λ along which time is defined:

$$\boxed{\frac{d}{dt} G_\mu = 0.}$$

We remark that the component:

$$G_\otimes = \int_\Sigma t_{\otimes\otimes}\, d\upsilon\,,$$

which is the integral of the mass density $t_{\otimes\otimes}$, represents the total mass (or energy) of the drop. As is well-known, that mass will vary with the choice of reference frame, since it is (up to *ic*) the fourth component of a quadri-vector. It then differs profoundly from the quantity with the analogous form:

$$J = \int_\Sigma j_\otimes\, d\upsilon$$

that we encountered in our kinematical study, and which represents the total *matter* that is contained in the drop, and which is, on the contrary, independent of the reference frame, since it is an invariant from the tensorial viewpoint. This is the first – but not the last – of the fundamental differences in variance between the kinematical and dynamical magnitudes.

As a result, one can likewise define an invariant upon starting with the momentum by simply taking the norm of the vector G_μ and setting:

$$G_\mu\, G_\mu = -M_0^2\, c^2\ .$$

(The – sign comes from the fact that G_μ is time-like.)

Since G_μ is constant in time, so is its square, and the quantity M_0 will represent an invariant and constant total proper mass of the drop. This quantity, which was introduced for the first time by Louis de Broglie [**16**], will be called the *proper mass of momentum*.

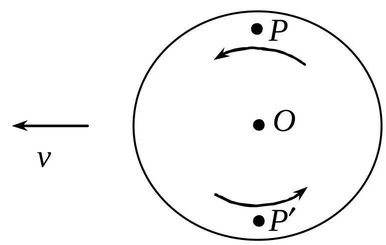
§ 5. The pseudo-center of mass. – In attempting to define a barycenter for the ensemble of the drop, we proceed by analogy with the kinematical study. We "weight" each point of the volume Σ with the mass density $t_{\otimes\otimes}$, and form the integrals:

$$\boxed{G_\otimes \xi_k = \int_\Sigma t_{\otimes\otimes}\, x_k\, d\upsilon} \qquad \text{with} \qquad G_\otimes = \int_\Sigma t_{\otimes\otimes}\, d\upsilon\,,$$

which define the coordinates relative the reference ΠΛ at a point ξ_k of the hyperplane Π that we call the *pseudo-center of mass*, because it is a point that is associated with the hyperplane Π and will change with it.

The fact that the pseudo-center of mass varies with the reference frame is well-known [**12**]. We shall show this by means of a simple physical example:

Consider a mass that spins about itself and appears to be a homogeneous sphere that spins about its fixed center *O* in a certain reference frame, at least, as far as its mass distribution is concerned. Obviously, the pseudo-center of mass will then be at *O*.



Next, consider another reference frame that moves to the left of the preceding one with a velocity of *v*. That velocity will add to the velocity of rotation of the points of the lower hemisphere (such as *P′*) and subtract from it for points of the upper hemisphere (such as *P*). If one is given the relativistic variation of the mass with velocity then it will result that the mass of the mower hemisphere will be greater in the new reference frame, and that the mass of the upper hemisphere will be smaller. From the geometric viewpoint, the two hemispheres will be transformed by the contraction into two halves of an ellipsoid of revolution that is flattened in the direction of *v*. It will thus remain geometrically symmetric, while becoming asymmetric as far as its mass is concerned. It will then result that in the new reference the pseudo-center of mass will not be found at the geometric center of the ellipsoid *O*, but will be shifted towards the bottom.

In order to make this mathematically precise, we first show that the defining formulas of the pseudo-center of mass relative to a certain hyperplane Π, in fact, characterize a quadri-vector. Indeed, the integral:

$$G_\otimes\, \xi_k = \int_\Sigma t_{\otimes\otimes} x_k \, d\upsilon$$

is taken with constant time: x_4 = constant = ξ_4 .

Thus, one can write an analogous formula:

$$G_\otimes\, \xi_4 = \xi_4 \int_\Sigma t_{\otimes\otimes}\, d\upsilon = \int_\Sigma t_{\otimes\otimes} \xi_4 \, d\upsilon = \int_\Sigma t_{\otimes\otimes} x_4 \, d\upsilon .$$

Thus, we will have the general formula:

$$G_\otimes\, \xi_\mu = \int_\Sigma t_{\otimes\otimes} x_\mu \, d\upsilon .$$

Therefore, the space-time vector ξ_μ defines the pseudo-center of mass *relative to the given hyperplane* Π intrinsically, and in any reference frame, but as we shall see, the right-hand side of the last equation is not a vector, so the equality will no longer be valid in the reference frame ΠΛ.

We remark that:

$$G_{\otimes}\ \xi_\mu = \frac{1}{ic}\ G_4\ \xi_\mu$$

has the form of the fourth component of a tensor. More precisely, one can consider the expression:

$$M_{\nu 4} = G_\nu\ \xi_4 - G_4\ \xi_\nu$$

to be the fourth component of a tensor $M_{\mu\nu}$, and later on we will confirm that it indeed relates to the total angular momentum:

$$M_{[\mu\nu]} = \int_\Sigma (x_\mu t_{\nu 4} - x_\nu t_{\mu 4})\, d\upsilon\,,$$

which is a tensor that is independent of the hyperplane, and which has the quantity $G_\nu\ \xi_4 - G_4\ \xi_\nu$ as its fourth component in every reference frame, where ξ_ν denotes the pseudo-center of mass *relative to the reference frame under consideration*.

Next, consider a hyperplane Π, the orthogonal axis Λ, and the pseudo-center of mass ξ_μ relative to Π. Choose a reference frame R^0 that has Λ for its time axis and is such that the axes will coincide with the point ξ_μ at time zero. Thus, one will need to have $\xi_k^0 = 0$ and $\xi_4^0 = 0$ in that coordinate system.

The component of the total angular momentum along the time axis Λ will be given by the general formula:

$$M_{k4}^0 = G_k^0\ \xi_4^0 - G_4^0\ \xi_k^0\,.$$

It will then be zero.

Now, perform a pure Lorentz transformation (with no rotation of the spatial axes), which we assume is infinitesimal. It will be expressed by:

$$x_k^1 = x_k^0 + \varepsilon_{k4} x_4^0\,, \qquad x_4^1 = x_4^0 + \varepsilon_{4k} x_k^0\,,$$

with $\varepsilon_{4k} = -\ \varepsilon_{k4}$.

This transformation will take us to another reference frame R^1 that simultaneously determines another hyperplane Π′ that intersects the hyper-tube in a cut Σ′ that will be different from Σ. A pseudo-center of mass will correspond to this cut, and if we place ourselves in the new reference from R^1 then the property that was pointed out for total angular momentum will result that one will have:

$$M_{k4}^1 = G_k^1 (\xi_4')^1 - G_4^1 (\xi_k')^1\,,$$

where M_{k4}^1, G_k^1, G_4^1 represent the *same* tensors as ever – viz., $M_{\mu\nu}$ and G_μ – relative to the new reference frame. On the other hand, if we look for the coordinates ξ_μ of the pseudo-center of mass relative to the old hyperplane Π in the reference frame R^1 then the Lorentz formulas will give us:

$$\xi_k^1 = \xi_k^0 + \varepsilon_{k4}\xi_4^0 = 0,$$
$$\xi_4^1 = \xi_4^0 + \varepsilon_{4k}\xi_k^0 = 0 .$$

It will then result from this that if the two pseudo-centers coincide – i.e., if one has:

$$(\xi'_k)^1 = (\xi_k)^1 = 0 \qquad \text{and} \qquad (\xi'_4)^1 = (\xi_4)^1 = 0$$

in the same reference R^1 then the fourth component of the total angular momentum, which we assumed was zero in the reference frame R^0, will be:

$$M_{k4}^1 = G_k^1\,\xi_4^1 - G_4^1\,\xi_k^1$$

here; i.e., it will also be zero in any reference frame R^1 that is obtained from R^0 by an arbitrary infinitesimal Lorentz transformation. That confirms what we saw qualitatively before: The variability of the pseudo-center of mass with the reference frame is connected with the rotation of the fluid mass. In general, every reference frame will have its own pseudo-center of mass for a rotating fluid mass, since will have its own special pseudo-center of matter. Moreover, it is obvious that the ways that the two pseudo-centers change under a change of reference frame will be profoundly different, since one of them involves the mass and its particular relativistic variance, while the other one involves only kinematical properties.

As we did for the pseudo-center of matter, we shall study the velocity of the pseudo-center of mass relative to its defining reference frame. We take the derivative of $G_\otimes\xi_k$ with respect to time and project it onto the axis Λ:

$$G_\otimes V_k = G_\otimes \frac{d}{dt}\xi_k = \frac{d}{dt}(G_\otimes\xi_k),$$

since the derivative of the components of G_μ with respect to nothing but time are zero.

Since:

$$G_\otimes\xi_k = \int t_{\otimes\otimes}x_k\,d\upsilon,$$

one will have:

$$G_\otimes V_k = \frac{d}{dt}\int_\Sigma t_{\otimes\otimes}x_k\,d\upsilon = ic\int_\Sigma \partial_4(t_{\otimes\otimes}x_k)\,d\upsilon = ic\int_\Sigma x_k\partial_4 t_{\otimes\otimes}\,d\upsilon = \int_\Sigma x_k\partial_4 t_{\otimes 4}\,d\upsilon .$$

Since:

$$\partial_\mu t_{\otimes\mu} = 0 = \partial_l t_{\otimes l} + \partial_4 t_{\otimes 4},$$

one will get:

$$G_\otimes V_k = -\int_\Sigma x_l\,\partial_4 t_{\otimes l}\,d\upsilon .$$

One can integrate this by parts and suppress the part of the integral that is taken over a space-like surface that is contained entirely within the vacuum:

$$G_\otimes V_k = \int_\Sigma t_{\otimes l} x_l \, d\upsilon = \int_\Sigma t_{\otimes l} \delta_{lk} \, d\upsilon = \int_\Sigma t_{\otimes k} \, d\upsilon = \int_\Sigma t_{k\otimes} \, d\upsilon$$

(since $t_{\mu\nu}$ is symmetric) or finally:

$$\boxed{G_\otimes V_k = G_k.}$$

The spatial velocity of the pseudo-center of mass relative to its defining reference frame is then given by:

$$V_k = \frac{G_k}{G_\otimes}.$$

Since G_k and $G_\otimes$ are constant in time, V_k will also be *constant*.

One sees that in every reference frame the pseudo-center of mass enjoys the fundamental property that characterizes the barycenter in non-relativistic dynamics: In the absence of external forces, its motion will be *uniform and rectilinear*.

One similarly exhibits the simple significance of the relation $G_\otimes V_k = G_k$.

Indeed, one can deduce the expression for the unit-length velocity w_μ of the pseudo-center of mass from it, and its components in the defining reference frame will be:

$$w_k = \alpha V_k \qquad \text{and} \qquad w_4 = ic\,\alpha, \qquad \text{with} \qquad \alpha = (1 - v^2 / c^2)^{-1/2},$$

$$v^2 = V_k V_k = -\,c^2 \frac{G_k G_k}{G_4^2},$$

$$1 - \frac{v^2}{c^2} = 1 + \frac{G_k G_k}{G_4^2} = \frac{G_\mu G_\mu}{G_4^2} = -\frac{M_0^2 c^2}{G_4^2} = \frac{M_0^2}{G_\otimes^2},$$

$$\alpha = \frac{G_\otimes}{M_0}, \qquad \text{from which} \qquad w_k = \frac{G_\otimes}{M_0}\frac{G_k}{G_\otimes} = \frac{G_k}{M_0} \quad \text{and} \quad w_4 = \frac{G_\otimes}{M_0} ic = \frac{G_4}{M_0},$$

which one can write as simply:

$$\boxed{M_0 w_\mu = G_\mu.}$$

This is the relativistic expression for the space-time momentum of a *material point* of proper mass M_0 .

Therefore, it seems that we can generalize the essential property of the barycenter of non-relativistic dynamics to a relativistic treatment, which consists of the possibility of describing certain characteristics of the global motion of a material system by considering a material *point* that is situated at the barycenter and has the total mass of the system for its mass. However, in reality, this is not the case, because we know that the pseudo-center of mass, despite the dynamical relations that it obeys, cannot be considered to be a material point in any way, since its coordinates are not covariant. It is a mathematical fiction, in the same sense as the pseudo-center of matter, and it does not play the role of either a material or a geometric "object."

Meanwhile, the covariant role that we just gave to the velocity invites us to extend the significance of the motion of the pseudo-center of mass. Consider the reference frame Σ. At each instant $t = x_4 / ic$, the drop will possess a pseudo-center of mass relative to that reference frame that is well-defined by its coordinates ξ_k . One can characterize it by a space-time point $\xi_\mu = (\xi_k, x_4)$ in the reference frame Σ.

However, this same point can be located in any other reference frame Σ′ by simply applying the Lorentz transformation that makes Σ go to Σ′ to the ξ_μ . ξ_μ will then be considered to be a space-time vector that defines an *intrinsic* point in all of the reference frames. Naturally, the point ξ_μ is only a pseudo-center of mass relative to the reference frame Σ in which it is defined. One will similarly define the other coordinates ξ'_μ in the reference frame Σ′, which will define a space-time point that is distinct from ξ_μ . This novel concept suggests that there exist an infinitude of pseudo-centers of mass. The covariant relation that we just established, namely:

$$M_0 \, w_\mu = G_\mu \,,$$

shows that all pseudo-centers of mass will have the same velocity. In particular, if one places oneself in the particular reference frame in which the space-time components G_k of the momentum are zero (viz., *the reference frame of inertia*) then one will see that all of the pseudo-centers of mass will have a zero spatial velocity: That is, all pseudo-centers of mass are at rest in the reference frame of inertia.

§ 6. Total and internal angular momentum. – Having introduced dynamical concepts that are defined by means of the energy-momentum tensor, we shall introduce the ones that are defined with the aid of the moment of rotation tensor density:

$$m_{[\mu\nu]\lambda} = x_\mu \, t_{\nu\lambda} - x_\nu \, t_{\mu\lambda} \,.$$

We defined the *total angular momentum* of an infinitesimal droplet above by:

$$M_{\mu\nu} = \int_{V_0} (x_\mu g_\nu - x_\nu g_\mu) \, d\upsilon_0 \,.$$

This expression can also be written, more generally, as:

$$M_{\mu\nu} = -\frac{1}{c^2} \int_{V_0} m_{\mu\nu\lambda} u_\lambda \, d\upsilon_0 \,,$$

and that is the form that we shall use. Indeed, upon taking into account the expressions $m_{[\mu\nu]\lambda} = x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}$ and $t_{\mu\nu} = g_\mu\, u_\nu + \theta_{\mu\nu}$ ($\theta_{\mu\nu}\, u_\nu = 0$) that we employed above, it will follow that:

$$M_{\mu\nu} = -\frac{1}{c^2}\int_{V_0}[(x_\mu g_\nu u_\lambda - x_\nu g_\mu u_\lambda)u_\lambda + (x_\mu \theta_{\nu\lambda} - x_\nu \theta_{\mu\lambda})u_\lambda]\,d\upsilon_0 .$$

The second term is zero ($\theta_{\nu\lambda}\, u_\lambda = \theta_{\mu\lambda}\, u_\lambda = 0$), and upon taking the fact that $u_\lambda\, u_\lambda = -\, c^2$ into account, what will remain is:

$$M_{\mu\nu} = \int_{V_0}(x_\mu g_\nu - x_\nu g_\mu)\,d\upsilon_0 .$$

As before, we remark that one cannot directly extend this definition to the microscopic case if one is given that there is no hyperplane Π that corresponds to the *proper* space at every point.

We refer the preceding definition to the proper reference frame of the infinitesimal droplet – i.e., to the same hyperplane over which we integrate:

$$M^0_{\mu\nu} = -\frac{ic}{c^2}\int_{V_0}\left(x^0_\mu t^0_{\nu 4} - x^0_\nu t^0_{\mu 4}\right)d\upsilon_0 = \int_{V_0} m^0_{\mu\nu\otimes}\,d\upsilon_0 ,$$

and we generalize to an arbitrary hyperplane Π by explicitly assuming that the tensors are referred to a reference frame that is defined by Π:

$$\boxed{M_{[\mu\nu]} = \int_\Sigma (x_\mu g_\nu - x_\nu g_\mu)\,d\upsilon_0 .}$$

In this form, which no longer involves proper space, the definition of the internal angular momentum can be generalized to a macroscopic fluid mass.

Since $\partial_\lambda\,(x_\mu\, t_{\nu\lambda} - x_\nu\, t_{\mu\lambda}) = \partial_\lambda\, m_{\mu\nu\lambda} = 0$, we once more meet up with the volume integral of the fourth component of a conservative tensor, and we can apply Møller's theorem (Appendix *A*). Moreover, one will immediately see that by virtue of the hypotheses that we made for surface tension, the hyper-boundary integral $\int_\Pi k_{\mu\nu} m_{\mu\nu\lambda}\, d\sigma_\lambda$ will be zero. Indeed, if one expresses the differential scalar element relative to the three axes $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ of the local proper system that we considered before then one will make the quantity:

$$m^0_{\mu\nu k} ds^0_k\, d\tau = (x^0_\mu\, t^0_{\nu k}\, ds^0_k - x^0_\nu\, t^0_{\mu k}\, ds^0_k)\ d\tau$$

appear, whose terms will both be separately equal to zero at every point, provided that one knows that energy-momentum of the surface tension that we previously introduced in the expression for the energy-momentum tensor on the surface.

Similarly, one easily shows that the scalar $k_{\mu\nu} M_{\mu\nu}$ (and as a consequence, the tensor $M_{\mu\nu}$) is constant in time. Indeed, one has:

$$\frac{d}{dt}(k_{\mu\nu}M_{\mu\nu}) = \int_S k_{\mu\nu}(m_{\mu\nu 4}v_k - ic\, m_{\mu\nu k})\, d\sigma_\lambda \ .$$

Upon evaluating the differential element in the proper reference frame, all that will remain is:

$$-\ k^0_{\mu\nu}\ ic\, m^0_{\mu\nu k}\ ds^0_k\ ,$$

which is a quantity that will be zero when one takes surface tension into account, as we just showed. One can thus conclude: On the one hand, the total angular momentum varies like a second-order, antisymmetric *tensor* that will be independent of the choice of hyperplane Π, provided that the spatial axes of the adopted reference frame are always chosen to be in that hyperplane. On the other hand, the tensor $M_{\mu\nu}$ will be *constant* in time for any time axis that is adopted:

$$\boxed{\frac{d}{dt}M_{\mu\nu} = 0.}$$

The physical significance of the spatial components:

$$M_{ij} = \int_\Sigma (x_i t_{j\otimes} - x_j t_{i\otimes})\, d\upsilon$$

is clear if one remembers that the components $t_{j\,\otimes}$ represent the momentum density in a relativistic fashion. M_{ij} is thus a *kinetic moment relative to the origin*, at least, if one assumes that the origin is found in the same spatial cut as the instantaneous Σ; i.e., that the constant time x_4 is zero, as one can always assume.

On the contrary, the role of the time components:

$$M_{i\,\otimes} = \int_\Sigma (x_i t_{\otimes\otimes} - x_\odot t_{i\otimes})\, d\upsilon$$

does not seem obvious on first glance. However, if one separates the two terms then one will see that $M_{i\,\otimes}$ involves the pseudo-center of mass.

$$\xi_\otimes \int_\Sigma t_{i\otimes}\ d\upsilon\ = \xi_\otimes\, G_i\ ,$$

so one will finally have $M_{i\,\otimes} = G_\otimes\ \xi_i - G_i\ \xi_\otimes$.

This is the relation that we utilized above in a manner that anticipated the proof that the pseudo-center of mass varies with the reference frame.

This relation can be simplified if one chooses the origin to belong to the hyperplane Π and changes the time origin to be $\xi_\otimes = 0$, and what will remain is:

$$\boxed{M_{i\otimes} = G_\otimes \xi_i,}$$

i.e., as is well-known [**1**], these are the components of the *barycentric moment with respect to the initial position of the pseudo-center of mass.*

Louis de Broglie has often insisted upon the fact that in relativity the kinetic moment with respect to an arbitrary origin of one's coordinate axes does not have to have any particular physical significance. The important quantity is the *proper* kinetic moment relative to a point that is connected with the fluid drop, translates with it, and is characterized by a well-defined physical property [**1**]. The choice of that point (which, from a certain viewpoint, ought to play the role of the "center" of the drop) presents some difficulties. We will address them shortly. For the moment, we assume that one has, in some fashion, defined a given point C that is connected with the drop and is independent of the reference frame. We shall call the tensor that is obtained in same manner as $M_{\mu\nu}$ the *internal angular momentum* $S_{\mu\nu}$, but we shall take the point C to be the origin, independently of the reference frame, and we will denote its coordinates by Y_μ, which will be covariant by definition. One will then have:

$$S_{\mu\nu} = \int_\Sigma [(x_\mu - Y_\mu)\, t_{\nu\otimes} - (x_\nu - Y_\nu)\, t_{\mu\otimes}]\, d\upsilon$$

in the reference frame that is determined by the hyperplane Π over which one integrates.

Upon separating the terms on x from the terms in Y, one will get:

$$S_{\mu\nu} = \int_\Sigma (x_\mu\, t_{\nu\otimes} - x_\nu t_{\mu\otimes})\, d\upsilon - Y_\mu \int_\Sigma t_{\nu\otimes}\, d\upsilon + Y_\nu \int_\Sigma t_{\mu\otimes}\, d\upsilon\,,$$

so:

(II.6) $$\boxed{S_{\mu\nu} = M_{\mu\nu} - Y_\mu G_\nu + Y_\nu G_\mu\,.}$$

Since the three terms on the right-hand side are tensors, independently of the chosen hyperplane Π, it will result that $S_{\mu\nu}$ is also a tensor.

§ 7. The fundamental equations. Center of mass. – If we endow the point C with the quality of being a "center" for the fluid mass then it will be natural for us to introduce the proper time τ of the point C as the *mean proper time* of the fluid. If the point C considered possesses a spatial velocity V_k in the reference frame $\Pi\Lambda$ then it will be easy to define:

$$\alpha = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

and to calculate the quantities $U_k = \alpha\, V_k$ and $U_4 = ic\,\alpha$, which will be the components of the unit-speed velocity U_μ of the point C in the reference frame $\Pi\Lambda$. The unit-speed velocity will then be a quadri-vector under the condition the C should be an intrinsic point whose coordinates transform according to the Lorentz formulas under a change of reference frame. One will then pass from the time t that relates to the reference frame $\Pi\Lambda$ to the proper time τ by way of:

$$\frac{dt}{d\tau} = \frac{U_4}{ic} = \alpha.$$

We shall now differentiate equation (6) *with respect to proper time* τ. Given that:

$$\frac{d}{d\tau}G_\mu = \frac{dt}{d\tau}\cdot\frac{d}{dt}G_\mu = 0,$$

$$\frac{d}{d\tau}M_{\mu\nu} = \frac{dt}{d\tau}\cdot\frac{d}{dt}M_{\mu\nu} = 0, \qquad \frac{d}{d\tau}Y_\mu = U_\mu\,,$$

if we denote the derivative with respect to proper time by a dot then it will follow that:

$$\boxed{\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu,}$$

which is a formula by which one will recover the second group of Frenkel-Weyssenhoff equations, to which, one can immediately append the first group, which is expressed by:

$$\boxed{\dot{G}_\mu = 0,}$$

as we just recalled.

We must point out a further important detail that concerms internal angular momentum: If one takes its time components in the reference frame $\Pi\Lambda$ then:

$$S_{k\otimes} = \int_\Sigma [(x_k - Y_k)\,t_{\otimes\otimes} - (x_\otimes - Y_\otimes)\,t_{k\otimes}]\,d\upsilon\,;$$

i.e.:

$$S_{k\otimes} = G_\otimes\,\xi_k - Y_k\,G_\otimes\,,$$

or furthermore:

(II.7) $$\boxed{S_{k\otimes} = G_\otimes(\xi_k - Y_k).}$$

Therefore, in an arbitrary reference frame, the time components of the proper angular momentum relative to an *arbitrary*, *intrinsic* point will constitute a spatial vector that is collinear with and proportional to the vector that connects that point to the pseudo-center of mass relative to the reference frame considered.

We seek to give a covariant form to that relation. In order to do that, we remark that since the point is an intrinsic point, its velocity U_μ will be a well-defined quadri-vector, and one can always find a reference frame Σ_0 (which is determined up to a spatial rotation) in which the point C is at rest. The hyperplane Π_0 will be orthogonal to the quadri-vector U_μ . We shall call this reference frame the *mean proper reference frame* of the fluid mass. If we write the preceding relation in the system Σ_0 then upon multiplying it by $U_4^0 = ic$, one will have:

$$S_{k\otimes}^0\,ic = G_\otimes^0\,ic\,(\xi_k^0 - Y_k^0)\,,$$

or furthermore:

$$S^0_{k\otimes}U^0_4 = G^0_4 U^0_4(\xi^0_k - Y^0_k),$$

if we denote the components with respect to the reference frame with the superscript 0.

One can further remark that:

$$S^0_{44} = 0 \qquad \text{(antisymmetry)}$$

and

$$\xi^0_4 - Y^0_4 = 0 \qquad \text{(simultaneity)},$$

which one can just as well write as:

$$S^0_{\mu 4}U^0_4 = G^0_4 U^0_4(\xi^0_\mu - Y^0_\mu).$$

However, since the spatial components U^0_k are zero in the system Σ_0, the quantities $S^0_{\mu 4}U^0_4$ and $G^0_4 U^0_4$ will be nothing but the expressions for the contracted tensor products $S_{\mu\nu}\,U_\nu$ and $G_\nu\,U_\nu$, respectively.

One can thus write:

$$(S_{\mu\nu}\,U_\nu)^0 = (G_\nu\,U_\nu)^0(\xi^0_\mu - Y^0_\mu).$$

In order to make this equation covariant, we must consider the ξ^0_μ to be the coordinates of an *intrinsic* point in the proper system that is defined all of the other systems by convenient Lorentz transformations, as we have said. That point will be called the *center of mass* X_μ, properly speaking, and will be the pseudo-center of mass only for a proper space cut. Moreover, we must remark that the "center" C of the drop will generally accelerate. The Galilean proper reference frame in which it is at rest at the instant t will not generally accelerate. Furthermore, at the instant t', the point C will no longer be at rest, so it will be necessary to consider a different proper reference frame. Since the center of mass at each instant is defined relative to the proper reference frame *relative to that instant*, it will result that the center of mass is not a pseudo-center of mass. It would be a gross error to apply the formulas for velocities that were established for pseudo-centers to such a point. In particular, it will be trivially at rest in the reference frame of inertia.

By means of these hypotheses, the equality:

$$(S_{\mu\nu}\,U_\nu)^0 = (G_\nu\,U_\nu)^0(X^0_\mu - Y^0_\mu)$$

will be, at the same time, a tensorial equation, and it will be true in any arbitrary reference frame. One can write it in the covariant fashion as:

$$S_{\mu\nu}\,U_\nu = G_\nu\,U_\nu\,(X_\mu - Y_\mu).$$

Finally, the contracted product $G_\nu U_\nu$ is an important invariant, which we, following Weyssenhoff, can write as:

$$\boxed{G_\nu U_\nu = -\mathfrak{M}_0 c^2,}$$

in which a new proper mass $\mathfrak{M}_0$ appears that differs from M_0 , and which is not generally constant in time, and which we shall call the *proper mass of inertia*.

It then ultimately follows that:

$$\boxed{S_{\mu\nu} U_\nu = \mathfrak{M}_0 c^2 (Y_\mu - X_\mu).}$$

It is remarkable that the two formula that were established for $\dot{S}_\mu$ and $S_{\mu\nu} U_\nu$ do not involve any properties are peculiar to the point C, other than the fact that it is an *intrinsic* point that is defined by covariant coordinates. It is no less obvious that the proper angular momentum can no longer have any physical significance when the point C is defined by starting with the properties of the fluid drop, or at least, when it is related to the "form" that this drop takes, in some arbitrary fashion. These are the problems that are provoked by a choice of the point C that remain for us to explore.

§ 8. A fundamental reference frame. – The study of points in relativistic hydrodynamics can play a role for a macroscopic fluid mass that is analogous to the role that is played by the center of gravity in non-relativistic dynamics, and which has been the object of numerous works [**12**, **13**]. They always collide with the facts that the integrals that define the coordinates of a point must necessarily be taken over a "volume" of the fluid mass that is the intersection of the world-tube with a spatial hyperplane that relates to the chosen reference frame, and that the coordinates thus-obtained will vary when one changes the reference frame, as we saw in the context of the pseudo-centers of matter and mass. It is never possible to reduce these integrals to merely covariant volume integrals that are the integrals of the fourth components of a conservative tensor (see Appendix *A*). That is why we shall adopt a different (and less ambitious) viewpoint.

We choose a point C that is defined in relation to the properties of the fluid drop *in a given reference frame*. Furthermore, we assume, from the outset, that this point is intrinsic – i.e., that we will obtain its coordinates in the other reference frames by means of the Lorentz transformation formulas, although we intend that it will no longer enjoy its defining property in the other reference frames. We already used this process in order to define the center of mass.

We therefore come back to the problem of the search for a *special reference system* that is defined by means of the properties of a fluid mass.

It is convenient to reason by analogy with the case of an infinitesimal droplet, because one will immediately recognize a reference frame for such a thing: viz., the *local proper reference frame*, which is orthogonal to the infinitesimal hyper-tube. That reference frame will be defined a spatial hyperplane Π_0 that is orthogonal to the time axis, whose properties we recapitulate, since they might have a different significance when one passes to the case of a macroscopic fluid mass:

1) The center of the droplet is at rest in the hyperplane Π_0 .

2) The hyperplane Π_0 is orthogonal to the local current j_μ .

3) The hyperplane Π_0 is orthogonal to the local momentum g_μ , which is collinear with j_μ .

Now, consider a macroscopic fluid mass. Cut the hyperplane Π in the direction of a domain Σ and see how one can generalize the three properties that we recalled above. Remember that a hyperplane Π that passes through a given space-time point M will be determined completely by three independent parameters: For example, one can define it by the three Lorentz coefficients $\beta_i = V_i / c$ of the pure Lorentz transformation that takes the laboratory reference frame to the reference frame $\Pi\Lambda$ that serves to fix the time axis Λ and to leave the spatial axes undetermined in the hyperplane Π.

1) One can restrict the "center" of the fluid mass to be immobile in the hyperplane Π.

Relative to the reference frame $\Pi\Lambda$, the role of "center" will be played by the pseudo-center of matter, and we know that its velocity must be expressed by $V_k = \frac{1}{J}\int_\Sigma j_k \, d\upsilon$ in the hyperplane Π in order for us to satisfy the three equations $V_k = 0$.

The V_k are complicated functions of six parameters: Three β_i determine the position of the axis Λ and three parameters (for example, the Euler angles θ_i) determine the three space axes in the hyperplane Π, while these last three parameters can remain undetermined. Three equations can then permit us to determine the three β_i , but the expressions that are found from them will contain the θ_i ; i.e., we will obtain different systems $\Pi_0\ \Lambda_0$ according to the *spatial* rotations that we allow in our reference frame. We cannot generally determine the reference frame $\Pi_0\ \Lambda_0$ in a unique fashion in that way.

2) One can restrict the spatial current to be zero in the mean. In general, one cannot directly consider a total current, because expressions such as $J_\mu = \int_\Sigma j_\mu \, d\upsilon$ are not tensorial. However, one can express the property in question in the case of the infinitesimal droplet by requiring that the hypersurface element $d\sigma_\mu$ that is cut out by the infinitesimal hyper-tube on a hyperplane Π should be collinear with the current:

$$j_\mu \, d\sigma_\nu = j_\nu \, d\sigma_\mu ,$$

which will lead us to say that the tensor $j_\mu \, d\sigma_\nu - j_\nu \, d\sigma_\mu$ is intrinsically zero.

If one integrates then one will get:

$$J_{[\mu\nu]} = \int_\Sigma (j_\mu d\sigma_\nu - j_\nu d\sigma_\mu) ,$$

which will be a tensor *for a given hyperplane* Π. Therefore, each hyperplane Π will correspond to a tensor $\mathfrak{J}_{[\mu\nu]}$ that is a function of three parameters θ_i . In order for the hyperplane Π to satisfy the same condition as the infinitesimal droplet does, one must have that $\mathfrak{J}_{[\mu\nu]}$ is intrinsically zero. However, that is not possible, in general, because that would involve *six* independent conditions, while we are provided with only three parameters. It seems fitting that if we are given the way that the current density varies then it will not be possible to define a system that is analogous to the proper reference frame of the infinitesimal droplet by addressing the kinematical properties of the fluid mass uniquely.

3) On the contrary, the third condition, which involves the dynamical properties of the fluid, immediately provides us with a reference frame. Indeed, as we know, there exists a vector G_μ – viz., the moment vector – that corresponds to the local momentum density vector for the aggregate of the fluid mass. If we direct the time axis parallel to that vector then the hyperplane Π will be orthogonal to the total momentum G_μ, just as the plane Π_0 of the infinitesimal droplet will be orthogonal to the local momentum density g_μ . We denote the special reference frame that is thus defined by $\Pi_1\ \Lambda_1$, and we call it the *reference frame of inertia* (Weyssenhoff).

It is useful to given the transformation formulas that permit us to pass from an arbitrary reference frame $\Pi\Lambda$ (e.g., the laboratory frame) to the reference frame of inertia $\Pi_1\ \Lambda_1$. Any vector with components G_μ in the reference frame $\Pi\Lambda$ will have the components:

$$G_\nu^1 = \lambda_{\mu\alpha}\, G_\alpha ,$$

in the reference frame $\Pi_1\ \Lambda_1$, and conversely:

$$G_\mu = l_{\mu\beta}\, G_\beta^1 ,$$

with

$$\lambda_{\alpha\beta} = l_{\beta\alpha} .$$

The coefficients $\lambda_{\alpha\beta}$ are given as functions of the relative velocity v_k of the system Π_1 with respect to the system Π by the well-known formulas [**3**]:

$$\lambda_{ik} = \delta_{ik} + \frac{\alpha^2 v_i v_k}{(1+\alpha)\, c^2}, \qquad \lambda_{4i} = -\lambda_{i4} = \frac{\alpha v_i}{ic},$$

$$\lambda_{4i} = \alpha \qquad \text{with} \qquad \alpha = (1 - v^2 / c^2)^{1/2}.$$

If the vector G_μ is the momentum vector, and if one restricts its spatial components to be zero in the system $\Pi_1\ \Lambda_1$ then one will have:

$$G_k^1 = \lambda_{ki}\, G_i + \lambda_{k4}\, G_4 = 0,$$

or, upon substituting the values of λ_{i4}, λ_{k4}:

$$\left[\delta_{ik} + \frac{\alpha v_i v_k}{(1+\alpha)c^2}\right] G_i = \frac{\alpha v_k}{ic} G_4 \equiv \alpha v_k G_\otimes .$$

Contracting this with v_k will give:

$$v_k v_k = v^2 \qquad \text{and} \qquad \alpha^2 v^2 = c^2 (\alpha^2 - 1),$$

so

$$\left(v_i + \frac{c^2(\alpha^2-1)}{c^2(\alpha^2+1)} v_i\right) G_i \equiv \alpha v_i G_i = \alpha G_\otimes v_i v_i .$$

That relation is satisfied by the velocity:

$$\boxed{v_i = \frac{G_i}{G_\otimes},}$$

(among others), which is a velocity that will effectively satisfy the complete equation when it is substituted in it. Upon substituting v_i in the transformation formulas, one will get:

$$\lambda_{ik} = \delta_{ik} + \frac{\alpha^2 G_i G_k}{G_\otimes^2 c^2 (1+\alpha)},$$

$$\lambda_{4i} = -\lambda_{i4} = \frac{\alpha G_i}{G_\otimes ic}, \qquad \lambda_{44} = \alpha.$$

One can further remark that:

$$G_4 = ic\, G_\otimes = l_{4i}\, G^1 + l_{44}\, G_4^1 = \alpha G_4^1 = \alpha\, ic\, M_0 ,$$

so

$$G_\otimes = \alpha M_0 ,$$

and finally:

(II.8) $$\boxed{\lambda_{ik} = \delta_{ik} + \frac{G_i G_k}{M_0 c^2 (M_0 + G_\otimes)}, \qquad \lambda_{4i} = -\lambda_{i4} = \frac{G_i}{ic M_0}, \qquad \lambda_{44} = \frac{G_\otimes}{M_0}.}$$

§ 9. The center of gravity. Møller’s theorem. – Once we have chosen the reference frame $\Pi_1\Lambda_1$, we can immediately consider two points: viz., its pseudo-centers of mass and matter, which, from what we discussed above, we consider to be intrinsic points, and from which we will obtain the coordinates in an arbitrary reference frame by means of the Lorentz transformation formulas. The first of these points will be called the *center of gravity* (Møller). We denote its (covariant) components by Z_μ . In the reference frame $\Pi_1\Lambda_1$, these components will be identified with those of the pseudo-center of mass $Z_\mu^1 = \xi_\mu^1$, and its spatial velocity will then assume the particular expression:

$$V_\mu = \frac{G_k^1}{G_\otimes};$$

i.e., it will *zero*. Since the hyperplane Π_1 is orthogonal to the vector G_μ, the spatial components will be zero. The center of gravity will then be determined by the fact that it is the only pseudo-center of mass that is *at rest* in its defining reference frame.

We can choose the center of gravity to be a point C, and endow it with the role of being "center" of the drop. That will lead us to take the internal angular momentum $S_{\mu\nu}$ relative to that point, to choose its proper time t to be the global proper time of the drop, and to attribute the unit-speed velocity U_μ of that point to the drop (taken as a unit), which is a unit-speed velocity that is, we repeat, orthogonal to Π_1; i.e., collinear to G_μ .

One will have:

$$G_k^1 = 0, \qquad\qquad G_4^1 = \sqrt{G_\mu G_\mu},$$

$$U_k^1 = 0, \qquad\qquad U_4^1 = ic$$

in the reference frame $\Pi_1\Lambda_1$; i.e.:

$$G_4^1 = M_0 U_4^1 \quad \text{and} \quad G_k^1 = M_0 U_k^1,$$

and thus, the covariant equation:

$$G_\mu^1 = M_0 U_\mu^1 .$$

The momentum is collinear with the velocity.

The two fundamental equations will give:

The first one: $\dot{G}_\mu = 0$, so $\dot{U}_\mu = 0$,

The second one: $\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu = 0$.

The unit-speed velocity and the proper angular momentum will be constant in the absence of external forces.

Therefore, if one condenses the fluid drop to a material point that is situated at its center of gravity then that material point will be in a completely classical state of motion, and the model will not present any great interest. After all, as we emphasized at the beginning of this chapter, it has not been a convenient way of representing a body that is animated with a rotation, since no points besides the center of gravity will enter into it explicitly. The material structure will reduce to a unique point with no distinct parts, and since it hardly makes sense to attribute a proper rotation to it, since angular momentum will no longer mean anything.

Meanwhile, that model has served to exhibit some of the important properties of global motion for a drop for Møller [**12**]. It will be useful for us to pause briefly. If we apply formula (7) to the reference frame of inertia then that will show us that the Møller's angular momentum (which shall denote by $\mathcal{L}_{\mu\nu}$, so as not confuse it with the one that we shall use in what follows) does not have non-vanishing temporal components in that reference frame. It can be condensed into an antisymmetric spatial tensor $\mathcal{L}^{\mathrm{I}}_{[\mu\nu]}$, and can also be just as well represented by its dual:

$$\lambda^{\mathrm{I}}_k = \tfrac{1}{2}\varepsilon_{ijk}\mathcal{L}^{\mathrm{I}}_{ij},$$

which is a vector that is situated entirely in inertial space, so inversely one will have:

$$\mathcal{L}^{\mathrm{I}}_{ij} = \varepsilon_{ijk}\lambda^{\mathrm{I}}_k .$$

As we will verify in the next chapter, this relation, which was established between the tensor $\mathcal{L}^{\mathrm{I}}_{ij}$ and the vector λ^{I}_k in the space of inertia, can be transformed into a covariant relation between Møller's angular momentum $\mathcal{L}_{\mu\nu}$ and a space-time vector λ_μ that we call the *Møller spin*, whose components along the axes of the system of inertia will be λ^{I}_k and 0. One will then have:

$$\lambda_\mu = \frac{i}{2c}\varepsilon_{\mu\nu\alpha\beta}\frac{G_\nu\mathcal{L}_{\alpha\beta}}{M_0},$$

and conversely:

$$\mathcal{L}_{\mu\nu} = \frac{i}{2c}\varepsilon_{\mu\nu\alpha\beta}\frac{G_\alpha\lambda_\beta}{M_0},$$

as one will easily very upon projecting onto the axes of the system of inertia. The expression for λ_μ shows that, in addition, this vector will remain *constant* in the course of its motion by virtue of the two laws:

$$\dot{G}_\mu = 0 \qquad \text{and} \qquad \dot{\mathcal{L}}_{\mu\nu} = 0.$$

Having said that, formula (7) will permit us to locate the pseudo-center of mass ξ_k that relates to an arbitrary reference frame $\Pi\Lambda$ with respect to the position Z_k of the center of gravity, which is situated in the same spacelike cut Π. One will have:

$$\mathcal{L}_{k4} = G_4\,(\xi_k - Z_k).$$

On the other hand, upon applying transformation (8) to the components of $\mathcal{L}_{\mu\nu}$ in the system of inertia, one will obtain:

$$\mathcal{L}_{k4} = l_{kp}\, l_{4q}\, \mathcal{L}^{\mathrm{I}}_{pq} = \frac{G_p}{ic\,M_0}\left[\delta_{pq} + \frac{G_k G_q}{M_0 c^2 (M_0 + G_\otimes)}\right]\varepsilon_{pqr}\mathcal{L}^{\mathrm{I}}_{pq}$$

for the $\mathcal{L}_{k4}$ components (upon considering only the spatial components of $\mathcal{L}^{\mathrm{I}}_{\mu\nu}$, of course), or, upon replacing $\mathcal{L}^{\mathrm{I}}_{pq}$ with its expression:

$$\mathcal{L}_{k4} = \frac{G_p}{ic\,M_0}\left[\delta_{pq} + \frac{G_k G_q}{M_0 c^2 (M_0 + G_\otimes)}\right]\varepsilon_{pqr}\lambda^{\mathrm{I}}_r .$$

The second term will disappear by antisymmetry:

$$\mathcal{L}_{k4} = G_4\,(\xi_k - Z_k) = \varepsilon_{pkr}\,\frac{G_p}{ic\,M_0}\lambda^{\mathrm{I}}_r .$$

If one denotes the spatial vector with the components $\xi_k - Z_k$ in the system ΠΛ by $\mathcal{A}$ then one will have:

$$\mathcal{A}_k = \frac{1}{M_0}\varepsilon_{prk}\frac{G_p}{G_\otimes}\lambda^{\mathrm{I}}_r .$$

This equation is not vectorial, because the quantities λ^{I}_r denote the components of a vector in the space of inertia. We can consider the vector $\mathcal{A}_k$ to be a space-time vector upon introducing the component $\mathcal{A}_4 = 0$ in the system ΠΛ. We then seek the components of this space-time vector in the system of inertia by applying the relations (8):

$$\mathcal{A}^{\mathrm{I}}_4 = \lambda_{4k}\,\mathcal{A}_k = \frac{G_k}{ic\,M_0}\frac{\varepsilon_{prk}}{M_0 c^2}\frac{G_p}{G_\otimes}\lambda^{\mathrm{I}}_r ;$$

$\mathcal{A}^{\mathrm{I}}_4$ is zero, by antisymmetry. The space-time vector $\mathcal{A}_\mu$ will then be contained in the space of the system of inertia. The two centers in question, when taken to be simultaneous in the system ΠΛ, will then be likewise simultaneous in the system of inertia.

Similarly, one will have:

$$\mathcal{A}^{\mathrm{I}}_j = \lambda_{jk}\,\mathcal{A}_k = \left[\delta_{jk} + \frac{G_j G_k}{M_0 c^2 (M_0 + G_\otimes)}\right]\frac{\varepsilon_{prk}}{M_0 c^2}\frac{G_p}{G_\otimes}\lambda^{\mathrm{I}}_r ;$$

Once again, one has a term that is zero by antisymmetry, and what will remain is:

$$\mathcal{A}_j^{\mathrm{I}} = \frac{\varepsilon_{prk}}{M_0 c^2} \frac{G_p}{G_\otimes} \lambda_r^{\mathrm{I}} .$$

However, the vector G_p / $G_\otimes$ is nothing but the velocity V_p of the system of inertia with respect to the system ΠΛ when it is represented along the spatial axes of the latter system. With the reservation that the axes of the two systems must be parallel and displaced without rotation with respect to each other, the same quantities V_p , with the opposite signs, will represent the components of the velocity of the system ΠΛ in the system of inertia. It then likewise represents a spatial vector $-\ V_p^{\mathrm{I}}$ in the system of inertia. The equation:

$$\mathcal{A}_j^{\mathrm{I}} = -\frac{\varepsilon_{prj}}{M_0 c^2} V_p^{\mathrm{I}} \lambda_r^{\mathrm{I}}$$

will then be a vectorial equation in the space of the system of inertia:

$$\boldsymbol{\mathcal{A}} = -\ \frac{\mathbf{v} \times \boldsymbol{\lambda}}{M_0 c^2} .$$

One then sees that for the various reference frames that correspond (by parallel displacement) to all of the possible velocities **v** (i.e., ones of norm less than c), the various pseudo-centers of mass that are simultaneous to the same center of gravity will be divided on a *disk* in the space of inertia that is perpendicular to Møller's spin (which is an invariant vector), and its radius will be obtained by giving the norm c to **v**, such that:

$$|\ \boldsymbol{\mathcal{A}}\ | \leq \frac{|\boldsymbol{\lambda}|}{M_0 c^2} .$$

That disk will be immobile, since the pseudo-centers of mass are at rest in the system of inertia. The special disposition of the pseudo-center relative to the spin is related to the fact that was pointed before that it is the rotation of the drop about itself that produces the variation of the pseudo-center of mass along the reference frame in which one defines it.

§ 10. The center of matter. The Bohm-Vigier model. One will arrive at more interesting results when one chooses the second remarkable point to be the center of mass for the fluid – viz., the pseudo-center of *matter* relative to the system of inertia, which shall call the *center of matter*, properly speaking, and whose (covariant) coordinates shall be denoted by Y_μ . In the reference frame $\Pi_1\Lambda_1$, these coordinates will be equal to those of the pseudo-center of matter $Y_\mu^{\mathrm{I}} = \eta_\mu^{\mathrm{I}}$, and its spatial velocity will take one the special expression:

$$V_k^{\mathrm{I}} = \frac{1}{J} \int_{\Sigma_1} J_k^{\mathrm{I}}\ d\upsilon .$$

However, these relations will no longer be true in another reference frame. It is with respect to this center of mass that we shall evaluate the internal angular momentum $S_{\mu\nu}$. Its proper time τ will be considered to be the mean proper time of the drop, and its unit-speed velocity $U_\mu = \dot{Y}_\mu$ will be the mean unit-speed velocity of the drop. One then sees immediately that $V_k^{\mathrm{I}} = \frac{1}{J}\int_{\Sigma_1} j_k^{\mathrm{I}}\, d\upsilon$ has no reason to be zero, so the center of matter will not be at rest in the reference frame of inertia, and the unit-speed velocity U_μ will not be collinear with the space-time momentum G_μ.

One will therefore have $G_\mu U_\nu \neq G_\nu U_\mu$, and the relation:

$$\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu \neq 0$$

will provide us with an internal angular momentum that will vary with time. One will then arrive at laws of motion that differ profoundly from the classical laws of Newtonian mechanics.

In order to specify the definitions upon which the model rests completely, it will be useful to give the formulas that express the velocity and the position of the center of matter relative to an arbitrary reference (e.g., the laboratory frame), which we characterize by the values of G_i, G_4, which provide the components of the total momentum in that reference frame, which will, as we know, permit us to write down the Lorentz transformations that take us to the reference frame of inertia.

As we know, the velocity of the center of matter is given by:

$$V_k^{\mathrm{I}} = \frac{1}{J}\int_{\Sigma_1} j_k^{\mathrm{I}}\, d\upsilon$$

in the reference frame of inertia. If one forms $V_k^{\mathrm{I}} V_k^{\mathrm{I}}$ and $\alpha = (1 - V_k^{\mathrm{I}} V_k^{\mathrm{I}} / c^2)^{-1/2}$ then one can write down the components of the unit-speed velocity U_μ in the system of inertia as:

$$U_k^{\mathrm{I}} = \alpha V_k^{\mathrm{I}}, \qquad U_4^{\mathrm{I}} = ic\, a.$$

One passes from this to the components in the laboratory system by transformation (8):

$$\boxed{U_k = \alpha\left(V_k^{\mathrm{I}} + \frac{G_i G_k}{M_0 c^2 (M_0 + G_\otimes)} V_k^{\mathrm{I}}\right),}$$

$$U_4 = ic\alpha\left(\frac{G_\otimes}{M_0} + \frac{G_i V_i^{\mathrm{I}}}{M_0 c^2}\right),$$

or furthermore:

$$\boxed{U_\otimes = \alpha\left(\frac{G_\otimes}{M_0} + \frac{G_i V_i^{\mathrm{I}}}{M_0 c^2}\right).}$$

These formulas then permit us to define the *proper* system by means of the Lorentz transformation $x_\mu^0 = L_{\mu\nu}\, x_\nu$, which takes us from the laboratory system to the proper system. One will then have the classical formulas:

$$L_{ik} = \delta_{ik} + \frac{U_i U_k}{(1+U_\otimes)\,c^2}, \qquad L_{4i} = -L_{i4} = \frac{U_i}{c}, \qquad L_{44} = U_\otimes ,$$

in which it will suffice to replace U_k and $U_\otimes$ with their values.

Finally, the coordinates of the center of mass in the system of inertia are:

$$Y_k^j(t^{\mathrm{I}}) = \frac{1}{J}\int_{\Sigma_1} j_\otimes^{\mathrm{I}}\, x_k^{\mathrm{I}}\, d\upsilon, \qquad Y_4^{\mathrm{I}} = ic\, t^{\mathrm{I}}.$$

If one passes once more to the laboratory system then:

$$Y_i(t^{\mathrm{I}}) = Y_i^{\mathrm{I}}(t^{\mathrm{I}}) + \frac{G_i G_k}{M_0 c^2 (M_0 + G_\otimes)} Y_k^{\mathrm{I}}(t^{\mathrm{I}}) + \frac{G_i}{M_0} t^{\mathrm{I}},$$

$$Y_\otimes(t^{\mathrm{I}}) = \frac{G_\otimes}{M_0} t^{\mathrm{I}} + \frac{G_k}{M_0 c^2} Y_k^{\mathrm{I}}(t^{\mathrm{I}}) .$$

These formulas provide the parametric equations of motion for the center of matter as a function of time in the system of inertia. It is more convenient to introduce proper time τ as a parameter. One has $t^{\mathrm{I}} = \alpha\tau$, so:

$$\boxed{\begin{aligned} Y_i &= Y_i^{\mathrm{I}}(\alpha\tau) + \frac{G_i G_k}{M_0 c^2 (M_0 + G_\otimes)} Y_k^{\mathrm{I}}(\alpha\tau) + \frac{G_i}{M_0}\alpha\tau, \\ Y_\otimes &= \frac{G_\otimes}{M_0}\alpha\tau + \frac{G_k}{M_0 c^2} Y_k^{\mathrm{I}}(\alpha\tau). \end{aligned}}$$

We have therefore defined three particular intrinsic points for any fluid drop:

1) The point Z_μ – or center of gravity – which is at rest in the system of inertia, for which it is the pseudo-center of mass.

2) The point Y_μ – or center of matter – which is the pseudo-center of matter for the system of inertia, and is at rest in the proper system by reason of the definition of that system.

3) The point X_μ – center of mass – which is the pseudo-center of mass for the proper system at the instant considered.

These three points will generally be distinct, and as we have seen, the space-time vector that joins the center of mass to the center of matter – which is a vector that we shall denote by Q_μ – is given by:

$$Q_\mu = X_\mu - Y_\mu = -\frac{1}{\mathfrak{M}_0 c^2} S_{\mu\nu} U_\nu .$$

One obviously has:

$$Q_\mu U_\mu = -\frac{1}{\mathfrak{M}_0 c^2} S_{\mu\nu} U_\nu U_\mu = 0$$

by antisymmetry, which signifies that Q_μ is a spatial vector in proper space, and translates into simply the fact that we have assumed that the two points are simultaneous relative to the proper system.

One can express the quadri-vector that joins the center of mass with the center of matter in an analogous fashion. Consider the expression for the time component of the proper angular momentum *in the reference frame of inertia:*

$$S^{\mathrm{I}}_{k\otimes} = \int_{\Sigma_1} [(x^{\mathrm{I}}_k - Y^{\mathrm{I}}_k)\, t^{\mathrm{I}}_{\otimes\otimes} - (x^{\mathrm{I}}_\otimes - Y^{\mathrm{I}}_\otimes)\, t^{\mathrm{I}}_{k\otimes}]\, d\upsilon ,$$

in which the integral is not taken with constant time relative to the reference frame of inertia: $x^{\mathrm{I}}_\otimes$ = constant = $Y^{\mathrm{I}}_\otimes$, which will annul the second term. What then remains will be:

$$S^{\mathrm{I}}_{k\otimes} = \int_{\Sigma_1} x^{\mathrm{I}}_k\, t^{\mathrm{I}}_{\otimes\otimes}\, d\upsilon - Y^{\mathrm{I}}_k \int_{\Sigma_1} t^{\mathrm{I}}_{\otimes\otimes}\, d\upsilon = G^{\mathrm{I}}_\otimes\, \xi^{\mathrm{I}}_k - Y^{\mathrm{I}}_k\, G^{\mathrm{I}}_\otimes ;$$

i.e.:

$$S^{\mathrm{I}}_{k\otimes} = G^{\mathrm{I}}_\otimes (\xi^{\mathrm{I}}_k - Y^{\mathrm{I}}_k) .$$

However, $G^{\mathrm{I}}_\otimes$ will be the norm M_0 of the quadri-vector G_μ in the system of inertia, and on the other hand, the pseudo-center of mass ξ^{I}_μ of the system of inertia will coincide with the intrinsic *center of gravity* Z_μ, so:

$$\boxed{S^{\mathrm{I}}_{k\otimes} = M_0 (Z^{\mathrm{I}}_\mu - Y^{\mathrm{I}}_\mu),}$$

or, if one remarks that $S_{4\otimes} = 0$ and $Z^{\mathrm{I}}_4 - Y^{\mathrm{I}}_4 = 0$ then:

$$S^{\mathrm{I}}_{\mu\otimes} = M_0 (Z^{\mathrm{I}}_\mu - Y^{\mathrm{I}}_\mu) .$$

This relation can be made tensorial by observing that the contracted product $S_{\mu\nu} G_\nu$, which is a quadri-vector, has $(S_{k\nu} G_\nu)^{\mathrm{I}} = S^{\mathrm{I}}_{k4}\, ic\, M_0$ and $(S_{4\nu} G_\nu)^{\mathrm{I}} = 0$ for its components in the system of inertia (where the components of G_ν will be 0, 0, 0, and $ic\, M_0$).

One then has:

$$(S_{\mu\nu}\,G_\nu)^{\mathrm{I}} = ic\,M_0\,S^{\mathrm{I}}_{\mu 4} = -\,c^2\,M_0^2(Z^{\mathrm{I}}_\mu - Y^{\mathrm{I}}_\mu)\,,$$

which is a tensorial relation that can be written in an arbitrary reference frame as:

$$\boxed{S_{\mu\nu}G_\nu = -M_0^2c^2(Z_\mu - Y_\mu).}$$

We denote the space-time vector that joins the center of gravity Z_μ to the center of matter Y_μ by $R_\mu = Y_\mu - Z_\mu$ so:

(II.9) $$\boxed{R_\mu = \frac{1}{M_0^2c^2}S_{\mu\nu}G_\nu.}$$

One immediately sees that $R_\mu\,G_\mu = 0$:

R_μ is a spatial vector in the system of inertia, which is natural, since the two points that it connects were defined to be simultaneous in that system.

The vectors Q_μ and R_μ are thus non-zero, in general. However, one can consider the particular cases in which they are zero.

1) $R_\mu = 0$: The center of gravity and the center of mass are the same: $S_{\mu\nu}\,G_\mu = 0$.

This is the case that was studied before. The droplet obeys the same dynamics as Møller's droplet:

$$\dot{U}_\mu = 0, \qquad \dot{S}_{\mu\nu} = 0.$$

2) $Q_\mu = 0$: The center of mass and the center matter are the same, which translates into the relation $S_{\mu\nu}\;G_\mu = 0$, which one recognizes to be the Frenkel-Weyssenhoff "auxiliary kinematical condition."

Thus, one sees that the Frenkel particle (or the Weyssenhoff drop) realizes a particular case of the general motion of a fluid drop in rotation, namely, the one in which *the center of matter and the center of mass are identical.*

Thus, the Bohm-Vigier drop provides a generalization of a relativistic spinning particle quite well, which was what we desired at the beginning of the present chapter. Upon endowing the internal angular momentum with time components in the proper system, components that express the separation of the center of matter from the center of mass, we introduced new parameters that one might employ, not only to represent the "classical" Dirac particle, but, more generally, to constitute fluids that are endowed with internal rotation that is capable of propagating the various waves of quantum mechanics.

In fact, in a recent paper [**51**], which we shall be content to merely point out, Vigier, Hillion, and Lochak succeeded in quantifying the general motion of the drop by making a fixed number of stable excitation states appear for which the internal motion was periodic. Thus, they recovered the classification of the elementary particles in such a way that provided the experiment and its result that allows us to show, for the first time, the relationship between the wave functions of the various elementary particles and the various states of the same concretely-described material structure, which permits us to attribute considerable theoretical significance to the model of the drop.

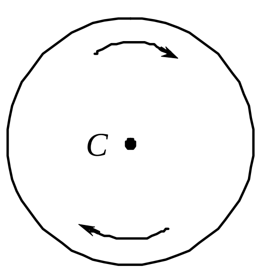
CHAPTER III

THE STUDY OF SOME PARTICULAR CASES OF MOTION

In the preceding chapter, we obtained a global expression for the motion of a classical fluid drop, which is an expression that we shall consider to be the basis for a new dynamics of material particles. It is a dynamic for which the classical dynamics of material points constitutes a first approximation, and which introduces several distinct points (three, in the general case) as a representation of a particle and shows itself to be capable of describing, without contradiction, the types of global motion that reflect the influence of the internal motions of the matter of the particle. Upon choosing a drop of classical fluid as a "model," in preference to a solid elastic mass or to any other type of body, one narrows the scope of the motion, but one gains the advantage of recovering the motions that were studied before, and have been used by various authors as special cases. One thus comes to an interesting generalization of this work that is, at the same time, an interpretation of their hypotheses that begins in classical hydrodynamical terms.

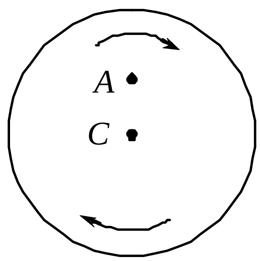
In the present chapter, we shall develop the new dynamics of particles in the general case, as well as in various special cases in which we confine ourselves to the case of the free particle in the absence of external forces. We shall commence by studying the case of the Frenkel-Weyssenhoff particle in detail, because it will provide us with some interesting relationships that we will not attempt to generalize.

§ 1. Weyssenhoff motion: its significance. If we are given the equations that we began with, which represent the evolution of the global quantities that characterize a drop entirely, one might demand that some of the properties of the local motion at each point should correspond to this or that special case. One might therefore propose the following interpretation of the Frenkel-Weyssenhoff motion that if one is given the condition $S_{\mu\nu} U_\nu = 0$ then that will translate into the idea that the center of matter and the center of mass are constantly identical [**14**].



By way of example, consider a homogeneous fluid sphere whose center C is immobile in a proper reference frame Σ, such that fluid rotates about an axis that passes through C and which we take to be perpendicular to the plane of the figure.

If the sphere rotates as a unit, or if the rotation is performed in a laminar fashion by spherical or cylindrical layers of varying velocities, then the point C, which is the center of matter, will also be the center of mass, by symmetry. Incidentally, in the absence of external forces, the total momentum relative to the proper system will be zero, and the motion of the point C will be a classical motion. The global kinematical properties will not be modified if the rotation of the fluid is more complicated – for example, if it involves a point, such as *A*, in a small vorticial structure that is carried along with a velocity **v**; i.e., *C* will still be the center of matter and the sphere will still be at rest in the system Σ, in the mean.



On the contrary, from the kinematical viewpoint, the vortex *A* will involve a supplementary energy *w* that will have the effect of:

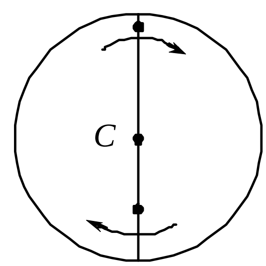
1) Displacing the center of mass towards the top, which will separate it from the center of matter, as a result.

2) Causing a supplementary momentum $w\mathbf{v} / c^2$ to appear that will, as a result, yield a residual momentum in the proper system that will therefore play the role of what we call the "transverse momentum."

Weyssenhoff motion corresponds to the case in which several vortices of the same species are arranged in such a fashion as to negate the first effect – in the event that the center of matter and the center of mass are identical – without negating the second one – in the event that one has a non-zero transverse momentum. It is indeed obvious that these two conditions are distinct. For example, consider two vortices A_1 and A_2 with energies w_1 and w_2 , respectively, that are situated on the same diameter on either side of *C* at distances r_1 and r_1 from that point.

The center of mass and the center of matter will be identical under the condition that $w_1 r_1 = w_2 r_2$. This condition translates into the Weyssenhoff condition that $S_{\mu\nu} U_\nu = 0$.

On the other hand, the transverse momentum will be:



$$\mathbf{p} = \frac{1}{c^2}\,(w_1\,\mathbf{v}_1 + w_2\,\mathbf{v}_2).$$

If the sphere spins as a unit like a solid body (up to vortices) with an angular velocity ω then the first condition $w_1 r_1 = w_2 r_2$ amounts to $w_1 \omega r_1 = w_2 \omega r_2$ or $w_1 \mathbf{v}_1 = w_2 \mathbf{v}_2$, and **p** is zero. However, it is natural to assume, as one usually does for fluid masses in rotation, that the velocity of rotation varies with the distance to the center, with the successive strata rotating about each other.

One has:

$$v_1 = r_1\omega_{(r_1)}, \qquad v_2 = r_2\omega_{(r_2)}, \qquad \text{and} \qquad p = \frac{1}{c^2}[w_1 r_1\omega_{(r_1)} - w_2 r_2\omega_{(r_2)}];$$

p might then be non-zero. More generally, if one has *n* vortices then the Weyssenhoff condition is written:

$$\sum_i w_i \mathbf{r}_i = 0,$$

and one has a momentum that is in proper space:

$$\mathbf{p} = \frac{1}{c^2}\sum_i w_i \mathbf{r}_i \omega(\mathbf{r}_i)\,.$$

It is quite obvious that these two conditions will remain constantly realized only if the vortices are permanent and are displaced by the current as if they were floating on water. Indeed, upon recalling the example of the vortices, we see that their orbital angular velocity will be different since **p** is non-zero. However, one of them will then displace with respect to the other one like the two hands of a watch. They will not lie on the same diameter, and the balance principle of Weyssenhoff will not be maintained. Conditions must then be imposed. For instance, the condition that the vortices must displace with the current, or, more generally, that there must be constant exchanges of energy between the vortices and the fluid layers in quasi-laminar rotation, so the vortices might disappear at one place and reappear at another in such a fashion that it would constantly maintain the coincidence of the center of mass and the center of matter.

Weyssenhoff motion might appear to be paradoxical in the sense that the center of mass remains constantly identical with the center of matter, because the center of matter is immobile in the proper system, while the center of mass is in motion in the same system. In reality, upon examining the Weyssenhoff case in detail, we will verify that the center of matter possesses a certain *acceleration* that is proportional to the transverse momentum, even in the absence of external forces. It will then be at rest in the proper system Σ_0, but accelerating, in such a way that at a later instant it will have acquired a certain velocity with respect to the system Σ_0, so Σ_0 will cease to be a proper system, since, as a Galilean system, it cannot be accelerating. This acceleration of the center of matter, which is characteristic of the non-Newtonian dynamics of the Weyssenhoff particle, therefore permits the center of mass, which is motion with respect to the successive proper reference frames, to remain in constant coincidence with the successive centers of matter, each of which is at rest in the proper reference frame to which it corresponds.

This same apparent paradox provides the key to the physical explanation for the Weyssenhoff motion. As we explained in the example of the vortices, the existence of a residual momentum in the proper system implies that the center of mass must be in motion in the proper system. However, this motion itself can be interpreted as only a certain internal deformation that the fluid undergoes, which will be a deformation that gives rise to antagonistic internal forces when one is given the existence of stresses. These will be forces whose effect, upon summation, translates into an acceleration that is applied to the center of matter. This acceleration, which obliges us to change to the proper reference frame at each instant, does not have a simple relationship with the center of mass, in general. On the contrary, it would be a special case in which it keeps the center of mass constantly identical with the center of matter. One then remains in the special case that is described by the Weyssenhoff equations.

§ 2. Weyssenhoff motion: the dynamical equations. To the two fundamental equations:

(III.1) $$\dot{G}_\mu = 0,$$

(III.2) $$\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu ,$$

we must add the two "auxiliary relations":

(III.3) $$U_\mu U_\mu = - c^2,$$
so
(III.4) $$U_\mu \dot{U}_\mu = 0,$$
and
(III.5) $$U_\mu \ddot{U}_\mu = - \dot{U}_\mu \dot{U}_\mu ,$$
and
(III.6) $$S_{\mu\nu} U_\nu = 0,$$
so
(III.7) $$S_{\mu\nu} \dot{U}_\nu = -\dot{S}_{\mu\nu} U_\nu .$$

One needs to use the important scalar quantities:

1) The proper mass of momentum M_0 : $-M_0^2 c^2 = G_\mu G_\mu$.

We know that this is *constant.*

2) The norm of the internal angular momentum: $\Sigma_0^2 = \frac{1}{2} S_{\mu\nu} S_{\mu\nu}$.

This norm is likewise *constant*. Indeed, one has:

$$\Sigma_0 \dot{\Sigma}_0 = \tfrac{1}{2} S_{\mu\nu} \dot{S}_{\mu\nu} = \tfrac{1}{2} S_{\mu\nu} (G_\mu U_\nu - G_\nu U_\mu).$$

The two terms go to zero separately, by virtue of relation (6), and one will therefore have:

$$\dot{\Sigma}_0 = 0 \qquad \text{and} \qquad \Sigma_0 = \text{const.}$$

3) The proper mass of inertia $\mathfrak{M}_0$: $- \mathfrak{M}_0 c^2 = G_\mu U_\mu$.

Equation (2), when contracted by U_ν, produces:

$$\dot{S}_{\mu\nu} U_\nu = - G_\mu c^2 + \mathfrak{M}_0 c^2 U_\mu ,$$

which, upon taking (7) into account, will give:

(III.8) $$\boxed{G_\mu = \mathfrak{M}_0 U_\mu + \frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu ,}$$

which is Weyssenhoff's first dynamical equation (recall that the vector on the left-hand side is constant).

Now, contract this equation with $\dot{U}_\mu$:

$$G_\mu \dot{U}_\mu = \mathfrak{M}_0 U_\mu \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu \dot{U}_\mu .$$

The first term on the right-hand side is zero, from (4). The second term is likewise zero, by reason of the antisymmetry of $S_{\mu\nu}$. What will then remain is:

(III.9) $$\boxed{G_\mu \dot{U}_\mu = 0.}$$

It results from this immediately, by virtue of (1), that the derivative of $G_\mu U_\mu$ – i.e., that of $\mathfrak{M}_0$ – is zero. The proper mass of inertia, as well as the proper mass of momentum, is *constant* in the course of motion. We may then differentiate equation (8), while taking (1) into account:

$$0 = \mathfrak{M}_0 \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu + \frac{1}{c^2} \dot{S}_{\mu\nu} \dot{U}_\nu .$$

The last term is zero:

$$\dot{S}_{\mu\nu} \dot{U}_\nu = 0.$$

One sees this by specifying that:

$$\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu ,$$

and upon taking (4) and (9) into account. What then remains is:

(III.10) $$\boxed{\mathfrak{M}_0 \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu = 0.}$$

The second term is zero, as a consequence of the antisymmetry of $S_{\mu\nu}$. It results from this that:

$$\dot{U}_\mu \ddot{U}_\mu = 0,$$

and, as a result, that the scalar square:

$$\dot{U}_\mu \dot{U}_\mu$$

is *constant*. We therefore make a fourth scalar appear, viz., the *norm of the spacetime acceleration:*

$$\gamma_0 = \sqrt{\dot{U}_\mu \dot{U}_\mu} ,$$

which is constant.

It is important to remark that equation (10) is of order two with respect to the unit-speed velocity of the center of mass (or of order three with respect to its coordinates). After integrating, it then remains for us to choose two systems of arbitrary constants.

One might therefore give it not only an initial velocity, but also an initial acceleration, which will then generally be non-zero, even though there are no external forces. One sees that one then finds oneself in the presence of a dynamic that is profoundly different from Newtonian mechanics.

§ 3. Weyssenhoff motion: spin. Following Frenkel, Weyssenhoff decomposed the internal angular momentum in the following fashion: In each reference frame, he considered the purely spatial components S_{12}, S_{23}, S_{31}, which constitute an antisymmetric space tensor, and then he took the spatial dual of this tensor, which is a space vector:

$$s_k = \tfrac{1}{2}\varepsilon_{ijk} S_{ij} .$$

On the other hand, he defined another vector from the temporal components of $S_{\mu\nu}$:

$$t_k = iS_{4k} .$$

Relation (6) then translates into the hypothesis that the vector t_k is zero in the proper reference frame. Louis de Broglie has remarked [**1**] that the vector s_k conveniently represents spin *in the proper system*, but this is no longer the case in any other system, because the decomposition considered is not covariant. One then regards Weyssenhoff's space vector:

$$\sigma_k^0 = (s_k)^0$$

relative to the particular decomposition in the *proper system* (which is a vector that expresses all of the non-zero components of $S_{\mu\nu}$ in the proper system) and considers it to be a *space-time* vector whose temporal component in the proper system is zero:

$$\sigma_4^0 = 0.$$

It is then possible to express the relation between this vector s_μ, which represents *spin*, and the internal angular momentum $S_{\mu\nu}$ [**18**]. In the proper system, one has:

(III.11) $$\sigma_k^0 = \tfrac{1}{2}\varepsilon_{ijk} S_{ij}^0 .$$

We then establish a correspondence between the permutations of the axes in space and the permutations of the axes in space-time by setting, by convention:

$$\varepsilon_{ijk} = \varepsilon_{ijk4} ,$$

so

$$\varepsilon_{ijk} = -\,\varepsilon_{4ijk} .$$

.

Multiply and divide the right-hand side of (11) by ic, which is the fourth (and only non-zero) component of the unit-speed velocity in the proper system. One will then get:

$$\sigma_k^0 = \frac{1}{2ic}\varepsilon_{ijk4}(S_{ij}^0 U_4)^0 .$$

However, this relation may also be written as:

$$\sigma_\alpha = \frac{1}{2ic}\varepsilon_{\mu\nu\alpha\beta}(S_{\mu\nu} U_\beta)^0,$$

because the terms in which $\beta \neq 4$ are all zero, and on the other hand, the fourth projection gives precisely $\sigma_4^0 = 0$.

In this form, the relation is covariant, and therefore valid in any system. One may therefore *define* the spin to be a space-time dual by setting:

(III.12)
$$\boxed{\sigma_\alpha = \frac{1}{2c}\varepsilon_{\beta\mu\nu\alpha}U_\beta S_{\mu\nu}.}$$

One will then have:

$$\sigma_\alpha U_\alpha = \frac{i}{2c}\varepsilon_{\mu\nu\alpha\beta}U_\beta U_\alpha S_{\mu\nu} = 0,$$

by antisymmetry.

Relation (12) then necessarily leads to the "auxiliary relation":

(III.13)
$$\sigma_\alpha U_\alpha = 0$$

that Costa de Beauregard introduced in his thesis as a postulate.

If we multiply both sides of (12) by U_λ and take duals then it will follow that:

$$\frac{i}{2}\varepsilon_{\alpha\lambda\gamma\rho}\,\sigma_\alpha U_\lambda = -\frac{1}{4c}\,\varepsilon_{\beta\mu\nu\alpha}\,\varepsilon_{\alpha\lambda\gamma\rho}\,U_\lambda U_\beta S_{\mu\nu},$$

$$= \frac{1}{4c}\delta_{\lambda\gamma\rho}^{\beta\mu\nu}U_\lambda U_\beta S_{\mu\nu},$$

where the generalized Kronecker symbols $\delta_{\lambda\gamma\rho}^{\beta\mu\nu}$ represents the sum of all products $\delta_\lambda^\beta\,\delta_\gamma^\mu\,\delta_\rho^\nu$ when one permutes the lower indices in every possible way, while prefixing a – sign for the odd permutations [**19**]. It then follows that:

$$\frac{i}{2}\varepsilon_{\alpha\lambda\gamma\rho}\,\sigma_\alpha U_\lambda = \frac{1}{4c}[U_\lambda S_{\gamma\rho}U_\lambda + U_\gamma S_{\rho\lambda}U_\lambda + U_\rho S_{\lambda\rho}U_\lambda$$
$$- U_\lambda S_{\rho\gamma}U_\lambda - U_\rho S_{\gamma\lambda}U_\lambda - U_\gamma S_{\lambda\rho}U_\lambda],$$

or, upon taking (3) and (6) into account:

$$\frac{i}{2}\,\varepsilon_{\alpha\lambda\gamma\rho}\,\sigma_\alpha\,U_\lambda = -\frac{2c^2}{4c}\,S_{\gamma\rho}\,,$$

or finally:

(III.14)
$$\boxed{S_{\gamma\rho} = \frac{i}{c}\,\varepsilon_{\gamma\rho\lambda\alpha}U_\lambda\sigma_\alpha.}$$

Therefore, for a given velocity field U_μ , one establishes a bijective correspondence between the antisymmetric, internal angular momentum tensors, which are orthogonal to the current, and the spin vectors, which are likewise orthogonal to the current. Thus, in order to express all of the components of the internal angular momentum, it is no longer necessary that it be just a spatial spin vector, but it can now be a *space-time* vector. In this new formulation, Weyssenhoff's vector t_k is intrinsically zero, and not just in the proper system. We add that it will reappear in the form of a covariant spacetime vector when we pass to the study of the general case, in which the center of mass is not identical with the center of matter, and in which $S_{\mu\nu}\,U_\nu \neq 0$.

It is easy to see that the norm of the spin is the same as that of the angular momentum. If one replaces $S_{\mu\nu}$ with its expression that one derives from (14) in the expression:

$$\Sigma_0^2 = \tfrac{1}{2}\,S_{\mu\nu}\,S_{\mu\nu}$$

then:

$$\Sigma_0^2 = -\frac{1}{2c^2}\,\varepsilon_{\mu\nu\alpha\beta}\,U_\alpha\,\sigma_\beta\,\varepsilon_{\mu\nu\lambda\rho}\,U_\lambda\,\sigma_\rho = -\frac{1}{2c^2}\,\delta^{\alpha\beta}_{\lambda\rho}\,U_\alpha\,\sigma_\beta\,U_\lambda\,\sigma_\rho\,,$$

where the generalized Kronecker symbols $\delta^{\alpha\beta}_{\lambda\rho}$ are simply $\delta^\alpha_\lambda\delta^\beta_\rho - \delta^\alpha_\rho\delta^\beta_\lambda$.

Thus:

$$\Sigma_0^2 = -\frac{1}{2c^2}\,(U_\alpha\,\sigma_\beta\,U_\alpha\,\sigma_\beta - U_\alpha\,\sigma_\beta\,U_\beta\,\sigma_\alpha),$$

or, upon taking (4) and (13) into account:

$$\Sigma_0^2 = \frac{1}{c^2}\,c^2\sigma_\beta\sigma_\beta = \sigma_0^2\,,$$

if we call the norm of the spin σ_0 .

One may generalize this calculation and thus obtain a very useful identity: Form the semi-contracted product:

$$\begin{aligned} S_{\mu\nu}\,S_{\mu\lambda} &= -\frac{1}{c^2}\,\varepsilon_{\mu\nu\alpha\beta}\,U_\alpha\,\sigma_\beta\,\varepsilon_{\mu\nu\gamma\rho}\,U_\gamma\,\sigma_\rho\,, \\ &= -\frac{1}{c^2}\,\delta^{\nu\alpha\beta}_{\lambda\gamma\rho}\,U_\alpha\,\sigma_\beta\,U_\gamma\,\sigma_\rho\,, \\ &= -\frac{1}{c^2}\,[\delta_{\nu\lambda}\,U_\alpha\,\sigma_\beta\,U_\alpha\,\sigma_\beta + U_\alpha\,\sigma_\lambda\,U_\nu\,\sigma_\rho + U_\lambda\,\sigma_\beta\,U_\beta\,\sigma_\nu\,, \end{aligned}$$

$$- \delta_{\nu\lambda}\, U_\alpha\, \sigma_\beta\, U_\beta\, \sigma_\alpha - U_\lambda\, \sigma_\beta\, U_\nu\, \sigma_\beta - U_\alpha\, \sigma_\lambda\, U_\alpha\, \sigma_\nu\,],$$

namely, upon taking (4) and (13) into account:

(III.15) $$\boxed{S_{\mu\nu}S_{\mu\lambda} = \sigma_0^2\left(\delta_{\nu\lambda} + \frac{U_\nu U_\lambda}{c^2}\right) - \sigma_\nu\sigma_\mu = \sigma_0^2\eta_{\nu\lambda} - \sigma_\nu\sigma_\lambda.}$$

Upon introducing the dual of the angular momentum:

$$\hat{S}_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\alpha\beta}\, S_{\alpha\beta} = \frac{1}{c}(\sigma_\mu\, U_\nu - \sigma_\nu\, U_\mu),$$

one may perform an analogous calculation. It will then follow that:

$$\hat{S}_{\mu\nu}\hat{S}_{\mu\lambda} = \sigma_0^2\,\frac{U_\nu U_\lambda}{c^2} - \sigma_\nu\,\sigma_\lambda\,,$$

which is a formula that gives us, in particular:

$$\tfrac{1}{2}\hat{S}_{\mu\nu}\hat{S}_{\mu\lambda} = -\,\sigma_0^2 \qquad \text{(one has that } \delta_{\nu\nu} = 4\text{)},$$

and similarly:

$$S_{\mu\nu}\hat{S}_{\mu\lambda} = 0.$$

Finally, one may contract $S_{\mu\nu}$ and $\hat{S}_{\mu\nu}$ with σ_ν:

$$S_{\mu\nu}\,\sigma_\nu = \frac{i}{c}\,\varepsilon_{\mu\nu\alpha\beta}\, U_\alpha\, \sigma_\beta\, \sigma_\nu = 0,$$

and, by antisymmetry:

$$\hat{S}_{\mu\nu}\,\sigma_\nu = \frac{1}{c}(\sigma_\mu\, U_\nu - \sigma_\nu\, U_\mu)\sigma_\nu = -\frac{\sigma_0^2}{c}\, U_\mu\,.$$

§ 4. Weyssenhoff motion: transverse momentum. We shall involve the spin in the expressing of the two laws of dynamics. The first law:

$$G_\mu = \mathfrak{M}_0 U_\mu + \frac{1}{c^2}\, S_{\mu\nu}\dot{U}_\nu$$

decomposes the momentum into a vector that is collinear with the current (and has the form the classical momentum) and a vector:

$$\frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu$$

that is orthogonal to the current, since $S_{\mu\nu} U_\nu = 0$.

It is useful to clarify the meaning of the latter vector, which represents non-classical momentum, or *transverse momentum*. We set:

(III.16) $$P_\mu = -\frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu ,$$

with

(III.17) $$G_\mu = \mathfrak{M}_0 U_\mu - P_\mu ,$$

and

(III.18) $$P_\mu U_\mu = 0.$$

Upon replacing $S_{\mu\nu}$ as a function of spin, it will then follow that:

(III.19) $$P_\mu = \frac{i}{c^3} \varepsilon_{\mu\nu\alpha\beta} U_\nu \dot{U}_\alpha \sigma_\beta .$$

It results immediately from this by antisymmetry that:

(III.20) $$P_\mu \sigma_\mu = 0,$$

and

(III.21) $$P_\mu \dot{U}_\mu = 0.$$

If one expresses this relation in the proper system then one will get:

$$P_k^0 = \frac{i}{c^3} \varepsilon_{k4ij} ic\, \gamma_i^0 \sigma_j^0 = \frac{i}{c^2} \varepsilon_{ijk}\, \sigma_i^0 \gamma_j^0 ,$$

which one may write in vectorial notation as:

$$\mathbf{P} = \frac{1}{c^2} \boldsymbol{\sigma} \times \boldsymbol{\gamma}.$$

Thus, the transverse momentum is represented by the vector product of the spin with the acceleration in the proper system.

On the other hand, the second dynamical law:

$$\mathfrak{M}_0 \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu = 0$$

gives:

$$\mathfrak{M}_0 \dot{U}_\mu + \frac{i}{c^3} \varepsilon_{\mu\nu\alpha\beta} U_\alpha \sigma_\beta \dot{U}_\nu = 0/$$

If one contracts this with σ_μ then the second term will go to zero, by antisymmetry, and what will remain is:

(III.22) $$\boxed{\dot{U}_\mu \sigma_\mu = 0.}$$

This relation completes the set of relations (4), (13), (20), (21), and shows that the four space-time vectors U_μ, $\dot{U}_\mu$, P_μ, and σ_μ are pair-wise orthogonal. Along the world-line that is described by the center of matter, they form a system of four orthogonal axes that generalizes the Darboux-Frenet moving frame. Then again, if one places oneself in the proper system then the three vectors – viz., $\dot{U}_\mu$, P_μ, and σ_μ – which are in proper space, will form a tri-rectangular trihedron. If one is given this arrangement then one can define three relations between these four vectors that are analogous to (19) and provide the expressions for U_μ, $\dot{U}_\mu$, and σ_μ, and likewise one may define relations between bivectors. For example, multiply the two sides of (19) by U_λ and take duals:

$$\frac{i}{c}\,\varepsilon_{\mu\nu\lambda\beta}\,P_\mu\,U_\lambda = -\frac{1}{2c^2}\,\varepsilon_{\mu\nu\lambda\beta}\,\varepsilon_{\mu\nu\lambda\beta}\,U_\nu\,\dot{U}_\alpha\,\sigma_\beta\,U_\lambda = -\frac{1}{2c^2}\,\delta^{\lambda\gamma\rho}_{\nu\alpha\beta}\;U_\nu\,U_\alpha\,\sigma_\beta\,U_\lambda\,,$$

which gives, upon taking into account the orthogonality relations:

$$\frac{i}{2}\,\varepsilon_{\mu\nu\lambda\beta}\,P_\alpha\,U_\beta = \frac{1}{2c}(\dot{U}_\gamma\sigma_\rho - \dot{U}_\rho\sigma_\gamma)\,.$$

There exists a natural relation between the norms of the four vectors considered, which are norms that are constant for all four of them. Indeed, we remark that relation (17), whose three vectors define a tri-rectangular trihedron, then gives us as a consequence that:

$$G_\mu\,G_\mu - \mathfrak{M}_0^2\,U_\mu\,U_\mu + P_\mu\,P_\mu\,.$$

Now, if one sets:

$$P_\mu\,P_\mu = \mathfrak{P}_0^2 c^2$$

(so $\mathfrak{P}_0$ now plays the role of a mass that expresses a non-classical energy in the proper system) then one will get:

(III.23) $$\mathfrak{P}_0^2 = \mathfrak{M}_0^2 - M_0^2 = \text{constant}.$$

If one then forms the scalar square of P_μ, when expressed as a function of $S_{\mu\nu}$, then one will get:

$$P_\mu\,P_\mu = \mathfrak{P}_0^2 c^2 = \frac{1}{c^4}\,S_{\mu\nu}S_{\mu\lambda}\dot{U}_\nu\dot{U}_\lambda\,,$$

or, from (15):

$$\mathfrak{P}_0^2 c^2 = \frac{1}{c^4}[\sigma_0^2\eta_{\mu\nu} - \sigma_\lambda\sigma_\nu]\dot{U}_\lambda\dot{U}_\nu = \frac{\sigma_0^2\dot{U}_\nu\dot{U}_\nu}{c^4} = \frac{\sigma_0^2\gamma_0^2}{c^4}\,,$$

when one takes (4) and (22) into account.

Therefore, the desired relation between the four invariants is:

$$\boxed{\sigma_0\gamma_0 = \mathfrak{P}_0 c^2.} \tag{III.24}$$

Finally, we construct the derivative of the spin in the form (12):

$$\dot{\sigma}_\mu = \frac{i}{2c}\varepsilon_{\nu\alpha\beta\mu}(\dot{U}_\nu S_{\alpha\beta} + U_\nu \dot{S}_{\alpha\beta}).$$

If one replaces $\dot{S}_{\alpha\beta}$ with its expression in (2) in the right-hand side then one will see that this term goes to zero by antisymmetry. If one replaces $S_{\alpha\beta}$ with (14) in the left-hand side then it will follow that:

$$\begin{aligned}\dot{\sigma}_\mu &= \frac{i}{2c}\varepsilon_{\nu\alpha\beta\mu}\dot{U}_\nu \frac{i}{c}\varepsilon_{\alpha\beta\gamma\rho}U_\gamma\sigma_\rho \\ &= \frac{1}{c^2}\delta^{\mu\nu}_{\gamma\rho}\dot{U}_\nu U_\gamma \sigma_\rho \\ &= \frac{1}{c^2}(\dot{U}_\nu U_\mu \sigma_\nu - \dot{U}_\nu U_\nu \sigma_\mu).\end{aligned}$$

The two sides go to zero, by virtue of (22) and (24).

What will then remain is:

$$\boxed{\dot{\sigma}_\mu = 0.}$$

The spin thus provides a second *constant* spacetime vector, in addition to momentum. Ultimately, this important property results directly from the fact that spin, which is a proper-space vector, is perpendicular to the acceleration. Indeed, if we take the proper system Σ_0 at a given instant to be our reference frame, and we consider the center of matter at the end of the time Δt in this system then this point, which is accelerating, will possess a certain velocity *in the direction of the acceleration vector*, and the Lorentz transformation will not involve any variation of the vector σ_k, since it will be perpendicular to the velocity.

§ 5. Weyssenhoff motion: integration. We may now transform the expression for the second dynamical law in such a fashion as to render it easily integrable. Contract (10) with $S_{\mu\nu}$:

$$-\mathfrak{M}_0\dot{U}_\mu S_{\lambda\mu} + \frac{1}{c^2}S_{\mu\nu}S_{\mu\lambda}\ddot{U}_\nu = 0,$$

or, from (15) and (16):

$$\mathfrak{M}_0\, c^2 P_\lambda + \frac{1}{c^2}\left[\sigma_0^2\left(\ddot{U}_\lambda + U_\lambda \frac{U_\nu \ddot{U}_\nu}{c^2}\right) - \sigma_\lambda \sigma_\lambda \ddot{U}_\nu\right] = 0.$$

Upon remarking that if one differentiates $\sigma_\nu \dot{u}_\nu = 0$ then it will follow that:

$$\sigma_\nu \ddot{U}_\nu + \dot{\sigma}_\nu \dot{U}_\nu = 0,$$

or, since:

$$\dot{\sigma}_\nu = 0, \qquad \sigma_\nu \ddot{U}_\nu = 0,$$

one sees that the last term is zero, and upon taking (5) into account, it will then follow that:

$$\mathfrak{M}_0\, c^2 P_\lambda + \frac{1}{c^2}\sigma_0^2\left(\ddot{U}_\lambda - U_\lambda \frac{\gamma_0^2}{c^2}\right) = 0,$$

or, from (24):

$$\sigma_0^2 \ddot{U}_\lambda + c^4(M_0^2\, U_\lambda - \mathfrak{M}_0\, G_\lambda) = 0.$$

One facilitates the integration by introducing the constant vector G_μ by means of (17) and (23):

$$\boxed{\sigma_0^2 \ddot{U}_\lambda + c^4(M_0^2 U_\lambda - \mathfrak{M}_0 G_\lambda) = 0.}$$

This equation may integrated simply by taking $M_0^2\, U_\lambda - \mathfrak{M}_0\, G_\lambda$ for the variable. One then obtains a sinusoidal function whose frequency Ω (as a function of *proper* time) is:

$$\Omega = \frac{M_0 c^2}{\sigma_0},$$

and the desired first integral is:

$$\boxed{M_0^2 U_\lambda = \mathfrak{M}_0 G_\lambda + A_\lambda e^{-i(\Omega\tau + \varphi_\lambda)}} \qquad \text{(no summation over } \lambda\text{);}$$

the A_λ and φ_λ are constants.

In order to succeed in carrying out this integration, one will benefit from constructing the radius vector R_μ that joins the center of matter to the center of *gravity*, which is fixed in the system of inertia. From (II.9), one has:

(III.25) $$M_0^2 c^2 R_\mu = S_{\mu\nu}\, G_\nu\, .$$

One knows that $R_\mu\, G_\mu = 0$. On the other hand, from (6), one has:

$$M_0^2 c^2 R_\mu\, U_\mu = S_{\mu\nu}\, G_\nu\, U_\mu = 0.$$

Therefore, the spacetime vector R_μ is simultaneously in proper space and the space of inertia. In other word, the positions of the center of mass and the center of gravity, which are simultaneous in the proper system, are also simultaneous in the system of inertia.

Since the radius vector is orthogonal to G_μ and U_μ, it is consequently also orthogonal to the transverse momentum P_μ. Finally, from (14), one has:

$$M_0^2 c^2 R_\mu \sigma_\mu = \frac{i}{c} \varepsilon_{\mu\nu\alpha\beta} U_\alpha \sigma_\beta G_\nu \sigma_\mu = 0$$

by antisymmetry.

The radius vector is therefore orthogonal to U_μ, P_μ, and σ_μ. It is therefore collinear to the fourth axis of our Frenet system; i.e., to $\dot{U}_\mu$. One gets this immediately upon substituting (8) in (25):

$$M_0^2 c^2 R_\mu = S_{\mu\nu} (\mathfrak{M}_0 U_\nu + \frac{1}{c^2} S_{\nu\lambda} \dot{U}_\lambda) = -\frac{1}{c^2} S_{\mu\nu} S_{\lambda\nu} \dot{U}_\lambda ,$$

or, from (15), and upon taking (4) and (22) into account:

$$\text{(III.26)} \qquad M_0^2 c^4 R_\mu = -\sigma_0^2 \dot{U}_\mu .$$

In particular, it results from this that the radius vector, like the acceleration, possesses a constant norm $\mathfrak{R}_0$:

$$\mathfrak{R}_0^2 = R_\mu R_\mu = \frac{\sigma_0^2 \gamma_0^2}{M_0^4 c^8} = \frac{\sigma_0^2 \mathfrak{P}_0^2 c^6}{M_0^4 c^8} ,$$

from (24). Thus, one finally has:

$$\boxed{\mathfrak{R}_0 = \frac{\sigma_0 \mathfrak{P}_0}{M_0^2 c}.}$$

If we differentiate the expression for R_μ then it follows that:

$$\begin{aligned} M_0^2 c^2 \dot{R}_\mu &= \dot{S}_{\mu\nu} G_\nu = (G_\mu U_\nu - G_\nu U_\mu) G_\nu \\ &= -\mathfrak{M}_0 c^2 G_\mu + M_0^2 c^2 U_\mu , \end{aligned}$$

such that:

$$\text{(III.27)} \qquad \boxed{\dot{R}_\mu = U_\mu - \frac{\mathfrak{M}_0}{M_0^2} G_\mu .}$$

It is easy to interpret and verify this relation by projecting it into the system of inertia. Indeed, one obtains:

$$\text{(III.28)} \qquad \dot{R}_k^I = U_k^I ,$$

since the spatial components of G_μ are zero. If we then consider the velocity v_k (with norm v_0) of the center of matter in the system of inertia, which is also the velocity of the proper system with respect to the system of inertia, and if one sets $\alpha = \left(1 - v_0^2 / c^2\right)^{-1/2}$ then one will have:

$$U_k^I = \alpha v_k \qquad \text{and} \qquad \dot{R}_k^I = \frac{d}{d\tau} R_k^I = \alpha \frac{d}{dt^I} R_k^I = v_k$$

(t^I is time, in the system of inertia), and relation (28) may be written:

$$\frac{d}{dt^I} R_k^I = v_k ,$$

which is obvious, since the origin of R_k^I is at rest in the system of inertia.

Incidentally, as long as one considers the system of inertia, one sees that R_k^I is orthogonal to the constant spatial vector σ_k^I, and its length $\mathfrak{R}_0$ is constant. The center of matter thus describes a circular motion around the center of gravity in a plane that is orthogonal to the spin. As we have seen that the unit-speed velocity is a sinusoidal function of time, the circular motion will be uniform, and the frequency Ω that we calculated will represent the angular velocity *as a function of proper time*. Furthermore, one will verify this immediately upon differentiating (28) and substituting the expression that is obtained for $\dot{U}_k^I$ into equation (26), when it is projected onto the spatial axes of the system of inertia.

It then follows that:

(III.29) $$M_0^2 c^2 R_k^I = - \sigma_0^2 \ddot{R}_k^I ,$$

which is an equation that can be integrated immediately, and provides a frequency:

$$\Omega = \frac{M_0 c^2}{\sigma_0} ,$$

as well.

However, equation (29) may also be expressed as a function of time t^I in the reference frame of inertia and integrated. In order to do this, one must calculate the Lorentz coefficient $\alpha = (1 - v_0^2 / c^2)^{-1/2}$.

One easily arrives at it upon contracting (27) by U_μ :

$$\dot{R}_\mu U_\mu = - c^2 + \frac{\mathfrak{M}_0^2}{M_0^2} c^2 = \frac{\mathfrak{M}_0^2 - M_0^2}{M_0^2} c^2 .$$

We then express the left-hand side in the system of inertia, where:

$$U_k^I = \alpha v_k , \qquad \dot{R}_k^I = \alpha v_k ,$$

and where, in addition, one has:

$$\dot{R}_4^I = \alpha ic - \frac{\mathfrak{M}_0}{M_0^2} G_4^I = ic \left(\alpha - \frac{\mathfrak{M}_0}{M_0} \right)$$

for the fourth component. It then follows that:

$$c^2 \frac{\mathfrak{M}_0^2 - M_0^2}{M_0^2} = \alpha^2 v_0^2 + \alpha ic \cdot ic \left(\alpha - \frac{\mathfrak{M}_0}{M_0} \right)$$

such that:

$$\frac{\mathfrak{M}_0^2}{M_0^2} - 1 = \alpha^2 \left(\frac{v^2}{c^2} - 1 \right) + \alpha \frac{\mathfrak{M}_0}{M_0},$$

or, since:

$$\alpha^2 \left(\frac{v^2}{c^2} - 1 \right) = -1,$$

$$\frac{\mathfrak{M}_0^2}{M_0^2} = \alpha \frac{\mathfrak{M}_0}{M_0},$$

one finally has:

(III.30) $$\boxed{\alpha = \frac{\mathfrak{M}_0}{M_0}.}$$

Substituting this result in equation (29) will then give:

$$M_0^2 c^4 R_k^I = -\sigma_0^2 \frac{d^2}{d(t^I)^2} R_k^I \alpha^2 = -\sigma_0^2 \frac{\mathfrak{M}_0^2}{M_0^2} \frac{d^2}{d(t^I)^2} R_k^I,$$

which, after integration, will give an angular velocity as a function of time t^I:

(III.31) $$\boxed{\omega^I = \frac{M_0^2 c^2}{\sigma_0 \mathfrak{M}_0}.}$$

This result was given before by Weyssenhoff [**2**], but the axiomatic basis for his theory does not permit one to specify that point as being fixed and spinning.

Meanwhile, we may give the form that one finds for the expression of the velocity in the system of inertia, because one deduces it immediately from formula (19). The relation:

$$G_\mu = \mathfrak{M}_0 U_\mu - P_\mu,$$

when projected onto the system of inertia, gives:

$$G_k = 0 = \mathfrak{M}_0\, U_k - P_k^I\,,$$

$$P_k^I = \mathfrak{M}_0 U_k^I = \alpha\, v_k\,.$$

One will get:

$$\mathfrak{M}_0\, \alpha\, v_k = \frac{i}{c^3}\, \varepsilon_{k4ij}\, \alpha\, ic \cdot \alpha^2\, \Gamma_i\, \sigma_j^I\,,$$

from this if one lets Γ_i denote the classical acceleration in the system of inertia:

$$\dot{U}_k^I = \alpha^2\, \Gamma_k\,,$$

and

$$\mathfrak{M}_0\, v_k = \frac{\alpha^2}{c^2}\varepsilon_{ijk}\sigma_i^I\Gamma_j = \frac{\mathfrak{M}_0^2}{M_0^2 c^2}\varepsilon_{ijk}\sigma_i^I\Gamma_j\,,$$

so:

$$v_k = \frac{\mathfrak{M}_0}{M_0^2 c^2}\varepsilon_{ijk}\sigma_i^I\Gamma_j\,,$$

which one may write simply as:

(III.32)
$$\boxed{\mathbf{v} = \frac{M_0}{M_0^2 c^2}\boldsymbol{\sigma}\times\boldsymbol{\Gamma}.}$$

The kinematic significance of this relation is immediate if one recalls that from (27) and (29), one has:

$$\dot{U}_k^I = \alpha^2\, \Gamma_k = \ddot{R}_k^I = -\frac{M_0^4 c^4}{\mathfrak{M}_0^2 \sigma_0^2}\mathbf{R}\,.$$

One may therefore replace the acceleration Γ with:

$$-\frac{1}{\alpha^2}\frac{M_0^2 c^4}{\mathfrak{M}_0 \sigma_0^2}\mathbf{R} = -\frac{M_0^4 c^4}{\mathfrak{M}_0^2 \sigma_0^2}\mathbf{R}\,.$$

Relation (32) then leads to:

$$\mathbf{v} = \frac{M_0^2 c^4}{\mathfrak{M}_0^2 \sigma_0^2}\mathbf{R}\times\boldsymbol{\sigma},$$

which is the classical expression for a uniform motion as a function of an angular velocity vector:

$$\boldsymbol{\omega} = \frac{M_0^2 c^2}{\mathfrak{M}_0 \sigma_0^2}\boldsymbol{\sigma},$$

namely:

(III.32′)
$$\mathbf{v} = \mathbf{R}\times\boldsymbol{\omega}.$$

The vector $\boldsymbol{\omega}$ is constant, since:

$$\dot{\sigma}_k^I = 0;$$

it has the quantity (31) for its norm, precisely.

§ 6. Wobble and gyration. We shall now study the general case, in which the angular momentum possesses non-zero time components in the proper system (e.g., the Bohm-Vigier drop [**59**, **60**]). We will follow the same plan as we did for the study of the Weyssenhoff drop. One has the two equations:

$$\dot{G}_\mu = 0,$$

(III.2) $$\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu ,$$

with the two auxiliary equations:

(III.3) $$U_\mu U_\mu = - c^2,$$

so:

(III.4) $$U_\mu \dot{U}_\mu = 0,$$

and $$U_\mu \ddot{U}_\mu = -\gamma_0^2 ,$$

and

(III.33) $$S_{\mu\nu} U_\nu = ct_\mu ,$$

so:

(III.34) $$t_\mu U_\mu = 0.$$

Meanwhile, recall that the proper space vector t_μ is related to the vector Q_μ that joins the center of matter to the center of mass in the proper space by:

$$t_\mu = - \mathfrak{M}_0 \, c \, Q_\mu .$$

In order to fix the vocabulary, we propose to call this vector the *wobble*, by analogy with certain notions in the classical mechanics of solid bodies.

Since the norm of G_μ is constant, that will permit us to define a proper mass of momentum:

$$M_0^2 c^2 = - G_\mu G_\mu = \text{constant}.$$

Similarly, one has the proper mass of inertia:

$$\mathfrak{M}_0 c^2 = - G_\mu U_\mu .$$

Upon contracting (2) with U_ν , one will get:

$$G_\mu = \mathfrak{M}_0 U_\mu + \frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu - \frac{1}{c^2} \dot{S}_{\mu\nu} U_\nu ,$$

or, upon differentiating (33):

(III.35) $$\boxed{G_\mu = \mathfrak{M}_0 U_\mu + \frac{1}{c^2} S_{\mu\nu} \dot{U}_\nu - \frac{1}{c^2} \dot{t}_\mu .}$$

The variation of the mass $\mathfrak{M}_0$ in time is given by:

(III.36) $$\dot{\mathfrak{M}}_0 c^2 = - G_\mu \dot{U}_\mu ,$$

since $\dot{G}_\mu = 0$.

The right-hand side may be calculated by contracting (35) with $\dot{U}_\mu$.

The first term is annulled, from (4).

The second term is annulled, by antisymmetry.

What remains is:

$$G_\mu \dot{U}_\mu = - \frac{1}{c} \dot{t}_\mu \dot{U}_\mu ,$$

so

(III.37) $$\boxed{\dot{\mathfrak{M}}_0 c^3 = - \dot{t}_\mu \dot{U}_\mu .}$$

Contrary to the case of the Weyssenhoff drop, the Bohm-Vigier drop has a mass of inertia that varies in the course of time, in general, which is a result that we will presently interpret.

Differentiating (35), which is the first dynamical equation, gives:

$$0 = \mathfrak{M}_0 \dot{U}_\mu + \dot{\mathfrak{M}}_0 U_\mu + \frac{1}{c^2} \dot{S}_{\mu\nu} \dot{U}_\nu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu - \frac{1}{c} \ddot{t}_\mu ,$$

or, upon taking (2) and (4) into account:

$$0 = \mathfrak{M}_0 \dot{U}_\mu + \dot{\mathfrak{M}}_0 U_\mu - \frac{1}{c^2} G_\nu U_\mu \dot{U}_\nu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu - \frac{1}{c} \ddot{t}_\mu .$$

From (36), one sees that the third term is two times the second one, so the second dynamical law will finally give:

$$\boxed{\mathfrak{M}_0 \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu = \frac{1}{c} \ddot{t}_\mu - 2\dot{\mathfrak{M}}_0 U_\mu .}$$

Upon contracting this with $\ddot{U}_\mu$ and taking (5) into account, one will get:

$$\mathfrak{M}_0 \gamma_0 \dot{\gamma}_0 = \frac{1}{c} \ddot{t}_\mu \ddot{u}_\mu + 2\dot{\mathfrak{M}}_0 \gamma_0^2 ,$$

in which the norm of the acceleration is no longer constant.

We shall now introduce the vector s_μ by an argument that is analogous to that of Weyssenhoff. In order to avoid any confusion with spin, properly speaking, which we will recall here shortly in an important special case that relates to quantum mechanics, in which this word consecrates a well-defined quantity by conscious intent, we shall propose to call the vector s_μ the *gyration*. Furthermore, we find that in the Weyssenhoff case the vector σ_μ that we have considered is identical to the vector that we call spin; hence, it is not convenient to employ the two words interchangeably in the case of the Weyssenhoff drop. If we consider the spatial components of the internal angular momentum *in the proper system* then it will constitute a spatial tensor S_{ij}^0, for which we take the dual in proper space:

$$s_k^0 = \tfrac{1}{2} \varepsilon_{ijk} S_{ij}^0 .$$

We then define the gyration to be the *space-time* vector that has s_1^0, s_2^0, s_3^0, and 0 for its components in the proper system, except that this vector will no longer involve all of the components of $S_{\mu\nu}$, since it does not take the temporal components into account. Following the same principle as Frenkel and Weyssenhoff, one can form a space vector $t_k^0 = iS_{k4}^0$ *in the proper system* by means of these temporal components, which will be a vector that is non-zero, this time. If one considers the *space-time* vector:

$$\boxed{t_\mu = \frac{1}{c} S_{\mu\nu} U_\nu}$$

then one will see that it coincides with the preceding vector in the proper system. We thus see the vector that Weyssenhoff defined from the temporal components of the angular momentum reappear in the general case. However, it now presents itself as a covariant spacetime vector that is not intrinsically zero, which is precisely what happened in the case that was treated by Weyssenhoff.

One may give the same covariant expression to the gyration:

(III.38) $$\boxed{s_\mu = \frac{i}{2c} \varepsilon_{\nu\alpha\beta\mu} U_\nu S_{\alpha\beta}}$$

by the same argument as above, so:

(III.39) $$s_\mu U_\mu = 0$$

and

(III.40) $$s_\mu \dot{U}_\mu = - \dot{s}_\mu U_\mu ,$$

and it is easy to see that, conversely, one can express the internal angular momentum in an unambiguous fashion as a function of the proper space vectors s_μ and t_μ .

Multiplying both sides of (38) by U_λ and taking duals gives:

$$\frac{i}{2}\,\varepsilon_{\mu\lambda\gamma\rho}\,s_\mu\,U_\lambda = \frac{1}{4c}\,\delta^{\nu\alpha\beta}_{\lambda\gamma\rho}\,U_\nu\,U_\lambda\,S_{\alpha\beta}\,,$$

or, upon taking (3) and (33) into account:

$$\frac{i}{2}\,\varepsilon_{\mu\lambda\gamma\rho}\,s_\mu\,U_\lambda = -\frac{c}{2}S_{\gamma\rho} + \frac{1}{2}(U_\gamma t_\rho - U_\rho t_\gamma)\,,$$

such that finally:

(III.41) $$\boxed{S_{\gamma\rho} = \frac{i}{c}\varepsilon_{\gamma\rho\mu\lambda}U_\mu s_\lambda + \frac{1}{2}(U_\gamma t_\rho - U_\rho t_\gamma).}$$

One sees that in a given space-time velocity field there is a two-to-one correspondence between the angular momenta that are represented by the *arbitrary* antisymmetric tensors and the systems of two proper space vector s_μ and t_μ. As we could have anticipated, we shall recover Weyssenhoff's proper space vectors s_k and t_k in proper space, but the latter vector will non-zero and the decomposition of $S_{\mu\nu}$ will be covariant.

For the moment, we may calculate various contracted products.

One has:

$$c\,S_{\mu\nu}\,s_\nu = (i\,\varepsilon_{\mu\nu\alpha\beta}\,U_\alpha\,s_\beta + U_\mu\;t_\nu - U_\nu\;t_\mu)\,s_\nu = U_\mu\,t_\nu\,s_\nu\,,$$

so

(III.42) $$S_{\mu\nu}\,s_\nu = \frac{t_\nu s_\nu}{c}U_\mu\,,$$

and similarly:

$$\hat{S}_{\mu\nu}t_\nu = -\frac{t_\nu s_\nu}{c}U_\mu\,.$$

One has, by analogy:

(III.43) $$S_{\mu\nu}\,t_\nu = \frac{1}{c}(i\,\varepsilon_{\mu\nu\alpha\beta}\,t_\nu\,U_\alpha\,s_\beta + t_0^2 U_\mu)$$

upon setting $t_\mu\,t_\mu = t_0^2$.

Similarly:

$$\hat{S}_{\mu\nu}s_\nu = \frac{1}{c}(-\,i\,\varepsilon_{\mu\nu\alpha\beta}\,t_\nu\,U_\alpha\,s_\beta \qquad s_0^2 U_\mu).$$

We may take the norm of the angular momentum tensor:

$$\Sigma_0^2 = \tfrac{1}{2}S_{\mu\nu}\,S_{\mu\nu} = \frac{1}{2c^2}\left[-2\delta^{\alpha\beta}_{\gamma\rho}U_\alpha s_\beta U_\gamma s_\rho + (U_\mu t_\nu - U_\nu t_\mu)^2\right],$$

because the diagonal terms are zero.

This being the case, and upon taking (3), (34), (39) into account, we have:

$$\Sigma_0^2 = \frac{1}{2c^2}(2c^2 s_0^2 - c^2 t_0^2 - c^2 t_0^2)\,,$$

or

$$\boxed{\Sigma_0^2 = s_0^2 - t_0^2.}$$

However, this time the norm is no longer constant. Indeed:

$$\Sigma_0 \dot{\Sigma}_0 = \tfrac{1}{2} S_{\mu\nu} \dot{S}_{\mu\nu}\,,$$

or, from (2) and (33):

$$\Sigma_0 \dot{\Sigma}_0 = \frac{2}{2}(ct_\mu\, G_\mu + ct_\nu\, G_\nu),$$

so

$$\boxed{\frac{d}{d\tau}\Sigma_0^2 \equiv 2\Sigma_0 \dot{\Sigma}_0 = 4ct_\mu G_\mu.}$$

Likewise, taking the norm of the dual gives:

$$\tfrac{1}{2}\hat{S}_{\mu\nu}\hat{S}_{\mu\nu} = t_0^2 - s_0^2 = -\Sigma_0^2\,.$$

Finally, one may form the semi-contracted product $S_{\mu\nu}\hat{S}_{\mu\nu}$, which will give, when one has completed all of the calculations:

$$S_{\mu\nu}\hat{S}_{\mu\nu} = S_{\nu\lambda}\, s_\mu\, t_\mu\,,$$

and which will give, in particular:

$$S_{\mu\nu}\hat{S}_{\mu\nu} = 4s_\mu\, t_\mu\,.$$

Finally, one may likewise construct the derivative of the scalar $\tfrac{1}{2} S_{\mu\nu}\hat{S}_{\mu\nu}$.

Upon taking the dual of (2), one will get:

$$\frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\dot{S}_{\alpha\beta} = \frac{d}{d\tau}\hat{S}_{\mu\nu} = i\ \varepsilon_{\mu\nu\alpha\beta}\, G_\alpha\, U_\beta\,.$$

Once all of the calculations have been completed, one will get from this that:

$$\boxed{\frac{d}{d\tau}\left(\tfrac{1}{2} S_{\mu\nu}\hat{S}_{\mu\nu}\right) = -4cs_\mu G_\mu.}$$

§ 7. Various relations in the general case. In order to examine the derivatives of the vector s_μ and t_μ, one differentiates (41), while taking (2) into account:

$$c(G_\mu U_\nu - G_\nu U_\mu) = i\,\varepsilon_{\mu\nu\alpha\beta}\,(\dot{U}_\alpha s_\beta + U_\alpha \dot{s}_\beta) + \dot{U}_\mu t_\nu - \dot{U}_\nu t_\mu + U_\mu \dot{t}_\nu - U_\nu \dot{t}_\mu\,.$$

Multiply the two sides by U_μ and take the duals, upon contracting with $i/2\ \varepsilon_{\mu\nu\lambda\rho}$. The left-hand side will go to zero by antisymmetry:

$$0 = -\frac{i}{2} 2\delta^{\alpha\beta}_{\lambda\rho}(\dot{U}_\alpha s_\beta + U_\alpha \dot{s}_\beta)U_\lambda + \frac{i}{2}\varepsilon_{\mu\nu\lambda\rho}(\dot{U}_\mu t_\nu - \dot{U}_\nu t_\mu + U_\mu \dot{t}_\nu - U_\nu \dot{t}_\mu)U_\lambda\,.$$

The last two terms go to zero by antisymmetry. Of the terms that are affected by $\delta^{\alpha\beta}_{\lambda\rho}$, all that will remain, upon taking (3), (4), and (39) into account, is:

$$c^2\dot{s}_\rho + U_\lambda \dot{s}_\lambda U_\rho = c^2\dot{s}_\rho - s_\lambda \dot{U}_\lambda U_\rho\,,$$

from (40), so:

$$\boxed{c^2\dot{s}_\rho = s_\lambda \dot{U}_\lambda \cdot U_\rho - i\varepsilon_{\mu\nu\lambda\rho}\dot{U}_\mu t_\nu U_\lambda.}$$

One immediately deduces from this [**20**] that:

$$\dot{s}_\mu \dot{U}_\mu = 0 \qquad \text{and} \qquad \dot{s}_\mu t_\mu = 0.$$

Upon contracting with s_ρ, one can also write:

$$\dot{s}_\mu s_\mu = \dot{s}_0 s_0 = \frac{i}{c^2}\varepsilon_{\mu\nu\alpha\beta} s_\mu U_\nu \dot{U}_\alpha t_\beta\,,$$

which shows that the norm of the gyration is not constant, in general.

As in the Weyssenhoff case, we introduce the transverse momentum P_μ , which is orthogonal to the current, by setting:

$$G_\mu = \mathfrak{M}_0 U_\mu - P_\mu \qquad (P_\mu U_\mu = 0).$$

The first dynamical equation (35) gives directly:

$$P_\mu = \frac{1}{c}\dot{t}_\mu - \frac{1}{c^2} S_{\mu\nu}\dot{U}_\nu\,,$$

or, upon replacing $S_{\mu\nu}$ with its expression in (41):

$$\boxed{c^3 P_\mu = i\varepsilon_{\mu\nu\alpha\beta} U_\nu \dot{U}_\alpha s_\beta + c^2 \dot{t}_\mu - t_\nu \dot{U}_\nu \cdot U_\mu.}$$

This expression simultaneously constitutes an expression for P_μ and $\dot{t}_\mu$.

Upon contracting it with $\dot{U}_\mu$ or s_μ it will then follow that:

$$P_\mu \dot{U}_\mu = \frac{1}{c}\dot{t}_\mu \dot{U}_\mu = c^2 \dot{\mathfrak{M}}_0 ,$$

from (37), and:

$$P_\mu \, s_\mu = \frac{1}{c}\dot{t}_\mu s_\mu .$$

One may express these two equalities by saying that the vector $cP_\mu - \dot{t}_\mu$ is orthogonal to the acceleration and to the gyration s_μ. On the contrary, the vector is not orthogonal to the current:

$$(cP_\mu - \dot{t}_\mu)U_\mu = t_\mu \dot{U}_\mu \neq 0.$$

Similarly, the gyration is not orthogonal to the space-time acceleration. The four vectors $P_\mu - \dot{t}_\mu / c$, U_μ, $\dot{U}_\mu$, s_μ do not form an orthogonal system of axes, in general, while at least $s_\mu \dot{U}_\mu$ and $t_\mu \dot{U}_\mu$ are non-zero. One may calculate the two scalar products by contracting the second dynamical equation:

(III.44) $$\mathfrak{M}_0 \dot{U}_\mu + \frac{1}{c^2} S_{\mu\nu} \ddot{U}_\nu = \frac{1}{c}\ddot{t}_\mu - 2\dot{\mathfrak{M}}_0 U_\mu .$$

First, contracting this with s_μ will give:

$$\mathfrak{M}_0 c^2 s_\mu \dot{U}_\mu = c^2 s_\mu \ddot{t}_\mu - c^2 S_{\mu\nu} s_\mu \ddot{U}_\nu .$$

Now, from (42), one has:

$$- c \, S_{\mu\nu} \, s_\mu \, \ddot{U}_\nu = t_\lambda \, s_\lambda \cdot U_\nu \, \ddot{U}_\nu = - t_\lambda \, s_\lambda \gamma_0^2 ,$$

from (5), so:

$$\mathfrak{M}_0 c^2 s_\mu \dot{U}_\mu = s_\mu (c^2 \ddot{t}_\mu - \gamma_0^2 t_\mu) .$$

Similarly, if one contracts (44) with t_μ then one will get:

$$\mathfrak{M}_0 c^3 t_\mu \dot{U}_\mu = c^2 \ddot{t}_\mu t_\mu - c S_{\mu\nu} t_\mu \ddot{U}_\nu .$$

Now, from (43):

$$\begin{aligned} - c \, S_{\mu\nu} \, s_\mu \, \ddot{U}_\nu &= - i \, \varepsilon_{\mu\nu\alpha\beta} \, t_\mu \, U_\alpha \, s_\beta \, \ddot{U}_\nu + t_\mu \, t_\mu \cdot U_\nu \, \ddot{U}_\nu \\ &= i \, \varepsilon_{\mu\nu\alpha\beta} \, \ddot{U}_\mu \, t_\nu \, U_\alpha \, s_\beta + t_\mu \, t_\mu \, \gamma_0^2 . \end{aligned}$$

Thus, one finally has:

$$\boxed{\mathfrak{M}_0 c^3 t_\mu \dot{U}_\mu = t_\mu (c^2 \ddot{t}_\mu - \gamma_0^2 t_\mu) + i\varepsilon_{\mu\nu\alpha\beta} \ddot{U}_\mu t_\nu U_\alpha s_\beta .}$$

One now sees the important role that is played in these relations by the vector $c^2 \ddot{t}_\mu - \gamma_0^2 t_\mu$ and the pseudo-scalar $i\varepsilon_{\mu\nu\alpha\beta} \ddot{U}_\mu t_\nu U_\alpha s_\beta$, In conclusion, we shall study the *radius vector* R_μ that connects the center of gravity with the center of mass in the space of inertia:

$$M_0^2 c^2 R_\mu = S_{\mu\nu} G_\nu \qquad \text{with} \qquad R_\mu G_\mu = 0.$$

One has:

$$M_0^2 c^2 R_\mu U_\mu = -c\, t_\nu G_\nu = c t_\nu P_\nu .$$

Similarly, from (42), one has:

$$M_0^2 c^2 R_\mu s_\mu = \mathfrak{M}_0 c\, s_\nu t_\nu .$$

The radius vector is no longer orthogonal to either the current or the gyration.

As in the Weyssenhoff case, the derivative of R_μ is:

(III.45) $$\boxed{\dot{R}_\mu = U_\mu - \frac{\mathfrak{M}_0}{M_0^2} G_\mu .}$$

If one contracts (45) with R_μ then it will follow that:

$$R_\mu \dot{R}_\mu = R_\mu U_\mu = \frac{t_\nu P_\nu}{M_0^2 c} .$$

Thus, the radius vector will no longer have constant norm.

Finally, if we contract (45) with U_μ then we will get:

$$\dot{R}_\mu U_\mu = -c^2 + \frac{\mathfrak{M}_0^2 c^2}{M_0^2}$$

or

(III.46) $$\boxed{\dot{R}_\mu U_\mu = \frac{\mathfrak{M}_0^2 - M_0^2}{M_0^2} c^2 .}$$

One can interpret these results by expressing (45) and (46) in the reference frame of inertia, where $G_k^I = 0$, and (45) gives us:

(III.47) $$\dot{R}_k^I = U_k^I = \alpha v_k ,$$

and

(III.48) $$\dot{R}_4^I = U_4^I - \frac{\mathfrak{M}_0}{M_0^2} ic M_0 = ic\left(\alpha - \frac{\mathfrak{M}_0}{M_0}\right),$$

upon setting $\alpha = (1 - v_0^2 / c^2)^{-1/2}$, in which v_0 is the norm of the velocity v_k of the center of matter relative to the system of inertia. Relation (47) is obvious, since the origin of R_k^I is at rest in the space of inertia.

Similarly, if one expresses the relation (46) as:

$$\dot{R}_k^I U_k^I + \dot{R}_4^I U_4^I = \alpha\, v_k\ \alpha\, v_k + ic\left(\alpha - \frac{\mathfrak{M}_0}{M_0}\right) ic\alpha = \alpha^2 v_0^2 - \alpha^2 c^2 + \alpha c \frac{\mathfrak{M}_0}{M_0}$$

$$= \frac{\mathfrak{M}_0^2 - M_0^2}{M_0^2} c^2,$$

since:

$$\alpha^2 v_0^2 - \alpha^2 c^2 = - c^2,$$

then it will follow that:

$$- c^2 + \alpha c^2 \frac{\mathfrak{M}_0}{M_0} = \frac{\mathfrak{M}_0^2}{M_0^2}\ c^2 - c^2,$$

so

$$\boxed{\alpha = \frac{\mathfrak{M}_0}{M_0}.}$$

This relation can also be anticipated by starting with the definition of the center of matter, so that point is necessarily and continually found in the same spatial intersection in the system of inertia as the center of gravity. Hence, one not only has $R_4^I = 0$, but also $\dot{R}_4^I = 0$; i.e., from (48), $\alpha = \mathfrak{M}_0 / M_0$.

Since $\alpha = (1 - v_0^2 / c^2)^{-1/2}$, one will immediately get the expression for the velocity v_0 :

$$\boxed{v_0^2 = \frac{\mathfrak{M}_0^2 - M_0^2}{\mathfrak{M}_0^2} c^2.}$$

These latter relations were found before in the Weyssenhoff case [**2**], where they were meaningful and generally significant. Amongst the school of Weyssenhoff, one sees that the constancy of the mass of inertia is related to the constant magnitude of the velocity of the center of matter in the system of inertia. On the contrary, the mass $\mathfrak{M}_0$ varies in the course of time in the general case; the same will be true for the velocity. On the other hand, one sees that one may write:

$$\mathfrak{M}_0 = \alpha M_0 ,$$

which is a relation that has an immediate significance: The mass of inertia is the mass, relative to the system of inertia, that is possessed by a point-like particle that is localized at the center of matter and has the (constant) mass of momentum of the drop for its *proper* mass.

§ 8. The classical Dirac particle. We have seen that the Frenkel-Weyssenhoff system of equations determines the global motion of the drop completely – up to initial conditions – by means of the motion of the center of matter. Furthermore, this results directly from considering the number of equations that must be satisfied by the 14 independent variables that are represented by the two vectors U_μ and G_μ, and the anti-symmetric tensor $S_{\mu\nu}$, namely:

One scalar equation: $U_\mu\, U_\mu = -\, c^2$,

One vector equation: $S_{\mu\nu}\, U_\nu = 0$,

which, however, express only three independent conditions, because one will obtain an identity upon contracting with U_μ, by reason of the anti-symmetry of $S_{\mu\nu}$:

A vectorial equation: $\dot{G}_\mu = 0$.

Finally, one has:

A tensorial equation: $\dot{S}_{\mu\nu} = G_\mu\, U_\nu - G_\nu\, U_\mu$

with six independent components.

One thus has precisely 14 distinct equations.

On the other hand, the general case that we just studied still remains largely indeterminate. More precisely, as one has simply suppressed the condition $S_{\mu\nu}\, U_\nu = 0$, the motion involves a *triple* indeterminacy. In order to go much further, it becomes necessary to introduce three distinct conditions, and for this, to formulate the new physical hypotheses that serve to determine the motion. We proceed by stages upon restricting the generality with only two conditions from the outset. We shall apply these two hypotheses to the vectors s_μ and t_μ into which the angular momentum decomposes.

The first restriction consists in supposing that the two vectors s_μ and t_μ are *collinear* [**61**]. Since they both belong to the proper space, that will serve to express the coincidence of the two directions in *space*, which implies two distinct conditions. The interest of this hypothesis is that it manages to rejoin this expression of motion with the well-known and very important formalism of quantum mechanics, namely, the Dirac formalism, or, more precisely, that which we have called the formalism of the classical Dirac particle.

One has that if one starts with the Dirac spinor ψ (and independently of the wave equation) then one can define a set of tensorial magnitudes that quantum mechanics interprets as the densities of the mean values for the particle of spin 1/2 [**55**]. Consider the Von Neumann matrices γ_μ and the matrices that are derived from them (in particular $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and the four matrices $\hat{\gamma}_\mu = i\, \gamma_\mu\, \gamma_5$.)

One remarks that the matrix γ_5, which anti-commutes with the four γ_μ, plays a role that is analogous to that of the symbol $\varepsilon_{\mu\nu\alpha\beta}$ when one applies it to an anti-symmetric product of γ matrices, while taking the commutation relations into account. One thus has:

$$\gamma_5\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu) = (\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\gamma_5 = -\,\varepsilon_{\mu\nu\alpha\beta}\,\gamma_\alpha\,\gamma_\beta .$$

If one sets:

$$\bar{\psi} = \psi^\dagger \gamma_5$$

($\psi^\dagger$ being the Hermitian conjugate of y) then one will thus define:

A scalar: $\Omega = \bar{\psi}\psi$,

A (pseudo) scalar: $\hat{\Omega} = i\,\bar{\psi}\,\gamma_5\,\psi$,

A vector: $S_\mu = i\,\bar{\psi}\,\gamma_\mu\,\psi$,

A (pseudo) vector: $\hat{S}_\mu = -\bar{\psi}\,\hat{\gamma}_\mu\,\psi = i\,\bar{\psi}\,\gamma_5\,\gamma_\mu\,\psi$,

An anti-symmetric tensor: $M_{\mu\nu} = -\dfrac{i}{2}\bar{\psi}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\psi$.

Contrary to the situation in the case of a scalar wave function, which one normalizes by simply setting $\psi^*\psi = 1$, one sees that one has two invariants Ω and $\hat{\Omega}$, so one normalizes the spinor ψ by setting:

$$\Omega^2 + \hat{\Omega}^2 = 1.$$

The physical interpretation of the Dirac formalism then leads to the following identifications *relative to a particle:*

The unit-speed velocity: $U_\mu = c\,S_\mu$,

The spin: $\sigma_\mu = \dfrac{\hbar}{2}\hat{S}_\mu$,

The electromagnetic moment: $\mu_{\alpha\beta} = -\dfrac{\hbar}{2}\dfrac{e}{m_0 c}M_{\alpha\beta}$,

which, by virtue of the Frenkel hypothesis, leads us to introduce:

An internal angular momentum: $S_{\alpha\beta} = \dfrac{m_0 c}{e}\mu_{\alpha\beta} = -\dfrac{\hbar}{2}M_{\alpha\beta}$.

Having said this, one knows that there exists a series of quadratic relations between the bilinear combinations that were defined above that are independent of any choice of wave equation, and result simply from the existence of certain identities between certain products of the elements of the γ matrices. These relations, which were established by Pauli and Koffinck [**56**, **57**, **58**], result immediately from the relations between the dynamical magnitudes that characterize the particle.

Therefore, the three variables:

$$S_\mu S_\mu = -(\Omega^2 + \hat{\Omega}^2), \qquad \hat{S}_\mu \hat{S}_\mu = \Omega^2 + \hat{\Omega}^2, \qquad S_\mu \hat{S}_\mu = 0$$

imply, for the unit-speed velocity and spin, that:

$$U_\mu U_\mu = -c^2, \qquad \sigma_\mu \sigma_\mu = \left(\frac{\hbar}{2}\right)^2, \qquad U_\mu \sigma_\mu = 0.$$

One has the relations:

$$M_{\mu\nu} S_\nu = -\hat{\Omega}\hat{S}_\mu, \qquad \text{and} \qquad \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta} M_{\alpha\beta} S_\nu = -\Omega \hat{S}_\mu$$

for the anti-symmetric tensor $M_{\mu\nu}$. In order to interpret this, one must express the two invariants that are related by the normalization relation as functions of a single variable, for which Takabayasi [**9**] chose an angle A. One sets:

$$\Omega = \cos A, \qquad \hat{\Omega} = \sin A,$$

and one then has:

$$\frac{2}{\hbar c} S_{\mu\nu} U_\nu = \frac{2}{\hbar} \sigma_\mu \sin A$$

and

$$\frac{2}{\hbar c}\frac{i}{2}\varepsilon_{\mu\nu\alpha\beta} S_{\alpha\beta} U_\nu = -\frac{2}{\hbar}\sigma_\mu \cos A.$$

However, if we refer to the formalism of the Bohm-Vigier drop then we will see that the relations translate into simply:

$$t_\mu = \sigma_\mu \sin A,$$
$$s_\mu = \sigma_\mu \cos A.$$

In other words, the vectors t_μ and s_μ are collinear, so one may write $t_\mu = \lambda\, s_\mu$, where the scalar λ is the tangent of Takabayasi's angle A. Therefore, upon assuming that the wobble and gyration are collinear, we place ourselves in a special case that allows us to rejoin the Dirac formalism, and we introduce, in addition to the Weyssenhoff variables, a new variable that corresponds to the mysterious Takabayasi angle A, which we interpret in terms of the classical dynamics of fluid drops as the connection between wobble and gyration.

§ 9. The collinearity of wobble and gyration: its significance. – One might further attempt to form an idea of the physical significance of this hypothesis by considering a model that was inspired by Poincaré's work on rotating fluid masses [**15**]. That scholar showed that when there exist internal stresses, fluid masses will tend toward stable configurations that are surfaces of revolution around an axis, such as, for example, a rotating torus. Suppose that one introduces internal vortices into such a mass [**14**], which will, as we have seen, destroy the symmetry of the mass distribution without modifying the distribution of matter, and recall the argument that we sketched out in the case of the Weyssenhoff drop upon a more general basis, while considering a gyration s_k^0 and a wobble t_k^0 that will be in the proper space of the geometric center of the torus, this time.

It is convenient to consider separately two types of asymmetry that might be involved with the distribution of vortices. On the one hand, one can have a *lateral* asymmetry that is produced by projecting all of the vortices onto the equatorial plane of the torus, upon which, one will find many more of them on one side than on the other (or at least, the sum $\sum \mathbf{w}_i \, \mathbf{r}_i$ will be greater for one half than for the other one). The center of mass will then be separate from the center of matter and elongated *in the equatorial plane* in the direction where $\sum \mathbf{w}_i \, \mathbf{r}_i$ is greatest. One then get a proper space vector t_k that is in the equatorial plane. As for the vector s_k, which is the spatial dual of the proper-space components S_{ij}^0 of the angular momentum, since it will have no vortices, by reason of symmetry, it can be directed along the axis of rotation. However, in general, it will be more likely that there will exist a certain angle between it and the axis of rotation, since vortices introduce a certain extra asymmetrical orbital momentum, to which, one can associate a proper moment of rotation that is due to the proper spins, in which case, they also be oriented in an asymmetrical fashion. One will then have a relatively complicated situation. Nevertheless, it is easy to expect that this state cannot be maintained for very long, in general, when the vortices are sufficiently numerous. Indeed, the fact that was pointed out above – viz., that the orbital rotation is generally performed with its angular velocity varying with the distance along the axis – leads to a change in the respective azimuths of the vortices, and if they are numerous – and for a stronger reason, if they are interacting – then the laws of statistics will teach us that after a few rotations they will lead to a uniform mean distribution along the axis, while for that situation, it will be their $\mathbf{w}_i \, \mathbf{r}_i$ that affects their spins. Thus, except for the case in which some special dynamical law maintains a certain asymmetry (whose necessity we have shown in the Weyssenhoff case), we will rapidly arrive at a center of mass that is once more indistinct from the center of matter, so its wobble t_k will be zero, while, on the other hand, its gyration s_k will lie along the axis of rotation.

On the contrary, suppose that the distribution of vortices varies *in the direction of the axis of the torus*, so – for example – upon cutting the torus with planes that are perpendicular to the axis, one will encounter a density of vortices that grows increasingly larger as one translates towards the extremities of the axis. On the contrary, assume, to simply, that the azimuthal distribution of vortices is one of revolution; for example, one that results from the statistical uniformization that we just described. The distribution of matter would then remain symmetric with respect to the equatorial plane, so the center of matter would remain at the intersection of that plane with the axis; the vector t_k would then be along the axis. As for the vector s_k , since the object is in a state of revolution, it will also be along the axis, and consequently, it will be collinear with t_k. Moreover, it is easy to see that this state of motion will be much less unstable than the preceding one. Indeed, since all of the motion is circular and carried out in planes that are perpendicular to the axis, on the surface of things, these planes will not influence each other in any way that would tend to rapidly re-establish the uniformity of the distribution. Without a doubt, the asymmetry will be created by internal forces of tension in the fluid that shift the center of mass to the center of matter, but it will be permissible to think that the action of those forces would be very slow with respect to the motion of the rotation of the fluid mass, and that after an appreciable time interval one will be dealing with a motion in which the vector s_k is on the axis of rotation, as well as the vector t_k. Hence, one will effectively have the relation that we assumed just now – viz., $t_k = \lambda\, s_k$ or $s_k = 1 / \lambda\, t_k$ – which permits us to set $\lambda = \tan A$ and to attribute a meaning to the variable $\tan A$.

It seems possible to justify the fact that the vector s_μ and t_μ are collinear for the Dirac particle by classical considerations, at least qualitatively.

By comparison, the condition:

$$\sigma_\mu\, \sigma_\mu = \left(\frac{\hbar}{2}\right)^2,$$

which implies that the norm of the spin (which is a proper-space vector) will remain constant in time, does not seem to admit an interpretation from the quantum viewpoint. That is because we introduced the first condition from the outset by just setting:

$$s_\mu = \sigma_\mu \cos A \quad \text{and} \qquad t_\mu = \sigma_\mu \sin A,$$

in which the vector σ_μ has a norm σ_0 that is variable, in principle. We therefore get a dynamic that is less general than that of the Bohm-Vigier drop, such that the Dirac particle represents a special case that is determined entirely by the condition that $\sigma_\mu\, \sigma_\mu = (\hbar / 2)^2$, and the Frenkel-Weyssenhoff particle constitutes a case that is even more special than that.

§ 10. The collinearity of wobble and gyration: characteristic relations. – The expression for the angular momentum and its dual, when given in the general case, can be written as:

$$S_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\, U_\alpha\, \sigma_\mu \cos A + (U_\mu\, \sigma_\nu - U_\nu\, \sigma_\mu) \sin A,$$

$$\frac{i}{2c}\, \varepsilon_{\mu\nu\alpha\beta}\, S_{\alpha\beta} = \frac{i}{c}\; \varepsilon_{\mu\nu\alpha\beta}\, U_\alpha\, \sigma_\beta \sin A - \frac{1}{c}(U_\mu\, \sigma_\nu - U_\nu\, \sigma_\mu) \sin A\,.$$

In this form, one recognizes two of the Pauli-Koffinck identities. Similarly, the contracted products of these two tensors with the spin σ_μ will give two other known identities:

$$S_{\mu\nu}\, \sigma_\nu = \frac{\sigma_0^2 \sin A}{c}\, U_\mu\,,$$

(III.50)

$$\frac{i}{2c}\,\varepsilon_{\mu\nu\alpha\beta}\,S_{\alpha\beta}\,\sigma_\nu = -\frac{\sigma_0^2\cos A}{c}\,U_\mu\,,$$

upon setting $\sigma_0^2 = \sigma_\mu\,\sigma_\mu$.

It is possible to calculate the contracted products $S_{\mu\nu}\,S_{\nu\lambda}$, $S_{\mu\nu}\hat{S}_{\mu\lambda}$, as we did in the Weyssenhoff case, by using the contracted products of the Levi-Civita symbols. One will then get:

$$S_{\mu\nu}\,S_{\nu\lambda} = -\frac{1}{c^2}\,\varepsilon_{\mu\nu\alpha\beta}U_\alpha\,\sigma_\beta\;\varepsilon_{\mu\lambda\gamma\rho}\,U_\gamma\,\sigma_\rho\,\cos^2 A + \frac{1}{c^2}\,(U_\mu\,\sigma_\nu - U_\nu\,\sigma_\mu)(\,U_\mu\,\sigma_\lambda - U_\lambda\,\sigma_\mu)\,\sin^2 A.$$

The squared terms will be zero, since expressions such as $\varepsilon_{\mu\nu\alpha\beta}U_\alpha\,\sigma_\beta\;U_\mu\,\sigma_\nu$ go to zero by antisymmetry.

The first term can be written:

$$-\frac{1}{c^2}\cos^2 A\;\delta^{\nu\alpha\beta}_{\lambda\gamma\rho}\;U_\alpha\,\sigma_\beta\;U_\gamma\,\sigma_\rho\;.$$

Upon developing this and taking the orthogonality of σ_μ and U_μ into account, all that will remain is:

$$\left(\sigma_0^2\delta_{\nu\lambda}+\sigma_0^2\,\frac{U_\nu U_\lambda}{c^2}-\sigma_\nu\sigma_\lambda\right)\,\cos^2 A\;.$$

Similarly, upon carrying out the second product, all that will remain is:

$$\left(\sigma_0^2\,\frac{U_\nu U_\lambda}{c^2}-\sigma_\nu\sigma_\lambda\right)\,\sin^2 A\;.$$

One thus finally gets:

(III.51)

$$\boxed{S_{\mu\nu}S_{\nu\lambda}=\sigma_0^2\left(\delta_{\nu\lambda}\cos^2 A+\frac{U_\nu U_\lambda}{c^2}\right)-\sigma_\nu\sigma_\lambda.}$$

This is another Koffinck identity, from which, we will extract, in particular (since $\delta_{\nu\nu}$ = 4)

$$S_{\mu\nu}\,S_{\mu\lambda} = \sigma_0^2\,(4\cos^2 A - 2) = 2\,\sigma_0^2\cos 2A,$$

which we can infer directly from the formula that was established in the general case.

Finally, the product $S_{\mu\nu}\hat{S}_{\mu\lambda}$ likewise gives:

$$-\frac{1}{c^2}\,\varepsilon_{\mu\nu\alpha\beta}\,U_\alpha\,\sigma_\beta\;\varepsilon_{\mu\lambda\gamma\rho}\,U_\gamma\,\sigma_\rho\;\cos A\sin A$$

$$= \begin{array}{l} -\dfrac{1}{c^2}(U_\mu\,\sigma_\nu - U_\nu\,\sigma_\mu)(U_\mu\,\sigma_\lambda - U_\lambda\,\sigma_\mu)\sin A\cos A \\ \left(\sigma_0^2\delta_{\nu\lambda} + \sigma_0^2\dfrac{U_\nu U_\lambda}{c^2} - \sigma_\nu\sigma_\lambda\right)\cos A\sin A \\ -\left(\sigma_0^2\dfrac{U_\nu U_\lambda}{c^2} - \sigma_\nu\sigma_\lambda\right)\cos A\sin A\,. \end{array}$$

All that remains is simply:

$$\boxed{S_{\mu\nu}\hat{S}_{\mu\nu} = \tfrac{1}{2}\sigma_0^2\,\delta_{\nu\lambda}\sin 2A,}$$

which is another Koffinck identity, from which, we infer, in particular, that:

$$S_{\mu\nu}\,\hat{S}_{\mu\nu} = 2\sigma_0^2\sin 2A\,.$$

The derivatives of the contracted products naturally take the form:

(III.52) $$\frac{d}{d\tau}\left(\tfrac{1}{2}S_{\mu\nu}S_{\mu\nu}\right) = -\,2c\;\sigma_\mu\,P_\mu\cos A,$$

(III.53) $$\frac{d}{d\tau}\left(\tfrac{1}{2}S_{\mu\nu}\hat{S}_{\mu\nu}\right) = -\,2c\;\sigma_\mu\,P_\mu\,\text{sins}\,A\,.$$

If we recall the expressions for the derivatives of s_μ and t_μ then it will follow that:

(III.54) $$c^2\,\dot{s}_\mu = i\;\varepsilon_{\mu\nu\alpha\beta}\,U_\mu\,\dot{U}_\alpha\,\sigma_\beta\sin A + \sigma_\nu\,\dot{U}_\nu\cos A\;U_\mu\,.$$

Upon contracting this with σ_μ, one will see that:

(III.55) $$\dot{s}_\mu\,\sigma_\mu = 0,$$

which signifies that not only is $\dot{s}_\mu$ is orthogonal to t_μ, as in the general case, but also that:

$$\boxed{\dot{s}_\mu s_\mu = 0.}$$

In other words, the norm s_0 of the gyration is *constant* in time; this is probably the most important special property of the motion under study:

$$s_0 = \sigma_0\cos A = \text{constant.}$$

One can likewise write:

(III.55′) $$c^2\,\dot{t}_\mu = -i\,\varepsilon_{\mu\nu\alpha\beta}\,U_\nu\,\dot{U}_\alpha\,\sigma_\beta\cos A + \sigma_\nu\,\dot{U}_\nu\,\sin A\,U_\mu + c^2\,P_\mu\,,$$

which will give us an expression for the way that the norm of the wobble varies.

On the other hand, one has:

(III.57) $$\dot{s}_\mu = \dot{\sigma}_\mu\cos A - \sigma_\mu\,\dot{A}\,\sin A,$$

(III.58) $$\dot{t}_\mu = -\,\dot{\sigma}_\mu\,\sin A + \sigma_\mu\,\dot{A}\,\cos A\,.$$

One can then find the derivatives of σ_μ and A:

$$\dot{\sigma}_\mu = \dot{s}_\mu\,\cos A + \dot{t}_\mu\,\sin A,$$
$$\dot{A}\,\sigma_\mu = -\,\dot{s}_\mu\,\sin A + \dot{t}_\mu\,\cos A,$$

which will give, upon employing equations (54) and (55′):

(III.59) $$\boxed{c^2\dot{\sigma}_\mu = \sigma_\nu\dot{U}_\nu U_\mu + c^2 P_\mu \sin A}$$

and

(III.60) $$\boxed{c^2\sigma_\mu\dot{A} = -i\varepsilon_{\mu\nu\alpha\beta}U_\nu\dot{U}_\alpha\sigma_\beta + c^3 P_\mu\cos A.}$$

These relations provide, by contracted multiplication:

The variation of the norm of the spin:

$$\frac{d}{d\tau}(\sigma_0^2) = 2\sigma_\mu\dot{\sigma}_\mu = 2c^3\,P_\mu\,\sigma_\mu\sin A,$$

two expressions for the variation of the angle A:

(III.61) $$\sigma_0^2\dot{A} = c\,P_\mu\,\sigma_\mu\cos A, \qquad \sigma_\mu\dot{U}_\mu\dot{A} = c\,P_\mu\,\dot{U}_\mu\,\cos A,$$

and finally, the norm of the transverse momentum:

$$c^4\,P_\mu\,P_\mu\cos^2 A = \gamma_0^2\sigma_0^2 - (\sigma_\nu\dot{U}_\nu)^2 + c^2\dot{A}^2\sigma_0^2\,.$$

On the other hand, we can use these expressions to specify the way that the mass of inertia varies in time, which will be equal to:

$$\dot{\mathfrak{M}}_0 c^3 = \dot{t}_\mu\dot{U}_\mu = \dot{\sigma}_\mu\dot{U}_\mu\sin A + \sigma_\mu\dot{U}_\mu\cos A\cdot\dot{A}\,,$$

in the general case, or, upon replacing $\dot{A}$ with its value (61):

(III.62) $$\boxed{\dot{\mathfrak{M}}_0 c^2 = \frac{1}{\sigma_0^2}\sigma_\mu \dot{U}_\mu \cdot \sigma_\nu U_\nu .}$$

This latter expression shows the particular case in which the mass of inertia of inertia will be constant in time.

In the various relations, we were involved with invariants that were zero in the Weyssenhoff case and which, in the present case, are the reason that certain vectors fact are longer orthogonal, so certain quantities will no longer be constant. This is notably the case for $\sigma_\mu \dot{U}_\mu$, and above all, for $\sigma_\mu P_\mu$. The latter once more enters into the relations that apply to the radius vector R_μ, which we content ourselves to merely transcribe: The angle that the radius vector makes with the unit-speed velocity depends upon the product:

(III.63) $$M_0^2 c^2 R_\mu U_\mu = c\ \sigma_\mu P_\mu \sin A,$$

which is, as one knows, the same quantity that enters into the variation of the norm of R_μ:

(III.64) $$\frac{d}{d\tau}(R_0^2) = 2R_\mu \dot{R}_\mu = \frac{2}{M_0^2 c}\ \sigma_\mu P_\mu \sin A.$$

Finally, the angle between the radius vector and the spin will depend upon the product:

(III.65) $$M_0^2\ c^3 R_\mu \sigma_\mu = \mathfrak{M}_0\ c\ \sigma_0^2 \sin A.$$

One will then see that the two vectors will be orthogonal only if $\sin A = 0$; i.e., when we are dealing with the Weyssenhoff case uniquely.

§ 11. The classical Dirac particle: the dynamical equations. All of the relations that just established suggest different interesting paths by which to choose a third relation that would serve to determine the dynamics of the particle. We now concern ourselves with the most interesting case in the context of the question with which we are occupying ourselves, namely, the one in which one constrains the particle to satisfy the third Dirac equation:

$$\sigma_\mu \sigma_\mu = \left(\frac{\hbar}{2}\right)^2 .$$

In the interest of preserving the greatest generality, we content ourselves with taking σ_0^2 to be constant, without giving it the value of the quantum of action, in particular.

We have already given the expression for the derivative of σ_0^2, namely:

(III-66) $$\frac{d}{d\tau}(\sigma_0^2) = 2c^3 P_\mu \sigma_\mu \sin A.$$

It is zero in two cases:

1. $\sin A = 0.$

This is simply the Weyssenhoff case; it does not interest us, here. Furthermore, we remark, in passing, that one also has $P_\mu \sigma_\mu = 0$ in the Weyssenhoff school. If the second factor of (66) is zero then the first one will be, as well.

However, the converse is not true.

2. $P_\mu \sigma_\mu = 0.$

This is the case that we studied that serves to define what we have called the classical Dirac particle [**62**]. We shall then recover our formulas by setting σ_0 = constant, or, in an equivalent fashion, $P_\mu \sigma_\mu = 0$.

The relation $P_\mu \sigma_\mu = 0$ implies that $G_\mu \sigma_\mu = 0$.

Spin is a spatial vector in the inertial system. One will thus have $P_k \sigma_k = 0$ in the space of that system.

Since, from (17), one has:

$$P_k = \mathfrak{M}_0 U_k - \mathfrak{M}_0 \alpha V_k$$

in that space, one will likewise have:

$$\sigma_k V_k = 0.$$

The velocity of the center of matter relative to the center of gravity is then orthogonal to the spin (which varies with time, moreover).

Relations (52) and (53) immediately show us that the contracted products:

$$S_{\mu\nu} S_{\mu\nu} \quad \text{and} \quad S_{\mu\nu} \hat{S}_{\mu\nu}$$

are both *constant.*

Since σ_0 is constant, the relation $s_0 = \sigma_0 \cos A$ = constant will show us that the angle A is just as constant. One sees this directly from the expression found for dA / dt, moreover. It results from this that the norm of the wobble $t_0 = \sigma_0 \sin A$ is also constant, as one confirms directly from the expression (56) that gives its derivative.

Furthermore, equation (62) gives us:

$$\dot{\mathfrak{M}}_0 c^2 = 0.$$

The Weyssenhoff proper inertial mass is likewise *constant.* It then results that:

$$G_\mu \dot{U}_\mu = 0.$$

The acceleration then belongs to the inertial space. It also results that the velocity of the center of matter relative to the inertial system will then have constant magnitude, since:

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{\mathfrak{M}_0}{M_0}.$$

Similarly, the norm of the transverse momentum:

$$P_\mu P_\mu = \mathfrak{P}_0^2 c^2 = (\mathfrak{M}_0^2 - M_0^2)\, c^2$$

is likewise constant. Its value may be easily deduced from the expression for P_μ, which, from (60), may be reduced to:

$$P_\mu = \frac{\varepsilon_{\mu\nu\alpha\beta} U_\nu \dot{U}_\alpha \sigma_\beta}{c^3 \cos A},$$

so one will get:

$$P_\mu P_\mu = \frac{\gamma_0^2 \sigma_0^2 - (\sigma_\nu \dot{U}_\nu)^2}{c^4 \cos^2 A}$$

upon setting:

$$\gamma_0^2 = \dot{U}_\mu \dot{U}_\mu .$$

as one does in the Weyssenhoff case.

Finally, contracting the expression (59) with dU_μ / dt gives us:

$$\dot{\sigma}_\mu \dot{U}_\mu = c\, P_\mu \dot{U}_\mu \sin A,$$

and since:

$$G_\mu \dot{U}_\mu = 0,$$

so one can immediately deduce that:

$$P_\mu \dot{U}_\mu = 0,$$

one will see that:

$$\dot{\sigma}_\mu \dot{U}_\mu = 0,$$

which implies, from (57) and (58), that:

$$\dot{s}_\mu \dot{U}_\mu = 0 \qquad \text{and} \qquad \dot{t}_\mu \dot{U}_\mu = 0.$$

Now, we introduce the radius vector R. From (63), one sees that $R_\mu U_\mu = 0$.

The radius vector (and, as a consequence, the center of gravity) is in proper space. As would be natural to suggest, we may localize the spin to the center of matter (which is instantaneously at rest) and the transverse momentum to the center of gravity. The relations:

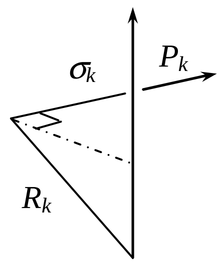
$$G_\mu R_\mu = 0 \qquad \text{and} \qquad P_\mu \sigma_\mu = 0$$

give:

$$P_k R_k = 0 \qquad \text{and} \qquad P_k \sigma_k = 0$$

in the proper system, respectively.

The transverse momentum is thus perpendicular to the plane of the vectors R_k and s_k. They have the disposition depicted below (see Figure), which shows that in the case considered there exists a close relationship between the "orbital" motion, which is characterized by a residual momentum in the system where the center of matter is fixed, and, on the other hand, by the "proper" rotation that is expressed by the spin.



One must recall, however, that the elements thus represented are *instantaneous* elements, since spin varies in the course of time, and that the center of matter is at rest only in the reference frame in question at the instant considered.

Relation (64) gives us:

$$R_\mu \dot{R}_\mu = 0.$$

If one places oneself in the system of inertia Σ_1, where the center of gravity, which is the origin of the radius vector, is always at rest, then one will have $R_k \dot{R}_k = 0$.

The center of matter remains at a constant distance from the center of gravity. It moves on a sphere whose radius R_0 is easy to calculate:

One has $R_\mu R_\mu M_0^2 c^4 = S_{\mu\nu} G_\nu S_{\mu\lambda} G_\lambda$, which gives, from (51):

$$M_0^2 c^4 R_0^2 = \left(\sigma_0^2 \, \delta_{\nu\lambda} \cos^2 A + \sigma_0^2 \frac{U_\nu U_\lambda}{c^2} - \sigma_\nu \sigma_\lambda \right) G_\nu G_\lambda ,$$

so:

$$\boxed{R_0^2 = \frac{\sigma_0^2 (\mathfrak{M}_0^2 - M_0^2 \cos^2 A)}{M_0^4 c^2}.}$$

Now, replacing G_ν by its expression (35) in the expression for the radius vector $M_0^2 c^4 R_\mu = S_{\mu\nu} G_\nu$, will give:

$$M_0^2 c^4 R_\mu = \mathfrak{M}_0 \, S_{\mu\lambda} \, U_\lambda - \frac{S_{\mu\lambda} \dot{t}_\lambda}{c} - \frac{S_{\mu\lambda} S_{\lambda\nu} \dot{U}_\nu}{c^2} .$$

The first term on the right-hand side of this is simply:

$$\mathfrak{M}_0 \, c \, t_\mu = \mathfrak{M}_0 \, c \sigma_\mu \sin A.$$

The second term gives $\frac{1}{c} S_{\mu\lambda}\dot{\sigma}_\lambda \sin A$ (since $\dot{A} = 0$).

Now, one has $S_{\mu\nu}\,\sigma_\lambda = \frac{1}{c}\sigma_0^2 \sin A\, U_\lambda$, from (50), and on the other hand, $\dot{S}_{\mu\lambda}\sigma_\lambda = (G_\mu U_\lambda - G_\lambda U_\mu)\,\sigma_\lambda = 0$, since $\sigma_\lambda U_\lambda = 0$ and $\sigma_\lambda G_\lambda = 0$. What then remains is:

$$S_{\mu\lambda}\dot{\sigma}_\lambda = \frac{1}{c}\sigma_0^2 \sin A\, \dot{U}_\lambda .$$

Finally, from (51), the third term gives $-\frac{1}{c^2}(\sigma_0^2\,\dot{U}_\mu \cos^2 A - \sigma_0^2 \sigma_\nu \dot{U}_\nu \sigma_\mu)$.

One then has:

(III.68)
$$M_0^2 c^2 R_\mu = \left(\mathfrak{M}_0 c \sin A + \frac{1}{c^2}\sigma_\nu \dot{U}_\nu\right)\sigma_\mu - \frac{\sigma_0^2}{c^2}\dot{U}_\mu .$$

This relation will provide the expression for R_0^2:

(III.69)
$$M_0^2 c^2 R_0^2 = \left(\mathfrak{M}_0 c \sin A + \frac{1}{c^2}\sigma_\nu \dot{U}_\nu\right)\sigma_\mu R_\mu - \frac{\sigma_0^2}{c^2}\dot{U}_\mu R_\mu .$$

One replaces R_0^2 with its express in (67), $\sigma_\mu R_\mu$ by its expression in (65), and $\dot{U}_\mu R_\mu$ with $-U_\mu \dot{R}_\mu$, which is given by formula (46). These identifications permit us to deduce the expression for the invariant $\sigma_\nu \dot{U}_\nu$ from the equality (69):

$$\boxed{\sigma_\nu \dot{U}_\nu = -\frac{(\mathfrak{M}_0^2 - M_0^2)\,c^3 \sin A}{\mathfrak{M}_0},}$$

which is an expression that is constant in time. Finally, upon substituting this value in the expression for $P_\mu P_\mu$, one will get a relation between the invariants:

$$\boxed{\gamma_0^2\,\sigma_0^2\,\mathfrak{M}_0^2 = (\mathfrak{M}_0^2 - M_0^2)(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)c^6,}$$

which is a relation that shows one that the acceleration $\dot{U}_\mu$ has a constant norm.

We know that $\dot{U}_\mu$ is spatial in the system of inertia. In that space, upon passing to the time in the system of inertia, we will be allowed to define an acceleration vector $\Gamma_k = \frac{M_0^2}{\mathfrak{M}_0^2}\dot{U}_k^{\mathrm{I}}$, which has constant magnitude $\Gamma_0 = \frac{M_0^2}{\mathfrak{M}_0^2}\gamma_0$, namely:

(III.70) $$\Gamma_0 = \frac{M_0^2 c^2}{\mathfrak{M}_0^2 \sigma_0} \sqrt{\mathfrak{M}_0^2 - M_0^2}\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin A}\,.$$

If we substitute the value that we found for $\sigma_\nu \dot{U}_\nu$ in expression (68) for the radius vector then it will follow that:

(III.71) $$R_\mu = \frac{\sigma_\mu \sin A}{\mathfrak{M}_0 c} - \frac{\sigma_0^2 \dot{U}_\mu}{M_0^2 c^4}\,.$$

Therefore, the acceleration and the radius vector will be coplanar in the space of inertia. Finally, if we differentiate this last equation then it will follow that:

(III.72) $$\dot{R}_\mu = \frac{\sin A}{\mathfrak{M}_0 c}\dot{\sigma}_\mu - \frac{\sigma_0^2}{M_0^2 c^4}\ddot{U}_\mu\,.$$

From the general theory, we have that $\dot{R}_\mu = U_\mu - (\mathfrak{M}_0 / M_0^2)\, G_\mu$.

On the other hand, upon substituting the expression that was found for $\sigma_\nu \dot{U}_\nu$ into equation (59), one will get:

(III.73) $$c^2 \dot{\sigma}_\mu = -\frac{(\mathfrak{M}_0^2 - M_0^2)\, c^3 \sin A}{\mathfrak{M}_0} U_\mu + c^3 P_\mu \sin A.$$

Equation (72) will then become:

$$U_\mu - \frac{\mathfrak{M}_0}{M_0^2} G_\mu = -\frac{\mathfrak{M}_0^2 - M_0^2}{\mathfrak{M}_0} \sin^2 A\, U_\mu - \frac{\sin^2 A}{\mathfrak{M}_0} P_\mu - \frac{\sigma_0^2}{M_0^2 c^4}\dot{U}_\mu\ ,$$

or furthermore, upon taking into account that $G_\mu = \mathfrak{M}_0\, U_\mu - P_\mu$:

(III.74) $$\boxed{M_0^2 c^4 U_\mu (\mathfrak{M}_0^2 - M_0^2 \sin^2 A) + \mathfrak{M}_0^2 \sigma_0^2 \ddot{U}_\mu = (\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, \mathfrak{M}_0 c^4 G_\mu,}$$

which will constitute the *equation of motion.*

This equation is of order two in U_μ or of order three relative to the coordinates of the center of matter, since the Mathisson equation constitutes a generalization of it. One is cautioned that the right-hand side is a constant vector, since $\dot{G}_\mu = 0$.

§ 12. The classical Dirac particle: integration of the motion. – We place ourselves in the system of inertia Σ_{I} , for which $G_k^{\mathrm{I}} = 0$.

We get the hodograph equation:

$$M_0^2 c^4 (\mathfrak{M}_0^2 - M_0^2 \sin^2 A) U_k^{\mathrm{I}} + \mathfrak{M}_0^2 \sigma_0^2 \ddot{U}_k^{\mathrm{I}} = 0,$$

or, upon taking into account that $(1 - v^2 / c^2)^{-1/2} = \mathfrak{M}_0 / M_0$, we will get:

(III.75)
$$\boxed{M_0^2 c^4 (\mathfrak{M}_0^2 - M_0^2 \sin^2 A) V_k + \mathfrak{M}_0^4 \sigma_0^2 \frac{d^2 V_k}{dt^2} = 0.}$$

It is obvious that the hodograph remains constantly in a plane that is determined by the initial orientation of the velocity and the acceleration, which is an initial orientation that is, one knows, arbitrary for both of the two vectors. On the other hand, since we have seen that the velocity and the acceleration both have constant magnitude, from (49), the hodograph will be a *circle* of radius v_0 :

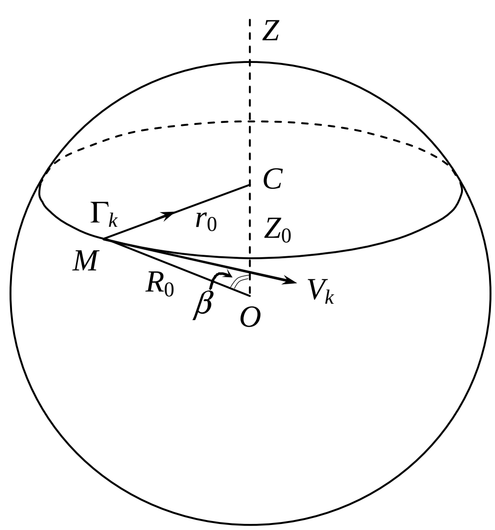
(III.76)
$$v_0^2 = c^2 \left(1 - \frac{M_0^2}{\mathfrak{M}_0^2}\right).$$

It performs a uniform motion whose angular velocity is deduced immediately from equation (75):

(III.77)
$$\boxed{\omega = \frac{M_0^2 \sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}{\mathfrak{M}_0^2 \, \sigma_0}.}$$

The motion of the center of matter is carried out on the sphere that has the center of gravity for its center and a radius of:

$$R_0 = \frac{\sigma_0}{M_0^2 c} \sqrt{\mathfrak{M}_0^2 - M_0^2 \cos^2 A}.$$



It will remain in a plane that is parallel to the hodograph. It is thus a uniform, circular motion that has ω for its angular velocity.

It is possible to determine its center and radius, since the orientation of the plane depends upon just the initial condition. If the radius of the circle is r_0 then one will have the relation $v_0 = r_0\,\omega$, from which, one will deduce r_0, since v_0 and ω are known, and it will follow that:

$$r_0 = \frac{c}{\mathfrak{M}_0}\sqrt{\mathfrak{M}_0^2 - M_0^2}\,\frac{\mathfrak{M}_0^2\sigma_0}{M_0^2c^2}\,\frac{1}{\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}},$$

so

$$r_0 = \frac{\mathfrak{M}_0\sigma_0}{M_0^2c}\sqrt{\frac{\mathfrak{M}_0^2 - M_0^2}{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}. \tag{III.78}$$

If one lets β denote the *constant* that the radius makes with the *fixed* axis Oz, around which its motion is performed, then one will easily find that:

$$\sin\beta = \frac{r_0}{R_0}, \qquad \tan^2\beta = \frac{r_0^2}{R_0^2 - r_0^2}.$$

Upon replacing r_0 and R_0 with their values, one will easily find that:

$$R_0^2 - r_0^2 = \frac{\sigma_0^2\sin^2 A\cos^2 A}{c^2(\mathfrak{M}_0^2 - M_0^2\sin^2 A)} = \overline{OC}^2 = Z_0^2, \qquad \tan\beta = \frac{2\mathfrak{M}_0\sqrt{\mathfrak{M}_0^2 - M_0^2}}{M_0^2\sin 2A}. \tag{III.79}$$

However, this axis of rotation is not trivially identical with the spin, as opposed to the Weyssenhoff case.

The position of the point C relative to the center of gravity is determined by the vector:

$$(\mathbf{OC})_k = Z_k = R_k + \frac{\Gamma_k}{\Gamma_0}\,r_0, \tag{III.80}$$

because the acceleration vector obviously passes through the center of motion. We calculate the space-time vector:

$$Z_\mu = R_\mu + \frac{\dot U_\mu}{\gamma_0}\,r_0,$$

and upon taking formulas (71) and (78) into account, one will have:

$$Z_\mu = \frac{\sigma_\mu\sin A}{\mathfrak{M}_0 c} - \frac{\dot U_\mu\sigma_0^2}{M_0^2c^4} + \frac{M_0\sigma_0\dot U_\mu}{c^3\sqrt{(\mathfrak{M}_0^2 - M_0^2)(\mathfrak{M}_0^2 - M_0^2\sin^2 A)}}\,\frac{\mathfrak{M}_0\sigma_0}{M_0^2c}\sqrt{\frac{\mathfrak{M}_0^2 - M_0^2}{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}$$

$$= \frac{\sigma_\mu \sin A}{\mathfrak{M}_0 c} + \frac{\dot{U}_\mu \sigma_0^2 M_0^2 \sin^2 A}{M_0^2 c^4 (\mathfrak{M}_0^2 - M_0^2 \sin^2 A)},$$

or

$$\text{(III.81)} \qquad \boxed{Z_\mu = \frac{\sigma_\mu \sin A}{\mathfrak{M}_0 c} + \frac{\dot{U}_\mu \sigma_0^2 \sin^2 A}{(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, c^4}.}$$

We show that this point is fixed. One differentiates the expression (81) by deducing $\dot{\sigma}_\mu$ from (73) and $\ddot{U}_\mu$ from (74):

$$\dot{Z}_\mu = \frac{\sin A}{\mathfrak{M}_0 c}\left(\frac{M_0^2 c U_\mu}{\mathfrak{M}_0} - cG_\mu\right)\sin A$$

$$+ \frac{\sigma_0^2 \sin^2 A}{(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, c^4 \mathfrak{M}_0^2 \sigma_0^2}[(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)(\mathfrak{M}_0 c^4 G_\mu - M_0^2 c^4 U_\mu]$$

$$= U_\mu \left[\frac{M_0^2}{\mathfrak{M}_0^2}\sin^2 A - \frac{M_0^2 c^4}{\mathfrak{M}_0^2}(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\frac{\sin^2 A}{(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, c^4}\right]$$

$$+ \left[-\frac{\sin^2 A}{\mathfrak{M}_0} + \frac{\sin^2 A}{(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, c^4 \mathfrak{M}_0^2}(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\mathfrak{M}_0 c^4\right].$$

The two components are annulled separately. One will then have $\dot{Z}_\mu = 0$, precisely.

It is easy to see that Z_μ is spatial in the system of inertia, and that the position of the fixed point C in that space is given by the vector:

$$\text{(III.82)} \qquad \boxed{Z_k = \frac{\sigma_k \sin A}{\mathfrak{M}_0 c} + \frac{\Gamma_k \mathfrak{M}_0^2 \sigma_0^2 \sin^2 A}{M_0^2 (\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\, c^4},}$$

where:

$$\Gamma_k = \frac{dV_k}{dt^{\mathrm{I}}} = \frac{\mathfrak{M}_0^2}{M_0^2}\dot{U}_k^{\mathrm{I}}$$

represents the acceleration in the system of inertia.

One easily verifies from this expression that Z_k is orthogonal to the velocity V_k and perpendicular to the acceleration Γ_k, and that the vector $R_k - Z_k$, which is the radius of the circle that is described by the center of matter, is precisely collinear with the acceleration.

One can seek to recover expressions for the velocity that are analogous to the ones that we obtained in the Weyssenhoff case by considering any vectors that are collinear with Z_k to be constant.

If we set, by analogy with (32):

$$\mathbf{V} = \frac{\mathfrak{M}_0}{M_0^2 c^2}\,\boldsymbol{\mathcal{A}} \times \boldsymbol{\Gamma}$$

then upon taking $\boldsymbol{\mathcal{A}}$ to be along the axis **Z** that is orthogonal to **V** and $\boldsymbol{\Gamma}$, one must get a norm:

$$V_0 = \frac{\mathfrak{M}_0}{M_0^2 c^2}\,\mathcal{A}_0\,\Gamma_0\,,$$

or, upon utilizing (76) and (70):

$$\frac{c}{\mathfrak{M}_0}\sqrt{\mathfrak{M}_0^2 - M_0^2} = \frac{\mathfrak{M}_0 M_0^2 c^3}{M_0^2\,\mathfrak{M}_0^3\,\sigma_0}\sqrt{\mathfrak{M}_0^2 - M_0^2}\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}\,\mathcal{A}_0\,,$$

so:

$$\mathcal{A}_0 = \frac{\mathfrak{M}_0\sigma_0}{\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}\,,$$

which will give:

$$\mathcal{A}_k = \frac{c\,\mathfrak{M}_0}{\sin A \cos A}\,Z_k\,,$$

upon taking the value of Z_0 in (79) into account, or upon defining a space-time vector:

$$\mathcal{A}_\mu = \frac{c\,\mathfrak{M}_0}{\sin A \cos A}\,Z_\mu = \frac{\sigma_\mu}{\cos A} + \frac{\mathfrak{M}_0\sigma_0^2 \sin A}{c^2(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)\cos A}\,\dot{U}_\mu\,.$$

This vector is obviously constant, as is Z_μ. It plays the same role in the present case that was played by spin in the Weyssenhoff case, to which it will be identical when one sets $A = 0$.

One can likewise define an angular velocity vector along the same axis by setting $\mathbf{V} = \mathbf{r} \times \boldsymbol{\omega}$, by analogy with (32′), where **r** is the radius of the circle.

One must therefore have $v_0 = r_0\,\omega_0$, in such a way that, from (78):

$$\frac{c}{\mathfrak{M}_0}\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A} = \frac{\mathfrak{M}_0\sigma_0}{M_0^2 c}\sqrt{\frac{\mathfrak{M}_0^2 - M_0^2}{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}\,\omega_0\,,$$

so

$$\omega_0 = \frac{M_0^2 c\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}{\mathfrak{M}_0^2\sigma_0^2}\,.$$

One can therefore define an angular velocity vector in the system of inertia that is a constant vector that is collinear with **Z**:

$$\boldsymbol{\omega} = \frac{M_0^2 c^3(\mathfrak{M}_0^2 - M_0^2 \sin^2 A)}{\mathfrak{M}_0^2\sigma_0^2 \sin A \cos A}\,\mathbf{Z}.$$

Naturally, the norm ω of this vector is equal to the value (77) of the angular velocity.

It remains for us to study the disposition and motion of the spin, and therefore, the wobble t_μ, which is collinear with it and which determines the position of the center of mass. We know that spin is a spatial vector in the system of inertia. Equation (80) shows that the three vectors G_k, R_k, and Z_k, to which the vector σ_k can be appended, from (82), are all in the same plane that rotates about Z_k with the angular velocity of ω that accompanies the center of matter. It is easy to see that spin makes a constant angle with Z_k, which shall call λ. We then have:

$$\cos\lambda = \frac{\sigma_k Z_k}{\sigma_0 Z_0}.$$

Since $Z_\mu \dot{U}_\mu = 0$, formula (81) that $Z_0^2 = Z_\mu Z_\mu = Z_\mu\ \sigma_\mu \sin A / \mathfrak{M}_0\, c$. Therefore:

$$\frac{Z_k \sigma_k}{Z_0} = \frac{Z_\mu \sigma_\mu}{Z_0} = \frac{M_0 c\, Z_0}{\sin A},$$

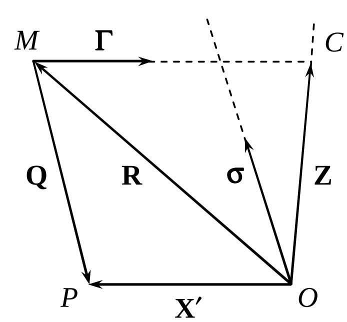
so finally:

$$\frac{Z_k \sigma_k}{Z_0 \sigma_0} = \frac{1}{\sigma_0}\frac{M_0 c}{\sin A}\frac{\sigma_0 \sin A \cos A}{c\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}},$$

namely:

$$\cos\lambda = \frac{\mathfrak{M}_0 \cos A}{\sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}.$$

Spin will then describe a cone whose axis is **Z**, along with the angular velocity $\boldsymbol{\omega}$. If one considers the fact that spin characterizes the *proper* rotation of the drop then one will see that in the system of inertia the axis of proper rotation will submit to a precession that is quite intimately coupled to the *orbital* rotation of the center of matter, since it accompanies it precisely, as it is simultaneously parallel and synchronous, rather like the case of the proper rotation of the moon. It is, moreover, easy to see that the cone that is described by spin is inside of the one that is described by the radius vector. It suffices to calculate the other angles that fix the position with respect to the three vectors $\boldsymbol{\sigma}$, **R**, and **Z**, by means of the cosines $\dfrac{Z_k R_k}{Z_0 R_0}$ and $\dfrac{\sigma_k R_k}{\sigma_0 R_0}$. One then sees that **s** is necessarily between **R** and **Z**.



The wobble is likewise in the space of inertia, where it satisfies:

$$t_k = \sigma_k \sin A.$$

Similarly, the vector Q_k that joins the center of matter M to the center of mass P can also be found in the space of inertia. One then has $Q_k = t_k / \mathfrak{M}_0\, c$.

If we would seek to localize the center of mass relative to the fixed center of gravity O then we would have to calculate the vector:

$$X_k = R_k + Q_k = R_k - \frac{\sigma_k \sin A}{\mathfrak{M}_0 c} \; .$$

However, if we consider formula (71) then it will give us:

$$R_k = \frac{\sigma_k \sin A}{\mathfrak{M}_0 c} - \frac{\sigma_0^2}{M_0^2 c^4} \frac{\mathfrak{M}_0^2}{M_0^2} \Gamma_k$$

in the system of inertia, and we will see that **R** is already decomposed into a vector **– Q** and a vector:

$$\boxed{X_k = -\frac{\sigma_0^2 \mathfrak{M}_0^2}{M_0^4 c^4} \Gamma_k .}$$

One then sees that the center of mass describes a circular motion at the same time as the center of matter, but *in the same plane with the center of gravity.*

The radius of the circle is:

$$X_0 = \frac{\sigma_0 \sqrt{\mathfrak{M}_0^2 - M_0^2} \sqrt{\mathfrak{M}_0^2 - M_0^2 \sin^2 A}}{M_0^2\, \mathfrak{M}_0 c} .$$

One remembers that in the Weyssenhoff case the center of mass, which is identical with the center of matter, describes a circular motion precisely that is centered at the center of gravity, but its radius $\dfrac{\sigma_0 \sqrt{\mathfrak{M}_0^2 - M_0^2}}{M_0^2\, c}$ is greater than X_0 .

In one case, as in the other one, one will have obviously discovered *Møller's disk* in the fixed plane where the circular motion of the center of mass is performed, which is related geometrically to the pseudo-centers of mass, and is at rest in the system of inertia. It will then be obvious that the fixed axis **Z** in the space of inertia, around which all of the motion takes place, is nothing but Møller's spin, which is orthogonal to the plane of the disk.

Therefore, when one considers a zero angle A, one will get the Weyssenhoff motion. In the reference frame of inertia, spin, which is an invariant, will be on the axis of orbital rotation **Z**. The center of matter and the center of mass rotate together in the plane Π with the center of gravity on a circle of radius $\dfrac{\sigma_0\sqrt{\mathfrak{M}_0^2 - M_0^2}}{M_0^2\,c}$ and with an angular velocity of:

$$\omega_1 = \frac{M_0^2 c^2}{\sigma_0 \mathfrak{M}_0}.$$

On the contrary, if the angle A is non-zero then spin will be detached from the axis **Z** and will precess around it. The center of mass will remain in the plane Π, but the radius of its orbit will become smaller. The center of matter will be separated from the center of mass and will leave the plane Π, while describing a circular motion that is parallel and coaxial to the preceding one, but will have a larger radius. The radii of the orbits of the centers of matter and mass relative to the radius $\mathfrak{R}_0$ of the Weyssenhoff motion will have the values:

$$r_0 = \frac{\mathfrak{M}_0\sigma_0\sqrt{\mathfrak{M}_0^2 - M_0^2}}{M_0^2\,c\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}} = \mathfrak{R}_0\frac{\mathfrak{M}_0}{\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}},$$

$$X_0 = \frac{\sigma_0\sqrt{\mathfrak{M}_0^2 - M_0^2}\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}{\mathfrak{M}_0 M_0^2\,c} = \mathfrak{R}_0\frac{\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}{\mathfrak{M}_0}.$$

such that one will have $r_0\,X_0 = \mathfrak{R}_0^2$.

Finally, the spin and the two centers rotate as a unit in the same plane that passes through the axis **Z** and in the same direction of that axis with an angular velocity:

$$\omega = \frac{M_0^2 c^2\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}{\mathfrak{M}_0^2\sigma_0},$$

which is equal to:

$$\omega = \omega_1\frac{\sqrt{\mathfrak{M}_0^2 - M_0^2\sin^2 A}}{\mathfrak{M}_0},$$

and as a consequence, it will be smaller than the velocity ω_1 of the Weyssenhoff motion. Naturally, this comparison will make sense only if the initial conditions are the same. More precisely, one has assumed that the motions both correspond to the same value of spin σ_0 and to the same value of the mass of inertia $\mathfrak{M}_0$.

Since that is given by $\mathfrak{M}_0 = M_0\sqrt{1 - v_0^2/c^2}$, that will amount to taking the same value for the velocity of the *center of matter*. From formulas (77) and (78), that hypothesis will lead to the fact that the expression for that velocity – namely, $r_0\ \omega_0$ – is completely independent of the angle A.

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CHAPTER IV

THE GENERAL THEORY OF HYDRODYNAMICAL MODELS

§ 1. Local velocity and matter density. Having begun to constitute fluids with molecular structure by means of the spinning particles that we have studied, we shall devote the present chapter to the study of representative fluids that one might constitute by starting with the wave functions of quantum mechanics, or more generally, the arbitrary wave functions that are at the basis of the classical theory of fields. In Appendix *C*, we show how, upon starting with a wave function of an arbitrary nature whose wave equation can be derived from a Lagrangian formalism one may deduce a system of conservative tensors, namely: a *current vector* j_μ, for which $\partial_\mu j_\mu = 0$, a second-order tensor $t_{\mu\nu}$ – viz., the *canonical energy-momentum tensor* – for which one has $\partial_\mu t_{\mu\nu} = 0$, and finally, a third-order tensor $f_{[\mu\nu]\lambda}$ – viz., the *Belinfante tensor* – which is antisymmetric in μ and ν, and obeys the equation:

$$\partial_\lambda f_{[\mu\nu]\lambda} = 0.$$

We will also show that one may subject the last two tensors to a very general gauge transformation that leaves the two fundamental equations invariant and does not modify certain global tensors that are integrated over the domain of the field; hence, one may consider the transformed tensors to represent the wave function just as well as the tensors $t_{\mu\nu}$ and $f_{[\mu\nu]\lambda}$, which comes about as a result of their indeterminacy up to a suitable choice of gauge.

We shall start with the tensorial formalism, and show that one may, in any case, deduce the variables that constitute a hydrodynamical model from it, because they obey general relations that one may interpret as fundamental dynamical equations.

Start with the current j_μ, for which we recall the expression:

(IV.1) $$j_\mu = i\left(\psi^r \frac{\partial\mathcal{L}}{\partial\psi^r_{,\mu}} - \psi^{r*} \frac{\partial\mathcal{L}}{\partial\psi^{r*}_{,\mu}}\right),$$

and give it the usual form of a fluid current in relativistic hydrodynamics. If one lets ρ denote the *matter density* (i.e., the number of "molecules" per unit *proper* volume) and lets u_μ denote the local *unit-speed velocity* of the fluid, for which one has $u_\mu\, u_\mu = -c^2$, then one can set:

$$\boxed{j_\mu = \hbar\rho\, u_\mu .}$$

The Planck constant intervenes as a dimensional factor. Indeed, in the formalism of quantum mechanics, Lagrangians always have the dimension of energy. In order for u_μ to the dimension of a velocity, one easily sees that one must divide the current by a factor that has the dimension of an action. It is a simple question of convenience, because it is quite certain that the significance of the hydrodynamical formalism will not be affected by any factor that one assume depends upon the density ρ. Therefore, one has $j_\mu\, j_\mu = -\hbar^2 \rho^2 c^2$, which will immediately provide the expressions for the two hydrodynamical quantities:

$$\rho = \frac{1}{\hbar c}\sqrt{-j_\mu j_\mu} \qquad \text{and} \qquad u_\mu = \frac{1}{\hbar \rho}\, j_\mu .$$

Since j_μ may be replaced with its expression in (1), one sees that the density of matter and the unit-speed velocity may be expressed as functions of the wave function ψ uniquely.

The significance of the conservation equation $\partial_\mu\, j_\mu = 0$ is immediate.

If one writes it as $\partial_\mu\, (\rho u_\mu) = 0$ then one will see that it is the derivative $\dot{\rho}$ of the quantity ρ along a streamline (cf., Appendix *A*), and one will see that the invariant matter density is conserved in time when one follows the same fluid element in the course of its motion. If one applies the general method of Appendix *A* then one will consider an infinitesimal droplet of fluid whose proper volume is V_0 . The quantity of matter that it contains may be expressed by:

$$Q = \int_{V_0} \rho\, dV_0 \; .$$

The droplet evolves between an instant t_1 and an instant t_2 , and therefore sweeps out a portion of a current tube Ω. One knows that one then has:

$$\int_\Omega \dot{\rho}\, d\omega = Q_2 - Q_1,$$

and since we have $\dot{\rho} = 0$ here, one sees that $Q_2 = Q_1$.

The conservation of current then signifies that the quantity of matter in the droplet is conserved in the course of its motion; there can then be no creation or annihilation of matter at each point of the fluid.

The determination of the unit-speed velocity permits us to analyze the tensors $t_{\mu\nu}$ and $f_{\mu\nu\lambda}$, along with their decomposition into hydrodynamical quantities, and then, as we just did, to follow a fluid drop in the course of its motion and to therefore specify the laws regarding the hydrodynamical quantities that are obtained.

§ 2. The decomposition of the energy-momentum tensor. We commence with the tensor $t_{\mu\nu}$:

$$t_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \psi^r_{,\nu}} \psi^r_{,\mu} - \delta_{\mu\nu}\mathcal{L} \, .$$

One then decomposes the tensor $t_{\mu\nu}$ into components that are collinear with or orthogonal to the current, following a method that was already pointed out (Chap. I). It results that:

$$\boxed{t_{\mu\nu} = \mu_0 u_\mu u_\nu - p_\mu u_\nu + q_\nu u_\mu + \theta_{\mu\nu},}$$

with:

$$p_\mu\, u_\mu = 0, \qquad q_\mu\, u_\mu = 0, \qquad \theta_{\mu\nu}\, u_\nu = \theta_{\mu\nu}\, u_\mu = 0.$$

The coefficients of this development are easy to calculate as functions of $t_{\mu\nu}$ and u_μ (and, as a consequence, of the wave functions ψ^r); one contracts with $u_\mu\, u_\nu$. It then results that:

$$\boxed{t_{\mu\nu} u_\mu u_\nu = \mu_0 c^4,}$$

from which, we obtain the scalar μ_0, which one calls the *proper mass density*; we shall soon see why. If one contracts with u_μ or u_ν then one will find that:

$$t_{\mu\nu}\, u_\nu = -\, c^2 \mu_0\, u_\mu + c^2 p_\mu\,,$$

hence:

$$\boxed{p_\mu = \mu_0 u_\mu + \frac{1}{c^2} t_{\mu\nu} u_\nu}$$

and

$$t_{\mu\nu}\, u_\mu = -\, c^2 \mu_0\, u_\nu - c^2 q_\nu\,,$$

hence:

$$\boxed{q_\mu = -\mu_0 u_\mu - \frac{1}{c^2} t_{\mu\nu} u_\nu\,,}$$

respectively. What ultimately remains is:

$$\boxed{\theta_{\mu\nu} = t_{\mu\nu} - \mu_0 u_\mu u_\nu + p_\mu u_\nu - q_\nu u_\mu\,.}$$

In the decomposition of $t_{\mu\nu}$, one recognizes two classical terms (cf., Appendix *B*): $\mu_0\, u_\mu\, u_\nu$, which expresses the dynamical part of the classical energy-momentum tensor (from which, one gets the interpretation of μ_0), and the proper space tensor $\theta_{\mu\nu}$, which provides the internal stress tensor. However, it may be remarked that, on the one hand, μ_0 may generally be put into only the classical form ρm_0, because one does not have:

$$\dot{\mu}_0 = 0,$$

and, on the other hand, the force density that is derived from the tensor $\theta_{\mu\nu}$, – viz., $\varphi_\mu = -\partial_\nu\, \theta_{\mu\nu}$ – is not generally contained in proper space. We shall now examine this point.

By hypothesis, one has:

$$\theta_{\mu\nu}\, u_\mu = 0,$$

from which, one infers that:

$$\partial_\nu (\theta_{\mu\nu} u_\mu) = 0,$$

and not that:

$$\varphi_\mu u_\mu = 0,$$

which would demand that:

$$u_\mu \partial_\nu \theta_{\mu\nu} = 0.$$

Now, decompose φ_μ into a component that is along the current, which we write as:

$$\frac{w_0}{c^2} u_\mu ,$$

and a proper-space component f_μ:

$$\varphi_\mu = \frac{w_0}{c^2} u_\mu + f_\mu , \qquad \text{with} \qquad f_\mu u_\mu = 0.$$

These two quantities may be calculated immediately by contracting with u_μ :

$$\varphi_\mu u_\mu = - w_0 , \qquad f_\mu = \varphi_\mu - \frac{w_0}{c^2} u_\mu .$$

As we will verify shortly, f_μ plays the classical role of the *force density* that is exerted by the internal stresses. It leads to the fact that the action of the internal stresses on a small parallelepiped (in the proper system, for example) is stronger on the one face than it is on the opposite face. On the other hand, one may make the expression for w_0 explicit:

$$w_0 = - \varphi_\mu u_\mu = - \partial_\nu \theta_{\mu\nu} u_\mu = - \partial_\nu(\theta_{\mu\nu} u_\mu) + \theta_{\mu\nu} \partial_\nu u_\mu .$$

The first term on the left-hand side is zero, and what will remain is:

(IV.2) $$\boxed{w_0 = \theta_{\mu\nu} \partial_\nu u_\mu .}$$

In order to interpret this quantity physically, we place ourselves in the proper system, where $\theta_{\mu\nu}$ possesses only spatial components. One then has:

$$w_0 = \theta_{ij}^0 (\partial_i u_j)^0 , \qquad \text{such that} \qquad w_0 = \theta_{ij}^0 (\partial_i v_j)^0 .$$

Since one must take into account the local variations of the velocity, the model of the infinitesimal drop, whose parts reasonably have the same velocity and was of service to us in Appendix *A* especially, will no longer suffice. One must consider a point *C* that is determined by the drop and describes a world-line $\mathfrak{L}$ with a velocity U_μ, and refer the drop to the proper system *at this point C*. The quantity that is studied thus brings other parts of the drop into play, relative to the axes defined at the point *C*. Having said that, we place ourselves to begin with in the simple case where the stress tensor (when referred to the proper system) has all of its off-diagonal components equal to zero, and its diagonal components equal to $\theta_{ij}^0 = p^0\,\delta_{ij}$.

We will then be in the case of the *perfect fluid*, so p^0 represents the pressure. Equation (2) leads to: $w_0 = p^0(\partial_i v_i)^0$. In order to interpret this, it suffices to integrate over the volume of the droplet in the proper system:

$$\int_{V_0} w_0 dV_0 = \int_{V_0} p^0 (\partial_i v_i)^0 dV_0 = p^0 \int_{V_0} (\partial_i v_i)^0 dV_0 \, ,$$

upon neglecting the variation of pressure on the scale of the droplet.

Finally, one has:

$$\int_{V_0} w_0 dV_0 = p^0 \int_{\Sigma_0} v_i^0 d\sigma_i^0 = p^0 \frac{\delta V_0}{\delta\tau} \, ,$$

and the integral, when multiplied by a proper time element $\delta\tau$ obviously represents the increase in volume in the droplet. The calculated expression then represents the effect of the force of pressure (per unit time) that is caused by a dilatation or contraction of the droplet.

In the general case, one must superpose another effect with the preceding one that is related to the off-diagonal components of the stresses. These components give us:

$$\theta_{ik}^0 \int_{V_0} (\partial_k v_i)^0 dV_0$$

for $i \neq j$, namely:

$$\theta_{ik}^0 \int_{\Sigma_0} v_i^0 d\sigma_k \; .$$

This time, we are dealing with the effect of the internal stresses on the components of the velocity that are tangent to the surface of the droplet. That brings into play the differences between the velocities of the fluid layers, which slide over each other; for example, due to the rotation of the drop in the system of axes that is defined by the point C, which is a rotation that gives rise to friction between the surface of the drop and its fluid environment. In this case, there is an effect that is produced by the existence of off-diagonal components in θ_{ij}^0, which amounts to assuming that one is dealing with a *viscous fluid*, which means that it should be subject to the laws of viscosity that one finds in material fluids (cf., Appendix B).

One can summarize the results of this discussion by considering the proper space stresses that provided our decomposition as being applied to the droplet globally. The stresses produce two entirely different types of effect on the droplet. On the one hand, there is the effect of a force f_μ (which is integrated over the *volume* of the droplet), which takes the form of an external force that is applied to a material point. Its only effect will be to increase the kinetic energy of the droplet. On the other hand, the work w_0 is due to the effects of dilatation and viscosity. The latter is related to the fact that the droplet is *not* a material point, because it essentially possesses a definite extension. The work w_0 does not translate into any global mechanical effect on the droplet. If it is positive then its energy will increase in a non-mechanical form that one may not express as a function of the parameters that were introduced. One is necessarily led to introduce a supplementary specific energy that naturally possesses a mass, but which may be expressed as a function of the motion of the fluid. We shall call this energy *heat*, following a suggestion of Takabayasi [**24**], as long as it is clearly understood that one cannot associate it with any of the usual interpretations of that word that relate to thermal agitation, entropy, and more generally, to the statistical structure of the fluid. The quantity w_0 will therefore be called the *proper caloric energy density.*

In order to interpret the other quantities that $t_{\mu\nu}$ is comprised of explicitly, we shall discuss the conservation relation:

$$\partial_\nu(\mu_0\, u_\mu\, u_\nu - p_\mu\, u_\nu + q_\nu\, u_\mu + \theta_{\mu\nu}) = 0.$$

Since the first two terms contain u_ν , the divergence will have the form of a derivative along a streamline (cf., Appendix *A*), which we denote by either $d\,/\,d\tau$ or a dot. As for $\partial_\nu\,\theta_{\mu\nu}$, we write it out in detail:

$$\frac{d}{d\tau}(\mu_0\, u_\mu - p_\mu) + \partial_\nu(q_\nu\, u_\mu) - \frac{w_0}{c^2}u_\mu - f_\mu = 0.$$

The last two terms have the form that one finds in the classical Euler formula, where one is led to consider the vector $\mu_0\, u_\mu - p_\mu$ as something that represents a generalized *momentum density* g_μ. One remarks that it is composed of a classical term $\mu_0 u_\mu$ that is collinear with the velocity and a term $- p_\mu$ that is in proper space, and which we shall call the *transverse momentum density* (see Chap III). One therefore has:

$$\boxed{g_\mu = -\frac{1}{c^2}t_{\mu\nu}u_\nu.}$$

This equation will then lead to:

(IV.3) $$\dot{g}_\mu + \partial_\nu(q_\nu u_\mu) - \frac{w_0}{c^2}u_\mu = f_\mu\,.$$

If we contract this with u_μ then it will follow, upon remembering that $f_\mu\, u_\mu = 0$, that:

$$\dot{g}_\mu u_\mu + u_\mu\, \partial_\nu(q_\nu\, u_\mu) + w_0 = 0\ ;$$

i.e.:

$$\frac{d}{d\tau}(g_\mu u_\mu) - g_\mu \dot{u}_\mu + \partial_\nu(-c^2 q_\nu) - q_\nu u_\mu \partial_\nu u_\mu + w_0 = 0.$$

Now, one knows that $g_\mu\, u_\mu = 0$ and $u_\mu\, u_\mu = - c^2$, so one will get:

$$u_\mu\, \partial_\nu\, u_\mu = 0,$$

and finally:

$$\dot{g}_\mu u_\mu = - p_\mu \dot{u}_\mu = \dot{p}_\mu u_\mu\,.$$

Thus, one finally gets:

$$\boxed{\dot{\mu}_0 c^2 + \dot{p}_\mu u_\mu = w_0 - c^2 \partial_\nu q_\nu .}$$

This relation obviously provides us with the accounting sheet for the "proper" energy. In the left-hand side, we recognize the variation of the proper mass density and a term that appears to be difficult to interpret, because one does not generally have:

$$\dot{p}_\mu u_\mu = 0 \qquad \text{or} \qquad p_\mu \dot{u}_\mu = 0.$$

In the right-hand side, we recover the caloric energy that is created by the work done by the internal stresses. However, the last term warns us that this caloric energy cannot be localized to the point where it was produced and that its migration is represented by the vector q_ν, which we – following Takabayasi [**24**] – call the *heat current density*. Its role will become precise immediately.

Now, consider equation (3), and integrate it over a portion of the tube that is swept out by a droplet in time $d\tau$.

One knows (cf., Appendix *A*) that if we consider the total momentum of the droplet:

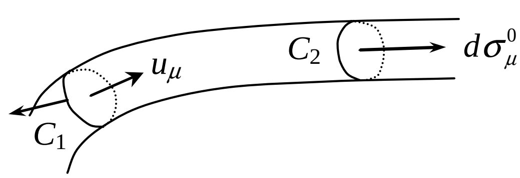
$$G_\mu = \int_{V_0} g_\mu dV_0$$

then the integral $\int_\Omega \dot{g}_\mu d\omega$ can be written as $\dot{G}_\mu \, d\tau$. Similarly, $\int_\Omega f_\mu d\omega$ can be written as:

$$d\tau \int_{V_0} f_\mu dV_0 = d\tau F_\mu ,$$

in which F_μ is the total force that is exercised by the stress forces.

The integral $\int_\Omega \partial_\nu (q_\nu u_\mu) d\omega$ reduces to $\int_\Sigma q_\nu u_\mu \, d\sigma_\nu$, which is an integral that is taken over the entire hypersurface that bounds that portion of the tube. The integrals over the two proper-space ends C_1 and C_2 will be zero since q_ν is in proper space, so one will have: $q_\nu^0 d\sigma_\nu^0 = 0$.



On the contrary, on the boundary one must say:

$$d\sigma_\lambda = \frac{i}{c} \varepsilon_{\mu\nu\rho\lambda} dx_\mu^{(1)} dx_\nu^{(2)} dx_\rho^{(3)} ,$$

where the three infinitesimal elements $dx_\mu^{(1)}$, $dx_\mu^{(2)}$, $dx_\mu^{(3)}$ are taken over the hyper-boundary.

One may choose one of them – for example, $dx_\mu^{(3)}$ – to point along the current, and calculate the product $q_\nu\, d\sigma_\nu$ in the proper system. One then has:

$$dx_i^{(3)} = 0, \qquad dx_4^{(3)} = ic\, d\tau,$$

$$d\sigma_4^0 = 0, \qquad d\sigma_k^0 = \frac{i}{c}\varepsilon_{ij4k}dx_i^{(1)}dx_i^{(2)}\, icd\tau,$$

or, upon setting $\varepsilon_{ij4k} = \varepsilon_{ijk}$, by convention:

$$d\sigma_k^0 = \varepsilon_{ijk}dx_i^{(1)}dx_i^{(2)}\, d\tau .$$

One therefore obtains simply the usual area element ds_k^0 over the surface of the droplet in proper space. We therefore obtain the integral:

$$d\tau\int_{S_0} u_\mu q_k^0 ds_k^0 ,$$

which, if one neglects the variations of u_μ at the level of the droplet, will give:

$$d\tau U_\mu \int_{S_0} q_k^0 ds_k^0 = d\tau\, U_\mu\, \Phi_0 ,$$

and we shall call Φ_0 the *heat flux*, or the flux of the heat current vector that passes through the drop in proper space. If this equation is integrated over the drop then that will give:

(IV.4) $$\boxed{G_\mu - U_\mu\left(\int_{V_0}\frac{w_0}{c^2}dV_0 - \int_{S_0} q_k^0 ds_k^0\right) = F_\mu .}$$

One sees that the total momentum of the droplet is subjected to two entirely different types of variations: On the one hand, the dynamical variation, which is measured by the force F_μ, as is the case for the classical material point. On the other hand, there is a "thermodynamic" variation that is due to the fact that one part of the energy that is contained in the drop is composed of the heat. This may increase as a result of the work that is done by the internal stresses and decrease as a result of one particular process of "conduction." It is only the difference between the total variation dG_μ / dt of the momentum and that of the supplementary momentum that is afforded by the heat that intervenes in the laws of dynamics.

Therefore, we find a justification for the interpretation that we have given to the two quantities $\theta_{\mu\nu}\, \partial_\nu\, u_\mu$ and q_ν, which, we believe, may not be interpreted in a purely dynamical fashion.

§ 3. The decomposition of the moment of proper rotation. We shall now use the projections onto the current in order to analyze the Belinfante tensor density of the moment of proper rotation, whose expression in terms of the operator of infinitesimal rotations we recall:

$$f_{[\mu\nu]\lambda} = \frac{\partial \mathcal{L}}{\partial \psi^r_{,\lambda}} \mathfrak{T}^{rs}_{[\mu\nu]} \psi^s \qquad \text{(Appendix } C\text{).}$$

One obtains a covariant decomposition by taking antisymmetry into account:

$$f_{[\mu\nu]\lambda} = A_{[\mu\nu]\lambda} + B_{\mu\nu} u_\lambda - B_{\mu\nu} u_\lambda + M_{[\mu\nu]} u_\lambda + c(T_\mu u_\nu - T_\nu u_\mu) u_\lambda ,$$

with

$$A_{\mu\nu\lambda} u_\mu = A_{\mu\nu\lambda} u_\nu = A_{\mu\nu\lambda} u_\lambda = 0,$$
$$B_{\mu\nu} u_\mu = B_{\mu\nu} u_\nu = 0,$$
$$M_{\mu\nu} u_\mu = M_{\mu\nu} u_\nu = 0,$$
$$T_\mu u_\mu = 0.$$

This decomposition remains very complex, and Takabayasi proposed to simplify it by profiting from the indeterminacy in the gauge. To that end, it helps to have in mind the physical idea that the abstract notion of "moment of proper rotation" in the kind of hydrodynamics under scrutiny must correspond to the existence of an intrinsic rotation that affects the elementary particles that the fluid may be regarded as composed of. Among the tensors into which $f_{[\mu\nu]\lambda}$ may be resolved, only one of them can conveniently represent the intrinsic rotation of a spinning particle. It is the antisymmetric proper-space tensor $M_{[\mu\nu]}$, to which Takabayasi gave the status of a proper angular momentum. However, we encountered more general motions in the analysis of the spinning particles that took the form of intrinsic rotations by using an antisymmetric tensor $S_{[\mu\nu]}$ that is no longer in proper space. If we would like to take into account the possibility of such a motion for the elements of our fluid then we would see that we must consider not only the term $M_{[\mu\nu]} u_\lambda$, but also the term $c(T_\mu u_\nu - T_\nu u_\mu) u_\lambda$ – i.e., the set of terms that contain u_λ as a factor. We are thus led to the following decomposition:

(IV.5) $$f_{\mu\nu\lambda} = \tfrac{1}{2} s_{[\mu\nu]} u_\lambda + {}^{+}f_{[\mu\nu]\lambda} ,$$

with:

$${}^{+}f_{[\mu\nu]\lambda} u_\lambda = 0.$$

The tensor 1/2 $s_{[\mu\nu]}$ may be decomposed into a proper-space part $M_{[\mu\nu]}$ that one may, like Takabaysi, interpret as being due to the gyrational momentum of the particles and a part $c\,(T_\mu u_\nu - T_\nu u_\mu)$ that is orthogonal to proper space and is due to the separation between the center of mass and the center of matter for each particle, if it exists. As for the rest of the tensor, it does not appear to be possible to give it a physical interpretation. Also, it is legitimate to make it disappear by a convenient choice of gauge, which, as we know, leaves not only the validity of the fundamental equations of conservation invariant, but also the value of the total momentum $\mathfrak{G}_\mu$ and the total angular momentum $\Gamma_{\mu\nu}$ when one integrates over the domain of the field. One therefore makes the transformation:

$$f'_{\mu\nu\lambda} = f_{\mu\nu\lambda} - {}^{+}f_{\mu\nu\lambda} ,$$

which must naturally accompany the transformation:

$$t'_{\mu\nu} = t_{\mu\nu} - \partial_\lambda \Phi_{\mu\nu\lambda} ,$$

with:

$$\Phi_{\mu\nu\lambda} = {}^{+}f_{\mu\nu\lambda} - {}^{+}f_{\mu\lambda\nu} - {}^{+}f_{\nu\lambda\mu} .$$

Of course, this transformation modifies all of the dynamical quantities that we deduced from $t_{\mu\nu}$, but not the equations to which they are related, nor, as a consequence, their hydrodynamical interpretations. Since, on the one hand, $s_{\mu\nu}$ and ${}^{+}f_{\mu\nu\lambda}$ are perfectly determined as functions of $f_{\mu\nu\lambda}$, and consequently, of the wave function, by the relation that is deduced from (5) upon contracting with u_λ :

$$\tfrac{1}{2} s_{\mu\nu} = -\frac{1}{c^2} f_{\mu\nu\lambda} u_\lambda ,$$

$${}^{+}f_{\mu\nu\lambda} = f_{\mu\nu\lambda} - \tfrac{1}{2} s_{\mu\nu} u_\lambda ,$$

one sees that all of the formalism continues to depend in a perfectly unambiguous fashion on the wave function.

By means of this gauge transformation, one has the simplified expression (upon suppressing the signs):

$$\boxed{f_{\mu\nu\lambda} = \tfrac{1}{2} s_{\mu\nu} u_\lambda}$$

for the tensor $f_{\mu\nu\lambda}$. It remains for us to justify the dynamical interpretation that makes us identify $s_{\mu\nu}$ with a density of proper angular momentum. The conservation equation for momentum gives us:

$$t_{\mu\nu} - t_{\nu\mu} = 2\partial_\lambda f_{\mu\nu\lambda} = \partial_\lambda (s_{\mu\nu} u_\lambda) = \dot{s}_{\mu\nu} ,$$

upon introducing the derivative of the density $s_{\mu\nu}$ along the streamline (Appendix *A*).

On the other hand, one can specify $t_{\mu\nu}$ as a function of the dynamical quantities:

$$t_{\mu\nu} - t_{\nu\mu} = \mu_0 u_\mu u_\nu - p_\mu u_\nu + q_\nu u_\mu + \theta_{\mu\nu} - \mu_0 u_\nu u_\mu + p_\nu u_\mu - q_\mu u_\nu - \theta_{\nu\mu} ,$$

$$\equiv (q_\nu + p_\nu) u_\mu - (q_\mu + p_\mu) u_\nu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu} .$$

One thus makes a generalized transverse momentum density vector appear:

$$p'_\mu = p_\mu + q_\mu ,$$

which permits us to give the equation the condensed form:

$$\boxed{p'_\nu u_\mu - p'_\mu u_\nu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu},}$$

which provides a generalization of the Frenkel-Weyssenhoff equation.

In order to utilize the customary procedure of integrating over an element of the tube, it is preferable to preserve the initial form:

$$g_\mu u_\nu - q_\nu u_\mu + q_\nu u_\mu - q_\mu u_\nu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu} .$$

Upon integrating, the right-hand side gives us $\dot{S}_{\mu\nu} d\tau$. For each term on the left-hand side, decompose the space-time element into $dV_0\, dt$, and integrate over the proper volume by assuming that it is small enough that the velocity varies only a little over it and may be taken out of the integral. One will then have:

$$G_\mu U_\nu - G_\nu U_\mu + U_\mu \int_{V_0} q_\nu dV_0 - U_\nu \int_{V_0} q_\mu dV_0 + 2 \int_{V_0} \theta_{<\mu\nu>} dV_0 = \dot{S}_{\mu\nu}$$

One may construct a *total heat current* vector:

$$Q_\mu = \int_{V_0} q_\mu dV_0$$

using the vector density of heat current q_μ . It will be a proper-space vector that represents a quantity of the same nature as the momentum G_μ .

Finally, $\theta_{<\mu\nu>}$ has the dimensions of a dipole density, since $\partial_\nu \theta_{\mu\nu}$ has the dimensions of a force density. One may thus define a *total* dipole:

$$N_{[\mu\nu]} = 2 \int_{V_0} \theta_{<\mu\nu>} dV_0 .$$

One may finally write the equation of the droplet as:

$$\text{(IV.6)} \qquad \boxed{(G_\mu - Q_\mu) U_\nu - (G_\nu - Q_\nu) U_\mu + N_{[\mu\nu]} = \dot{S}_{\mu\nu}.}$$

One sees that this equation expresses the classical theorem of the kinetic moment for the droplet. As we found in all of the situations in which we were concerned with a relativistic spinning particle, the variation of the total angular momentum is composed of an orbital term $U_\mu G_\nu - U_\nu G_\mu$ that expresses the "curvature" of the global motion of the droplet and a supplementary term $\dot{S}_{[\mu\nu]}$ that defines the existence of a "proper" angular momentum, and thus justifies our interpretation of $s_{\mu\nu}$ as a proper angular momentum density. On the other hand, this variation is equal to the quantity $N_{[\mu\nu]}$, which, as the integral of the antisymmetric part of the internal stress tensor, obviously plays the role of a dipole of torsion.

It remains for us to interpret the appearance of heat. One sees that it presents itself here in the form of a heat current:

$$Q_\mu = \int_{V_0} q_\mu dV_0 \ ,$$

whereas in the force equation it takes the form of the quantity:

$$A_0 = \int_{V_0} \frac{w_0}{c^2} dV_0 - \int_{S_0} q_\mu ds_\mu^0 \ .$$

In both cases, the total momentum G_μ expresses the motion of the totality of the energy, which is composed of caloric energy here. In equation (4), it plays the role of the *variation* of the momentum during a time $d\tau$. It is composed of a dynamical part that is due to the variation of the velocity under the action of the force and a non-dynamical part that is due to the variation of the proper mass during a time $d\tau$, which is a variation that results in the creation of heat by the work that was done by the internal stresses and the loss or acquisition of heat by conduction. The second part of the variation of the momentum does not have any dynamical cause. It does not depend upon the external force, which acts upon only the dynamical part. In order to obtain the latter, one must then *subtract* the non-dynamical variation $A_0\ U_\mu$ from the total variation $\dot{G}$. On the contrary, in the dipole equation one is involved with the expression $G_\nu\ U_\mu - G_\mu\ U_\nu$, which represents the variation of the orbital angular momentum of *all* of the energy, which is composed of heat here. Now, this will experience a double displacement: On the one hand, there is one that relates to the matter that increases its inertia and is involved with the current and contributes to the dynamical moment of rotation of it by the same right as mass, properly speaking, from which, it will not be discernible, moreover. On the other hand, in the interior of the matter current it will submit to a thermal "migration" that has no dynamical cause, so there will result an apparent angular momentum that must not be accounted for in the dipole equation and thus, for that reason, one must *subtract* it from the total orbital angular momentum in order that only the dynamical part should remain, which will thus take the form:

$$(G_\nu\ U_\mu - G_\mu\ U_\nu) - (Q_\nu\ U_\mu - Q_\mu\ U_\nu).$$

It is for the same reason that we have been led to write the equation for the momentum density:

$$p'_\nu u_\mu - p'_\mu u_\nu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu} \ ,$$

upon introducing a generalized transverse momentum $p'_\mu = p_\mu + q_\mu$, which is in proper space. If one contracts with u_μ then it will follow that:

(IV.7) $$p'_\mu = p_\mu + q_\mu = \frac{1}{c^2} \dot{s}_{\mu\nu} u_\mu \ .$$

This relation shows, among other things, that if the fluid does not possess a proper angular momentum then one will have $p_\mu + q_\mu = 0$.

One thus remarks that the existence of a transverse momentum (or the fact that the momentum g_μ is not collinear with the current) is not necessarily an indication of a proper angular momentum. It likewise results from the existence of a heat current, and it can be used in the decomposition into two terms that gives the momentum as:

$$g_\nu = \mu_0\, u_\nu + q_\nu - \frac{1}{c^2}\,\dot{s}_{\mu\nu} u_\mu .$$

As we did for the particle quantities in Chapter III, we can decompose the angular momentum $s_{\mu\nu}$ in a covariant fashion into two space-time vectors:

$$s_\mu = \frac{i}{2c}\,\varepsilon_{\nu\alpha\beta\mu}\, u_\nu\, s_{\alpha\beta} \qquad \text{or the gyration density,}$$

and

$$t_\mu = \frac{1}{c}\, s_{\mu\nu}\, u_\nu \qquad \text{or the wobble density,}$$

and one has:

$$s_{\mu\nu} = \frac{i}{c}\,\varepsilon_{\mu\nu\alpha\beta}\, u_\alpha\, s_\beta + \frac{1}{c}(t_\mu\, u_\nu - t_\nu\, u_\mu).$$

One may express the derivative $\dot{s}_{\mu\nu}$ that factors in the dipole equation as:

$$\dot{s}_{\mu\nu} = \frac{i}{c}\,\varepsilon_{\mu\nu\alpha\beta}\,(u_\alpha \dot{s}_\beta + \dot{u}_\alpha s_\beta) + \frac{1}{c}(u_\mu \dot{t}_\nu - u_\nu \dot{t}_\mu + \dot{u}_\mu t_\nu - \dot{u}_\nu t_\mu) .$$

Equation (7) then yields:

$$p'_\mu = \frac{i}{c^3}\,\varepsilon_{\mu\nu\alpha\beta} u_\nu \dot{u}_\alpha s_\beta + \frac{1}{c}\dot{t}_\mu + \frac{\dot{t}_\nu u_\nu}{c^3} u_\mu ,$$

which, upon contracting with $\dot{u}_\mu$ and s_μ, respectively, will provide the two equalities:

$$p'_\mu \dot{u}_\mu = \frac{1}{c}\dot{t}_\mu \dot{u}_\mu ,$$

$$p'_\mu s_\mu = \frac{1}{c}\dot{t}_\mu s_\mu .$$

Otherwise speaking, the space-time vector:

$$p'_\mu - \frac{1}{c}\dot{t}_\mu = p_\mu + q_\mu - \frac{1}{c}\dot{t}_\mu$$

is simultaneously orthogonal to $\dot{u}_\mu$ and s_μ.

Naturally, the same thing will be true for the vector $g_\mu + q_\mu - \frac{1}{c}\dot{t}_\mu$.

§ 4. Another decomposition that was proposed by Takabayasi. We now point out that Takabayasi [**24**] likewise used another decomposition of the tensor $f_{\mu\nu\lambda}$ (which is equivalent to the one that we proposed above [**25-27**]). We will then explain why it does not seem satisfactory to us. One causes the completely antisymmetric part of the product $s_{[\mu\nu]}u_\lambda$ to appear, which is defined as it was in the foregoing. Upon taking the antisymmetry of $s_{\mu\nu}$ into account, it will be easy to see that the tensor:

$$c\sigma_{[\mu\nu\lambda]} = s_{[\mu\nu]}\, u_\lambda + s_{[\nu\lambda]}\, u_\mu + s_{[\lambda\mu]}\, u_\nu$$

is completely antisymmetric. (We remark that under this operation, the part of $s_{\mu\nu}$ that is not located in proper space will disappear, and we will have, for that matter:

$$\tfrac{1}{2}c\sigma_{[\mu\nu\lambda]} = M_{[\mu\nu]}\, u_\lambda + M_{[\nu\lambda]}\, u_\mu + M_{[\lambda\mu]}\, u_\nu ,$$

which is, moreover, the expression that was considered by Takabayasi.) One may then attribute the significance of *spin* to $\sigma_{[\mu\nu\lambda]}$ or its dual σ_α, which are related by:

$$\sigma_\alpha = \frac{i}{6}\varepsilon_{\alpha\mu\nu\lambda}\, \sigma_{\mu\nu\lambda} , \qquad\qquad \sigma_{[\mu\nu\lambda]} = i\varepsilon_{\mu\nu\lambda\alpha}\, \sigma_\alpha .$$

One then performs a gauge transformation that is different from the one that was proposed previously. The decomposition:

$$f_{\mu\nu\lambda} = \tfrac{1}{2}s_{\mu\nu}u_\lambda + {}^{+}f_{\mu\nu\lambda}$$

leads to:

$$f_{\mu\nu\lambda} = \tfrac{1}{2}c\sigma_{[\mu\nu\lambda]} - \tfrac{1}{2}s_{\nu\lambda}u_\mu - \tfrac{1}{2}s_{\lambda\mu}u_\nu + {}^{+}f_{\mu\nu\lambda} ,$$

and one takes the quantity:

$${}^{++}f_{\mu\nu\lambda} = {}^{+}f_{\mu\nu\lambda} + \tfrac{1}{2}s_{\lambda\nu}u_\mu - \tfrac{1}{2}s_{\lambda\mu}u_\nu$$

to be a gauge, in such a fashion that one will have simply:

$$f_{\mu\nu\lambda} = \frac{c}{2}\,\sigma_{[\mu\nu\lambda]} = \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\alpha}\,\sigma_\alpha .$$

Under these conditions, the torque equation will become:

$$p'_\nu u_\mu - p'_\mu u_\nu + 2\theta_{<\mu\nu>} = ic\varepsilon_{\mu\nu\lambda\alpha}\,\partial_\lambda\,\sigma_\alpha .$$

The appearance of the intrinsic rotation in the form of the dual of the rotation of the spin is correct in the case of the Dirac equation, and in effect the hydrodynamical representation that results from the gauge transformation that we have envisioned in the Dirac case will correspond precisely to the hydrodynamics of the Dirac fluid that was obtained by Takabayasi in a very natural manner [**9**]. However, one will encounter very grave difficulties in its interpretation in the general case when one integrates the equation over an element of the tube: One will find the same expression on the left-hand side that we just found:

$$[(G_\mu - Q_\mu)\, U_\nu - (G_\nu - Q_\nu)\, U_\mu + N_{[\mu\nu]}]\, d\tau.,$$

and with the same interpretation. In order to deal with the right-hand side, we place ourselves (as Takabayasi did) in the case where $s_{\mu\nu} u_\nu = 0$.

One will then have (see Chap. III):

$$s_{\mu\nu} = \frac{i}{c}\, \varepsilon_{\mu\nu\lambda\alpha}\, u_\alpha\, \sigma_\beta .$$

We have to calculate the integral:

$$\int_\Omega ic\,\varepsilon_{\mu\nu\lambda\alpha} \partial_\lambda \sigma_\alpha d\omega = \int_\Sigma ic\,\varepsilon_{\mu\nu\lambda\alpha} \sigma_\alpha d\mathfrak{A}_\lambda ,$$

which is taken over the domain of the hypersurface that bounds that tube element (we denote the hypersurface element by $\mathfrak{A}_\lambda$ in order to avoid confusion with spin). At the ends C_1 and C_2 in proper space, one has:

$$d\mathfrak{A}_\lambda = -\frac{u_\lambda}{c^2} dV_0$$

and

$$d\mathfrak{A}_\lambda = +\frac{u_\lambda}{c^2} dV_0 ,$$

respectively.

Thus, the contribution:

$$-\int_{C_1} \frac{i}{c} \varepsilon_{\mu\nu\lambda\alpha} \sigma_\alpha u_\lambda dV_0 + \int_{C_2} \frac{i}{c} \varepsilon_{\mu\nu\lambda\alpha} \sigma_\alpha u_\lambda dV_0$$

will become:

$$d\tau \frac{d}{d\tau} \int_{V_0} \frac{i}{c} \varepsilon_{\mu\nu\lambda\alpha} \sigma_\alpha u_\lambda dV_0 ,$$

which is nothing but:

$$d\tau \frac{d}{d\tau} \int_{V_0} s_{\mu\nu} dV_0 = dt\ \dot{S}_{\mu\nu} .$$

One then recovers the correct term in the Frenkel-Weyssenhoff equation precisely. We have seen that on the hyper-boundary one may, upon placing oneself in the proper system, express $d\mathfrak{A}_k^0$ in terms of the proper space surface element ds_k^0 as $d\mathfrak{A}_k^0 = ds_k^0\,d\tau$; one thus has the integral:

$$I_{[\mu\nu]} = ic\;\varepsilon_{\mu\nu\lambda\alpha}\int_{S_0}\sigma_\mu^0 d\mathfrak{A}_\lambda^0 .$$

The pure space components I_{ij}^0 are zero because their integrands contain the factors σ_4^0 or $d\mathfrak{A}_4^0$, which are zero. The remaining components are:

$$I_{i4}^0 = d\tau\,\frac{ic}{2}\varepsilon_{i4jk}\int_{S_0}(\sigma_k^0 ds_j^0 - \sigma_j^0 ds_k^0) = d\tau\,\frac{ic}{2}\int_{S_0}\varepsilon_{ijk}(\sigma_k^0 ds_j^0 - \sigma_j^0 ds_k^0) .$$

Now one knows (Chap. III) that in proper space one has:

$$\sigma_i^0 = \varepsilon_{ijk}\; s_{jk}^0 ,$$

so:

$$s_{jk}^0 = \tfrac{1}{2}\varepsilon_{ijk}\;\sigma_i^0 .$$

One thus has:

$$I_{i4}^0 = d\tau\,\frac{ic}{2}\int_{S_0}(\tfrac{1}{2}s_{ij}^0 ds_j^0 - \tfrac{1}{2}s_{ki}^0 ds_k^0) = d\tau\,\frac{ic}{2}\int_{S_0}s_{ij}^0 ds_j^0 .$$

One therefore sees the flux of the proper angular momentum appear, which is more conveniently transformed into a volume integral:

$$I_{i4}^0 = d\tau\,\frac{ic}{2}\int_{V_0}\partial_j s_{ij}^0\,dV_0 .$$

If one remarks that $ic = U_4^0$ then one can give the antisymmetric form:

$$I_{[i4]}^0 = \frac{1}{2}d\tau\left(U_4^0\int_{V_0}\partial_j^0 s_{ij}^0 dV_0 - U_i^0\int_{V_0}\partial_j^0 s_{4j}^0 dV_0\right)$$

to I_{i4}^0, since the second term is zero, or similarly:

$$I_{[j4]} = \frac{1}{2}d\tau\left(U_4^0\int_{V_0}\partial_\lambda^0 s_{i\lambda}^0 dV_0 - U_i^0\int_{V_0}\partial_\lambda^0 s_{4\lambda}^0 dV_0\right),$$

which amounts to adding the zero terms $\partial_4^0\,s_{j4}^0$. (Recall that the s_{i4}^0 components are zero in the proper system.) One then sees that with this form for $I_{[i4]}^0$, since all of the spatial components are equal to:

$$I^0_{[jk]} = \frac{1}{2} d\tau \left(U^0_k \int_{V_0} \partial^0_\lambda s^0_{j\lambda} dV_0 - U^0_j \int_{V_0} \partial^0_\lambda s^0_{k\lambda} dV_0 \right),$$

which are zero because the U^0_k are, one can construct a tensor whose components in the proper system are all identical to those of the covariant tensor:

$$I_{[\mu\nu]} = -\frac{1}{2} d\tau \left(U_\mu \int_{V_0} \partial_\lambda s_{\nu\lambda} dV_0 - U_\nu \int_{V_0} \partial_\lambda s_{\mu\lambda} dV_0 \right),$$

or, if one sets:

$$D_\mu = \frac{1}{2} \int_{V_0} \partial_\lambda s_{\nu\lambda} dV_0$$

then we will have:

$$I_{\mu\nu} = - d\tau (U_\mu D_\nu - U_\nu D_\mu).$$

Hence, the equation of the drop will be:

$$\dot{S}_{\mu\nu} = (G_\mu - Q_\mu)\, U_\nu - (G_\nu - Q_\nu)\, U_\mu + N_{\mu\nu} + (U_\mu D_\nu - U_\nu D_\mu).$$

It must be strongly emphasized that the quantity D_μ is a function of only the distribution of the density of the proper angular momentum in the middle of the drop.

If we would like to interpret this relation then we must first remark that in making the internal stresses tend towards zero (which all but obliges us to suppress the heat current, as well, since there will no longer be any production of caloric energy specifically), one must, in any case, be led to the suppression of any type of action on the drop on the part of the rest of the fluid, and one must thus arrive at the equations of motion for a free, isolated drop. In that case, one will have:

$$\dot{S}_{\mu\nu} = G_\mu U_\nu - G_\nu U_\mu + U_\mu D_\nu - U_\nu D_\mu .$$

Now, in all of the analysis of the free drop in a state of rotation that we did previously, we never found any terms, other than $\dot{S}_{\mu\nu}$, for which the proper angular momentum intervened. Furthermore, in the case of "pure matter," we were justified in regarding a hydrodynamical model that provided us with such terms with distrust. Naturally, it might happen that by reason of the form of the Lagrangian, certain particular wave functions will provide us with internal stresses that are determined, at least in part, by the Dirac wave function. In that case, the term in D_μ might enter into the torque $N_{\mu\nu}$. However, we would then be dealing with only one particular case, whereas here we are seeking a general method that would be applicable to any wave function. At the very most, one can accept that one is restricted to a particular class of hydrodynamical models for which the proper angular momentum has a conservative flux in the absence of an antisymmetric part to the stress tensor. The vector D_μ will then be annulled at the same time as the torque $N_{\mu\nu}$, and in the case of "pure matter," we will recover the usual equation for the free, spinning particle.

§ 5. The classification of hydrodynamical models. The foregoing considerations provide us with the framework for a logical classification of hydrodynamical models. To begin with, one may distinguish the fluids that are given an internal angular momentum from the ones that are not.

On the one hand, the former fluids may be characterized by the properties of the internal angular momentum. As for the more general case, for which we have the two fundamental equations:

$$\boxed{\begin{aligned} &\dot{g}_\mu + \partial_\nu (q_\nu u_\mu) = -\partial_\nu \theta_{\mu\nu} = f_\mu + \frac{w_0}{c^2} u_\mu , \\ &(g_\mu - q_\mu) u_\nu - (g_\nu - q_\nu) u_\mu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu} , \end{aligned}}$$

to which, the decomposition relation for $s_{\mu\nu}$ may be adjoined:

$$\boxed{s_{\mu\nu} = \frac{i}{c} \varepsilon_{\mu\nu\alpha\lambda} u_\alpha s_\lambda + \frac{1}{c}(u_\mu t_\nu - u_\nu t_\mu),}$$

we may, as we were inspired to do in Chapter III, consider two restrictions:

1) *The two vectors s_μ and t_μ are collinear.* They are expressed as functions of a unique proper space vector – viz., the *spin* σ_μ – and a scalar variable A:

$$s_\mu = \sigma_\mu \cos A, \qquad t_\mu = \sigma_\mu \sin A.$$

If one makes the moment of proper rotation intervene:

$$f_{\mu\nu\lambda} = \tfrac{1}{2}\, s_{\mu\nu}\, u_\lambda ,$$

which is considered after making the gauge transformation that makes the other terms disappear, then one will have:

$$s_{\mu\nu} = -\, \frac{2}{c^2} f_{\mu\nu\rho}\, u_\rho ,$$

$$s_\mu = \frac{i}{2c}\, \varepsilon_{\nu\alpha\beta\mu}\, u_\nu\, s_{\alpha\beta} = -\frac{i}{c^3}\, \varepsilon_{\nu\alpha\beta\mu}\, u_\nu\, f_{\alpha\beta\rho}\, u_\rho ,$$

$$t_\mu = \frac{1}{c}\, s_{\mu\nu}\, u_\nu = -\frac{2}{c^3}\, f_{\mu\nu\rho}\, u_\nu\, u_\rho .$$

One then expresses the idea that s_μ and t_μ are collinear by means of the relation $\varepsilon_{\alpha\beta\mu\nu}$ $s_\mu\, t_\nu = 0$, which annuls their exterior product, such that:

$$\varepsilon_{\alpha\beta\mu\nu}\, \varepsilon_{\lambda\sigma\tau\mu}\; u_\lambda\, s_{\sigma\tau}\, s_{\nu\rho}\, u_\rho = 0,$$
$$\delta^{\alpha\beta\nu}_{\lambda\sigma\tau}\; u_\lambda\, s_{\sigma\tau}\, s_{\nu\rho}\, u_\rho = 0.$$

The left-hand side gives us:

$$s_{\alpha\beta}\, u_\nu\, s_{[\nu\rho]}\, u_\rho + s_{\beta\nu}\, u_\alpha\, s_{\nu\rho}\, u_\rho + s_{\nu\alpha}\, u_\beta\, s_{\nu\rho}\, u_\rho$$

and three similar terms. The first of these terms is zero by antisymmetry. What will remain is:

$$s_{\nu\rho}\, u_\rho\, (s_{\nu\alpha}\, u_\beta - s_{\nu\beta}\, u_\alpha) = 0,$$

or, when expressed as a function of the moment of proper rotation:

$$\boxed{f_{\nu\rho\lambda} u_\rho u_\lambda (f_{\nu\alpha\beta} - f_{\nu\beta\alpha}) = 0.}$$

2) In a more restrictive fashion, one might have the case in which the *angular momentum is in proper space:*

$$t_\mu = 0 \qquad \text{or} \qquad s_{\mu\nu}\, u_\nu = 0.$$

For the moment of proper rotation, this translates into:

$$\boxed{f_{\mu\nu\lambda} u_\nu u_\lambda = 0,}$$

once one has made the appropriate gauge transformation.

In this case, one knows that the angular momentum can be represented in an equivalent fashion in terms of spin, which is a proper space vector:

$$\sigma_\mu = \frac{i}{2c}\, \varepsilon_{\nu\alpha\beta\mu}\; u_\nu\, s_{\alpha\beta},$$

with

$$s_{\mu\nu} = \frac{i}{c}\, \varepsilon_{\mu\nu\alpha\beta}\; u_\alpha\, \sigma_\beta.$$

The generalized transverse momentum can be written:

$$p'_\mu = p_\mu + q_\mu = \frac{1}{c^2}\, \dot{s}_{\mu\nu}\, u_\nu = -\frac{1}{c^2}\, s_{\mu\nu}\, \dot{u}_\nu = -\frac{i}{c^3}\, \varepsilon_{\mu\nu\alpha\beta} u_\alpha \sigma_\beta \dot{u}_\nu ,$$

and one has:

$$p'_\mu \dot{u}_\mu = 0 \qquad \text{and} \qquad p'_\mu \sigma_\mu = 0.$$

It is orthogonal to the current, the spin, and the space-time acceleration.

Other restrictions that are independent of the preceding ones might pertain to the energy-momentum tensor. To begin with, one might assume that *there is no heat convection*; i.e., $q_\mu = 0$.

One would then have that $t_{\mu\nu}\, u_\mu = -\,\mu_0 c^2\, u_\mu$ is collinear with the current, which is a condition that can be written as:

$$\varepsilon_{\alpha\beta\mu\nu}\, t_{\rho\nu}\, u_\rho\, u_\mu = 0,$$

such that:

$$t_{\rho\nu}\, u_\rho\, \eta_{\nu\lambda} \equiv t_{\rho\nu}\, u_\rho \left(\delta_{\nu\lambda} + \frac{u_\nu u_\lambda}{c^2} \right) = 0.$$

With these conditions, the transverse momentum:

$$p'_\mu = \frac{1}{c^2} \dot{s}_{\mu\nu}\, u_\nu$$

is solely due to the angular momentum. It is determined by the gyration and the wobble of the spin, which obey the relations that we just wrote for the quantity $p'_\mu = p_\mu + q_\mu$.

The two fundamental equations can then be written:

$$\dot{g}_\mu = -\,\partial_\nu\, \theta_{\mu\nu}\,,$$
$$g_\mu\, u_\nu - g_\nu\, u_\mu + 2\,\theta_{<\mu\nu>} = \dot{s}_{\mu\nu}\,.$$

An even more special case is the one in which there is *neither convection nor production of heat.* Thus, along with the preceding condition, we will also have:

$$\theta_{\mu\nu}\, \partial_\nu\, u_\mu = 0,$$

which will give us:

$$\theta_{ij}^0 (\partial_i v_j)^0 = 0$$

in the proper system.

If we exclude the case in which only the global condition is satisfied (which is a case that is particularly difficult to interpret) then we must have, on the one that, that:

$$\theta_{ij}^0 (\partial_i v_j)^0 = 0 \qquad \text{for} \qquad i \neq j,$$

which signifies that there is no viscosity. Therefore, the fluid must be *perfect*, and the non-relativistic stress tensor will then be written as:

$$\theta_{ij}^0 = \delta_{ij}\, p^0.$$

Moreover, one must have that:

$$\delta_{ij}\, p^0 (\partial_i\, v_j) = 0 \qquad \text{or} \qquad p^0 (\partial_i\, v_i)^0 = 0,$$

which is possible only in two cases:

1) $(\partial_i\, v_i)^0 = 0$, which is *the case of a perfect, incompressible fluid.*

2) $p^0 = 0$, so there is no internal stress.

This is the case of a *pure matter fluid*. This last case realizes the Weyssenhoff hydrodynamics when one has, moreover, that $s_{\mu\nu}\, u_\nu = 0$, or else it realizes the generalizations that are suggested by the Bohm-Vigier particle dynamics.

Finally, one may consider a *fluid that is devoid of any internal angular momentum*. It is not necessary that the Belinfante tensor $f_{\mu\nu\lambda}$ must be identically zero. It suffices that one should have $f_{\mu\nu\lambda}\, u_\lambda = 0$ – i.e., that $f_{\mu\nu\lambda}$ should reduce to the part that we have seen fit to eliminate by a gauge transformation in all cases.

It will then result from this hypothesis that, on the one hand, the energy-momentum tensor (after a gauge transformation) is *symmetric*, and, on the other hand, that $p_\mu + q_\mu = 0$.

The existence of a transverse momentum is solely due to the heat current. The fundamental equations then take the form:

$$\dot{g}_\mu + \partial_\nu(q_\mu u_\nu) = - \partial_\nu\, \theta_{\mu\nu}\,,$$

$$(g_\mu - q_\mu)\, u_\nu - (g_\nu - q_\nu)\, u_\mu = 2\theta_{<\mu\nu>}\,.$$

One may recover the hypotheses that we made regarding the energy-momentum tensor for this type of fluid.

If there is no heat current – i.e., $q_\mu = 0$ – then the momentum g_μ will be collinear with the current. The kinematical part of the energy-momentum will then reduce to a symmetric term $\mu_0\, u_\mu\, u_\nu$, so the stress tensor will likewise be symmetric, and the fundamental equations will reduce to:

$$g_\mu = \mu_0\, u_\mu\,,$$

$$\dot{g}_\mu = - \partial_\nu\, \theta_{\mu\nu}\,.$$

Finally, if one assumes that there is no production of heat, moreover, then one will recover the two classical cases of the perfect, incompressible fluid and the pure matter fluid.

We remark that the angular momentum intervenes in the equations of motion only by means of the derivative $\dot{s}_{\mu\nu}$. If the proper angular momentum is not zero along the streamline, but only constant (which is a case that we encountered above when studying the Møller drop) then one will recover the same equations as the ones above.

§ 6. The representation of the Schrödinger wave function. We shall now apply the preceding considerations to the principal wave functions of quantum mechanics. First of all, we may reconsider the hydrodynamical representation of the Schrödinger equation that we began with in our Introduction (i.e., the Madelung fluid) in terms of the general formalism that we just discussed. Although it is essentially a non-relativistic equation, the transposition is immediate.

As we know, the Schrödinger equation can be derived from a non-relativistic Lagrangian:

$$\mathcal{L} = \frac{i\hbar}{2}(\dot{\psi}^*\psi - \psi^*\dot{\psi}) + \frac{\hbar^2}{2m}\partial_k\psi^*\partial_k\psi + V\psi^*\psi\,.$$

ψ^* denotes the complex conjugate of ψ and $\dot{\psi}$ denotes the ordinary derivative with respect to time. Upon writing the Euler-Lagrange equation that relates to the two groups of variables t and x_k, one will obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) + \partial_k\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*_{,k}}\right) = \frac{\partial\mathcal{L}}{\partial\psi^*},$$

such that we have:

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\partial_k\partial_k\psi + V\psi = 0;$$

i.e., the Schrödinger equation. One has an analogous equation for ψ^*.

One easily deduces the components of the current from this Lagrangian. Corresponding to the spatial variables, one has:

$$j_k = i\left(\psi\frac{\partial\mathcal{L}}{\partial\psi_{,k}} - \psi^*\frac{\partial\mathcal{L}}{\partial\psi^*_{,k}}\right) = \frac{i\hbar^2}{2m}(\psi\,\partial_k\,\psi^* - \psi^*\,\partial_k\,\psi),$$

and corresponding to time, one has:

$$j_\otimes = i\left(\psi\frac{\partial\mathcal{L}}{\partial\dot{\psi}} - \psi^*\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) = \hbar\,\psi^*\psi\,;$$

this last component obviously represents the fluid density (up to the factor $\hbar$).

We now pass on to real functions by making the transformation:

$$\psi = R\,e^{iS/\hbar}\,.$$

This makes the Lagrangian take the form:

$$\mathcal{L} = P\left(\frac{\partial S}{\partial t} + \frac{\partial_k S\,\partial_k S}{2m} + V\right) + \frac{\hbar^2}{8m}\frac{\partial_k P\,\partial_k P}{P}$$

(if we set $R^2 = P$).

The Euler-Lagrange equations will then be:

$$(J) \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}\partial_k S\,\partial_k S + V - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0,$$

$$(C) \qquad \frac{\partial P}{\partial t} + \partial_k \left(\frac{P \partial_k S}{m} \right) = 0 \qquad \text{(see Introduction)},$$

which, when substituted in the Lagrangian, gives it the new form:

$$\mathcal{L} = \frac{\hbar^2}{2m} (R\,\Delta R + \partial_k R\,\partial_k R) = \frac{\hbar^2}{4m} \Delta P.$$

The current components are:

$$j_k = \frac{\hbar P}{m} \partial_k S , \qquad j_\otimes = \hbar P ,$$

which corresponds to a matter density:

$$\rho = \frac{j_\otimes}{\hbar} = P$$

and a velocity:

$$v_k = \frac{\partial_k S}{m} ,$$

conforming to the previously-chosen hypotheses. It immediately results from equation (C) that one has:

$$\frac{\partial \rho}{\partial t} + \partial_k (\rho v_k) = \frac{d\rho}{dt} = 0.$$

There is conservation of matter in the course of this motion. One easily constructs a set of quantities that comprise the space-time tensor of energy-momentum in the non-relativistic case. They are:

1) A spatial energy-momentum tensor:

$$t_{ij} = \frac{\partial \mathcal{L}}{\partial P_{,j}} \partial_i P + \frac{\partial \mathcal{L}}{\partial S_{,j}} \partial_i S - \delta_{ij} \mathcal{L} = \frac{\hbar^2}{4m} \frac{\partial_i P \partial_j P}{P} + P \frac{\partial_i S\, \partial_j S}{m} - \frac{\hbar^2}{4m} \delta_{ij} P \Delta P ,$$

or

$$\boxed{t_{ij} = \rho m v_i v_j + \frac{\hbar^2}{2m} [2 \partial_i R\, \partial_j R - \delta_{ij} [R \Delta R + \partial_k R\, \partial_k R].}$$

This is a symmetric tensor.

2) A momentum density vector:

$$g_k \equiv t_{k\otimes} = \frac{\partial \mathcal{L}}{\partial \dot{S}} \partial_k S = P\, \partial_k S,$$

or

$$\boxed{g_\mu = \rho m v_k.}$$

3) An energy density current vector:

$$t_{\otimes k} = \frac{\partial \mathcal{L}}{\partial S_{,k}}\dot{S} = P\frac{\partial_k S}{m}\dot{S},$$

or, from (J):

$$\boxed{-t_{\otimes k} = \rho E\, v_k.}$$

4) Finally, one has an energy density scalar:

$$t_{\otimes\otimes} = \frac{1}{c^2}\frac{\partial \mathcal{L}}{\partial \dot{S}}\dot{S} = \frac{1}{c^2}P\dot{S},$$

or

$$\boxed{-t_{\otimes\otimes} = \rho\frac{E}{c^2}.}$$

The relativistic equation of conservation $\partial_\nu\, t_{\mu\nu} = 0$ is thus subdivided into:

– An equation of momentum conservation:

$$\partial_j\, t_{ij} + \frac{\partial}{\partial t}t_{i\otimes} = 0,$$

such that:

$$\partial_j(\rho\, m v_i v_j) + \frac{\partial}{\partial t}(\rho\, m v_i) + \partial_j \theta_{ij} = 0,$$

or, upon recalling that $\rho m\, v_i$ is the momentum g_i, and that $-\partial_j \theta_{ij} = \varphi_j$ is the density of the stress forces, it will then happen that:

$$\frac{d}{dt}g_i = \varphi_i\,.$$

Similarly, one has an equation of energy conservation:

$$\partial_j\, t_{\otimes j} + \frac{\partial}{\partial t}t_{\otimes\otimes} = 0,$$

such that:

$$-\,\partial_j(\rho E\, v_\otimes) - \frac{\partial}{\partial t}(\rho E) = -\frac{d}{dt}(\rho E) = 0.$$

Hence, since $d\rho\,/\,dt = 0$, one will have:

$$\frac{dE}{dt} = 0.$$

Thus, the energy is invariant in the absence of external forces.

Therefore, from the equation of momentum conservation, it results that the second term of the tensor t_{ij} represents the internal stress tensor:

$$\theta'_{ij} = \frac{\hbar^2}{2m}[2\,\partial_i R\,\partial_j R - \delta_{ij}(R\,\Delta R + \partial_k R\,\partial_k R)],$$

or furthermore:

$$\theta'_{ij} = \frac{\hbar^2}{2m}\left(\frac{\partial_i \rho\,\partial_j \rho}{\rho} - \delta_{ij}\Delta\rho\right).$$

Meanwhile, this tensor, which is always symmetric, differs completely from the one that one obtains by the Madelung method (see, the Introduction), namely:

$$\theta_{ij} = \frac{\hbar^2}{2m}(R\,\partial_i\partial_j R - \partial_i R\,\partial_j R),$$

or furthermore:

$$\theta_{ij} = \frac{\hbar^2}{4m}\rho\,\partial_i\,\partial_j \log \rho.$$

Similarly, the internal pressures that one deduces by contracting have different expressions, namely:

$$p' = \frac{1}{3}\theta'_{ii} = -\frac{\hbar^2}{6m}(3R\,\Delta R + \partial_i R\,\partial_i R),$$

while for the Madelung case one has:

$$p = \frac{1}{3}\theta_{ii} = -\frac{\hbar^2}{6m}(-R\,\Delta R + \partial_i R\,\partial_i R).$$

By comparison, the only physical quantity in the expression for the force density that is deducible from the tensor θ_{ij} that intervenes in the hydrodynamical equations explicitly is:

$$\varphi'_i = -\partial_j\theta_{ij} = -\frac{\hbar^2}{2m}[2\partial_i\partial_j R\,\partial_j R + 2\partial_i R\,\Delta R - \partial_i R\,\Delta R - R\,\partial_i\Delta R - 2\partial_i\partial_j R\,\partial_j R]\;;$$

i.e.:

$$\varphi'_i = \frac{\hbar^2}{2m}(R\,\partial_i\Delta R - \Delta R\,\partial_i R) = R^2\partial_i\left(\frac{\hbar^2}{2m}\frac{\Delta R}{R}\right),$$

which is the same expression that we started with in order to derive the Madelung stresses. Thus, the two models are physically equivalent.

§ 7. The Klein-Gordon wave function: the representation of de Broglie and Takabayasi. We now pass on to the relativistic wave functions. One knows that they must all satisfy the Klein-Gordon equation:

$$\partial_\mu \partial_\mu \Phi = \frac{m_0^2 c^2}{\hbar^2} \Phi ,$$

and that in the case of a scalar wave function this condition will suffice to completely determine the wave equation. We shall begin to study this case by first recalling the procedure that was employed by Louis de Broglie [**16**, **42**], and reprised by Takabayasi [**43**], which is a procedure that generalizes that of Madelung.

Upon setting $\Phi = R\, e^{iS/\hbar}$, the Klein-Gordon equation will split into:

$$(C) \qquad \partial_\mu (R^2 \partial_\mu S) = 0$$

$$(J) \qquad \partial_\mu S\, \partial_\mu S + m_0^2 c^2 - \hbar^2 \frac{\Box R}{R} = 0.$$

Here, one may not simply set:

$$\boxed{u_\mu = \frac{\partial_\mu S}{m_0}.}$$

because the condition $u_\mu u_\mu = -c^2$ will not be satisfied, in general. One forms the scalar $\partial_\mu S\, \partial_\mu S$ and sets:

$$\partial_\mu S\, \partial_\mu S = -M_0^2 c^2$$

upon introducing a scalar M_0, which is variable, in general, in lieu of the proper mass m_0 that appears in the non-relativistic equation. One may thus set:

$$u_\mu = \frac{\partial_\mu S}{M_0} .$$

Thus, equation (J) leads to:

$$-M_0^2 c^2 + m_0^2 c^2 = \hbar^2 \frac{\Box R}{R} .$$

Hence, the expression for M_0 is:

$$\boxed{M_0^2 = m_0^2 - \frac{\hbar^2}{c^2} \frac{\Box R}{R}.} .$$

One sees that M_0 differs from m_0 by only a correction term that is simultaneously quantum and relativistic. One has, furthermore:

$$M_0 \approx m_0 - \frac{1}{c^2}\frac{\hbar^2}{2m_0}\frac{\Box R}{R},$$

approximately, which is a formula in which one recognizes the relativistic generalization of the quantum potential:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}.$$

As Takabayasi pointed out, the introduction of the quantity M_0 is possible only under the condition that:

$$\frac{\hbar^2}{c^2}\frac{\Box R}{R} \leq m_0^2,$$

which is a relation that might be found wanting for particles of vanishing mass or in regions where $\Box R / R$ becomes very large. On the other hand, note that equation (*J*) may be written:

$$\partial_\mu S\, \partial_\mu S \equiv M_0\, u_\mu\, \partial_\mu S = M_0 \dot{S} = -M_0^2 c^2$$

(upon introducing the derivative along the streamline), so if one desires to preserve the significance of the action functional for S then $-\dot{S}$ will be the proper energy, and one will have:

$$M_0 \dot{S} = -M_0^2 c^2 \equiv -E_0\, M_0$$

or

$$E_0 = M_0\, c^2,$$

which will serve to establish the significance of M_0, which will then appear to be a *total proper mass.*

If one takes the gradient of equation (*J*) then one will get:

$$2\partial_\mu S\, \partial_\mu \partial_\nu S = \partial_\nu\left(\hbar^2 \frac{\Box R}{R}\right).$$

Upon dividing both sides by $2M_0$, one will get:

$$u_\mu\, \partial_\mu \partial_\nu S = \frac{d}{d\tau}(M_0\, u_\nu) = \frac{1}{2M_0}\partial_\nu\left(\hbar^2 \frac{\Box R}{R}\right) = -\frac{1}{2M_0}\partial_\nu (M_0^2 c^2) = -\partial_\nu (M_0 c^2),$$

such that:

$$\boxed{\frac{d}{d\tau}(M_0 u_\nu) = -\partial_\nu (M_0 c^2).}$$

Therefore, the quantity M_0 will simultaneously play the role of a variable proper mass and that of a scalar potential of relativistic forces that corresponds to the quantum potential of the Schrödinger wave function.

One sees that (for reasons of homogeneity) one must set:

$$\rho = R^2 \frac{M_0}{m_0},$$

and one will then have:

$$\dot{\rho} \equiv \partial_\mu (u_\mu \rho) = 0 \qquad \text{(see Appendix } A\text{)}$$

One then writes equation (J) in the form:

$$u_\mu \partial_\mu S = - M_0 c^2.$$

Multiply this by ρ and then take the gradient. The left-hand side will then give us:

$$\begin{aligned} \partial_\nu (\rho u_\mu \partial_\mu S) &= \rho u_\mu \partial_\nu \partial_\mu S + \partial_\mu \partial_\nu S(\rho u_\mu) \\ &= \partial_\mu (\rho u_\mu \partial_\nu S) - \partial_\nu S \partial_\mu (\rho u_\mu) + M_0 u_\mu \partial_\nu (\rho u_\mu). \end{aligned}$$

The second term is zero, on account of equation (C). What will remain is:

$$\partial_\nu (\rho M_0 u_\nu u_\mu) + M_0 u_\mu \rho \partial_\nu u_\mu + M_0 u_\mu u_\mu \partial_\nu \rho.$$

The second term will be zero, since $u_\mu u_\mu = - c^2$.

Thus, one finds that:

$$\partial_\mu (\rho M_0 u_\nu u_\mu) - M_0 c^2 \partial_\nu \rho = - \partial_\nu (\rho M_0 c^2),$$

or furthermore:

$$\partial_\mu (\rho M_0 u_\nu u_\mu) = - \rho \partial_\nu (M_0 c^2).$$

Transforming the right-hand side by expressing ρ and $M_0^2 c^2$ as functions of R will give:

$$- \rho \partial_\nu (M_0 c^2) = - R^2 \frac{M_0}{m_0} \partial_\nu (M_0 c^2) = - \frac{R^2}{2m_0} \partial_\nu (M_0^2 c^2)$$

$$= \frac{R^2}{2m_0} \partial_\nu \left(\hbar^2 \frac{\Box R}{R} \right) = \frac{\hbar^2}{2m_0} (R \partial_\nu \Box R - \partial_\nu R \Box R).$$

We may transform the term in parentheses as we did in the case of the Madelung stress tensor:

$$R \partial_\nu \partial_\mu \partial_\mu R - \partial_\nu R \partial_\mu \partial_\mu R \equiv \partial_\mu (R \partial_\nu \partial_\mu R) - \partial_\mu (\partial_\nu R \partial_\mu R).$$

The equation can ultimately be written as:

$$\partial_\mu (\rho M_0 \, u_\nu \, u_\mu) = \frac{\hbar^2}{2m_0} \partial_\mu (R \, \partial_\nu \, \partial_\mu R - \partial_\nu R \, \partial_\mu R).$$

In this form, the equation then expresses the idea that the divergence will vanish for an energy-momentum tensor that expressed in the form:

$$\boxed{t_{\mu\nu} = \rho M_0 u_\mu u_\nu + \frac{\hbar^2}{2m_0} (\partial_\nu R \, \partial_\mu R - R \partial_\nu \partial_\mu R).}$$

Moreover, upon remarking that:

$$R \, \partial_\nu \, \partial_\mu R - \partial_\nu R \, \partial_\mu R = R^2 \partial_\nu \left(\frac{\partial_\mu R}{R} \right) = R^2 \, \partial_\nu \, \partial_\mu \, \log R,$$

and upon introducing the density:

$$\rho = R^2 \frac{M_0}{m_0},$$

we will get:

$$\boxed{t_{\mu\nu} = \rho M_0 u_\mu u_\nu - \rho \frac{\hbar^2}{2M_0} \partial_\nu \partial_\mu \log R.}$$

By generalizing Lichnerowicz's theory of classical fluids, it is easy to show that the Klein-Gordon fluid is a holonomic fluid, and that the above form for the energy-momentum tensor makes the pseudo-mass M_0 appear, along with the pressure tensor:

$$\pi_{\mu\nu} = \frac{\hbar^2}{2m_0} (\partial_\nu R \, \partial_\mu R - R \, \partial_\nu \, \partial_\mu R) \qquad \text{(cf., Appendix } B\text{)}.$$

One has, indeed:

$$- \partial_\nu \, \pi_{\mu\nu} = \frac{\hbar^2}{2m_0} (\Box R \partial_\mu R + \partial_\nu R \, \partial_\nu \partial_\mu R - \partial_\nu R \, \partial_\nu \partial_\mu R - R \partial_\mu \Box R) = \frac{\hbar^2}{2m_0} R^2 \partial_\mu \left(\frac{\Box R}{R} \right),$$

or, upon utilizing equation (J):

$$- \partial_\nu \, \pi_{\mu\nu} = \frac{R^2}{2m_0} \partial_\mu (m_0^2 - M_0^2) \, c^2 = - R^2 \frac{M_0}{m_0} \partial_\mu M_0 \, c^2 = - \partial_\mu M_0 \, c^2 .$$

Therefore, the internal force per unit pseudo-mass is:

$$K_\mu = - c^2 \, \partial_\mu \log M_0 .$$

This takes the form of a gradient precisely, and upon disposing of the indeterminacy in the coefficient, one will immediately get the index:

$$\boxed{F = \frac{M_0}{m_0}.}$$

One therefore has a third interpretation for de Broglie's variable mass: It corresponds to the index of a holonomic fluid, in the Lichnerowicz sense. Moreover, equation (J) permits us to express F as a function of the variable R:

$$\boxed{F^2 = 1 - \frac{\hbar^2}{m_0^2 c^2} \frac{\Box R}{R}}$$

or:

$$F \approx 1 - \frac{\hbar^2}{2m_0^2 c^2} \frac{\Box R}{R},$$

approximately.

One then sees the significance of the vector $\partial_\mu S \,/\, m_0$ in the causal interpretation, as well: It represents Lichnerowicz's weighted velocity vector C_μ (see Appendix B); i.e., the unit-speed velocity relative to the Riemannian metric that is associated with the fluid. This vector:

$$\boxed{C_\mu = \frac{M_0}{m_0} u_\mu = \frac{\partial_\mu S}{m_0}}$$

will be a gradient, as opposed to the unit velocity u_μ . Its rotation, which is Lichnerowicz's vorticity tensor, will therefore be zero. Thus, the Klein-Gordon motion is essentially *irrotational*.

The vector C_μ also allows us to express the relativistic compressibility:

$$\boxed{\partial_\mu C_\mu = -\frac{1}{m_0} \Box S.}$$

The case in which the Klein-Gordon fluid is incompressible is thus expressed by $\Box S = 0$; in other words, the phase wave propagates with the velocity c, which implies a vanishing proper mass. One may likewise remark that from equation (C) one has:

$$\partial_\mu R \, \partial_\mu S = -\tfrac{1}{2} R \Box S .$$

The case of the incompressible fluid is thus further characterized by the fact that the gradient $\partial_\mu R$ is orthogonal to the gradient $\partial_\mu S$; i.e., it is situated in proper space.

Once more, we consider the energy-momentum tensor, and remark that the part that we have called a pressure tensor, namely:

$$- \rho \frac{\hbar^2}{2M_0} \partial_\mu \partial_\nu \log R,$$

possesses components along u_μ, as well as along u_ν. If one is to obtain an internal stress tensor that is contained in proper space then it must be decomposed, and this will make a transverse momentum and a heat current appear.

In order to rapidly calculate these vectors, it is convenient to remark [**9**] that the projection relative to the index β of a tensor $A_{\alpha\beta}$ on the current is:

$$A_{\alpha\nu} \left(-\frac{u_\nu u_\beta}{c^2} \right),$$

whereas the projection onto proper space is:

$$A_{\alpha\nu} \left(\delta_{\nu\beta} + \frac{u_\nu u_\beta}{c^2} \right) = \eta_{\nu\beta} A_{\alpha\nu} .$$

We will then obtain the momentum g_μ immediately upon projecting $t_{\mu\nu}$ onto u_ν:

$$g_\mu \, u_\nu = t_{\mu\lambda} \left(-\frac{u_\lambda u_\nu}{c^2} \right) = \rho M_0 \, u_\mu \, u_\nu + \frac{\rho \hbar^2}{2M_0 c^2} u_\lambda \, \partial_\lambda \, \partial_\mu \log R \cdot u_\nu ,$$

$$g_\mu = \rho M_0 \, u_\mu + \rho \, \frac{\hbar^2}{2M_0 c^2} \, u_\lambda \, \partial_\lambda \, \partial_\mu \log R.$$

One obtains the transverse momentum $- p_\mu$ by projecting g_μ onto proper space – i.e., $- p_\mu = g_\lambda \, \eta_{\lambda\mu}$ – namely:

$$\boxed{p_\mu = -\rho \frac{\hbar^2}{2M_0 c^2} u_\sigma \partial_\sigma \partial_\lambda \log R \cdot \eta_{\lambda\mu} .}$$

In the absence of proper angular momentum, the heat current is, as one knows, equal to the transverse momentum:

$$\boxed{q_\mu = +\rho \frac{\hbar^2}{2M_0 c^2} u_\sigma \partial_\sigma \partial_\lambda \log R \cdot \eta_{\lambda\mu} .}$$

The proper mass density μ_0 may be obtained by projecting $t_{\mu\nu}$ onto the current for each of the two indices:

$$\mu_0 \, u_\mu \, u_\nu = t_{\lambda\sigma} \left(-\frac{u_\lambda u_\nu}{c^2} \right) \left(-\frac{u_\sigma u_\nu}{c^2} \right),$$

so:

$$\boxed{\mu_0 = \rho\left(M_0 - \frac{\hbar^2}{2M_0c^4}u_\lambda u_\sigma \partial_\lambda \partial_\sigma \log R\right).}$$

Finally, the proper space stress tensor is obtained by projecting $t_{\mu\nu}$ onto proper space for each of the two indices:

$$\boxed{\theta_{\mu\nu} = -\rho\frac{\hbar^2}{2M_0}\partial_\lambda\partial_\sigma \log R\cdot\eta_{\lambda\mu}\eta_{\sigma\nu}.}$$

We may deduce two quantities from this expression that physically translate into the effect of quantum stresses by forming the vector $-\partial_\nu\,\theta_{\mu\nu}$ and then projecting it onto proper space and the current:

$$-\partial_\nu\,\theta_{\mu\nu} = \frac{\hbar^2}{2m_0}\,\eta_{\lambda\mu}\,\partial_\nu(\eta_{\nu\sigma}R^2\,\partial_\lambda\,\partial_\sigma \log R) + \frac{\hbar^2}{2m_0}R^2\,\partial_\lambda\partial_\sigma \log R\cdot\eta_{\nu\sigma}\partial_\nu\eta_{\lambda\mu}\,.$$

The first term is explicitly a proper space vector, since it involves the $\eta_{\lambda\mu}$. As for the second term, we see that:

$$\partial_\nu\,\eta_{\lambda\mu} = \partial_\nu\left(\frac{u_\lambda u_\mu}{c^2}\right) = \frac{1}{c^2}(u_\lambda\,\partial_\nu\,u_\mu + u_\mu\,\partial_\nu\,u_\lambda),$$

which is an expression whose first term is orthogonal to the current – i.e., $u_\mu\,\partial_\nu\,u_\mu = 0$ – and must therefore contribute to the proper-space force f_μ, and whose second term is along the current u_μ, and must therefore provide an energy w_0 that is produced per unit time by the internal stresses in the form of heat. One therefore has, in summation:

$$f_\mu = \frac{\hbar^2}{2m_0}\left\{\eta_{\lambda\mu}\partial_\nu(R^2\partial_\lambda\partial_\sigma \log R\cdot\eta_{\nu\sigma}) + u_\lambda\frac{\partial_\nu u_\mu}{c^2}R^2\partial_\lambda\partial_\sigma \log R\cdot\eta_{\nu\sigma}\right\}$$

or

$$\boxed{f_\mu = \frac{\hbar^2}{2m_0}\left\{u_\lambda\partial_\nu\left(\frac{R^2}{c^2}u_\mu\partial_\lambda\partial_\sigma \log R\cdot\eta_{\nu\sigma}\right) + \partial_\nu(R^2\partial_\mu\partial_\sigma \log R\cdot\eta_{\nu\sigma}\right\}.}$$

One sees that the force involves a non-relativistic principal term $\hbar^2/2m_0\,\partial_\nu(R^2\partial_\mu\partial_\nu \log R)$, a term in $\hbar^2/c^2$, and a term in $\hbar^2/c^4$. On the other hand, the energy w_0 is given by:

$$\boxed{w_0 = \frac{\hbar^2}{2M_0c^4}\rho\,\partial_\lambda\partial_\sigma \log R\cdot\eta_{\nu\sigma}\partial_\nu u_\lambda,}$$

whose principal part has the same order as the quantum part of the proper mass μ_0 , namely:

$$\frac{\hbar^2}{2M_0c^4}\rho\,\partial_\nu u_\lambda\cdot\partial_\nu\partial_\lambda\log R\,.$$

Finally, one likewise expresses the internal pressure π – viz., $\pi = 3\theta_{\mu\mu}$ – as:

$$\pi = -\rho\frac{\hbar^2}{6M_0}\left(\Box\log R+\frac{1}{c^2}u_\lambda u_\sigma\partial_\lambda\partial_\sigma\log R\right),$$

namely:

$$\boxed{\pi = --\rho\frac{\hbar^2}{6M_0}\eta_{\lambda\sigma}\partial_\lambda\partial_\sigma\log R.}$$

One may perform these various operations by possibly taking the conservation relation into account, namely, $\partial_\mu(M_0R^2u_\mu) = 0$.

We confine ourselves to a few remarks:

1) As we know, de Broglie’s variable proper mass M_0 differs from the constant proper mass m_0 by a quantity:

$$-\frac{\hbar^2}{2m_0c^2}\frac{\Box R}{R},$$

which is a quantity that has order two from the quantum viewpoint, as well as from the relativistic viewpoint. It is only this mass that appears in the case of the Takabayasi decomposition. The term that we were led to add to it, namely:

$$-\frac{\hbar^2}{2M_0c^4}u_\lambda\,u_\sigma\partial_\lambda\,\partial_\sigma\log R,$$

is considerably smaller, because it is of fourth order from the relativistic viewpoint.

2) Upon examining the Weyssenhoff particle, we were led to define another proper mass M_0 by the relation:

$$\frac{g_\mu}{\rho}\frac{g_\mu}{\rho} = -M_0'^2c^2\,.$$

The expression that was found for g_μ gave us:

$$-M_0'^2c^2 = \left(M_0u_\mu+\frac{\hbar^2}{2M_0c^2}u_\lambda\partial_\lambda\partial_\mu\log R\right)^2,$$

namely:

$$M_0'^2 \approx M_0^2 - \frac{2\hbar^2}{2M_0c^4} M_0 u_\mu u_\nu \partial_\mu \partial_\nu \log R\,;$$

upon neglecting the term in $\hbar^4 / c^6$, which will give, to the same approximation:

$$M_0'^2 \approx M_0 - \frac{\hbar^2}{2M_0c^4} u_\mu u_\lambda \partial_\mu \partial_\lambda \log R\,;$$

i.e., the same expression that we deduced from $g_\mu / \rho \cdot u_\mu$.

One remembers that in the Weyssenhoff case, the difference between the two masses was given by:

$$M_0' - \mathfrak{M}_0 = \frac{1}{2} \frac{\gamma_0^2 s_0^2}{\mathfrak{M}_0 c^6},$$

where s_0 was the norm of the spin, namely $\hbar / 2$, after quantization. The difference between the two masses then has order $\hbar^2 / c^6$, whereas in the Klein-Gordon case it is of order $\hbar^4 / c^6$. One sees that it is considerably smaller, obviously by reason of the absence of proper angular momentum.

3) In a general fashion, the difference between the mass of momentum and the mass of inertia is due to the fact that the momentum is not collinear with the current. The magnitude of this difference is related to that of the transverse momentum. Now, in the Weyssenhoff case, the transverse momentum had the norm $s_0\, g_0 / c^2$. It was essentially related to spin, and its order of magnitude was $\hbar / c^2$. On the contrary, in the present case, the transverse momentum is due to the heat current; it has order $\hbar^2 / c^2$. The phenomena that are taken into account by our decomposition (and that do not appear in Takabayasi's decomposition) are thus weaker than the ones that we focused upon in the case of fluids that are given an internal angular momentum.

4) Finally, the stress tensor involves a principally quantum, but not relativistic, term, namely:

$$-\rho\, \frac{\hbar^2}{2M_0} \partial_\lambda \partial_\sigma \log R \cdot \delta_{\lambda\mu}\, \delta_{\sigma\nu} = -\rho\, \frac{\hbar^2}{2M_0} \partial_\mu \partial_\nu \log R,$$

which is the one in Takabayasi's decomposition, and two new terms, one of which is of order $\hbar^2 / c^2$, and the other of which is much smaller and of order $\hbar^2 / c^4$. These two terms obviously disappear in the non-relativistic approximation.

§ 8. The Klein-Gordon wave function: another representation and the non-relativistic approximation. Up till now, the decomposition of the energy-momentum tensor onto the current and proper space has been performed in a covariant form. It is nonetheless obvious that our expressions will take on a much simpler form if we refer the tensors to the proper axes themselves. One then sets:

$$u_k^0 = 0, \qquad u_4^0 = ic.$$

The derivatives of u_μ may be obtained easily (see Appendix B):

$$(\partial_j u_k)^0 = (\partial_j v_k)^0, \qquad (\partial_4 u_k)^0 = \frac{1}{ic}\frac{\partial}{\partial t} v_k^0, \qquad (\partial_j u_4)^0 = (\partial_4 u_4)^0 = 0.$$

The continuity relation $\partial_\mu (M_0 R^2 u_\mu) = 0$ can be put into the form:

$$2\frac{\partial^0}{\partial t}\log R + \frac{\partial^0}{\partial t}\log M_0 + (\partial_k v_k)^0 = 0,$$

and the tensors that characterize the fluid will become:

$$t_{jk}^0 = -\rho\frac{\hbar^2}{2M_0}\partial_j^0\partial_k^0 \log R,$$

$$t_{j4}^0 = t_{4j}^0 = \frac{i}{c}\rho\frac{\hbar^2}{2M_0}\partial_j^0\frac{\partial^0}{\partial t}\log R,$$

$$t_{44}^0 = -\rho c^2 M_0,$$

$$\mu_0 = \rho M_0 - \rho\frac{\hbar^2}{2M_0 c^4}\left(\frac{\partial^2}{\partial t^2}\right)^0 \log R,$$

$$-p^0 = q^0 = -\rho\frac{\hbar^2}{2M_0 c^2}\frac{\partial^0}{\partial t}(\nabla \log R),$$

$$\theta_{jk}^0 = -\rho\frac{\hbar^2}{2M_0 c^2}\partial_j^0\partial_k^0 \log R,$$

$$\varphi_k^0 = \frac{\hbar^2}{2m_0}\left\{\partial_j^0\left[R^2\left(\partial_k^0\partial_j^0 \log R + \frac{u_j^0}{c^2}\partial_k^0\frac{\partial^0}{\partial t}\log R + \frac{u_k^0}{c^2}\partial_j^0\frac{\partial^0}{\partial t}\log R\right)\right]\right.$$
$$\left. + \frac{\partial^0}{\partial t}\left(R^2\frac{u_j^0}{c^2}\partial_k^0\partial_j^0 \log R\right)\right\},$$

so:

$$\varphi_k^0 = f_k^0 = \frac{\hbar^2}{2m_0}\Big\{\partial_j^0(R^2\partial_k^0\partial_j^0 \log R)$$

$$+ \frac{R^2}{c^2}\left[\partial_k^0 \frac{\partial^0}{\partial t}\log R(\partial_j v_j)^0 + \partial_j^0 \frac{\partial^0}{\partial t}\log R(\partial_j v_k)^0 + \partial_k^0\partial_j^0 \log R\left(\frac{\partial v_k}{\partial t}\right)^0\right]\Big\},$$

so that finally:

$$w_0 = \frac{\hbar^2}{2M_0c^4}\,\rho\,\partial_k^0\partial_j^0 \log R(\partial_j v_i)^0$$

and

$$\pi = -\rho\frac{\hbar^2}{6M_0}\left(\Box^0 \log R + \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\right)^0 \log R\right) \equiv -\rho\frac{\hbar^2}{6M_0}\Delta^0 \log R\,.$$

In order to compare these results with the hydrodynamical representation of the Schrödinger equation, we make the non-relativistic approximation. In order to do this, we must subtract the proper energy m_0c^2, which does not intervene in non-relativistic physics, from the relativistic energy. This amounts to subtracting m_0c^2 from the derivative $-\dot{S}$, or furthermore, taking the action function to be $S' = S + m_0c^2t$, from which, one will find that:

$$\frac{\partial S}{\partial t} = \frac{\partial S'}{\partial t} - m_0c^2, \qquad \frac{\partial^2 S}{\partial t^2} = \frac{\partial^2 S'}{\partial t^2}\,.$$

One then projects the Klein-Gordon equations onto the spatial axes and the time axis, and upon neglecting the terms in $1/c^2$, it will then follow that:

$$\partial_\mu S\,\partial_\mu S = (\nabla S')^2 - \frac{1}{c^2}\left(\frac{\partial S'}{\partial t} - m_0c^2\right)^2 = (\nabla S')^2 + 2m_0\frac{\partial S'}{\partial t} - m_0^2c^2,$$

$$\Box S = \Delta S', \qquad \Box R = \Delta R, \qquad \partial_\mu R\,\partial_\mu S = \partial_i R\,\partial_i S + m_0\frac{\partial R}{\partial t}\,.$$

The Klein-Gordon equations (J) and (C) give:

$$\nabla S' + 2m_0\frac{\partial S'}{\partial t} - \hbar^2\frac{\Delta R}{R} = 0,$$

$$2R\left(\partial_i R\partial_i S' + m_0\frac{\partial R}{\partial t}\right) + R^2\Delta S' \equiv \partial_i(R^2\,\partial_i S') + m_0\frac{\partial R^2}{\partial t} = 0,$$

which are the Schrödinger equations (J) and (S). De Broglie's variable proper mass is given by:

$$-M_0^2c^2 \equiv \partial_\mu S\, \partial_\mu S = (\nabla S')^2 + + 2m_0 \frac{\partial S'}{\partial t} - m_0^2 c^2 ,$$

so from equation (J), one will have:

$$-M_0^2 c^2 = - m_0^2 c^2 + \hbar^2 \frac{\Delta R}{R} .$$

The difference between the two masses is given by:

$$m_0^2 - M_0^2 = \frac{\hbar^2}{c^2}\frac{\Delta R}{R}, \qquad\qquad M_0 \approx m_0\left(1 - \frac{\hbar^2}{2m_0^2c^2}\frac{\Delta R}{R}\right).$$

This difference must be negligible in the non-relativistic approximation. One can therefore identify M_0 with m_0 and set:

$$\mathbf{v} = \frac{\nabla S'}{m_0}, \qquad \rho = R^2.$$

The components of the energy-momentum tensor then become:

$$t_{ij} = R^2 m_0 v_i v_j + \frac{\hbar^2}{2m_0}(\partial_i R\, \partial_j R - R\, \partial_i \partial_j R).$$

The second term is precisely the Madelung stress tensor:

$$t_{k4} \equiv g_k\, ic = R^2 m_0\, v_k\, ic + \frac{\hbar^2}{2m_0}\left(\partial_k R \cdot \frac{1}{ic}\frac{\partial R}{\partial t} - R\partial_k \frac{1}{ic}\frac{\partial R}{\partial t}\right),$$

so that:

$$t_{k\otimes} \equiv g_k = R^2 m_0\, v_k ,$$

since the second terms is of order $1/c^2$, and:

$$t_{44} = R^2 m_0(-c^2) + \frac{\hbar^2}{2m_0}\left(\frac{1}{ic}\frac{\partial R}{\partial t}\frac{1}{ic}\frac{\partial R}{\partial t} + R\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2}\right),$$

so that:

$$t_{\otimes\otimes} \equiv -\frac{R^2 E_0}{c^2} = - R^2 m_0 .$$

One thus recovers all of the characteristics of the Madelung fluid precisely.

We shall now re-examine the same problem by means of the general method that was described at the beginning of this chapter. We start with the Klein-Gordon Lagrangian as a function of the scalar wave function Φ:

$$\mathcal{L} = \frac{\hbar^2}{2m_0}\partial_\mu\Phi^*\partial_\mu\Phi + \frac{m_0c^2}{2}\Phi^*\Phi .$$

One immediately sees that the Euler-Lagrange equations are:

$$\partial_\mu\,\partial_\mu\Phi = \frac{m_0^2c^2}{\hbar^2}\Phi, \qquad \partial_\mu\,\partial_\mu\Phi^* = \frac{m_0^2c^2}{\hbar^2}\Phi^*,$$

namely, the Klein-Gordon equations.

From our general formalism, we obtain a current:

$$j_\mu = \frac{i\hbar^2}{2m_0}(\Phi\partial_\mu\Phi^* - \Phi\partial_\mu\Phi)$$

and an energy-momentum tensor:

$$t_{\mu\nu} = \frac{\hbar^2}{2m_0}(\partial_\mu\Phi\,\partial_\nu\Phi^* + \partial_\nu\Phi\,\partial_\mu\Phi^*) - \delta_{\mu\nu}\,\mathcal{L}.$$

Since the wave function is a scalar, it is, by definition, indifferent to a rotation of the axes, and there will be no moment of proper rotation.

If we introduce the real fields R and S then the Lagrangian will become:

$$\mathcal{L} = \frac{\hbar^2}{2m_0}\partial_\lambda R\,\partial_\lambda R + \frac{R^2}{2m_0}(\partial_\lambda S\,\partial_\lambda S + m_0^2c^2),$$

or, upon taking equation (J) into account:

$$\mathcal{L} = \frac{\hbar^2}{2m_0}(\partial_\lambda R\,\partial_\lambda R + R\Box R),$$

which is more conveniently written as:

$$\mathcal{L} = \frac{\hbar^2R^2}{2m_0}(2\partial_\lambda \log R\;\;\partial_\lambda \log R + \Box\log R).$$

The current becomes:

$$j_\mu = \frac{i\hbar^2}{2m_0}R^2\left(-\frac{2i}{\hbar}\partial_\mu S\right),$$

namely:

$$j_\mu = \frac{\hbar}{m_0} R^2 \partial_\mu S = \rho \hbar u_\mu \,.$$

Hence, one has a matter density:

$$\rho = R^2 \sqrt{\frac{\partial_\mu S \, \partial_\mu S}{-m_0^2 c^2}} \,.$$

or, upon setting $\partial_\mu S \, \partial_\mu S = -M_0^2 c^2$:

$$\rho = R^2 \frac{M_0}{m_0} \,,$$

and a unit-speed velocity:

$$u_\mu = \frac{\partial_\mu S}{M_0} \,.$$

These are the results that we obtained before.

The energy-momentum tensor becomes:

$$t_{\mu\nu} = \frac{\hbar^2}{2m_0} 2\left(\frac{R^2}{\hbar^2} \partial_\mu S \, \partial_\nu S + \partial_\mu R \, \partial_\nu R \right) - \delta_{\mu\nu} \mathcal{L},$$

or, upon introducing ρ, M_0, and u_μ :

$$\boxed{t_{\mu\nu} = \rho M_0 u_\mu u_\nu + \rho \frac{\hbar^2}{2M_0} [2\partial_\mu \log R \, \partial_\nu \log R - \delta_{\mu\nu} (2\partial_\lambda \log R \, \partial_\lambda \log R + \Box \log R)].}$$

One sees that the quantum term differs from that of the de Broglie and Takabayasi model, as was the case before for the Schrödinger fluid. Meanwhile, it again plays the role of an internal pressure tensor for a holonomic fluid that corresponds to the pseudo-mass M_0:

$$\begin{aligned} -\partial_\nu \pi_{\mu\nu} &= -\frac{\hbar^2}{2m_0} [2\partial_\mu R \Box R + 2\partial_\mu \partial_\nu R \cdot \partial_\nu R - \partial_\lambda R \cdot \partial_\mu \partial_\lambda R - \partial_\mu R \Box R - R \partial_\mu \Box R] \\ &= \frac{\hbar^2}{2m_0} [R \partial_\mu \Box R - \partial_\mu R \Box R] = \frac{\hbar^2}{2m_0} R^2 \partial_\mu \left(\frac{\Box R}{R} \right). \end{aligned}$$

From this viewpoint, there is nothing to change in what we said about the de Broglie-Takabayasi fluid. One obtains the same index:

$$F = \frac{M_0}{m_0}$$

and the same weighted velocity:

$$C_\mu = \frac{\partial_\mu S}{m_0},$$

and the motion will likewise be irrotational.

By contrast, the decomposition onto the proper axes gives different quantities; we shall simply present the results.

One finds a transverse momentum:

$$p_\mu = -\rho \frac{\hbar^2}{M_0 c^2} u_\lambda \partial_\lambda \log R \, \partial_\nu \log R \cdot \eta_{\mu\nu},$$

in which one may naturally replace $u_\lambda \partial_\lambda \log R$ with $-\Box S / 2M_0$.

The heat current will naturally be:

$$q_\mu = \rho \frac{\hbar^2}{M_0 c^2} u_\lambda \partial_\lambda \log R \, \partial_\nu \log R \cdot \eta_{\mu\nu}.$$

The proper mass density will be:

$$\mu_0 = \rho M_0 + \rho \frac{\hbar^2}{M_0 c^2} (2 \, \partial_\mu \log R \, \partial_\nu \log R \; \eta_{\mu\nu} + \Box \log R).$$

The proper-space stress tensor will be:

$$\theta_{\mu\nu} = \frac{\rho\hbar^2}{2M_0} [2 \, \partial_\alpha \log R \, \partial_\beta \log R \; \eta_{\alpha\mu} \, \eta_{\beta\nu} - (\Box \log R + 2\partial_\lambda \log R \, \partial_\lambda \log R) \; \eta_{\mu\nu}].$$

Its divergence, namely, $-\partial_\nu \theta_{\mu\nu}$, will provide us with an internal stress density:

$$\varphi_\mu = -\frac{\hbar^2}{m_0 c^2} \left[u_\alpha \partial_\nu \left\{ \frac{u_\mu}{c^2} R^2 [2\partial_\alpha \log R \, \partial_\beta \log R \, \eta_{\beta\nu} - (\Box \log R + 2\partial_\lambda \log R \, \partial_\lambda \log R) \delta_{\alpha\nu}] \right\} \right.$$
$$\left. + \; \partial_\nu \left\{ R^2 [2\partial_\mu \log R \, \partial_\beta \log R \, \eta_{\beta\nu} - (\Box \log R + 2\partial_\lambda \log R \, \partial_\lambda \log R) \delta_{\mu\nu} \right\} \right],$$

and a caloric energy that is produced per unit time:

$$w_0 = -\frac{\rho\hbar^2}{2M_0 c^4} [\partial_\alpha \log R \, \partial_\beta \log R \, \eta_{\beta\nu} - (\Box \log R - 2\partial_\lambda \log R \, \partial_\lambda \log R) \delta_{\alpha\nu}] \partial_\nu u_\alpha,$$

and finally, one will deduce an internal pressure from this:

$$\pi = \frac{\rho\hbar^2}{2M_0}\left(\frac{2}{3}\partial_\alpha \log R\,\partial_\beta \log R\,\eta_{\alpha\beta} - \Box\log R - 2\partial_\lambda \log R\,\partial_\lambda \log R\right).$$

As we did before, we express these quantities relative to the proper axes. Calculation gives:

$$-p^0 = q^0 = \rho\,\frac{\hbar^2}{M_0c^2}\frac{\partial^0}{\partial t}\log R\,\nabla^0 \log R,$$

$$\mu_0 = \rho M_0 + \rho\,\frac{\hbar^2}{2M_0c^2}\,(2\,\nabla^0 \log R\,\nabla^0 \log R + \Box \log R),$$

$$\theta^0_{jk} = \rho\frac{\hbar^2}{2M_0}[2\partial^0_j \log R\,\partial^0_k \log R - \delta_{jk}(2\partial_\lambda \log R\partial_\lambda \log R + \Box\log R)].$$

Thus, upon setting:

$$\frac{\rho\hbar^2}{2M_0}(2\partial_\lambda \log R\partial_\lambda \log R + \Box\log R) = \mathcal{L},$$

the force φ^0_j will becomes:

$$\varphi^0_k = -\frac{\hbar^2}{m_0}\left\{\partial^0_k R\left[\Delta^0 R + \frac{1}{c^2}\left(\frac{\partial^0 R}{\partial t}\partial^0_j v_j + \partial^0_j R\frac{\partial^0 v_j}{\partial t}\right)\right] + \partial^0_j R\left(\partial^0_j\partial^0_k R + \frac{1}{c^2}\frac{\partial^0 R}{\partial t}\partial^0_j v_k\right)\right\}$$
$$+\ \partial^0_k\mathcal{L} + \frac{1}{c^2}\mathcal{L}\frac{\partial^0 v_k}{\partial t},$$

and the caloric energy will become:

$$w_0 = -\rho\,\frac{\hbar^2}{2M_0c^4}(\partial^0_j \log R\,\partial^0_k \log R - \delta_{jk}\mathcal{L}).$$

In this simplified form, one sees that one does not obtain all of the same expressions that one got for the de Broglie-Takabayasi fluid.

In order to express these differences precisely, we make the non-relativistic approximation on the components of the energy-momentum tensor:

$$t_{ij} = R^2 m_0 v_i v_j + R^2\frac{\hbar^2}{2m_0}\,[2\,\partial_i \log R\,\partial_j \log R - \delta_{ij}\,(2\,\partial_k \log R\,\partial_k \log R + \Delta \log R)],$$

such that:

$$t_{ij} = R^2 m_0 v_i v_j + \frac{\hbar^2}{2m_0}[2\,\partial_i R\,\partial_j R - \delta_{ij}\,(2\,\nabla R\,\nabla R + R\,\Delta\,R)],$$

$$t_{k\otimes} \equiv g_k = R^2 m_0 v_k\,, \qquad\qquad t_{\otimes\otimes} = -\,R^2 m_0\,.$$

Upon applying our general procedure to the Schrödinger equation, one will see that one recovers the expressions that we obtained exactly, which are expressions that differ from the ones that describe the Madelung fluid by the form of the quantum stress tensor. Moreover, we know that the (non-relativistic) force of stress is the same in both cases. In the rigorous, relativistic formulation, one sees that they are not the same, and that the two expressions for φ_μ differ noticeably. One is therefore dealing with two distinctly different fluids.

§ 9. The Dirac wave function: its tensorial representation. We now apply our method to the case of the Dirac equations for particles of spin 1/2. We will not give any details, but merely content ourselves with rediscovering the results that that were given in the fundamental treatise of Takabayasi [**9**] by our own method. One knows that the Dirac wave function is a spinor. We will now deal with the problem of giving it a tensorial hydrodynamical representation. This is why is seems useful for us to rapidly recall the classical relations between spinors and tensors here. The variance of spinors with four components (which one sometimes calls *4-spinors*) is closely linked with the properties of the four Dirac matrices, which are matrices that we define by the commutation rule:

(IV-8) $$\gamma_\mu\,\gamma_\nu + \gamma_\nu\,\gamma_\mu = 2\delta_{\mu\nu}\,.$$

It is unnecessary to specify the chosen representation. We content ourselves with assuming that we have chosen a *Hermitian* representation:

$$\gamma_\mu^\dagger = \gamma_\mu\,.$$

The index μ takes on the four values 1, 2, 3, 4, corresponding to the axes of Minkowski spacetime.

There exists a fundamental relationship between the canonical transformations that operate on these matrices and the Lorentz transformations that act on the axes of Minkowski space. We proceed to associate an infinitesimal matrix transformation:

$$T = 1 + \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu}$$

with every *infinitesimal* Lorentz transformation $L_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$ ($\omega_{\mu\nu} = -\omega_{\mu\nu}$ being infinitesimal). One easily shows that the transformation T acts on the matrix products $\gamma_4\gamma_\mu$ in a manner that is equivalent to the action of the corresponding Lorentz transformation on each of the matrices $\gamma_4\gamma_\mu$, when they are considered to be the four components of a vector:

$$T^{\dagger}(\gamma_4\gamma_\mu)T = (\delta_{\mu\nu} + \omega_{\mu\nu})(\gamma_4\gamma_\mu).$$

Indeed, take the Hermitian conjugate of the defining relation:

$$T = 1 + \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu}, \qquad\qquad T^{\dagger} = 1 + \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega^{*}_{\mu\nu}.$$

Upon taking care that ω_{ij} are real, and that the ω_{k4} and ω_{4k} are pure imaginary, one will have:

$$T^{\dagger} = 1 + \tfrac{1}{4}\,\gamma_j\,\gamma_i\,\omega_{ij} - \tfrac{1}{4}\,\gamma_4\,\gamma_k\,\omega_{k4} - \tfrac{1}{4}\,\gamma_k\,\gamma_4\,\omega_{4k},$$

and since ω_{ij} is anti-symmetric, the commutation relations will give:

$$T^{\dagger} = 1 - \tfrac{1}{4}\,\gamma_j\,\gamma_i\,\omega_{ji} - \tfrac{1}{4}\,\gamma_4\,\gamma_k\,\omega_{k4} - \tfrac{1}{4}\,\gamma_k\,\gamma_4\,\omega_{4k}.$$

In order to obtain a covariant form, one right-multiplies this by γ_4, and upon taking the commutation relation into account, it will follow that:

$$T^{\dagger}\gamma_4 = \gamma_4 - \tfrac{1}{4}\,\gamma_j\,\gamma_i\,\gamma_4\,\omega_{ji} - \tfrac{1}{4}\,\gamma_4\,\gamma_k\,\gamma_4\,\omega_{k4} - \tfrac{1}{4}\,\gamma_4\,\gamma_4\,\gamma_k\,\omega_{4k},$$

or finally:

$$T^{\dagger}\gamma_4 = \gamma_4(1 - \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu}).$$

Upon remarking that the term in parentheses is simply the matrix t that is inverse to T, one will get:

$$T^{\dagger}\gamma_4 = \gamma_4\,t \qquad \text{with} \qquad t = 1 - \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu}.$$

One may now express the transformation $T^{\dagger}\gamma_4\gamma_\mu\,T$ as:

$$T^{\dagger}\gamma_4\gamma_\lambda\,T = \gamma_4\,(1 - \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu})\;\gamma_\lambda\,(1 + \tfrac{1}{4}\,\gamma_\mu\,\gamma_\nu\,\omega_{\mu\nu}),$$

or, upon neglecting the term in $(\omega_{\mu\nu})^2$, as:

$$T^{\dagger}\gamma_4\gamma_\lambda\,T = \gamma_4\,[\gamma_\lambda + \tfrac{1}{4}\,\omega_{\mu\nu}\,(\gamma_\lambda\,\gamma_\mu\,\gamma_\nu - \gamma_\mu\,\gamma_\nu\,\gamma_\lambda)].$$

From (8), the parentheses yield:

$$(-\gamma_\mu\,\gamma_\lambda + 2\delta_{\mu\lambda})\;\gamma_\nu - \gamma_\mu\,(-\gamma_\lambda\,\gamma_\nu + 2\delta_{\lambda\nu}) = 2(\delta_{\mu\lambda}\gamma_\nu - \delta_{\nu\lambda}\,\gamma_\mu),$$

$$T^{\dagger}\gamma_4\gamma_\lambda\,T = \gamma_4\,[\gamma_\mu + \tfrac{1}{2}(\omega_{\lambda\nu}\gamma_\nu - \omega_{\mu\lambda}\gamma_\mu)] = \gamma_4\gamma_\lambda + \omega_{\lambda\nu}\gamma_4\gamma_\nu = (\delta_{\lambda\nu} + \omega_{\lambda\nu})\;\gamma_4\gamma_\nu.$$

This is precisely the Lorentz transformation that we promised.

Since the Lorentz transformations and the transformations $t\;\gamma_\lambda\;T$ constitute two groups, this property can be generalized to an arbitrary transformation that is considered to be the product of infinitesimal transformation: One can define a transformation T such that:

$$T^{\dagger}\gamma_4\gamma_\lambda T = L_{\mu\nu}\,\gamma_4\gamma_\nu$$

corresponds to each Lorentz transformation $L_{\mu\nu}$ uniquely.

Consider a spinor with four components then. The notation ψ represents a set of four components that are expressed in the form of a column matrix and are subject to the ordinary rules of matrix algebra:

$$\psi = \begin{bmatrix} \psi_{(1)} \\ \psi_{(2)} \\ \psi_{(3)} \\ \psi_{(4)} \end{bmatrix}.$$

One similarly defines the complex conjugate ψ^*, which is formed from the complex conjugates of the $\psi_{(k)}$:

$$\psi^* = \begin{bmatrix} \psi^*_{(1)} \\ \psi^*_{(2)} \\ \psi^*_{(3)} \\ \psi^*_{(4)} \end{bmatrix}.$$

The Hermitian conjugate $\psi^{\dagger}$ is the row matrix that is composed of the $\psi^*_{(k)}$:

$$\psi^{\dagger} = \left|\psi^*_{(1)}\ \ \psi^*_{(2)}\ \ \psi^*_{(3)}\ \ \psi^*_{(4)}\right|.$$

Finally, we define Dirac *adjoint* spinor – or simply the *adjoint spinor* – by the product:

$$\bar{\psi} = \psi^{\dagger}\gamma_4 .$$

We then fix the variance of the spinor by the following rule: Every Lorentz transformation that acts on the reference system will correspond to the action that was defined above of the operator T on the spinor. While a vector A_μ transforms as $A'_\mu = L_{\mu\nu} A_\nu$, a spinor ψ will transform as:

$$\psi' = T\,\psi .$$

The conjugate spinor $\psi^{\dagger}$ transforms as $\psi'^{\dagger} = \psi^{\dagger}T^{\dagger}$, and the adjoint spinor $\bar{\psi} = (\psi^{\dagger}\gamma_4)$ transforms as:

$$\bar{\psi}' = \psi^{\dagger}T^{\dagger}\gamma_4 = \psi^{\dagger}\,\gamma_4\,t = \bar{\psi}\ t.$$

This being the case, once one has chosen a representation for the γ_μ, with the aid of a spinor ψ, one can form the quantities:

$$A_\mu = \bar{\psi}\,\gamma_\mu\,\psi ,$$

which are, in fact, bilinear combinations of the components of the spinor. If one operates on it with a Lorentz transformation then it will follow that:

$$A'_\mu = \bar{\psi}' \gamma_\mu \psi' = \bar{\psi}\ t\ \gamma_\mu T\psi = \psi^\dagger \gamma_4\ t\ \gamma_\mu T\ \psi = \psi^\dagger T^\dagger \gamma_4 \gamma_\mu T\psi,$$

or, upon applying the fundamental property of T:

$$A'_\mu = \psi^\dagger L_{\mu\nu} \gamma_4 \gamma_\nu \psi = L_{\mu\nu} \bar{\psi} \gamma_\nu \psi = L_{\mu\nu} A_\nu .$$

One sees that the quantities A_μ are *vectors*.

This conclusion immediately extends to the case of an arbitrary product of the γ_μ matrices. Therefore, the quantity:

$$\tfrac{1}{2}\bar{\psi}\,(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)\ \psi$$

is an anti-symmetric tensor of second rank. One can, moreover, choose operators of such a sort that the tensor magnitudes are real – or, at least, that their spatial components are real and their temporal components are pure imaginary. Therefore, if one sets:

$$S_\mu = i\bar{\psi}\,\gamma_\mu\ \psi$$

then the complex conjugate of a spatial component:

$$S_k = i\psi^\dagger \gamma_4 \gamma_k\ \psi$$

will be:

$$S_k^* = -\,i\psi^\dagger \gamma_k \gamma_4\ \psi,$$

or, on account of (8):

$$S_k^* = i\psi^\dagger \gamma_4 \gamma_k\ \psi = S_k .$$

One will have:

$$S_4 = i\psi^\dagger \gamma_4 \gamma_k\ \psi, \qquad S_4^* = -\,i\psi^\dagger \gamma_4 \gamma_4\ \psi = -\,S_4$$

for the temporal component.

One may then form a set of real tensorial magnitudes that correspond to a complete set of matrices that are formed from the γ_μ. One therefore has:

A scalar: $\boxed{\Omega = \bar{\psi}\psi,}$

which corresponds to the identity matrix,

A vector: $\boxed{S_\mu = i\bar{\psi}\gamma_\mu\psi,}$

which corresponds to the γ_μ matrices, and

An anti-symmetric tensor: $\boxed{M_{[\mu\nu]} = -\frac{i}{2}\bar{\psi}[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]\psi,}$

which corresponds to the six independent matrices $\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu$ that one derives from the products $\gamma_\mu\,\gamma_\nu$ while taking the commutation rules into account.

One may also form the (pseudo) tensor:

$$\boxed{\hat{M}_{[\mu\nu]} = -\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\bar{\psi}\gamma_\alpha\gamma_\beta\psi.}$$

The tensor $M_{\mu\nu}$ will then be dual to $\hat{M}_{\mu\nu}$:

$$\hat{M}_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,M_{\alpha\beta}\,.$$

Thus, these two tensors express the same basic information.

The products of three matrices lead to only four new, completely anti-symmetric, independent matrices. If one replaces them with their duals:

$$\boxed{\hat{\gamma}_\mu = \frac{i}{3!}\varepsilon_{\mu\nu\alpha\beta}\gamma_\nu\gamma_\alpha\gamma_\beta.}$$

then one can define:

A (pseudo) vector: $\boxed{\hat{S}_\mu = -\bar{\psi}\,\hat{\gamma}_\mu\psi,}$

and finally:

A (pseudo) scalar: $\boxed{\hat{\Omega} = i\bar{\psi}\gamma_5\psi}$ $(\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4)$.

One remarks that when takes the commutation relations into account the matrix γ_5, which commutes with the four γ_μ matrices, will play a role that is analogous to that of the $\varepsilon_{\mu\nu\alpha\beta}$ symbol when one applies it to the anti-symmetric product of the γ_μ matrices. Therefore, one has:

$$\gamma_5\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu) = (\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\gamma_5 = -\,\varepsilon_{\mu\nu\alpha\beta}\,\gamma_\alpha\,\gamma_\beta,$$

so:

$$\hat{M}_{\mu\nu} = \frac{1}{2}\,\bar{\psi}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\gamma_5\,\psi\,.$$

Similarly:

$$\hat{\gamma}_\mu = \frac{i}{3!}\,\varepsilon_{\mu\nu\alpha\beta}\,\gamma_\nu\gamma_\alpha\gamma_\beta = i\,\gamma_\mu\,\gamma_5,$$

so:

$$\hat{S}_\mu = i\,\bar{\psi}\,\gamma_5\gamma_\mu\psi\,.$$

Of course, these tensorial magnitudes, which are determined by the eight variables that are implied by the four complex components of the spinor, are not independent. The relations between them may be obtained by using a fundamental identity that was proved by Pauli [**56**] between the elements of the γ_μ matrices. One has, upon specifying the row and column index of each element by a superscript:

(IV.8) $$\boxed{\gamma_\mu^{\rho\sigma}\gamma_{\mu'}^{\rho'\sigma'} = -(\delta^{\rho\sigma}\delta^{\rho'\sigma'} + \gamma_5^{\rho\sigma}\gamma_5^{\rho'\sigma'}) + 2\delta^{\rho\sigma'}\delta^{\rho'\sigma} - 2(B^{-1})^{\rho\rho'}B^{\sigma\sigma'}.}$$

B denotes a matrix that was introduced by Pauli and which transforms each γ_μ into its matrix transpose γ_μ^T:

$$\gamma_\mu^T = B\gamma_\mu B^{-1}.$$

This matrix is antisymmetric, just like $B\gamma_\mu$ and $B\gamma_5$. Upon multiplying this relation by an element of the matrix $\gamma_\mu^{\rho\sigma}$ or $\gamma_{\mu'}^{\rho'\sigma'}$ one or more times and contracting the multiplication with respect to the one of the upper indices, one will obtain a series of other identities.

If one now multiplies one of these relations by two components $\psi^\sigma\psi^{\sigma'}$ of a spinor and two components $\bar{\psi}^\rho\bar{\psi}^{\rho'}$ of its adjoint and sums over the four indices then one will obtain an identity between the tensorial magnitudes. For example, if one simply multiplies the fundamental relation (9) by them then one will see contracted products appear that can be written, in spinorial notation, as:

$$\psi^\sigma\psi^{\sigma'}\bar{\psi}^\rho\bar{\psi}^{\rho'}\gamma_\mu^{\rho\sigma}\gamma_{\mu'}^{\rho'\sigma'} = \bar{\psi}^\sigma\gamma_\mu^{\rho\sigma}\psi^\sigma\bar{\psi}^{\rho'}\gamma_{\mu'}^{\rho'\sigma'}\psi^{\sigma'} = \bar{\psi}\gamma_\mu\psi\cdot\bar{\psi}\gamma_\mu\psi, \qquad \text{etc.}$$

The last term disappears, in any case, on account of the antisymmetry of $B^{\rho\rho'}$ and $(B^{-1})^{\rho\rho'}$, since $\bar{\psi}^\rho\bar{\psi}^{\rho'}$ and $\psi^\sigma\psi^{\sigma'}$ are symmetric.

Finally, it follows that:

$$\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma_\mu\psi = -(\bar{\psi}\psi\,\bar{\psi}\psi + \bar{\psi}\gamma_5\psi\,\bar{\psi}\gamma_5\psi) + 2\bar{\psi}\psi,$$

such that:

$$-S_\mu S_\mu = \Omega\cdot\Omega + \hat{\Omega}\cdot\hat{\Omega}.$$

In this fashion, one will obtain the following four relations, which we call "identities of the first kind":

(IV.10) $$S_\mu S_\mu = -(\Omega^2 + \hat{\Omega}^2),$$

(IV.11) $$\hat{S}_\mu\hat{S}_\mu = (\Omega^2 + \hat{\Omega}^2),$$

(IV.12) $$S_\mu\hat{S}_\mu = 0,$$

(IV.13) $$(\Omega^2 + \hat{\Omega}^2)\, M_{\mu\nu} = \hat{\Omega}(\hat{S}_\mu S_\nu - \hat{S}_\nu S_\mu) - i\Omega\varepsilon_{\mu\nu\alpha\beta}\hat{S}_\alpha S_\beta\,.$$

This last relation determines $M_{\mu\nu}$ completely as a function of the quantities Ω, $\hat{\Omega}$, S_μ, $\hat{S}_\mu$. Upon multiplying by $-\, i/2\ \varepsilon_{\mu\nu\lambda\rho}$ one will obtain an analogous expression for $\hat{M}_{\mu\nu}$:

(IV.14) $$(\Omega^2 + \hat{\Omega}^2)\, \hat{M}_{\mu\nu} = -\,\Omega\,(\hat{S}_\mu S_\nu - \hat{S}_\nu S_\mu) - i\hat{\Omega}\varepsilon_{\mu\nu\alpha\beta}\hat{S}_\alpha S_\beta\,.$$

One may derive two other interesting identities from these two relations.

Multiply (13) by $\hat{\Omega}$, (14) by $-\,\Omega$, and take the adjoint. Since $\Omega^2 + \hat{\Omega}^2$ is a factor, it will follow that:

(IV.15) $$\hat{\Omega}M_{\mu\nu} - \Omega\hat{M}_{\mu\nu} = \hat{S}_\mu S_\nu - \hat{S}_\nu S_\mu\,.$$

Similarly, upon multiplying (13) by Ω and (14) by $\hat{\Omega}$, one will have:

(IV.16) $$\Omega M_{\mu\nu} + \hat{\Omega}\hat{M}_{\mu\nu} = -\, i\ \varepsilon_{\mu\nu\alpha\beta}\ \hat{S}_\alpha S_\beta\,.$$

Therefore, we have to consider that only the two vectors S_μ and $\hat{S}_\mu$, and the two scalars Ω and $\hat{\Omega}$ – namely, ten quantities – are basic tensorial magnitudes. Furthermore, they are subject to three identities (10), (11), and (12). We therefore have seven independent quantities, while the spinor that describes the wave function involves eight real independent quantities. That will therefore oblige us to appeal to some other tensorial quantities that we shall derive from the spinor in a different way.

If we consider the transformation of the gradient $\partial_\mu\psi$ of the wave function under a Lorentz transformation then it will be obvious that the operator ∂_μ submits to the vectorial transformation $L_{\mu\nu}$ and the function ψ, to the spinorial transformation T. The gradient becomes $L_{\mu\nu}\partial_\mu\, T\psi$. One can then repeat the proof that we made for quantities of the type:

$$A_\mu = \bar{\psi}\gamma_\mu\psi \qquad \text{(quantities of the first kind)}$$

and apply it to the quantities of the type:

$$B_{\nu\mu} = \bar{\psi}\,\gamma_\mu\,\partial_\nu\psi \quad \text{or} \quad C_{\nu\mu} = \partial_\nu\bar{\psi}\,\gamma_\mu\psi \qquad \text{(quantities of the second kind).}$$

One then shows that these quantities are once more *tensors* that are formed by means of the function ψ. Moreover, we remark that:

$$\partial_\nu\bar{\psi}\,\gamma_\mu\psi + \bar{\psi}\,\gamma_\mu\partial_\nu\psi = \partial_\nu(\bar{\psi}\,\gamma_\mu\psi)\,.$$

The sum $B_{\mu\nu} + C_{\mu\nu}$ is therefore the gradient of the vector of the first kind $\bar{\psi}\gamma_\mu\psi$; it will not provide us with any new quantities. By contrast, the difference:

$$C_{\nu\mu} - B_{\nu\mu} = \partial_\nu \bar{\psi}\, \gamma_\mu \psi - \bar{\psi}\, \gamma_\mu \partial_\nu \psi$$

does not directly involve any quantities of the first kind. It is such differences that we will consider. We shall use only three of them:

Two vectors: $\boxed{J_\mu = i(\partial_\mu \bar{\psi}\, \psi - \bar{\psi}\, \partial_\mu \psi) = i\bar{\psi}[\partial_\mu]\psi}$

and

$$\boxed{\hat{J}_\mu = -(\partial_\mu \bar{\psi}\gamma_5\, \psi - \bar{\psi}\, \gamma_5 \partial_\mu \psi) = i\bar{\psi}[\partial_\mu] i\gamma_5 \psi.}$$

A tensor: $\boxed{T_{\mu\nu} = -(\partial_\mu \bar{\psi}\gamma_\nu\, \psi - \bar{\psi}\, \gamma_\nu \partial_\mu \psi) = -\bar{\psi}[\partial_\mu]\gamma_\nu \psi.}$

The symbol $[\partial_\mu]$ indicates that one takes the difference of the two terms thus obtained by first differentiating the adjoint spinor $\bar{\psi}$ and then the spinor ψ. The factor i has the purpose of making the spatial components real and the temporal components pure imaginary. Therefore, if we take the complex conjugate of the components of J_μ, it will follow that:

$$J_\mu^* = -\, i(\bar{\psi}\, \partial_\mu^* \psi - \partial_\mu^* \bar{\psi} \psi) \;=\; i\, \bar{\psi}[\partial_\mu]^* \psi\, ,$$

such that, since $\partial_k^* = -\, \partial_k$ and $\partial_4^* = -\, \partial_4$:

$$J_k^* = i\, \bar{\psi}[\partial_k]\psi \;= J_k\, ,$$
$$J_4^* = -\, i\, \bar{\psi}[\partial_4]\psi \;= -\, J_4\, .$$

It is, moreover, possible to form other tensorial magnitudes of the second kind by means of the various combinations of the γ_μ, but we shall not use them.

Then again, the Pauli identities regarding the elements of the γ_μ matrices allow identities to appear among the magnitudes of the second kind. It suffices to operate on the products of the type $\partial_\mu \bar{\psi}^\rho\, \psi^\sigma \bar{\psi}^{\rho'}\, \psi^{\sigma'}$ by contracted multiplication and to subtract the relation that one gets by contracting the same identity for $\bar{\psi}^\rho \partial_\mu\, \psi^\sigma \bar{\psi}^{\rho'}\, \psi^{\sigma'}$ from the relation thus obtained. The terms that contain the doubly-anti-symmetric product $B^{\sigma\sigma'}$ $(B^{-1})^{\rho\rho'}$ will once more disappear because it remains a symmetric product, such as $\psi^\sigma\, \psi^{\sigma'}$ or $\bar{\psi}^\rho \bar{\psi}^{\rho'}$. One will thus obtain a series of identities that express all of the quantities of the second kind as functions of the only the two vectors J_μ and $\hat{J}_\mu$, along with quantities of the first kind. The only one that we will have to consider relates to the tensor $T_{\mu\nu}$:

(IV.17) $$\Omega T_{\mu\nu} = J_\mu\, S_\nu + \partial_\mu \hat{\Omega}\, \hat{S}_\nu \; + \partial_\mu\, S_\lambda\, M_{\nu\lambda}\, .$$

Furthermore, the vectors J_μ and $\hat{J}_\mu$ are related by the relation that is obtained from (12) in the same manner:

(IV.18) $$\hat{\Omega}\, J_\mu - \Omega\, \hat{J}_\mu = \hat{S}_\lambda \partial_\mu S_\lambda = -\, S_\lambda \partial_\mu \hat{S}_\lambda \,.$$

Of the eight quantities that are expressed by J_μ and $\hat{J}_\mu$, only four of them are independent. One can represent them by the new vector:

(IV.19) $$\boxed{K_\mu = \Omega\, J_\mu + \hat{\Omega}\, \hat{J}_\mu,}$$

by means of which, one may express J_μ and $\hat{J}_\mu$ upon taking (18) into account, and, as a consequence, all of the magnitudes of the second kind: Multiply (18) by $\hat{\Omega}$ and (19) by Ω, and take their adjoints:

$$(\Omega^2 + \hat{\Omega}^2)\, J_\mu = \Omega K_\mu + \hat{\Omega}\, \hat{S}_\lambda \partial_\mu S_\lambda \,.$$

Multiplying (18) by $-\,\Omega$ and (19) by $\hat{\Omega}$ will give:

$$(\Omega^2 + \hat{\Omega}^2)\, \hat{J}_\mu = \hat{\Omega}\, K_\mu - \Omega\, \hat{S}_\lambda \partial_\mu S_\lambda \,.$$

Furthermore, Takabaysi [**9**] has shown that when one starts with an identity of the second kind, K_μ will be restricted by a "kinematic condition of the second kind," which we shall not describe, except to say that one may characterize it as a "quasi-irrotationality" condition. It results from this that, in reality, K_μ (and, indeed, the whole set of quantities of the second kind) contains *only one independent quantity*, which is a quantity of the second kind, and which brings the number of independent components that are expressed by all of the tensorial formalism to exactly eight.

It is useful to derive some other identities from equation (17) that we will need. Upon multiplying it by $\hat{\Omega}$, one will have:

$$\hat{\Omega}\, \Omega\, T_{\mu\nu} = \hat{\Omega}\, J_\mu\, S_\nu + \hat{\Omega} \partial_\mu \hat{\Omega}\, \hat{S}_\nu + \hat{\Omega}\, \partial_\mu\, S_\lambda\, M_{\lambda\nu} \,.$$

Now, from (15), one has:

$$\hat{\Omega}\, M_{\nu\lambda} = \Omega\, \hat{M}_{\mu\nu} + \hat{S}_\nu S_\lambda - \hat{S}_\lambda S_\nu \,.$$

Upon multiplying this by $\partial_\mu S_\lambda$, one will cause to appear, on the one hand, $S_\lambda\, \partial_\mu\, S_\lambda$, which, from (10), is equal to $-\,\Omega\, \partial_\mu \Omega - \hat{\Omega} \partial_\mu \hat{\Omega}$, and, on the other hand, $\hat{S}_\lambda \partial_\mu S_\lambda$, which, from (18), is equal to $\Omega\, \hat{J}_\mu - \hat{\Omega}\, J_\mu$. One will therefore have:

$$\hat{\Omega}\, \Omega T_{\mu\nu}$$

$$= \hat{\Omega}\, J_\mu\, S_\nu + \hat{\Omega} \partial_\mu \hat{\Omega}\, \hat{S}_\nu + \Omega \partial_\mu\, S_\lambda \hat{M}_{\nu\lambda} - \Omega \partial_\mu \Omega\, \hat{S}_\nu - \hat{\Omega} \partial_\mu \hat{\Omega}\, \hat{S}_\nu + \Omega\, \hat{J}_\mu S_\nu - \hat{\Omega}\, J_\mu\, S_\nu,$$

such that after reducing this and dividing by Ω, one will have:

(IV.20) $$\hat{\Omega}\, T_{\mu\nu} = \hat{J}_\mu S_\nu - \partial_\mu \Omega\, \hat{S}_\nu + \partial_\mu S_\lambda\, \hat{M}_{\nu\lambda}.$$

Finally, upon multiplying (17) by Ω and (20) by $\hat{\Omega}$, one will get:

$$(\Omega^2 + \hat{\Omega}^2) T_{\mu\nu} = (\Omega\, J_\mu + \hat{\Omega}\, \hat{J}_\mu) S_\nu + (\Omega \partial_\mu \hat{\Omega} - \hat{\Omega} \partial_\mu \Omega)\, \hat{S}_\nu + \partial_\mu S_\lambda (\hat{\Omega} \hat{M}_{\nu\lambda} + \Omega M_{\nu\lambda}).$$

One thus sees the previously-defined vector K_μ appear in the first term, and in the second one, a quantity that is derived from the quantities of the first kind that we denote by:

$$Q_\mu = \Omega\, \partial_\mu \hat{\Omega} - \hat{\Omega}\, \partial_\mu \Omega,$$

and finally, in the third term, one sees a quantity appear that, from (16), is equal to – $i\, \varepsilon_{\nu\lambda\alpha\beta}\, \hat{S}_\alpha S_\beta$. Thus, one gets:

(IV.21) $$\boxed{(\Omega^2 + \hat{\Omega}^2) T_{\mu\nu} = K_\mu S_\nu + Q_\mu \hat{S}_\nu - \partial_\mu S\, i\varepsilon_{\nu\lambda\alpha\beta} \hat{S}_\alpha S_\beta.}$$

§ 10. The Dirac wave function: a hydrodynamical model. Current and spin. It is essential to point out that the definition of all these tensorial magnitudes and the identities that we established between them are entirely independent of the Dirac theory, and are derived solely from the fact that the wave function is a 4-spinor, and that its variance makes the γ matrices intervene. One may apply this formalism to other spinorial equations that are like that of Dirac [**66**]. However, we shall now pass on to the Dirac case and introduce new quantities and hydrodynamical equations that are essentially related to the Dirac theory; i.e., they are applicable to only the wave function of the electron.

We use the (real) Dirac spinor Lagrangian, in its Von Neumann form [**71**]:

$$\mathcal{L} = \frac{\hbar c}{2} (\bar{\psi}\, \gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\, \gamma_\mu \psi + 2\kappa \bar{\psi} \psi),$$

with:

$$\kappa = \frac{mc}{\hbar}.$$

One immediately derives the Dirac equations from this:

$$\begin{aligned} \gamma_\mu \partial_\mu \psi &= -\kappa\, \psi, \\ \partial_\mu \bar{\psi}\, \gamma_\mu &= +k\, \bar{\psi}. \end{aligned}$$

It results from these equations that the Lagrangian is zero: $\mathcal{L} = 0$.

We shall deduce the fundamental magnitudes of our hydrodynamics from this Lagrangian, conforming to the method that was described at the beginning of this chapter.

The *current* is given by:

$$j_\mu = i\frac{\hbar c}{2}(\bar\psi\gamma_\mu\psi + \psi\gamma_\mu\bar\psi) = i\hbar c\,\bar\psi\gamma_\mu\psi\,.$$

One forms the vector:

$$S_\mu = i\,\bar\psi\gamma_\mu\psi\,,$$

so one will find that:

$$j_\mu = \hbar c\, S_\mu\,.$$

Furthermore, it immediately results from the Dirac equation that j_μ is conservative:

$$i\hbar c\partial_\mu(\bar\psi\gamma_\mu\psi) = \partial_\mu j_\mu = i\hbar c(\bar\psi\,\gamma_\mu\partial_\mu\psi + \partial_\mu\bar\psi\,\gamma_\mu\psi) = 0.$$

The matter density ρ is given by:

$$j_\mu\, j_\mu = -\rho^2\, c^2\, \hbar^2 = \hbar^2\, c^2\, S_\mu\, S_\mu\,,$$

$$\rho^2 = -S_\mu\, S_\mu\,.$$

Now, relation (10) gives us:

$$\boxed{\rho^2 = \Omega^2 + \hat\Omega^2.}$$

Hence, one directly deduces the *unit-speed velocity:*

$$\boxed{u_\mu = \frac{j_\mu}{\rho\hbar} = \frac{cS_\mu}{\rho} = \frac{cS_\mu}{\sqrt{\Omega^2 + \hat\Omega^2}},}$$

which gives us precisely:

$$u_\mu\, u_\mu = -c^2.$$

We now pass on to the Belinfante tensor:

$$f_{[\mu\nu]\lambda} = \frac{\partial\mathcal{L}}{\partial\psi_{,\lambda}}\mathfrak{T}_{[\mu\nu](\text{op.})}\psi + \frac{\partial\mathcal{L}}{\partial\bar\psi_{,\lambda}}\bar{\mathfrak{T}}_{[\mu\nu](\text{op.})}\bar\psi\,.$$

We know that, for an infinitesimal Lorentz transformation $\omega_{[\mu\nu]}$ the spinor ψ is subjected to the transformation $\psi' = T\psi$, with:

$$T = 1 + \frac{1}{4}(\gamma_\mu\,\gamma_\nu)\,\omega_{\mu\nu}\,,$$

or, to take into account the anti-symmetry of $\omega_{\mu\nu}$:

$$T = 1 + \frac{1}{8}(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\;\omega_{\mu\nu},$$

For the spinor ψ, the operator $\mathfrak{T}_{[\mu\nu]}$ is thus:

$$\mathfrak{T}_{[\mu\nu]\,(\text{op.})} = \frac{1}{8}(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,.$$

We have the transformation $\bar{\psi}' = \bar{\psi}\,t$ for the adjoint spinor $\bar{\psi} = \psi^\dagger\,\gamma_4$, where t is the inverse of T, which will give us an operator:

$$\bar{\mathfrak{T}}_{[\mu\nu](\text{op.})} = -\frac{1}{8}(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,.$$

All totaled, one thus has:

$$f_{[\mu\nu]\lambda} = \frac{\hbar c}{2}\left[\bar{\psi}\,\gamma_\lambda\frac{1}{8}(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\;\psi + \bar{\psi}\,\frac{1}{8}(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\;\gamma_\lambda\psi\right].$$

Upon applying the commutation relations, one will get:

$$\begin{aligned}\gamma_\lambda\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu) &= \quad(\gamma_\mu\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\lambda\gamma_\mu) + 2\delta_{\lambda\mu}\,\gamma_\nu - 2\delta_{\lambda\nu}\,\gamma_\mu\,,\\ (\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\gamma_\lambda &= -(\gamma_\mu\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\lambda\gamma_\mu) + 2\gamma_\mu\,\delta_{\lambda\nu} - 2\gamma_\nu\,\delta_{\lambda\mu}\,.\end{aligned}$$

Upon adding the two operators, only the terms inside the parentheses will remain, from which, one finds that:

$$f_{[\mu\nu]\lambda} = -\frac{\hbar c}{8}\bar{\psi}\,(\gamma_\mu\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\lambda\gamma_\mu)\;\psi.$$

From the original formula, one knows that the only non-zero components of this tensor will be the ones for which $\mu \neq \nu$. One may consider the components for which:

$$f_{\mu\nu\mu} = -\frac{\hbar c}{8}\bar{\psi}\,(\gamma_\mu\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\mu)\;\psi = 0,$$

and the same thing will be true when $\lambda = \nu \neq \mu$.

Thus, the only components to consider are the ones for which:

$$\lambda \neq \mu \neq \nu,$$

and it will then result from the commutation relations that the tensor $f_{\mu\nu\lambda}$ is completely anti-symmetric, and one may write:

$$f_{[\mu\nu]\lambda} = \frac{\hbar c}{4} \bar{\psi}\, \gamma_\mu \gamma_\nu \gamma_\lambda\, \psi \qquad (\mu \neq \nu \neq \lambda).$$

One may likewise introduce the fourth matrix γ_α ($\mu \neq \nu \neq \lambda \neq \alpha$) and write (with no summation):

$$f_{\mu\nu\lambda} = \frac{\hbar c}{4} \bar{\psi}\, \gamma_\mu \gamma_\nu \gamma_\lambda\, \gamma_\alpha \gamma_\alpha\, \psi\,.$$

One then observes that one has $\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\alpha = \varepsilon_{\mu\nu\lambda\alpha}\, \gamma_5$, so one will find that:

$$\begin{aligned} f_{\mu\nu\lambda} \quad &= \frac{\hbar c}{4} \varepsilon_{\mu\nu\lambda\alpha}\, \bar{\psi}\, \gamma_5 \gamma_\alpha\, \psi \\ &= -\frac{i\hbar c}{4} \varepsilon_{\mu\nu\lambda\alpha}\, \bar{\psi}\, i\gamma_5 \gamma_\alpha\, \psi\,, \end{aligned}$$

which will bring the following vector into play:

$$\hat{S}_\alpha = i\bar{\psi}\;\; \gamma_5 \gamma_\alpha\, \psi\,.$$

One finally gets:

$$\boxed{f_{\mu\nu\lambda} = -\frac{i\hbar c}{4} \varepsilon_{\mu\nu\lambda\alpha} u_\lambda \hat{S}_\alpha.}$$

The internal angular momentum $s_{[\mu\nu]}$ may be obtained immediately from:

$$f_{\mu\nu\lambda}\, u_\lambda = -\, c^2 \tfrac{1}{2} s_{\mu\nu} = -\, \frac{i\hbar c}{4} \varepsilon_{\mu\nu\lambda\alpha}\, \hat{S}_\alpha\, u_\lambda\,,$$

so

(IV.22)
$$\boxed{s_{\mu\nu} = \frac{i\hbar}{2c} \varepsilon_{\mu\nu\lambda\alpha} u_\lambda \hat{S}_\alpha.}$$

One sees at once that $s_{\mu\nu}\, u_\nu = 0$: The Dirac-Takabayasi fluid is then a Weyssenhoff fluid. As in the Weyssenhoff case (see Chapter III), one can put the internal angular momentum into the form:

(IV.23)
$$s_{\mu\nu} = \frac{i}{c} \varepsilon_{\mu\nu\lambda\alpha}\, u_\lambda\, \sigma_\alpha\,,$$

in which, σ_α denotes the spin density, which is orthogonal to u_α . Since equation (12) shows us that $\hat{S}_\alpha$ is indeed orthogonal to u_α, we can identify equations (22) and (23) upon setting:

$$\boxed{\sigma_\alpha = \frac{\hbar}{2} \hat{S}_\alpha.}$$

Relation (11) will then show us that:

$$\sigma_0^2 = \sigma_\alpha\,\sigma_\alpha = \frac{\hbar^2}{4}\rho^2,$$

so the norm of the spin density is $\sigma_0 = \hbar\rho/2$.

In other words, the "particle" spin is constant and equal to $\hbar/2$, which is a property that the particles that might constitute our fluid share with the electron.

One may remark that the tensor $f_{\mu\nu\lambda}$ can be reduced to just the term $s_{\mu\nu}\,u_\lambda$. One has the development:

$$f_{\mu\nu\lambda} = s_{\mu\nu}\,u_\lambda - \frac{i\hbar c}{2}\varepsilon_{\mu\nu\kappa\alpha}\hat{S}_\alpha\left(\frac{u_\kappa u_\lambda}{c^2} + \delta_{\kappa\lambda}\right).$$

Normally, we must make the last term disappear by a gauge transformation and operate on the energy-momentum tensor, which remains for us to express, with the equivalent transformation. Meanwhile, since we would like to simply summarize the theory of Takabayasi here, we content ourselves with applying the "second gauge procedure," despite the objections that we made at the beginning of the present chapter. From this point of view, one does not have to perform any gauge transformation since the tensor $f_{\mu\nu\lambda}$ can be expressed exclusively in terms of the spin.

Of course, the expression for $f_{\mu\nu\lambda}$, viz.:

$$f_{\mu\nu\lambda} = -\frac{ic}{2}\varepsilon_{\mu\nu\lambda\alpha}\,\sigma_\alpha,$$

is not the Weyssenhoff expression. The Weyssenhoff torque equation:

$$t_{\mu\nu} - t_{\nu\mu} = \dot{s}_{\mu\nu},$$

which translates into simply the general conservation law:

$$t_{\mu\nu} - t_{\nu\mu} = 2\,\partial_\lambda f_{\mu\nu\lambda},$$

must be replaced with:

$$\boxed{t_{\mu\nu} - t_{\nu\mu} = -ic\,\varepsilon_{\mu\nu\lambda\alpha}\partial_\lambda\sigma_\alpha.}$$

§ 11. The Dirac wave function: the energy-momentum tensor and hydrodynamical equations. We now pass on to the study of the energy-momentum tensor. One has:

$$t_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\psi_{,\nu}}\partial_\mu\psi + \partial_\nu\bar{\psi}\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{,\mu}} - \delta_{\mu\nu}\mathcal{L},$$

$$= \frac{\hbar c}{2}(\bar{\psi}\,\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\,\gamma_\nu\psi).$$

One recognizes a magnitude of the second kind:

$$t_{\mu\nu} = -\frac{\hbar c}{2}\bar{\psi}[\partial_\mu]\gamma_\nu\psi = \frac{\hbar c}{2}T_{\mu\nu}\,.$$

One may therefore express this by using the identity (21) in the form:

$$t_{\mu\nu} = \frac{\hbar c}{2\rho^2}\left[\frac{\rho}{c}K_\mu u_\nu + \frac{2}{\hbar}Q_\mu\sigma_\nu - \frac{2\rho}{\hbar c^2}i\varepsilon_{\nu\gamma\alpha\beta}\partial_\mu(\rho u_\lambda)\sigma_\alpha u_\beta\right].$$

In order to express the quantity $Q_\mu = \Omega\partial_\mu\hat{\Omega} - \hat{\Omega}\partial_\mu\Omega$, one must specify the invariants Ω and $\hat{\Omega}$ as functions of ρ and a new variable that we represent by an angle A, as in the study of the classical Dirac particle (Chap. III).

We set:

$$\Omega = \rho\cos A, \quad \hat{\Omega} = \rho\sin A,$$

which gives:

$$\Omega^2 + \hat{\Omega}^2 = \rho^2,$$

precisely.

We will then have:

$$Q_\mu = \rho\cos A\,(\rho\cos A\,\partial_\mu A + \sin A\,\partial_\mu\rho) - \rho\sin A\,(-\rho\sin A\,\partial_\mu A + \cos A\,\partial_\mu\rho),$$

such that:

$$\boxed{Q_\mu = \rho^2\partial_\mu A.}$$

On the other hand, if one develops $\partial_\mu(\rho u_\lambda)$ in the expression for $t_{\mu\nu}$ then the term $u_\lambda\,\partial_\mu\rho$ will go to zero, by anti-symmetry. The last term in brackets will then become:

$$-\frac{2\rho}{\hbar c^2}i\,\varepsilon_{\lambda\nu\alpha\beta}\,\rho\,\partial_\mu u_\lambda\,\sigma_\alpha u_\beta\,,$$

namely:

$$+\frac{2\rho}{\hbar c^2}\partial_\mu u_\lambda\,s_{\nu\lambda}\,.$$

The energy-momentum tensor then takes the form:

$$t_{\mu\nu} = \frac{\hbar}{2\rho}K_\mu u_\nu + c\,\partial_\mu A\,\sigma_\nu + s_{\nu\lambda}\,\partial_\mu u_\lambda\,.$$

We deduce the momentum from this by way of:

$$-c^2 g_\mu \equiv t_{\mu\nu}u_\nu = -c^2\frac{\hbar}{2\rho}K_\mu\,,$$

so

$$\boxed{g_\mu = \frac{\hbar}{2}\frac{K_\mu}{\rho},}$$

which will give $t_{\mu\nu}$ the form:

(IV.24) $$\boxed{t_{\mu\nu} = g_\mu u_\nu + c\partial_\mu A\sigma_\nu + s_{\nu\lambda}\partial_\mu u_\lambda .}$$

The non-kinetic part, which is comprised of the last two terms, permits us to define:

A heat current: $-c^2 q_\nu = (c\partial_\mu A\ \sigma_\nu + s_{\nu\lambda}\partial_\mu u_\lambda)\, u_\mu$

namely:

$$\boxed{q_\nu = -\frac{1}{c}\dot{A}\sigma_\nu - \frac{1}{c^2}s_{\nu\lambda}\dot{u}_\lambda .}$$

An internal stress tensor:

$$\theta_{\mu\nu} = c\,\partial_\mu A\sigma_\nu + s_{\nu\lambda}\,\partial_\mu u_\lambda + \frac{1}{c}\dot{A}\sigma_\nu u_\mu + \frac{1}{c^2}s_{\nu\lambda}\dot{u}_\lambda u_\mu\ ,$$

which may be written more simply as:

$$\boxed{\theta_{\mu\nu} = \eta_{\mu\lambda}(c\partial_\lambda A\sigma_\nu + s_{\nu\alpha}\partial_\lambda u_\alpha)} \qquad \left(\eta_{\mu\lambda} = \delta_{\mu\lambda} + \frac{u_\mu u_\lambda}{c^2}\right).$$

If we would like to decompose and analyze this momentum more closely then we would have to introduce a supplementary relation besides, since our formalism involves a variable A that cannot be interpreted from the hydrodynamical point of view, and which is not, as a consequence, governed by any dynamical law. We must borrow from the wave equations themselves. (We remark, in passing, that this fact is exclusive of the fact that Takabayasi deduced *all* of the hydrodynamical equations.) We simply start with the fact that the wave equations annul the Lagrangian, as we have mentioned before. On the other hand, the Lagrangian can be expressed as a function of the fundamental tensorial quantities as:

$$\begin{aligned}\mathcal{L} &= \frac{\hbar c}{2}(\bar{\psi}\gamma_\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma_\mu\psi + 2\kappa\bar{\psi}\psi)\\ &= \frac{\hbar c}{2}(T_{\mu\mu} + 2\kappa\Omega),\end{aligned}$$

or, as a function of the hydrodynamical variables:

$$\mathcal{L} = t_{\mu\mu} + \rho m_0 c^2 \cos A.$$

One thus has, by virtue of the wave equations:

$$t_{\mu\mu} + \rho m_0 c^2 \cos A = 0,$$

which will give:

$$g_\mu u_\mu + c\sigma_\mu \partial_\mu A + s_{\mu\lambda} \partial_\mu u_\lambda + \rho m_0 c^2 \cos A = 0$$

upon replacing $t_{\mu\mu}$ with its expression.

One infers from this the expression for the proper mass density, which was given for the first time by Yvon [**72**]:

$$\boxed{\mu_0 = \rho m_0 \cos A + \frac{1}{c}\sigma_\mu \partial_\mu A + \frac{1}{c^2} s_{\mu\lambda}\partial_\mu u_\lambda.} \tag{IV.25}$$

One sees the proper particle mass m_0 figure in this, but it is affected with a coefficient $\cos A$ that might lead us to consider the variable A to be something that expresses some sort of mixture of two fluids that carry, on the one hand, a positive energy, and on the other, a negative energy. A also intervenes by way of its gradient in the second term. As for the third term, it may also be written:

$$\frac{1}{2c^2} s_{\mu\lambda} (\partial_\mu u_\lambda - \partial_\lambda u_\mu),$$

and in this form one will see that it expresses an energy that couples the angular momentum with the vorticity of the current. One may also remark that the "internal pressure" that one may derive from the stress tensor that we have written is:

$$\pi = \frac{1}{3}\theta_{\mu\mu} = c\, \sigma_\mu \partial_\mu A + s_{\mu\alpha} \partial_\mu u_\alpha .$$

Therefore, the Yvon formula expresses the idea that the energy $\rho m_0 c^2 \cos A$ that one adds to the potential energy is due to the internal pressure of the fluid.

We now write the torque equation, while reminding ourselves that we must place ourselves in the second Takabayasi gauge:

$$t_{\mu\nu} - t_{\nu\mu} = 2\, \partial_\lambda f_{\mu\nu\lambda} = - i\, c\varepsilon_{\mu\nu\lambda\alpha} \partial_\lambda \sigma_\alpha .$$

One will have, from (24):

$$\begin{aligned} g_\mu u_\nu - g_\nu u_\mu + c(\partial_\mu A\, \sigma_\nu - \partial_\nu A\, \sigma_\mu) + s_{\nu\lambda} \partial_\mu u_\lambda - s_{\mu\lambda} \partial_\nu u_\lambda \\ = - i\, c\varepsilon_{\mu\nu\lambda\alpha} \partial_\lambda \sigma_\alpha . \end{aligned} \tag{IV.26}$$

Contracting this with u_λ will give:

$$- c^2 g_\mu + m_0 c^2 u_\mu - c \dot{A} \sigma_\mu - s_{\mu\lambda} \dot{u}_\lambda = - i\, c\varepsilon_{\mu\nu\lambda\alpha} u_\nu \partial_\lambda \sigma_\alpha .$$

This gives the expression for momentum as:

(IV.27) $$g_\mu = m_0\, u_\mu - \frac{1}{c}\dot{A}\sigma_\mu - \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha} u_\nu \partial_\lambda \sigma_\alpha .$$

This expression may be transformed by using an identity that is a consequence of the Pauli-Koffinck relations that Vigier, Lochak, and myself pointed out a few years ago [**26**]. One has:

(IV.28) $$\boxed{\frac{1}{c^2}(s_{\mu\lambda}u_\nu - s_{\nu\lambda}u_\mu)\partial_\nu u_\lambda + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}\sigma_\alpha \partial_\lambda u_\nu = 0.}$$

Takabayasi showed [**9**] that such a vectorial relation is equivalent to the three relations that are obtained by contracting with u_μ, σ_μ, and $s_{\mu\nu}$. We successively verify these three relations:

1. Contracting with u_μ gives:

$$s_{\nu\lambda}\, \partial_\mu\, u_\lambda + \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\, u_\mu\, \sigma_\alpha\, \partial_\lambda\, u_\nu = 0,$$

because:

$$\frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}\, u_\mu\, \sigma_\alpha = - s_{\lambda\nu} .$$

2. Contracting with σ_μ annuls each term separately:

3. Contracting with:

$$s_{\mu\beta} \equiv \frac{i}{c}\varepsilon_{\mu\beta\gamma\rho}\, u_\gamma\, \sigma_\rho$$

gives:

$$\frac{1}{c^2}(\sigma_0^2\eta_{\lambda\beta} - \sigma_\lambda\sigma_\beta)\dot{u}_\lambda - \frac{1}{c^2}\delta^{\nu\lambda\alpha}_{\beta\gamma\rho} u_\gamma \sigma_\rho \sigma_\alpha \partial_\lambda u_\nu$$

$$= \frac{1}{c^2}[\sigma_0^2\dot{u}_\beta - \sigma_\beta\sigma_\lambda\,\dot{u}_\lambda - u_\lambda\sigma_\rho\sigma_\rho\partial_\lambda u_\beta + u_\lambda\sigma_\rho\sigma_\beta\partial_\lambda u_\beta] = 0.$$

Identity (28) is thus verified. Upon substituting this in the expression (27) for g_μ, it will follow that:

(IV.29) $$g_\mu = \mu_0\, u_\mu - \frac{i}{c}\dot{A}\sigma_\mu + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}u_\nu\partial_\lambda\sigma_\alpha + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}u_\alpha\partial_\lambda\sigma_\nu - \frac{1}{c^2}s_{\nu\lambda}\partial_\nu u_\lambda u_\mu ,$$

which will give, upon replacing μ_0 with its expression in (25):

$$g_\mu = \rho\mu_0 \cos A\, u_\mu + \frac{1}{c}\sigma_\nu\,\partial_\nu A\, u_\mu - \frac{1}{c}\dot{A}\sigma_\mu + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}(u_\nu\,\partial_\lambda\,\sigma_\alpha + u_\alpha\,\partial_\lambda\,\sigma_\nu),$$

such that finally:

$$\boxed{g_\mu = \rho m_0 \cos A \cdot u_\mu + \frac{1}{c}\partial_\nu A(\sigma_\nu u_\mu - \sigma_\mu u_\nu) - \partial_\lambda s_{\mu\lambda}.}$$

Formula (29) permits the appearance of a transverse momentum along with the classical momentum $\mu_0\, u_\mu$, which is collinear with the current:

$$\begin{aligned} p_\mu &= \frac{1}{c}\dot{A}\sigma_\mu + \partial_\lambda s_{\mu\lambda} + \frac{1}{c^2} s_{\nu\lambda}\partial_\nu u_\lambda u_\mu \\ &= \frac{1}{c}\dot{A}\sigma_\mu - \partial_\nu s_{\nu\mu} - \frac{1}{c^2} u_\lambda u_\mu \partial_\nu s_{\nu\lambda}, \end{aligned}$$

or finally:

$$\boxed{p_\mu = \frac{1}{c}\dot{A}\sigma_\mu - \eta_{\mu\lambda}\partial_\nu s_{\nu\lambda}.}$$

Upon successively contracting the torque equation (26) with:

$$u_\nu\, \sigma_\mu\,, \qquad \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\sigma_\alpha\,, \qquad \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}u_\alpha\,,$$

one will obtain a fundamental system of equations of evolution for the hydrodynamical magnitudes that is equivalent to (26):

Contracting (26) with $u_\nu\, \sigma_\mu$, or simply contracting (27) with σ_μ gives:

$$g_\mu\, u_\mu = -\frac{i}{c}\dot{A}\sigma_0^2 + \frac{i}{c}\varepsilon_{\mu\nu\lambda\alpha}\sigma_\mu u_\nu \partial_\lambda \sigma_\alpha\,.$$

One immediately obtains the evolution equation for the variable A:

$$\boxed{\dot{A} = -\frac{c}{\sigma_0^2}(\sigma_\mu g_\mu + s_{\lambda\alpha}\partial_\lambda \sigma_\alpha).}$$

Contracting with $\frac{i}{2c}\varepsilon_{\mu\nu\gamma\beta}\,\sigma_\gamma$ gives:

$$\frac{i}{c}\varepsilon_{\mu\nu\gamma\beta}\sigma_\gamma g_\mu u_\nu + \frac{i}{c}\varepsilon_{\mu\nu\gamma\beta}\sigma_\gamma s_{\nu\lambda}\partial_\mu u_\lambda - \frac{1}{2}2\delta^{\lambda\alpha}_{\gamma\beta}\sigma_\gamma \partial_\lambda \sigma_\alpha = 0.$$

The first term becomes simply $s_{\mu\beta}\, u_\mu$; upon specifying $s_{\nu\lambda}$, the second one will become:

$$\frac{1}{c^2}\delta^{\mu\nu\beta}_{\lambda\alpha\rho}u_\alpha \sigma_\rho \sigma_\gamma \partial_\mu u_\lambda = \frac{1}{c^2}(\sigma_\gamma \partial_\mu u_\gamma \sigma_\mu u_\beta + \sigma_0^2 \dot{u}_\beta - \sigma_0^2 u_\beta \partial_\mu u_\mu - \sigma_\gamma \partial_\mu u_\gamma u_\mu \sigma_\beta)\,,$$

Following Takabayasi, we can introduce the vector z_μ :

$$\rho\,\frac{\hbar}{2}\,z_\mu \equiv \sigma_\gamma \partial_\mu\,\sigma_\gamma,$$

which represents a special quantity. On the other hand, we can replace σ_0^2 with its value $\rho^2\hbar^2/4$, which will give:

$$\frac{1}{c^2}\left[\rho^2\,\frac{\hbar^2}{4}(u_\mu\partial_\mu u_\beta - u_\beta\partial_\mu u_\mu) + \rho\frac{\hbar}{2}z_\mu(\sigma_\mu u_\beta - \sigma_\beta u_\mu)\right].$$

We remark that:

$$\rho\,u_\mu\,\partial_\mu\,u_\beta - \rho\,u_\beta\,\partial_\mu\,u_\mu = u_\mu\,\partial_\mu\,(\rho\,u_\beta),$$

due to the relation $\partial_\mu\,(\rho\,u_\beta) = 0$.

The equation will then take the form:

$$\frac{1}{c^2}\left[\rho\frac{\hbar^2}{4}u_\mu\partial_\mu(\rho u_\beta) + \rho\frac{\hbar}{2}z_\mu(\sigma_\mu u_\beta - \sigma_\beta u_\mu)\right] = s_{\beta\mu}\,g_\mu + \sigma_\gamma\,\partial_\mu\,\sigma_\beta - \sigma_\gamma\,\partial_\beta\,\sigma_\gamma\,,$$

or, since:

$$\sigma_\mu\,\partial_\beta\,\sigma_\mu = \frac{\hbar^2}{4}\rho\,\partial_\beta\,\rho\,,$$

one will have:

$$\boxed{-\sigma_\mu\partial_\mu\sigma_\beta + \rho\frac{\hbar^2}{4}\left[\frac{1}{c^2}u_\mu\partial_\mu(\rho u_\beta) + (\partial_\beta\rho)\right] = s_{\beta\mu}g_\mu - \frac{1}{c^2}\rho\frac{\hbar}{2}z_\mu(\sigma_\mu u_\beta - \sigma_\beta u_\mu).}$$

Finally, we contract equations (26) with:

$$\frac{i}{2c}\varepsilon_{\mu\nu\gamma\beta}u_\gamma\,,$$

which will give:

$$\frac{i}{c}\varepsilon_{\mu\nu\lambda\beta}(c)\partial_\mu A\sigma_\nu u_\gamma + \frac{i}{c}\varepsilon_{\mu\nu\lambda\beta}s_{\nu\lambda}\partial_\mu u_\lambda u_\gamma - \frac{1}{2}(2)\delta^{\lambda\alpha}_{\gamma\beta}\partial_\lambda\sigma_\alpha u_\gamma = 0.$$

The first term is:

$$c\,\partial_\mu\,A\,s_{\beta\mu}\,.$$

The second one will give:

$$\frac{1}{c^2}\delta^{\gamma\beta\mu}_{\lambda\alpha\rho}u_\alpha\sigma_\rho\partial_\mu u_\lambda u_\gamma = -\,\partial_\mu\, u_\mu\;\sigma_\beta + u_\mu\,\partial_\mu\;\sigma_\beta = \sigma_\beta\, u_\mu\,\partial_\mu \log\rho + \sigma_\mu\,\partial_\mu\, u_\beta,$$

when one uses the relation $\partial_\mu(\rho\, u_\mu) = 0$.

One therefore has:

$$c\;\partial_\mu A\; s_{\beta\mu} + \sigma_\beta\, u_\beta\,\partial_\mu \log\rho + \sigma_\mu\,\partial_\mu\, u_\beta - u_\gamma\partial_\gamma\,\sigma_\beta + u_\gamma\partial_\beta\,\sigma_\gamma = 0,$$

or finally:

$$\boxed{\sigma_\mu\partial_\mu u_\beta - u_\mu\partial_\mu\sigma_\beta = \frac{\hbar}{2}\rho z_\beta + c\partial_\mu A\, s_{\mu\beta} + \sigma_\beta u_\mu\partial_\mu \log\rho.}$$

It is appropriate to remark that all of this hydrodynamical formalism that we constructed from the Dirac wave function differs completely from the considerations of Chapter III concerning what we called the "classical Dirac particle." We were led to attribute an internal angular momentum that was not situated in proper space to the latter notion, in whose expression the angle *A* intervened essentially. Here, in the continuous fluid, the hydrodynamical representation for the regular wave (which, from the ideas of the causal interpretation, constitutes the extended part – or "wave-like aspect" – of the Dirac electron) is a Weyssenhoff fluid; i.e., its internal angular momentum is situated in proper space. Its expression does not involve the angle *A*, whose role in the present model seems to be concerned mainly with the distribution of energy, and the expression for the proper mass density μ_0 might possibly become negative, and therefore produce a momentum that is directed against the current (which Takabayasi referred to picturesquely as "ass-like behavior"). The considerations in Chapter III that were founded on the Dirac equation thus conveniently suggest an interesting viewpoint for studying a particular dynamic of the spinning particle. However, this dynamic does not seem to enter into the hydrodynamical representation of the Dirac equation in any manner.

In conclusion, we point out that the application of our general method to other wave equations has been carried out (in work that is unpublished, at least, to our knowledge) for the case of the Maxwell equation (spin 1 particle) by Phillippe Laruste at the Institut Henri Poincaré, and for the Duffin-Kemmer-Petiau equation (spin 0 particle) by Otsuka of Nagoya.

CHAPTER V

SPINNING FLUIDS WITH MOLECULAR STRUCTURE

§ 1. Generalities. – We shall now use the particles that defined the dynamics of Chapters I, II, and III in order to constitute fluids that we shall now characterize by a certain number of continuous magnitudes at each point that are obtained by the taking the mean of each particle magnitude over a large number of particles. That will mean that from now on we shall place ourselves at another scale, and that quantities r, GM, etc., will now relate to means that are taken over a large number of particles, each of which will relate to only the *global* properties (which we continue to denote by capital letters), and no longer, as in Chapter II, to the local properties of a "sub-fluid" that constitutes each particle.

In the present chapter, we shall confine ourselves to fluids that are composed of Weyssenhoff particles, so the center of mass will be identical the center of matter for each of them.

We first study the case of "pure matter" in the absence of fields in detail, and then we shall introduce suitable interactions between the particles that translate into the presence of well-defined internal stresses on the fluid.

We thus begin with particles that obey the three Weyssenhoff equations:

$$\begin{aligned} S_{\mu\nu}\, U_\nu &= 0, \\ \dot{G}_\mu &= 0, \\ \dot{S}_{\mu\nu} &= G_\mu\, U_\nu - G_\nu\, U_\mu\,, \end{aligned}$$

which are equations for which we have proposed an interpretation in Chapter II. We assume that the distribution of the particles is continuous; i.e., that all of their global properties vary only slightly at the scale of the distance between particle neighborhoods. Therefore, the mean value of any of these properties, when taken over a domain that gets smaller and smaller around a given point, will attain a well-defined value in practice, while the dimensions of the domain will be such that it will contain a large number of particles. It is these limiting values that we shall use for the local hydrodynamical properties for the fluid. In particular, the local unit-speed velocity at a point will be the mean unit-speed velocity around the point considered at the *center of matter* of each particle; i.e., the quadri-vector U_μ, which is a mean velocity that we will denote by u_μ in order to stipulate that we are dealing with a continuous hydrodynamical quantity.

We introduce a *matter density* ρ that expresses the number of particles per unit volume, which will be a number that is calculated in a very small volume and in the local *proper* system. It is thus the "invariant matter density" of classical relativistic hydrodynamics. In order to express the idea that the number of particles is conserved, we must subject this density to the conservation condition $\partial_\nu\,(\rho\, u_\nu) = 0$, which will be appropriate when one conventionally considers each particle to be "localized" to its center of matter.

We can then define the hydrodynamical properties of the fluid, such as the *momentum density* $g_\mu = \rho\, \bar{G}_\mu$ and the density of proper angular momentum $s_{\mu\nu} = \rho\, \bar{S}_{\mu\nu}$.

We can likewise introduce the derivatives of these quantities along a streamline, which will be derivatives that we denote by a dot, and which, since we will be concerned with densities, will have the expressions:

$$\dot{g}_\mu = \partial_\nu (u_\nu\, g_\mu) \qquad \text{and} \qquad \dot{s}_{\mu\nu} = \partial_\nu (u_\nu\, s_{\mu\nu}) \qquad \text{(Appendix } A\text{).}$$

Upon using the conservation relation $\dot{\rho} = \partial_\nu (u_\nu\, \rho) = 0$, the particle equations that were referred to above will yield the hydrodynamical equations:

$$\boxed{\begin{array}{ll} u_\mu u_\mu = -c^2, & s_{\mu\nu} u_\mu = 0, \\ \dot{g}_\mu = 0, & \dot{s}_{\mu\nu} = g_\mu u_\nu - g_\nu u_\mu. \end{array}}$$

These are identical to Weyssenhoff's axiomatic hydrodynamical equations, which should not be surprising, since as we have remarked before, one can always construct a Weyssenhoff fluid from Weyssenhoff particles or consider a Weyssenhoff particle to be a droplet of Weyssenhoff fluid, for that matter, since we are dealing with a "pure matter" fluid.

One therefore defines:

A density of proper mass of inertia:

$$-\mu_0\, c^2 = g_\nu\, u_\nu\,,$$

A density of proper mass of momentum:

$$-\, m_0^2 c^2 = g_\nu\, g_\nu\,.$$

One sees that the density of matter ρ can be eliminated from all of the formulas, and it might seem that its conservation relation is not involved with the formalism. In reality, one must take care that the derivative along the streamline, which we have denoted by a dot, should represent a different operation for which one has a density, such as μ_0, m_0, G_M, or $s_{\mu\nu}$ (and similarly, the transverse momentum density $p_\mu = \mu_0\, u_\mu - G_M$ and the spin density $\sigma_\mu = (i\, /\, 2c)\ \varepsilon_{\mu\nu\alpha\beta}\, u_\nu\, s_{\alpha\beta}$), or, on the contrary, a particle magnitude, such as the velocity u_μ. For example, one has:

$$\dot{g}_\mu = \partial_\nu (g_\mu\, u_\nu),$$

and, on the contrary:

$$\dot{u}_\mu = u_\nu\, \partial_\nu\, u_\mu\,.$$

This fact goes back to the matter density that is hidden in all of the density magnitudes and whose conservation relation will allow us to apply only the ordinary rules to this type of derivation. Therefore, the derivative of a product, such as $s_{\mu\nu}\, u_\nu$, will be:

$$\frac{d}{d\tau}(s_{\mu\nu}u_\nu) = \frac{d}{d\tau}(\rho S_{\mu\nu}u_\nu) = \partial_\lambda (u_\lambda \rho S_{\mu\nu} u_\nu) = u_\lambda \rho \partial_\lambda (S_{\mu\nu} u_\nu) + S_{\mu\nu} u_\nu \partial_\lambda (u_\lambda \rho).$$

The second term is zero due to the conservation law. One therefore has:

$$\frac{d}{d\tau}(s_{\mu\nu}u_\nu) = u_\lambda \rho (u_\nu \partial_\lambda S_{\mu\nu} + S_{\mu\nu} \partial_\lambda u_\nu) ,$$

and by virtue of the same law, one can put $u_\lambda \rho$ into the derivative:

$$\frac{d}{d\tau}(s_{\mu\nu}u_\nu) = u_\lambda \partial_\lambda (u_\lambda \rho S_{\mu\nu}) + u_\lambda \rho S_{\mu\nu} \partial_\lambda u_\nu .$$

One can then replace $\rho S_{\mu\nu}$ with $s_{\mu\nu}$ and the derivatives $\partial_\lambda (u_\lambda s_{\mu\nu})$ and $u_\lambda \partial_\lambda u_\nu$ with $\dot{s}_{\mu\nu}$ and $\dot{u}_\nu$, so that:

$$\frac{d}{d\tau}(s_{\mu\nu}u_\nu) = \dot{s}_{\mu\nu}u_\nu + s_{\mu\nu}\dot{u}_\nu .$$

Nonetheless, this rule applies here (and one must wary of this) only because the product includes just *one density*. Therefore, the relation $g_\mu G_\mu = - m_0^2c^2$, which involves the *square* of a density, will lead to:

$$g_\mu \dot{g}_\mu = - m_0\dot{m}_0c^2$$

precisely, as one will verify by replacing the dot with the complete operator, but the two sides of the second equation will be the derivatives of the two sides of the previous one.

By observing these preceding precautions, one will recover results that are completely formally identical to the ones that we obtained in the study of the Weyssenhoff particle. If one applies them to densities then these results will sometimes take on a different significance. Therefore, the integration of these equations will provide us with the law of motion for a vector:

$$R_\mu = \frac{1}{M_0^2c^2} S_{\mu\nu} G_\nu ,$$

which is a law that translates into a uniform, circular motion around a center of gravity on a circle of definite radius in the reference frame of inertia. In the present case, we shall similarly get a vector:

$$R_\mu = \frac{1}{m_0c^2} s_{\mu\nu} g_\nu ,$$

which rotates with a uniform motion in the local space of inertia, and upon expressing m_0, $s_{\mu\nu}$, and G_M in terms of the of the density ρ and the particle quantities, it will be easy to see that the radius of the circle will be the same as it is for a single particle. The paradoxical appearance of a finite length that is related to the dimensions of the particles in an apparently continuous hydrodynamical context has been pointed out before by Weyssenhoff, namely, the radius of a helicoidal motion that appears for a droplet and is independent of the dimensions of that droplet. This fact is related to the paradox of Costa de Beauregard that was recalled in our introduction, and which obliges us to constitute fluids that are endowed with an internal rotation density from a finite number of particles.

§ 2. The Lagrangian formulation of a pure matter fluid without spin. – In order to treat the pure matter fluid, and above all, to extend that treatment to fluids that are endowed with internal stresses, it will be useful to construct a Lagrangian formalism. Recall how one constructs this formalism in the case of the *classical* pure matter fluid, in order to extend the procedure to the fluid with spin.

The Lagrangian first includes the classical term of proper energy $\rho\,\mathfrak{M}_0\,c^2$. One adds terms to the Lagrangian that are destined to imply the two conditions:

$$u_\mu\,u_\mu = -\,c^2 \qquad \text{and} \qquad \partial_\mu\,(\rho\,u_\mu) = 0.$$

The Lagrangian will then take the form:

$$\boxed{\mathcal{L} = \rho M_0 c^2 + \rho u_\mu \partial_\mu S - \frac{\lambda}{2}(u_\mu u_\mu + c^2),}$$

in which S and λ are Lagrange multipliers. One will get the equations:

$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0, \qquad \text{which gives:} \quad u_\mu\,u_\mu = -\,c^2,$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial S_{,\mu}}\right) = 0, \qquad \text{which gives} \quad \partial_\mu\,(\rho\,u_\mu) = 0,$$

and

(V.1) $$\frac{\partial\mathcal{L}}{\partial\rho} = 0, \qquad \text{namely,} \qquad \mathfrak{M}_0\,c^2 + u_\mu\,\partial_\mu S = 0.$$

Therefore, it results that $\mathcal{L} = 0$, as well.

Finally:

$$\frac{\partial\mathcal{L}}{\partial u_\mu} = 0$$

gives:

(V.2) $$\rho\,\partial_\mu S - \lambda\,u_\mu = 0.$$

Upon contracting this with u_μ, one will get $\rho\,u_\mu\,\partial_\mu S + \lambda c^2 = 0$, so, from (1), one will have:

$$\lambda = -\frac{1}{c^2}\rho\, u_\mu\, \partial_\mu S \;=\; \frac{1}{c^2}\rho\, \mathfrak{M}_0\, c^2 = \rho\, \mathfrak{M}_0\,.$$

If we substitute this value into (2) then it will follow that:

$$\rho\, \partial_\mu S - \rho\, \mathfrak{M}_0\, u_\mu = 0, \qquad\qquad \partial_\mu S = \mathfrak{M}_0\, u_\mu\,.$$

That will make it apparent that S is the Hamilton-Jacobi function for a particle. One can then form the energy-momentum tensor:

$$t_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial S_{,\nu}}\,\partial_\mu S = \rho\, u_\mu\, \partial_\mu S = \rho\, \mathfrak{M}_0\, u_\mu\, u_\nu\,.$$

The conservation equation $\partial_\nu\, t_{\mu\nu} = 0$ yields the classical equation $\rho\, \mathfrak{M}_0\; \dot{u}_\mu = 0$.

It is easy to extend this formalism to the "perfect fluid" case if one assumes the existence of an "equation of state." In order to do that, we replace the expression $\rho\, \mathfrak{M}_0$ in the proper energy term of the Lagrangian (which is characteristic of the "pure matter" case since it employs only the mass of the material particles for its mass) with the more general quantity μ_0, which we use to denote the "total proper mass density," and we assume (this is the equation of state hypothesis) that μ_0 is determined completely by the conservative matter density ρ; i.e., that it depends upon the coordinates and time only by the intermediary of ρ. Now, apply the preceding method to the Lagrangian:

$$\mathcal{L} = \mu_0(\rho)\, c^2 + \rho\, u_\mu\, \partial_\mu S - \frac{\lambda}{2}(u_\mu\, u_\mu + c^2)\,.$$

The condition:

$$\frac{\partial \mathcal{L}}{\partial \rho} = 0 \qquad \text{will give} \qquad \mu_0'\, c^2 + u_\mu\, \partial_\mu S = 0,$$

if we denote the derivative of μ_0 with respect to ρ by μ_0'. Therefore:

$$u_\mu\, \partial_\mu S = -\,\mu_0'\, c^2.$$

The Lagrangian is no longer zero:

$$\mathcal{L} = (\mu_0 - \rho\mu_0')\, c^2,$$

so

(V.3) $$\frac{\partial \mathcal{L}}{\partial u_\mu} = 0 \qquad \text{gives} \qquad \rho\, \partial_\mu S - \lambda\, u_\mu = 0,$$

which, when contracted with u_μ, will yield $\rho\, u_\mu\, \partial_\mu S + \lambda\, c^2 = 0$, or furthermore, $\lambda = \rho\mu_0'$, which will give $\partial_\mu S = \mu_0'\, u_\mu$, when it is substituted into (3).

Finally, the expression for the energy-momentum tensor that is derived from the Lagrangian is:

$$t_{\mu\nu} = \rho\, u_\nu\, \partial_\mu S - \delta_{\mu\nu}\, \mathcal{L} = \rho\mu_0'\; u_\mu\, u_\nu - \delta_{\mu\nu}\,(\mu_0 - \rho\mu_0')\, c^2.$$

The momentum density g_μ is given by:

$$-c^2\, g_\mu = t_{\mu\nu}\, u_\nu = \rho\mu_0'\, u_\mu\, u_\nu - \mu_0\, c^2\, u_\mu + \rho\mu_0'\, c^2 u_\mu\,,$$

$$g_\mu = \mu_0\, u_\mu\,,$$

which is a relation that is characteristic of a *classical* fluid, and which shows that the variable quantity μ_0 has precisely the hydrodynamical significance of a proper mass density.

The internal stress tensor:

$$\theta_{\mu\nu} = \rho\mu_0'\, u_\mu\, u_\nu - \delta_{\mu\nu}\,(\mu_0 - \rho\mu_0')\, c^2 - \mu_0\, u_\mu\, u_\nu = c^2(\rho\mu_0' - \mu_0)\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right)$$
$$\equiv \eta_{\mu\nu}\, c^2(\rho\mu_0' - \mu_0)\,.$$

This is identically orthogonal to the current, and one will see that it has the form of a stress tensor for a *perfect* fluid whose pressure is expressed by:

$$\boxed{\pi = (\rho\mu_0' - \mu_0)\, c^2.}$$

In a perfect fluid that has an equation of state, one will see that there exists a relation between the pressure and the variation of the mass density as a function of the matter density, which is reasonable, since the variation of the mass density in the course of the motion depends essentially upon the pressure, which is itself related to the matter density by the equation of state. In fact, if one expresses that variation by starting with the conservation equations for $t_{\mu\nu}$ then one will get:

$$\left(\mu_0 + \frac{\pi}{c^2}\right)\dot{u}_\mu + \dot{\mu}_\mu u_\mu + \frac{\dot{\pi}}{c^2}u_\mu + \partial_\mu \pi = 0,$$

so that, upon contracting with u_μ:

$$-\dot{\mu}_0 c^2 = \dot{\pi} - u_\mu \partial_\mu \pi = \partial_\mu\,(u_\mu\, \pi) - u_\mu\, \partial_\mu\, \pi = \pi\, \partial_\mu\, u_\mu\,.$$

If one expresses the idea that μ_0 is a function of only ρ then one will have:

$$\dot{\mu}_0 = \partial_\mu\,(u_\mu\, \mu_0) = u_\mu\, \partial_\mu\, \mu_0 + \mu_0\, \partial_\mu\, u_\mu = u_\mu \mu_0' \partial_\mu \rho + \mu_0 \partial_\mu u_\mu\,,$$

or, upon taking into account that $\partial_\mu\,(u_\mu\, \rho) = 0$:

$$\dot{\mu}_0 = -\rho\mu_0'\, \partial_\mu u_\mu + \mu_0 \partial_\mu u_\mu\,.$$

Therefore, the equation of mass variation will become:

$$\rho\mu_0' c^2\,\partial_\mu u_\mu - \mu_0 c^2 \partial_\mu u_\mu = \pi\,\partial_\mu\, u_\mu\,,$$

or, upon dividing by the scalar $\partial_\mu\, u_\mu$:

$$(\rho\mu_0' - \mu_0)\,c^2 = \pi,$$

which is precisely the equation that we found.

We can introduce Lichnerowicz's pseudo-mass density (see Appendix *B*) by setting:

$$t_{\mu\nu} = \mu\, u_\mu\, u_\nu + \delta_{\mu\nu}\nu\,\pi,$$

so

$$\mu = \mu_0 + \frac{\pi}{c^2} = \rho\mu_0'\,.$$

The internal force field is then:

$$K_\alpha = -\,\frac{\partial_\mu \pi}{\mu} = -\,\frac{c^2\partial_\mu(\rho\mu_0' - \mu_0)}{\rho\mu'} = c^2\left[\frac{\partial_\alpha \mu_0}{\rho\mu_0'} - \partial_\alpha \log(\rho\mu')\right],$$

and since one can write $\partial_\mu\, \mu_0 = \mu_0'\,\partial_\alpha\rho$:

$$K_\alpha = -\,c^2\,\partial_\alpha\,[\log\,(\rho\mu_0') - \log\rho] = -\,c^2\,\partial_\alpha \log\mu_0'\,.$$

Finally, the index of the fluid is given by:

$$K_\alpha = -\,c^2\,\partial_\alpha \log F,$$

so one has simply:

$$F = C\mu_0'\,.$$

One can use the "ideal gas" as an example of this type of fluid, which is characterized by an equation of state:

$$\pi = K\,c^2\mu_0 \qquad \text{(Mariotte's law)}$$

$$K\mu_0 = \rho\mu_0' - \mu_0\,,$$

and which will give:

$$\frac{d\,\mu_0}{\mu_0} = (K+1)\,\frac{d\,\rho}{\rho}\,, \qquad \mu_0 = C\,\rho^{K+1}.$$

One sees that if the coupling constant *K* tends to zero then one will arrive at the case of pure matter with $\mu_0 = \rho\,\mathfrak{M}_0$.

The integration constant C can thus be taken to be equal to the particle mass $\mathfrak{M}_0$, but things will be different in the case where $K \neq 0$.

The pseudo-mass density will then be:

$$\mu = \rho\mu'_0 = (K + 1)\,\mu_0\,.$$

The energy-momentum tensor will be:

$$t_{\mu\nu} = (K + 1)\,C\,u_\mu\,u_\nu + \delta_{\mu\nu}\,K\,c^2\mu_0\,.$$

One easily derives the differential equation of the streamlines from this:

$$\dot{u}_\mu = -\,K\,c^2\left(\delta_{\alpha\mu} + \frac{u_\alpha u_\mu}{c^2}\right)\partial_\alpha \log\rho \equiv -\,\eta_{\alpha\mu}\,K\,c^2\,\partial_\alpha \log\rho\,.$$

This will bring only the gradient of the *matter* density into play.

§ 3. The Lagrangian formulation of the pure matter fluid with spin. – We shall extend these considerations to the case of a fluid that is endowed with an internal angular momentum density following some papers of Takabayasi (*) and Vigier and Unal [**52**]. Recall that Frenkel endeavored to give a Lagrangian formulation for the dynamics of a spinning particle before, and that in the absence of variables that are adapted to the representation of rotations his Lagrangian was not an exact integral (Chapter I). The variables that are appropriate here are Einstein and Kramer's "tetrapodes" (Ger. *Vierbeine*) that were also used recently in order to represent the Dirac field [**49**, **53**, **54**]. One attaches a system of four orthonormal vectors a_μ^1, a_μ^2, a_μ^3, a_μ^4 to each point of the fluid. One assumes that each of these vectors can be expressed in terms of the other three by means of the formula:

(V.4) $$a_\mu^\xi = \varepsilon^{\xi\eta\zeta\iota}\,\varepsilon_{\mu\nu\alpha\beta}\,a_\nu^\eta\,a_\alpha^\zeta\,a_\beta^\iota\,.$$

The upper Greek letters indicate the numbers of the vectors here. They vary from 1 to 4, and the formula above orients the vierbein. One therefore has:

$$a_\mu^1 = \varepsilon_{\mu\nu\alpha\beta}\,a_\nu^2\,a_\alpha^3\,a_\beta^4\,,$$

but:

$$a_\mu^4 = -\,\varepsilon_{\mu\nu\beta\alpha}\,a_\nu^1\,a_\alpha^2\,a_\beta^3\,.$$

Formula (4) permits us to establish some relations between the various bivectors of the vierbein, so:

(V.5) $$\varepsilon_{\alpha\beta\mu\nu}\,a_\mu^3\,a_\nu^4 = a_\alpha^1\,a_\beta^2 - a_\alpha^2\,a_\beta^1\,.$$

(*) Manuscript communicated by the author.

One sees that if one contracts (4) with one of the four vectors, such as a_μ^κ, then one will find that the norm $a_\mu^\xi a_\mu^\xi = 1 = \det \| a_\mu^\xi \|$ if $\kappa = \xi$ and zero if $\kappa \neq \xi$, since a_μ^ξ and a_μ^κ are orthogonal, which the formula shows due to the antisymmetry of $\varepsilon_{\mu\nu\alpha\beta}$.

One then has:

(V.6) $$a_\mu^\xi a_\mu^\eta = \delta^{\xi\eta}.$$

Formula (4), when multiplied by a_λ^ξ and summed over ξ, likewise shows that:

(V.7) $$a_\mu^\xi a_\lambda^\xi = \delta_{\mu\lambda},$$

which is a relation that was less obvious.

It then suits us to choose $a_\mu^4 = (1 / ic)\, u_\mu$ to be collinear with the current and $a_\mu^2 = (1 / \rho h_0)\ \sigma_\mu$ to be collinear with the spin. (σ_μ denotes the spin density here, while h_0 represents the norm of the *particle's* spin.) This hypothesis is always possible since the spin is essentially orthogonal to the current.

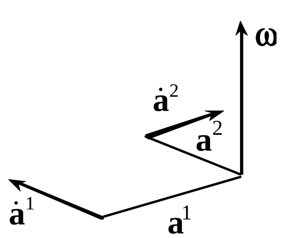
As for the other two vectors, in modern research, their choice is related to the wave functions by means of spinors [**53**]; we shall not specify them. However, any pair of vectors that are connected with the particle in proper space and are orthogonal to spin might seem suitable. The choice of a_μ^3 and a_μ^4 permits us to express the angular momentum density (see Chapter III) by:

$$s_{\mu\nu} = -\rho\, h_0\, \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^4 a_\beta^3 .$$

From (5), one also has immediately:

$$s_{\mu\nu} = -\rho\, h_0\, (a_\mu^1 a_\nu^2 - a_\nu^1 a_\mu^2) .$$

The motion of these vectors permits to characterize the rotation at each point. First, consider a particle in its proper system. If the spin is fixed (we know that this is the case for the free Weyssenhoff particle) then the rotation of the vectors a_μ^3 and a_μ^4 will characterize the "proper rotation." The velocities are $\dot a_k^1$ and $\dot a_k^2$ equal, orthogonal, and situated in the plane $a_k^1 a_k^2$. One knows that the spatial angular velocity vector will then be collinear with the spin, and since $\mathbf{a}^1$ and $\mathbf{a}^2$ have unit speed, it will be given by the vector product $\dot{\mathbf{a}}^1 \times \mathbf{a}^1$, or by $\dot{\mathbf{a}}^2 \times \mathbf{a}^2$, for that matter. Therefore:



$$\omega_i = \varepsilon_{ijk}\ \dot a_j^1\ a_k^1 = \varepsilon_{ijk}\ \dot a_j^2\ a_k^2 = \tfrac{1}{2}\varepsilon_{ijk}(\dot a_j^1\, a_k^1 + \dot a_j^2 a_k^2) .$$

If the spin is not fixed then the velocity of the extremity of $\mathbf{a}^3$ will describe the precession of the axis of proper rotation. The two preceding terms will cease to be coplanar and equal, and a term $\dot{\mathbf{a}}^3 \times \mathbf{a}^3$ and must be added:

$$\omega_i = \tfrac{1}{2}\varepsilon_{ijk}(\dot{a}_j^1\, a_k^1 + \dot{a}_j^2\, a_k^2 + \dot{a}_j^3\, a_k^3)\,.$$

One can finally add a similar term $\dot{a}_j^4\, a_k^4$, where the vector $\dot{a}_j^4$ is zero in the proper system. One will then have:

$$(\omega_i)^0 = \tfrac{1}{2}\varepsilon_{ijk}(\dot{a}_j^\xi\, a_k^\xi)^0 \qquad \text{(summation over } \xi\text{)}.$$

One knows that the angular velocity vector is an axial vector that is dual to an antisymmetric tensor that appears precisely in the formula above. From the tensorial viewpoint, it is the latter itself that must be used. One then sets:

$$(\omega_{kj})^0 = \tfrac{1}{2}(\dot{a}_j^\xi\, a_k^\xi - a_j^\xi\, \dot{a}_k^\xi)^0\,.$$

This expression can be made covariant from the relativistic viewpoint by introducing the corresponding components in the proper system:

$$(\omega_{j4})^0 = \tfrac{1}{2}(\dot{a}_j^\xi\, a_4^\xi - a_j^\xi\, \dot{a}_4^\xi)^0\,.$$

Since, from (7), one has:

$$a_j^\xi\, \dot{a}_4^\xi = 0.$$

One can replace:

$$a_j^\xi\, \dot{a}_4^\xi \text{ with } -\, a_j^\xi\, \dot{a}_4^\xi\,,$$

so:

$$(\omega_{j4})^0 = (a_j^\xi\, \dot{a}_4^\xi)^0\,.$$

In this form, the significance of the time components in the proper system will appear immediately. Indeed, the components a_4^ξ will be zero for $\xi = 1, 2, 3$. What will then remain is the term:

$$(\dot{a}_j^4\, a_4^4)^0 = (\dot{a}_j^4)^0 = \frac{1}{ic}\dot{u}_j^0 = \frac{1}{ic}\left(\frac{dv_j}{dt}\right)^0.$$

Therefore, the spatial components of the tensor $\omega_{\mu\nu}$ in the proper system represent the usual angular velocity tensor, and the temporal components represent the linear acceleration of the particle. One then proposes the covariant formulation:

$$\boxed{\omega_{[\mu\nu]} = \tfrac{1}{2}(\dot{a}_\mu^\xi\, a_\nu^\xi - a_\mu^\xi\, \dot{a}_\nu^\xi).}$$

In order to form the Lagrangian that relates to this generalized rotation, one is inspired by the classical formula that gives the non-relativistic energy of a body that rotates in space, namely:

$$T = \tfrac{1}{2} I_{[ij]}\ \omega_i\ \omega_j\ .$$

$I_{[ij]}$ is the "inertia tensor" that generalizes the elementary notion of the moment of inertia. From the relativistic viewpoint, since the angular velocity is a tensor, one must introduce a fourth-order inertia tensor and write:

$$T = \tfrac{1}{2} I_{[\alpha\beta][\mu\nu]}\ \omega_{[\alpha\beta]}\ \omega_{[\mu\nu]}\ .$$

Just as one defines the non-relativistic kinetic moment by $I_{[ij]}\ \omega_j$, one can identify the internal angular momentum tensor $I_{[\alpha\beta][\mu\nu]}\ \omega_{[\alpha\beta]}$, and we write the energy:

$$T = \tfrac{1}{2} s_{\mu\nu}\ \omega_{\mu\nu}\ .$$

The Lagrangian of proper rotation will then be:

$$-\tfrac{1}{2}\rho\, h_0\, \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^4\, a_\beta^3 \cdot \tfrac{1}{2}(\dot{a}_\mu^\xi\, a_\nu^\xi - a_\mu^\xi\, \dot{a}_\nu^\xi)\ ,$$

or, upon taking (7) into account:

$$\tfrac{1}{2}\rho\, h_0\, \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^3\, a_\beta^4 \cdot (\dot{a}_\mu^\xi\, a_\nu^\xi)\ .$$

We remark that this expression will be annulled by antisymmetry for $\xi = 3, 4$. What will then remain is:

$$\tfrac{1}{2}\rho\, h_0\, \varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^3\, a_\beta^4 (\dot{a}_\mu^1\, a_\nu^1 + \dot{a}_\mu^2\, a_\nu^2)\ .$$

However, we know that $\varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^3\, a_\beta^4\, a_\nu^1 = -\ a_\mu^2$ and that $\varepsilon_{\mu\nu\alpha\beta}\, a_\alpha^3\, a_\beta^4\, a_\nu^2 = +\ a_\mu^1$.

The Lagrangian must therefore be:

$$\tfrac{1}{2}\rho\, h_0\, (\dot{a}_\mu^1\, a_\mu^2 - a_\mu^2\, \dot{a}_\mu^1) = \rho\, h_0\ ic\ a_\lambda^4\, a_\mu^1\, \partial_\lambda a_\mu^2 .$$

In order to form the Lagrangian, as in the classical case, we introduce the proper energy, which will be $\rho\, \mathfrak{M}_0\, c^2$ here, plus two Lagrange multiplier terms. The first one – namely, $-\ ic\ \rho\ a_\mu^4\, \partial_\mu S$ – assures the conservation relation:

$$\dot{\rho}\ = \partial_\mu (\rho\, u_\mu) = 0,$$

while the second one – namely, $\lambda_{\mu\nu}(a_\mu^\xi a_\nu^\xi - \delta_{\mu\nu})$ – replaces and generalizes the classical term $\lambda\ (u_\mu\ u_\mu + c^2)$ and provides the four vectors of the vierbein with the conditions of normality and orthogonality that are expressed by relations (7). The quantities $\partial_\mu S$ and

$\lambda_{\mu\nu}$, which is obviously *symmetric* in μ and ν, are the Lagrange multipliers. One then has:

$$\boxed{L = \rho\mathfrak{M}_0 c^2 + ic\,\rho\, a_\mu^4 \partial_\mu S + ic\,\rho h_0\, a_\lambda^4\, a_\mu^1\, \partial_\lambda a_\mu^2 + \lambda_{\mu\nu}(a_\mu^\xi a_\nu^\xi - \delta_{\mu\nu}).}$$

We assume that the proper mass of the particle $\mathfrak{M}_0$ is constant.

We begin the search for the Belinfante tensor $f_{[\mu\nu]\lambda}$. It involves only the third term, which is the only one that contains the gradient $\partial_\mu a_\lambda^2$, and is thus capable of changing as a result of a Lorentz transformation. One has:

$$\frac{\partial\mathcal{L}}{\partial a_{\alpha,\lambda}^2} = ic\,\rho\, h_0\;a_\lambda^4\, a_\lambda^4 .$$

Since we are concerned with a vector, the operator of infinitesimal rotation will be:

$$\mathfrak{L}_{\mu\nu}^{\alpha\beta} = \tfrac{1}{2}\delta_{\alpha\beta}^{\mu\nu} = \tfrac{1}{2}\,(\delta_{\alpha\mu}\,\delta_{\beta\nu} - \delta_{\alpha\nu}\,\delta_{\beta\mu}),$$

and one will therefore have:

$$\begin{aligned} f_{[\mu\nu]\lambda} &= ic\,\rho\, h_0\;a_\lambda^4 a_\alpha^1 \cdot \tfrac{1}{2}\delta_{\alpha\beta}^{\mu\nu} a_\beta^2 \\ &= \tfrac{1}{2} ic\,\rho\, h_0\;(a_\mu^1 a_\nu^2 - a_\nu^2 a_\mu^1)\,. \end{aligned}$$

One knows that the proper angular momentum is derived from $f_{\mu\nu\lambda}$ by way of:

$$f_{\mu\nu\lambda}\, u_\lambda = -\tfrac{1}{2} c^2\, s_{\mu\nu}\,.$$

That will give $s_{\mu\nu} = \rho\, h_0\;(a_\mu^1 a_\nu^2 - a_\nu^2 a_\mu^1)$.

This is precisely what we proposed to begin with. We then see that the tensor $f_{\mu\nu\lambda}$ can be reduced to the term $s_{\mu\nu}\, u_\lambda$. We will not have to perform a gauge transformation.

Now, we shall describe the Lagrangian with respect to the various variables. In addition to the two Lagrange conditions:

$$\dot\rho = 0 \qquad\qquad \text{and} \qquad a_\mu^\xi a_\nu^\xi = \delta_{\mu\nu}\,,$$

we will obtain, upon differentiating with respect to ρ:

(V.8) $$\mathfrak{M}_0\, c^2 + ic\;a_\mu^4 \partial_\mu S + ic\, h_0\, a_\mu^4 a_\lambda^1 \partial_\mu a_\lambda^2 = 0,$$

so

$$\dot S = -\,\mathfrak{M}_0\, c^2 - h_0\, a_\lambda^1 \dot a_\lambda^2\,.$$

It then results from this relation that the Lagrangian is zero $\mathcal{L} = 0$. Upon differentiation with respect to a_μ^1:

(V.9) $$\rho h_0\,\dot{a}_\mu^2 + 2\lambda_{\alpha\beta}a_\alpha^1 = 0.$$

Upon differentiating with respect to a_μ^2:

$$\partial_\lambda (ic\,\rho h_0\, a_\lambda^4 a_\mu^1) = 2\lambda_{\mu\alpha}a_\alpha^2,$$

so

(V.10) $$-\rho\, a_\mu^1 \dot{h}_0 - \rho h_0\, \dot{a}_\mu^1 + 2\lambda_{\mu\alpha}a_\alpha^2 = 0.$$

Upon differentiating with respect to a_μ^3:

(V.11) $$2\lambda_{\mu\alpha}a_\alpha^3 = 0.$$

Finally, upon differentiation with respect to a_μ^4:

(V.12) $$ic\,\rho\,\partial_\mu S + ic\,\rho\, h_0\, a_\lambda^1 \partial_\mu a_\lambda^2 + \lambda_{\alpha\beta}a_\alpha^4 = 0.$$

In order to use the orthogonality relation, multiply (9), (10), (11), and (12) by a_ν^1, a_ν^2, a_ν^3, and a_ν^4, respectively, and add them:

$$-\rho h_0\, a_\nu^2 a_\mu^1 + \rho h_0\,(a_\nu^1 \dot{a}_\mu^2 - a_\nu^2 \dot{a}_\mu^1) + ic\rho\,(a_\nu^4 \partial_\mu S + h_0 a_\nu^4 a_\lambda^1 \partial_\mu a_\nu^3) + 2\lambda_{\mu\alpha}\delta_{\nu\alpha} = 0.$$

If we exchange μ and ν and subtract the equation that is thus obtained from the preceding one then the last term will disappear by reason of the symmetry of $\lambda_{\mu\nu}$, and what will remain is:

(V.13) $$\begin{aligned}&\rho\dot{h}_0 (a_\nu^1 a_\mu^2 - a_\mu^1 a_\nu^2) + \rho h_0 (a_\nu^1 \dot{a}_\mu^2 - a_\mu^1 \dot{a}_\nu^2 - a_\nu^2 \dot{a}_\mu^1 + a_\mu^2 \dot{a}_\nu^1) \\ &+ ic\rho(a_\nu^4\,\partial_\mu S - a_\mu^4\,\partial_\nu S) + ic\,\rho h_0\, a_\lambda^4 (a_\nu^4\,\partial_\mu a_\lambda^2 - a_\mu^4\,\partial_\nu a_\lambda^2) = 0.\end{aligned}$$

The first two terms are simply the derivative of the internal angular momentum $\dot{s}_{\nu\mu}$. If we contract this relation with a_μ^4 then we will get:

$$\dot{s}_{\nu\mu}\dot{a}_\mu^4 + ic\,\rho\,(a_\nu^4 a_\mu^4 \partial_\mu S - \partial_\nu S) + ic\,\rho h_0\, a_\lambda^1 (a_\nu^1 a_\mu^1 \partial_\mu a_\lambda^2 - \partial_\nu a_\lambda^2) = 0,$$

so, upon taking (8) into account, we will get:

$$-\dot{s}_{\nu\mu}\dot{a}_\mu^4 - \rho\,(a_\nu^4\,\mathfrak{M}_0 c^2 + a_\nu^4\, h_0\, a_\lambda^1 \dot{a}_\lambda^2 + ic\,\partial_\nu S) + \rho h_0\, a_\lambda^1 (a_\nu^4 \dot{a}_\lambda^2 - ic\,\partial_\nu a_\lambda^2) = 0,$$

which will provide us with the expression for $\partial_\nu S$:

(V.14) $$ic\,\rho\,\partial_\nu S = -\rho\,a_\nu^4\,\mathfrak{M}_0 c^2 - ic\,\rho h_0\,a_\lambda^1\,\partial_\nu a_\lambda^2 - s_{\nu\lambda}\dot{a}_\lambda^4 .$$

We can now form the energy-momentum. It is composed of two parts:

1) $$\frac{\partial\mathcal{L}}{\partial S_{,\nu}}\partial_\nu S = ic\,\rho\;a_\nu^4\,\partial_\mu S\;,$$

so that, upon taking (14) into account, we will get:

$$\frac{\partial\mathcal{L}}{\partial S_{,\nu}}\partial_\nu S = -\,\rho\,\mathfrak{M}_0\,c^2\;a_\nu^4\,a_\mu^4 - ic\,\rho\,h_0\,a_\lambda^1\,a_\nu^4\,\partial_\mu a_\lambda^4 - s_{\mu\lambda}\dot{a}_\lambda^4\,a_\nu^4 ,$$

$$= \rho\,\mathfrak{M}_0\,u_\mu\,u_\mu - ic\,\rho\,h_0\;a_\lambda^1 a_\nu^4 a_\lambda^2 + \frac{1}{c^2}s_{\mu\nu}\dot{u}_\mu u_\nu .$$

2) $$\frac{\partial\mathcal{L}}{\partial a_{\lambda,\nu}^2}\partial_\mu a_\lambda^2 = ic\,\rho\,h_0\;a_\nu^4 a_\lambda^1\,\partial_\mu a_\lambda^2 ,$$

which annuls the second term of the first part:

$$t_{\mu\nu} = \rho\,\mathfrak{M}_0\,u_\mu\,u_\nu + \frac{1}{c^2}s_{\mu\lambda}\dot{u}_\lambda u_\nu .$$

One sees that $t_{\mu\nu}$ includes u_ν as a factor, which is characteristic of a *pure matter* field. The momentum g_μ is given by:

$$g_\mu = \rho\,\mathfrak{M}_0\,u_\mu + \frac{1}{c^2}s_{\mu\lambda}\dot{u}_\lambda .$$

This has precisely the form that we sought in the Weyssenhoff case, with a classical term $\rho\,\mathfrak{M}_0\,u_\mu$ that is collinear with the current and a transverse term:

$$-\,p_\mu = \frac{1}{c^2}s_{\mu\lambda}\dot{u}_\lambda .$$

The conservation for the tensor $t_{\mu\nu}$ gives immediately:

$$\dot{g}_\mu = 0.$$

The conservation relation for the total moment of rotation $x_\mu\,t_{\nu\lambda} - x_\nu\,t_{\mu\lambda} + f_{\mu\nu\lambda}$ gives us the second equation:

$$\dot{s}_{\mu\nu} = g_\mu\,u_\nu - g_\nu\,u_\mu = p_\mu\,u_\nu - p_\nu\,u_\mu .$$

Moreover, this can be obtained directly by substituting the value for $\partial_\mu S$ into (14).

§ 4. The Bohm-Vigier droplet in an external force field. – Since our intention is to generalize the hydrodynamics of Weyssenhoff fluids by introducing forces of interaction between the particles, we shall begin with the preceding Lagrangian and generalize it, on the one hand, by assuming that the proper mass $\mathfrak{M}_0$ and the norm of the spin h_0 for each particle can be variable (which is a generalization of the method that gave us the classical perfect fluid), and on the other hand, by adding supplementary energy terms. However, in order to prepare ourselves for the interpretation of the results that we thus obtain, we must first extend the general dynamics of free, spinning particles that we elaborated upon in our first chapter to the case of particles that are subjected to external forces, and thus connect with the hydrodynamics of fluids with internal stresses that we encountered before in the case of fluids that represent quantum wave functions. We must then re-evaluate the considerations of Chapter II for the model of the Bohm-Vigier droplet.

The theory of a drop in an external force field is much less satisfying than that of a free drop. We hope to arrive at a formulation that is analogous to that of the classical relativistic dynamics of point-like matter; i.e., to first define (and in a covariant fashion) the global dynamical quantities that characterize the drop, then to express that laws that determine the evolution of these quantities under the action of external forces. Now, on the one hand, the reasons by which we proved the covariant character of the momentum G_μ and the internal angular momentum $S_{\mu\nu}$ fall short here, because the energy-momentum tensor and its moment are no longer conservative. On the other hand, as Møller showed [**3**], it is impossible to separate the dynamical characteristics of the system from those of the external field completely. They will necessarily be included in the definition of the latter. As a result of the external force to which the drop is subjected, the *definitions* of the moment and angular momentum of a given drop will not be the same.

We introduce our definitions while being careful that we must revert to the same results as in the preceding study in the absence of forces.

We further assume that the fluid is classical; i.e., that its energy-momentum tensor is symmetric and satisfies the relation:

$$t_{\mu\nu}\, u_\nu = k\, u_\mu \,.$$

The local hydrodynamical equations are:

$$\partial_\mu\, t_{\mu\nu} = f_\mu \,, \tag{V.15}$$

$$\partial_\lambda\, m_{\mu\nu\lambda} = x_\mu\, f_\nu - x_\nu\, f_\mu \,, \tag{V.16}$$

in which f_μ is the external force density. It is obvious that the second equation is a consequence of the first one and the symmetry of $t_{\mu\nu}$. As in the preceding analysis, we shall consider an intrinsic center C for the drop that is animated with a unit-speed velocity U_μ, but we shall save the question of how it can be defined for a later examination. As it is impossible to derive covariant volume integrals from $t_{\mu\nu\lambda}$ and $m_{\mu\nu\lambda}$, we can place ourselves in a particular reference frame – namely, the *proper reference frame* $\Pi_0\, \Lambda_0$ of the point C. We define the momentum and angular momentum by the same expressions that we used for the free drop, but we will refer them to the proper reference frame explicitly:

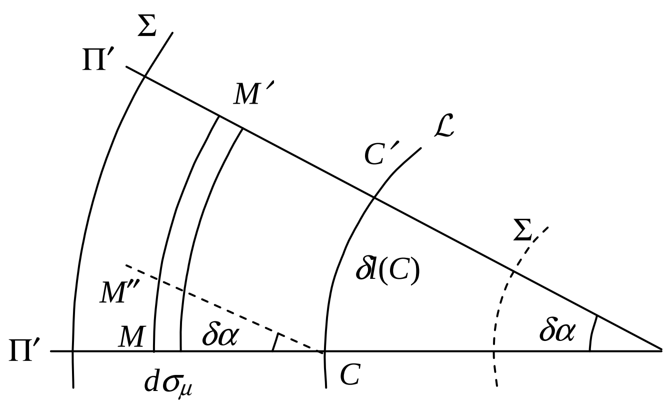
$$G_\mu^0 = \int_{\Sigma_0} t_{\mu\otimes}^0 \, d\sigma_4^0 \equiv -\frac{1}{c^2}\int_{\Sigma_0} t_{\mu 4}^0 \, d\sigma_4^0 = -\frac{1}{c^2}\int_{\Sigma_0} t_{\mu\nu}^0 \, d\sigma_\nu^0 \,,$$

$$M_{\mu\nu}^0 = \int_{\Sigma_0} m_{\mu\nu\otimes}^0 \, d\upsilon^0 \equiv -\frac{1}{c^2}\int_{\Sigma_0} m_{\mu\nu 4}^0 \, d\sigma_4^0 = -\frac{1}{c^2}\int_{\Sigma_0} m_{\mu\nu\lambda}^0 \, d\sigma_\lambda^0 \,.$$

In this form, the various elements will be tensorial, and one can go to arbitrary axes, but this time with the condition that we must assume that the domain of integration remains the section Σ_0 of the tube by the hyperplane Π_0 of *proper* space. Under these conditions, one can write:

$$\boxed{\begin{aligned} G_\mu &= -\frac{1}{c^2}\int_{\Sigma_0} t_{\mu\nu} d\sigma_\nu, \\ M_{\mu\nu} &= -\frac{1}{c^2}\int_{\Sigma_0} m_{\mu\nu\lambda} d\sigma_\lambda, \end{aligned}}$$

which are expressions that coincide with the ones for the free drop only in the proper reference frame. Moreover, that is the only reference in which they will have a clear significance, because it is the only one in which they will take the form of volume integrals.



In order to write the global dynamical equations, one integrates the two local equations (15) and (16) over a domain $\delta\Omega$ that is bounded by the hyper-boundary Σ of the tube and two proper spaces hyperplanes Π and Π' that correspond to two positions C and C' of the center along its world-line $\mathcal{L}$ and are separated by an infinitesimal interval of proper time $\delta\tau$. Upon cutting way the portion of the hyperplane Π that is occupied by matter in elementary domains, such as $d\sigma_\mu$, whose center is at M, it will be obvious that the domain $\delta\Omega$ is composed of elementary hyper-tubes of area $d\sigma_\mu$ and length $\delta l(M)$.

If the motion of C is uniform and rectilinear then the hyperplanes Π and Π' will be parallel, and δl will be the same for all points M. However, in general, the world-line of C will be curved, so its unit-speed velocity U_μ will not be constant, and the hyperplanes Π and Π' will have a certain "inclination," which is represented in the figure above by the angle $\delta\alpha$. At the same time, it is obvious that at a point such as M, one will have to consider a length such as $MM' = MM'' + M''M'$, where $M''M' = \delta l(M)$ and $M''M' = CM$ $\delta\alpha$, when $\delta\alpha$ is small.

If one considers the problem in two dimensions, as illustrated in the figure (where the point C is assumed to describe the curve $\mathcal{L}$ with a velocity of constant norm v), then one will have:

$$\delta l_{(M)} = \delta l_{(C)} + \delta\alpha, \qquad \text{where} \quad \delta l_{(C)} = v\ \delta t,$$

and in which $\delta\alpha$ is given by the classical formula of elementary kinematics:

$$da = \frac{\gamma}{v}\,\delta t, \qquad \text{so} \qquad \delta l_{(M)} = \left(v + \overline{CM}\,\frac{\gamma}{v}\right)\delta t.$$

Moreover, since the acceleration $\boldsymbol{\gamma}$ is collinear with the vector **CM**, and in the opposite sense (as is the case in the figure), it will be obvious that one can write $\delta l = \left(v + \frac{\mathbf{CM}\cdot\boldsymbol{\gamma}}{v}\right)\delta t$, and that for very small δt, this formula will be identical with the vectorial formula $\boldsymbol{\delta l} = \mathbf{v}_{(C)}\,dt\left(1 - \frac{\mathbf{CM}\cdot\boldsymbol{\gamma}}{v_{(C)}^2}\right)$.

In that form, the formula will extend without modification to the case of space; i.e., to the points M that are not in the plane through C that is perpendicular to the dihedral $\Pi\,\Pi'$ (i.e., the plane of the figure), and likewise to case of the relativistic space-time. In the latter case, the constant norm of the velocity of C is ic, and one will have:

$$\delta l_\nu = U_\nu\,\delta t\left[1 - \frac{(x_\mu - Y_\mu)\dot{U}_\mu}{-c^2}\right] = U_\nu\,\delta t\left(1 + \frac{x_\mu - Y_\mu}{c^2}\dot{U}_\mu\right),$$

if one lets Y_μ denote the coordinates of the point C and assumes that the points C and M are simultaneous in the proper system $x_4^0 = Y_4^0$ or $(x_\mu - Y_\mu)\,U_\mu = 0$.

In order to simplify, we set $1 + \frac{x_\mu - Y_\mu}{c^2}\dot{U}_\mu = \gamma(x)$, and remark that one can also write:

$$\gamma = 1 + \frac{x_\mu - Y_\mu}{c^2}\dot{U}_\mu = 1 - \frac{\dot{x}_\mu - \dot{Y}_\mu}{c^2}U_\mu = -\frac{\dot{x}_\mu U_\mu}{c^2},$$

since:

$$\dot{Y}_\mu = U_\mu .$$

Finally, upon evaluating γ in the proper system, one will get:

$$\gamma = -\frac{ic}{c^2}\dot{x}_4^0 = \dot{x}_\otimes^0 .$$

Therefore, if one considers the same interval $\delta\tau$ of *proper time for the point C* for all points of the drop then one will have a space-time element:

$$d\omega = \gamma(x)\, U_\nu\, d\sigma_\nu\, \delta\tau$$

for each point, which is a differential scalar that one can express in the proper system, where one has to consider only the components:

$$U_4^0 = ic, \qquad d\sigma_4^0 = ic\, d\upsilon_0\,.$$

One finally has:

$$\boxed{d\omega = -c^2\gamma(x)\, d\upsilon_0\, \delta\tau = -c^2\left[1 + \frac{x_\mu - Y_\mu}{c^2}\dot{U}_\mu\right] d\upsilon_0\, \delta\tau.}$$

One can see by means of this expression, which relates each of the space-time elements at the various points of the drop to the other ones, that $\delta\tau$ will appear as a factor in any integral that is taken over a space-time domain $\delta\Omega$, and one will no longer have to take a hyper-partition integral – i.e., that one will find oneself reverting to a proper volume element.

Integrate the first hydrodynamical equation $\partial_\nu t_{\mu\nu} = f_\nu$ over the domain $\delta\Omega$ according to this principle.

For the left-hand side, one has:

$$\int_{\delta\Omega} \partial_\nu t_{\mu\nu} d\omega = \int_{\Pi_0'} t_{\mu\nu} d\sigma_\nu - \int_{\Pi_0} t_{\mu\nu} d\sigma_\nu = \frac{d}{d\tau}\left(\int_{\Sigma_0} t_{\mu\nu} d\sigma_\nu\right) \delta\tau = -c^2\, \dot{G}_\mu\, \delta\tau\,.$$

We remark that we have assumed that the hyper-boundary terms is zero. We know that this amounts to postulating the existence of appropriate surface tensions on the surface of the drop. It is, moreover, obvious that these tensions will depend upon the external force, which one must account for in the equation of equilibrium, of a surface element.

The right-hand side of the equation gives:

$$\int_{\delta\Omega} f_\mu d\omega = -c^2\, \delta\tau \int_{\Sigma_0} f_\mu\, \gamma(x)\, d\upsilon_0\,.$$

We can then pose, by definition, the following expression:

$$\boxed{F_\mu = \int_{\Sigma_0} f_\mu \left[1 + \frac{x_\nu - Y_\nu}{c^2}\dot{U}_\nu\right] d\upsilon_0}$$

for the *global force*. One sees that this expression refers to a particular intersection of the tube by a *proper* space hyperplane. However, once that intersection is defined, F_μ will become a space-time vector. By means of this hypothesis, one will get:

$$\boxed{\dot{G}_\mu = F_\mu}$$

as a first global equation, namely, the classical equation. (Note, however, that G_μ is not generally collinear with U_μ.)

It is easy to integrate the torque equation:

$$\partial_\lambda m_{\mu\nu\lambda} = x_\mu f_\nu - x_\nu f_\mu$$

in a similar fashion.

We introduce a density of *internal* rotational moment $s_{\mu\nu\lambda}$ by taking the moment of $t_{\mu\nu}$ *with respect to the center C of the drop*, and we further define the internal angular momentum the drop by means of the integral:

$$S_{\mu\nu} = -\frac{1}{c^2}\int_{\Sigma_0} s_{\mu\nu\lambda}\, d\sigma_\lambda \,.$$

One will then have:

$$\int_{\partial\Omega} \partial_\lambda m_{\mu\nu\lambda} = -\, c^2 \,\delta\tau\, \dot{M}_{\mu\nu}$$

for the left-hand side. Now:

$$M_{\mu\nu} = S_{\mu\nu} - \frac{1}{c^2}\int_{\Sigma_0} (Y_\mu t_{\nu\nu} - Y_\nu t_{\mu\lambda})\, d\sigma_\lambda \,,$$

so

$$\dot{M}_{\mu\nu} = \dot{S}_{\mu\nu} + U_\mu G_\nu - U_\nu G_\mu + Y_\mu \dot{G}_\nu - Y_\nu \dot{G}_\mu \,,$$

or, upon taking the first equation into account:

$$-\, c^2 \delta\tau\, \dot{M}_{\mu\nu} = -\, c^2 \delta\tau\, \left(\dot{S}_{\mu\nu} + U_\mu G_\nu - U_\nu G_\mu\right) + Y_\mu \int_{\partial\Omega} f_\nu\, d\omega - Y_\nu \int_{\partial\Omega} f_\mu\, d\omega,$$

One can make the last two terms on the right-hand side vanish, which will then leave:

$$\int_{\partial\Omega} [(x_\mu - Y_\mu)\, f_\nu - (x_\nu - Y_\nu)\, f_\mu]\, d\omega = -\, c^2 \delta\tau \int_{\Sigma_0} [(x_\mu - Y_\mu)\, f_\nu - (x_\nu - Y_\nu)\, f_\mu]\; \gamma(x)\, d\upsilon_0 \,.$$

One defines the *global torque* by:

$$\boxed{\Gamma_{\mu\nu} = \int_{\Sigma_0} [(x_\mu - Y_\mu)\, f_\nu - (x_\nu - Y_\nu)\, f_\mu] \left[1 + \frac{x_\lambda - Y_\lambda}{c^2} \dot{U}_\lambda\right] d\upsilon_0 \,,}$$

and one will therefore have the second dynamical equation:

$$\boxed{\dot{S}_{\mu\nu} + U_\mu G_\nu - U_\nu G_\mu = \Gamma_{\mu\nu} \,,}$$

which generalizes the second Frenkel-Weyssenhoff equation.

If one considers the expressions that define the force F_μ and the torque $\Gamma_{\mu\nu}$ that act upon the drop globally then it is important to remark that neither of them is to be found in proper space, in general. Similarly, if the external field acts in such a fashion as to produce a force density f_μ at each point of the drop that is orthogonal to the *local* current (which will be the case for the electromagnetic field and the hypotheses that one generally makes in classical hydrodynamics) then the integrals of that force and its moment over all of the volume of the drop will have no reason to be orthogonal to the unit-speed velocity *at the center of matter*, which defines the global proper system. It is the one important difference that one must not lose sight of between our dynamics of the spinning particle and that of the classical Newtonian particle. Indeed, the proper mass will be constant for the latter. The dynamical equations $\dot{G}_\mu = F_\mu$ can be written $\mathfrak{M}_0\, \dot{U}_\mu = F_\mu$, so, upon contracting with U_μ, one will get $F_\mu U_\mu = 0$.

On the contrary, in the case of the spinning particle, one will have:

$$\dot{G}_\mu U_\mu = -\ \dot{\mathfrak{M}}_0 c^2 - G_\mu \dot{U}_\mu .$$

We will see that we will be led to confine ourselves to the case in which G_μ is orthogonal to, but it will still be true that $F_\mu U_\mu = -\ \dot{\mathfrak{M}}_0 c^2$.

Indeed, the inertial mass of the drop is variable, in general, and its variation will be equal to the temporal component of F_μ in the proper system – i.e., to the *work* that is performed by the force. The variability of the proper mass is related to the existence of numerous internal degrees of freedom that we have attributed to our particle explicitly by being endowed with an extended structure, which permits the work that is done by the force to have other effects than just the pure and simple augmentation of the kinetic energy of translation (which is all that enters into consideration when one considers the classical particle). These effects, which are essentially the variations of the kinetic energy that are due to the *internal* motions of the matter in the drop, translate into the variations of the global mass, likewise relative to the proper system, with respect to which, the classical energy of translation will be constantly zero, by definition; there will be variation of the *proper* mass.

It remains for us to specify the definition of the point C. Naturally, we shall once more define it to be a *center of matter*, but the procedure that we employed in the case of the free drop to choose the particular reference in which it will be the pseudo-center of matter will be forbidden to us. Indeed, the vector G_μ, which permits the unambiguous definition of the reference frame of inertia, will no longer be defined in an intrinsic fashion; it presupposes the choice of a center of matter and a proper system. Moreover, Møller has shown that as long as one considers only the properties of the fluid, independently of the external field (and therefore not a closed system), it will be vain to hope to define a privileged vector in a covariant fashion. By contrast, if one considers the *total* energy-momentum, which includes the part that is manifested by the external field, then one will once more find oneself in the case of applying Møller's theorem (viz., a general, closed system). It is then possible to define a *total* momentum vector $\mathfrak{G}_\mu$, which we distinguish from the momentum G_μ, that one can call the *internal momentum* of the drop.

We limit ourselves to the case of a *holonomic* external force field; i.e., one whose force density can be expressed as a divergence $f_\mu = -\partial_\nu \tau_{\mu\nu}$.

This case includes the case of a relativistic gradient $f_\mu = -\partial_\mu V$, with $\tau_{\mu\nu} = \delta_{\mu\nu} V$, and therefore the case of an electromagnetic field $F_{[\alpha\beta]}$ that obeys the Maxwell equations. One will then have the Lorentz force density:

$$f_\mu = F_{\mu\nu} j_\nu \qquad (j_\nu \text{ is the electric current density}).$$

If one sets:

$$\tau_{\mu\nu} = F_{\alpha\nu} F_{\alpha\nu} - \tfrac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta}$$

then one will get:

$$-\partial_\mu F_{\mu\nu} = F_{\mu\nu}\; j_\nu$$

by taking the two Maxwell equations into account:

$$\partial_\nu F_{\nu\mu} = j_\mu \qquad \text{and} \qquad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

Finally, when we constitute our fluid model with interacting droplets, it will be just such forces that we will be led to introduce.

In any case, the hydrodynamical equation $\partial_\nu \tau_{\nu\mu} = f_\mu$ will take the form $\partial_\nu (t_{\nu\mu} + \tau_{\nu\mu}) = 0$, in which a *total energy-momentum tensor* appears – viz., $\mathfrak{T}_{\mu\nu} = t_{\mu\nu} + \tau_{\mu\nu}$ – that is conservative. Under these conditions, and the restriction that tensor $\tau_{\mu\nu}$ must be zero at infinity in any spacelike cut, one can apply Møller's argument and define a *total momentum:*

$$\mathfrak{G}_\mu = \int_\infty \mathfrak{T}_{\mu\otimes}\, d\upsilon ,$$

which will be a *vector*, independently of the spacelike hyperplane over which one integrates, under the condition that one index must refer to the component of $\mathfrak{T}_{\mu\nu}$ along the time axis that is orthogonal to the hyperplane in question. Moreover, the vector $\mathfrak{G}_\mu$ will be constant time.

This proof necessitates several precautions that relate to hyper-boundary integrals. One can decompose the divergence integral into two parts: One part relates to the tensor $t_{\mu\nu}$, namely $\int_\infty \partial_\nu t_{\mu\nu}\, d\omega$, and is taken over two separate domains. One the one hand, one has the interior of the tube that is swept out by the drop, and Gauss's theorem can be applied with no difficulty, since the boundary term will be annulled by the presence of surface tension. On the other hand, one has the rest of spacetime, where $t_{\mu\nu}$ will be zero everywhere, so the integral will not contribute anything. As for the term $\int_\infty \partial_\nu \tau_{\mu\nu}\, d\omega$, it will be integrated directly over all of space-time. Gauss's theorem can be applied with no special precautions, since the functions $\tau_{\mu\nu}$ are continuous, and as a result, the boundary term will disappear (as in Møller's original argument) when the hypersurface is pushed out to infinity, where $\tau_{\mu\nu}$ is zero everywhere. We note that although $\mathfrak{G}_\mu$ can also be decomposed into $\int_\infty t_{\mu\otimes}\, d\upsilon + \int_\infty \tau_{\mu\otimes}\, d\upsilon$, it will be only the sum that is tensorial. A Lorentz transformation that accompanies a corresponding change of cut will transform each of the two integral in a non-tensorial fashion, but the changes that are due to the Lorentz formula will cancel when they are summed by reason of the conservative character of the *total* tensor.

The total momentum vector thus expresses the idea that the energy-momentum of matter and that of the field are connected indissolubly, which conforms to the thinking of general relativity. However, it gives us a privileged reference frame – viz., the *reference frame of inertia* $\Pi_1\ \Lambda_1$ – in which its spatial components are zero, and which one can express relative to the laboratory reference frame, in which the components are $\mathfrak{G}_k$, $\mathfrak{G}_4$, by means of the formulas that we established in the case of the free drop.

That reference frame will serve to define a pseudo-center of mass for us that can be expressed as an intrinsic point in any arbitrary reference frame. It will be the *center of gravity* Z_μ . It will no longer be either at rest in the reference frame of inertia $\Pi_1\ \Lambda_1$ or in uniform motion, as it was for the free drop. Of course, the energy with which we weight each point of the drop is uniquely that of matter:

$$Z_k^{\mathrm{I}} \int_{\Sigma_{\mathrm{I}}} t_{\otimes\otimes}^{\mathrm{I}}\, d\upsilon^{\mathrm{I}} = \int_{\Sigma_{\mathrm{I}}} t_{\otimes\otimes}^{\mathrm{I}}\, x_k^{\mathrm{I}} d\upsilon^{\mathrm{I}} .$$

Since the integrals are taken over a cut Σ_{I} by the spacelike hyperplane of inertia, unlike in the case of the free drop, we cannot associate the characteristic of the center of gravity with the components of the *internal* momentum, for which the integrals are taken over *proper*-space sections.

However, above all, the reference frame of inertia will permit us to define a pseudo-center of matter that we take to be the intrinsic *center of matter C* by calculating its coordinates in an arbitrary reference frame by means of a suitable Lorentz transformation. The transformation formulas, as well as the ones that make us pass to the *proper system* that is attached to the point *C*, are the ones that we gave in the context of the free drop, with the condition that we should take the components $\mathfrak{G}_k$ and $\mathfrak{G}_4$ of the *total* momentum in the laboratory system to be G_k and G_4. We can then take the reference frame in which *C* is at rest to be the proper system and make cuts through the drop in that reference frame over which we will take the volume integrals that define the internal momentum and the internal angular momentum of the drop.

One sees, as we have announced, that the external field intervenes by the intermediary of the choice of the reference frame of inertia, which is related to the total momentum G_μ, in the definition of the global dynamical quantities that characterize the drop. One remarks that it can make the energy-momentum of the drop negligible, as opposed to that of the field, which can extend very far into space, such that, in fact, the privileged direction $\mathfrak{G}_\mu$ can be independent of the drop, in practice, and the system of inertia will play the role of a sort of absolute reference for it. It can also be the case that there are other drops in that space whose energy-momentum will contribute to the determination of the reference frame of inertia, or similarly (as will be the case in fluids that are composed of droplets) that the field is created uniquely by the forces that are exerted by the other droplets. In the latter case, the reference frame considered will play the role of a collective reference frame that brings into play the global energy of all of the drops that constitute the fluid; viz., the internal energy of each drop and the external energy that is related to the their interactions.

Since the definitions of the centers of gravity and mass appeal to integrals that are taken over all of space, the vector R_μ that joins these two points will no longer be given by the expression $S_{\mu\nu}\, G_\nu / M_0^2 c^2$, as it was in the case of the free drop, because the tensors that constitute that expressions will result from integrals that are taken over the proper volume.

By contrast, if we define a *center of mass* X_μ in the proper system then we can easily see that the relation that we found in the case of the free drop, namely:

$$S_{\mu\nu}\, U_\nu = \mathfrak{M}_0 c^2 (Y_\mu - X_\mu),$$

persists, since the two sides of the equations employ only proper space integrals. Moreover, the decomposition of the internal angular momentum that we derive from this relation will have a completely different character and will take on considerable significance.

§ 5. Generalities on fluids with spin that have internal stresses. – We can now consider a collection of particles of the same type that we have been considering that has a continuous distribution of magnitudes that characterize them and constitute a *fluid* from them in the same way that we did for free particles. Along with the ordinary densities of matter ρ, momentum g_μ, spin s_μ, and internal angular momentum $s_{\mu\nu}$, we will also consider a volumetric force density f_μ and a volumetric torque density $\gamma_{\mu\nu}$. (One must take special care that these symbols should no longer have the same significance as they did in the consideration of the drop. We shall henceforth go on to another scale that is much wider in scope where one must consider means that are taken over volumes $d\upsilon$ that contain a large number of spinning particles. The present force density f_μ is obtained by taking the sum of the global forces F_μ that act on all of the particles that are contained in the volume $d\upsilon$ and forming the quotient: $f_\mu = \sum F_\mu \,/\, d\upsilon$. On the contrary, in the preceding paragraphs, we considered a force density that acted at the various points of one of the drops, which we now consider to be particles, and formed the global force F_μ by *integrating* that density over the volume of the drop.) Once we have accepted the remarks that we made about the global force and torque, we will be free to consider force or torque densities that are not situated in proper space.

The global equations of the drop:

$$\dot{G}_\mu = F_\mu\,,$$
$$\dot{S}_{\mu\nu} + G_\nu\, U_\mu - G_\mu\, U_\nu = \Gamma_{\mu\nu}$$

immediately give us the fundamental hydrodynamical equations:

$$\dot{g}_\mu = f_\mu ,$$
$$\dot{s}_{\mu\nu} + g_\nu u_\mu - g_\mu u_\nu = \gamma_{\mu\nu} ,$$

to which, we append the equation matter conservation:

$$\dot{\rho} = 0.$$

The derivatives that are denoted by a dot have the significance that recalled at the beginning of the chapter.

The forces and torques that interest us are not the ones that are due to the action of anything outside the fluid. They are the ones that result from interactions between particles. The force F_μ and the torque $\Gamma_{\mu\nu}$ to which each particle is subjected individually are the results of the action of the set of all other particles upon that particle. It is expressed completely in terms of these other particles; i.e., in terms of the local hydrodynamical magnitudes and their derivatives. f_μ and $\gamma_{\mu\nu}$ thus constitute auxiliary hydrodynamical properties (such as ρ, u_μ, g_μ, σ_μ, and $s_{\mu\nu}$), if one assumes that one can express the dynamical equations in such a way that f_μ and $\gamma_{\mu\nu}$ determine the evolution of these fundamental magnitudes by their motion.

In order to introduce different types of interactions, the most appropriate method is to postulate the form of a coupling term that is represented by an energy density, and then add that term to the Lagrangian of the "pure matter" fluid. In that way, the usual method will yield an energy-momentum tensor that involves a kinetic part and another part that we interpret as an internal stress tensor $\theta_{\mu\nu}$. We then derive a force density $f_\mu = -\partial_\mu \theta_{\mu\nu}$ from this tensor, as well as a torque density $\gamma_{\mu\nu} = \theta_{\mu\nu} - \theta_{\nu\mu} = 2\ \theta_{<\mu\nu>}$.

Therefore, that will be how we always operate, although it will probably result in certain restrictions on the kinds of interaction forces and torques at which we shall arrive. However, we believe that this method will still subsume all of the truly interesting cases.

One can encounter a difficulty in the course of making the comparison that we propose to make between the fluids that are obtained by way of particles that are related by forces and torques and fluids that are derivable from a Lagrangian formulation. Indeed, for the latter, we have seen that the general decomposition of the energy-momentum tensor will give us:

$$t_{\mu\nu} = \mu_0\, u_\mu\, u_\nu - p_\mu\, u_\nu + q_\nu\, u_\mu + \theta_{\mu\nu} ,$$

in which the tensors p_μ, q_μ, and $\theta_{\mu\nu}$ are orthogonal to the current.

The Lagrangian treatment then gives a quantity q_μ that seems difficult to interpret, in addition to an internal stress tensor that is situated in proper space completely, which will then imply restrictive conditions on the force and torque. Indeed, if we refer to q_μ as a "heat current" in the general case of the hydrodynamical representation of the wave functions that is defined by a Lagrangian formulation, which is a current that takes the form of an energy that flows in the fluid independently of the matter and without the aid of any mechanical force in the present case, then, on the contrary, we will have no other energy that is localized to the particles and moves with them nor other exchanges of energy than ones that take place by means of internal stress forces. If we construct an energy-momentum tensor by separating the particles and the internal forces then we will find a kinetic part $g_\mu u_\nu$ and a stress tensor $\theta_{\mu\nu}$, but not a tensor of heat current. In order to revert to the general formalism, one must then dispose of the quantity q_μ .

To that end, one might be inspired by the physical consideration that we developed in Chapter IV in the context of the dynamics of a fluid drop, which was a dynamic that can be summarized by the following two global equations:

$$\dot{G}_\mu - U_\mu \left(-\int_{S_0} q_i^0 ds_i^0 + \int_{V_0} \frac{w_0}{c^2} d\upsilon_0 \right) = F_\mu ,$$

$$\dot{S}_{\mu\nu} + \left(G_\nu - \int_{V_0} q_\nu d\upsilon_0 \right) U_\mu - \left(G_\mu - \int_{V_0} q_\mu d\upsilon_0 \right) U_\nu = 2\int_{V_0} \theta_{<\mu\nu>} d\upsilon_0 .$$

We said that these equations make the global momentum G_μ , which was obtained by the usual method, appear as a partially-fictitious quantity, and that the *dynamical* quantities, whose evolution is determined entirely by the force and torque through the usual dynamical equations of spinning particles, are:

$$G_\mu - \int_{V_0} q_\mu d\upsilon_0$$

for the torque equation and:

$$\dot{G}_\mu - U_\mu \left(-\int_{S_0} q_i^0 \, ds_i^0 + \int_{V_0} \frac{w_0}{c^2} d\upsilon_0 \right)$$

for that of the forces. One can apply these considerations to the hydrodynamical equations and change the definition of the momentum density and the stress tensor in such a fashion as to eliminate the heat current, and at the same time, to make magnitudes appear uniquely that will produce dynamical quantities that have some physical significance after one integrates them over the drop.

§ 6. Spinning fluids with no heat current. – We first define a *purely kinetic momentum density* g'_μ by subtracting the heat current from the generalized momentum density g_μ : $g'_\mu = g_\mu - q_\mu$.

This amounts to taking the transverse momentum p_μ , which we have seen to result, at the same time, from internal rotation (in the case of the Weyssenhoff equation) and the heat current (in the case of the Klein-Gordon), which amounts to taking a purely kinematic transverse momentum:

$$p'_\mu = p_\mu + q_\mu ,$$

which is no longer related to the internal rotation, since, as we showed in Chapter IV, we will have, in any case, the Weyssenhoff relation:

$$p'_\mu = p_\mu + q_\mu = -\frac{1}{c^2} s_{\mu\nu} \dot{u}_\nu \, .$$

Since p_μ is orthogonal to the current, we do not need to modify the proper mass density $g'_\mu u_\mu = g_\mu u_\mu - \mu_0 c^2$, and the momentum will then take on the form that was given by Weyssenhoff:

$$g'_\mu = \mu_0 u_\mu + \frac{1}{c^2} s_{\mu\nu} \dot{u}_\nu \, , \tag{V.17}$$

independently of the internal stresses, which shows that it expresses a property that is attached to the particles when the interactions are not taken into account.

It will likewise result from this equation (which is a consequence of the torque equation) that g'_μ is orthogonal to the space-time acceleration $\dot{u}_\mu$ and to the spin σ_μ, as one will show by contracting with these two vectors, respectively, and upon taking into account that $s_{\mu\nu}\,\sigma_\nu = 0$ and that $s_{\mu\nu}\dot{u}_\mu \dot{u}_\nu = 0$ (antisymmetry), we will get:

$$\boxed{g'_\mu \dot{u}_\mu = 0,} \qquad \boxed{g'_\mu \sigma_\mu = 0.} \tag{V.18}$$

If we compare these equations with the results that we found in Chapter III in the case of pure matter then we will see that the system of four vectors u_μ, $\dot{u}_\mu$, p'_μ, σ_μ all form right angles with each other, except for σ_μ and $\dot{u}_\mu$. Relation (18) results from the contraction of the torque equation:

$$g'_\mu u_\nu - g'_\nu u_\mu + 2\theta_{<\mu\nu>} = \dot{s}_{\mu\nu} \, . \tag{V.19}$$

If one follows the Takabayasi method then one can get two other equations by contracting with $ic\varepsilon_{\mu\nu\alpha\beta}\, u_\alpha$ and $ic\ \varepsilon_{\mu\nu\alpha\beta}\ \sigma_\alpha$, respectively. These two equations, along with equation (18), will form a system that is equivalent to (19).

Upon contracting with $ic\varepsilon_{\mu\nu\alpha\beta}\, u_\alpha$, the two terms $g'_\mu u_\nu$ and $g'_\nu u_\mu$ will disappear. It follows that:

$$2ic\varepsilon_{\mu\nu\alpha\beta}\, \theta_{\mu\nu}\, u_\alpha = ic\varepsilon_{\mu\nu\alpha\beta}\, \dot{s}_{\mu\nu}\ u_\alpha .$$

If one replaces $s_{\mu\nu}$ by its expression as a function of spin and velocity then the right-hand side of the equation will give:

$$\begin{aligned} ic\varepsilon_{\mu\nu\alpha\beta}\, \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\, \frac{d}{d\tau}(u_\gamma\, \sigma_\rho)\, u_\alpha &= -2\ \delta^{\alpha\beta}_{\gamma\rho}\, \frac{d}{d\tau}(u_\gamma\, \sigma_\rho)\, u_\alpha \\ &= -2\left[(\dot{u}_\alpha \sigma_\beta + u_\alpha \dot{\sigma}_\beta)u_\alpha - (\dot{u}_\beta \sigma_\alpha + u_\beta \dot{\sigma}_\alpha)u_\alpha\right] \\ &= -2\ (-c^2 \dot{\sigma}_\beta - u_\alpha \dot{\sigma}_\alpha \cdot u_\beta) = 2\ (c^2 \dot{\sigma}_\beta - \dot{u}_\alpha \sigma_\alpha \cdot u_\beta)\, . \end{aligned}$$

One will therefore get the first equation:

(V.20)
$$\boxed{ic\,\varepsilon_{\mu\nu\alpha\beta}\theta_{\mu\nu}u_{\alpha} = c^2\dot{\sigma}_{\beta} - \dot{u}_{\alpha}\sigma_{\alpha}\cdot u_{\beta} \equiv \eta_{\alpha\beta}c^2\dot{\sigma}_{\alpha},}$$

which is an equation that expresses the evolution law for spin.

Finally, contract this with $i / c\ \varepsilon_{\mu\nu\alpha\beta}\,\sigma_{\alpha}$:

$$2\frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}g'_{\mu}u_{\nu}\sigma_{\alpha} + \frac{2i}{c}\varepsilon_{\mu\nu\alpha\beta}\theta_{\mu\nu}\sigma_{\alpha} = \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\dot{s}_{\mu\nu}\sigma_{\alpha}.$$

The expression for $s_{\mu\nu}$ appears in the left-hand side, and that left-hand side will become:

$$2s_{\mu\beta}g'_{\mu} + 2\frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\theta_{\mu\nu}\sigma_{\alpha}.$$

The right-hand side, when transformed as before, will give:

$$-2\ \delta^{\alpha\beta}_{\gamma\rho}\frac{d}{d\tau}(u_{\gamma}\,\sigma_{\rho})\,\sigma_{\alpha} = -\frac{2}{c^2}(\sigma_{\alpha}\dot{u}_{\alpha}\cdot\sigma_{\beta} - \sigma_0^2\dot{u}_{\beta} - \sigma_0\dot{\sigma}_0 u_{\beta}),$$

when one introduces the norm of the spin $\sigma_0^2 = \sigma_{\alpha}\,\sigma_{\alpha}$ and its derivative $\dot{\sigma}_0 = \sigma_{\alpha}\dot{\sigma}_{\alpha} / \sigma_0$. One then has:

$$-\ s_{\mu\beta}g'_{\mu} = \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\ \theta_{\mu\nu}\ \sigma_{\alpha} = -\ \frac{2}{c^2}(\sigma_{\alpha}\dot{u}_{\alpha}\cdot\sigma_{\beta} - \sigma_0^2\dot{u}_{\beta} - \sigma_0\dot{\sigma}_0 u_{\beta}).$$

This equation can be transformed by introducing the expression (17) for g'_{μ}, and upon taking relation (III.15) into account to express the product $s_{\mu\beta}\,s_{\mu\nu}$, the left-hand side will become:

$$-\frac{1}{c^2}\left[\sigma_0^2\left(\delta_{\mu\nu} + \frac{u_{\beta}u_{\nu}}{c^2}\right) - \sigma_{\beta}\sigma_{\nu}\right]\dot{u}_{\nu} = -\frac{1}{c^2}\left[\sigma_0^2\dot{u}_{\beta} - \sigma_{\nu}\dot{u}_{\nu}\sigma_{\beta}\right],$$

which gives the equation:

$$\sigma_{\nu}\dot{u}_{\nu}\sigma_{\beta} - \sigma_0^2\dot{u}_{\beta} = ic\ \varepsilon_{\mu\nu\alpha\beta}\ \theta_{\mu\nu}\ \sigma_{\alpha} + \ \sigma_{\alpha}\dot{u}_{\alpha}\sigma_{\beta} - \sigma_0^2\dot{u}_{\beta} - \sigma_0\dot{\sigma}_0 u_{\beta},$$

so one finally gets:

(V.21)
$$\boxed{\sigma_0\dot{\sigma}_0 u_{\beta} = ic\varepsilon_{\mu\nu\alpha\beta}\theta_{\mu\nu}\sigma_{\alpha},}$$

which is an equation that gives us an expression for the current.

Finally, upon contracting (20) with σ_{β} or contracting equation (21) with u_{β}, one will get the important relation:

$$ic\, \varepsilon_{\mu\nu\alpha\beta}\, \theta_{\mu\nu}\, \sigma_\alpha\, u_\beta = -\, c^2\, \sigma_0 \dot{\sigma}_0\,,$$

so that

$$\boxed{\sigma_0 \dot{\sigma}_0 = \theta_{\mu\nu} s_{\mu\nu}\,,}$$

which will yield the variation of the norm of the spin, and which will show that the condition for it to be constant is:

$$\theta_{\mu\nu}\, s_{\mu\nu} = 0.$$

If we introduce g'_μ, in place of g_μ, then the energy-momentum tensor will obviously take the form:

$$t_{\mu\nu} = g'_\mu\, u_\nu + q_\mu\, u_\nu + q_\nu\, u_\mu + \theta_{\mu\nu}\,,$$

and we will thus make the two "heat" terms enter into the stresses when we set $\theta'_{\mu\nu} = \theta_{\mu\nu} + q_\mu\, u_\nu + q_\nu\, u_\mu$.

Under these conditions, one will have:

$$t_{\mu\nu} = g'_\mu\, u_\nu + \theta'_{\mu\nu}\,,$$

and the conservation relation can be written:

$$\dot{g}'_\mu = -\, \partial_\nu \theta'_{\mu\nu} = \varphi'_\mu\,. \tag{V.22}$$

Here, the new stress tensor will not be in proper space, since one will have $\theta'_{\mu\nu}\, u_\nu = -\, c^2\, q_\mu$ and $\theta'_{\mu\nu}\, u_\mu = -\, c^2\, q_\nu$.

We remark that the terms that were added to $\theta'_{\mu\nu}$ are symmetric in μ and ν, so one can simply replace $\theta_{\mu\nu}$ with $\theta'_{\mu\nu}$ in the torque equation and in the relations that result. Incidentally, it results from this that the moment of the torque $\theta'_{<\mu\nu>}$ will be in space, as in the old formalism.

We can derive two important relations from equation (22) by contracting it with u_μ and σ_μ; on will then has $\dot{g}'_\mu\, u_\mu = \varphi'_\mu\, u_\mu$.

The left-hand side expression the variation $-\dot{\mu}_0 c^2$ of the proper mass density, since $g'_\mu\, \dot{u}_\mu = 0$.

The second relation relates to the temporal component of force density in the proper system:

$$\boxed{\mu'_0 = \frac{1}{c^2} u_\mu \partial_\nu \theta'_{\mu\nu}.}$$

This relation, which translates into the conservation of energy, differs in two respects from the corresponding relation in Takabayasi's theory: One the one hand, the heat current no longer appears explicitly. The energy exchanges in the proper system will translate into just variations of the proper mass of the particles, which will be variations that are, as have said, coupled to some new parameters that relate to the structure of the particles, and which will translate into modifications of the kinetic energy of the internal motions of each particle. On the other hand, if one decomposes the force density φ'_μ into a proper-space component f'_μ ($f'_\mu u_\mu = 0$) and a component along the current $(A_0 / c^2)\, u_\mu$:

$$\varphi'_\mu = f'_\mu + \frac{A_0}{c^2} u_\mu$$

then one will have:

$$\dot{\mu}_0 = -\frac{1}{c^2} \varphi'_\mu u_\mu = \frac{A_0}{c^2},$$

but if one compares A_0 with the corresponding term in the old formalism:

$$w_0 = -\varphi_\mu u_\mu = \theta_{\mu\nu} \partial_\nu u_\mu$$

then one will see that A_0 involves heat terms $-c^2 \partial_\nu q_\nu$ and $-u_\mu u_\nu \partial_\nu q_\mu$, in addition to w_0, and those terms can be an order of magnitude larger that w_0, due to the factor of c^2. This is related to the fact that was pointed out before in the context of the drop at the beginning of the present chapter that the variation of internal kinetic energy of the drop can correspond to the time component in the proper system for the force that acts upon that drop, and that time component will have a non-relativistic order of magnitude, while the Takabayasi force, which is the gradient of a proper space stress $\theta_{\mu\nu}$, will have only a time component in $1 / c^2$ in the proper system.

We can use this relation to write down the differential equation of the streamlines:

$$\dot{g}'_\mu = \dot{\mu}_0 u_\mu + \mu_0 \dot{u}_\mu + \frac{1}{c^2} s_{\mu\lambda} \ddot{u}_\lambda + \frac{1}{c^2} \dot{s}_{\mu\lambda} \dot{u}_\lambda = \varphi'_\mu .$$

If we substitute the value that we found for $\dot{\mu}_0$ and the expression for $\dot{s}_{\mu\nu}$ then it will become:

$$\frac{1}{c^2} u_\nu \partial_\lambda \theta'_{\nu\lambda} u_\mu + \mu_0 \dot{u}_\mu + \frac{1}{c^2} s_{\mu\lambda} \ddot{u}_\lambda + \frac{1}{c^2} (g'_\mu u_\lambda - g'_\lambda u_\mu + 2\theta'_{<\mu\nu>})\, \dot{u}_\lambda = -\partial_\lambda \theta'_{\mu\lambda} .$$

The first two terms in the parenthesis go to zero, and one will have, upon collecting the similar terms:

$$\boxed{\left(\mu_0 \delta_{\mu\nu} + \frac{2}{c^2} \theta'_{<\mu\nu>}\right) \dot{u}_\nu + \frac{1}{c^2} s_{\mu\nu} \ddot{u}_\nu = -\eta_{\mu\lambda} \partial_\nu \theta'_{\lambda\nu} .}$$

This equation, in which one recognizes the expression $f_\mu = \eta_{\mu\lambda}\, \varphi_\lambda$ for the proper-space force in the left-hand side, generalizes the Mathisson equation for the pure matter fluid (see Chapter III).

Finally, upon contracting (22) with σ_μ, and taking (18) into account, one will get:

$$\dot{g}'_\mu \sigma_\mu = -\dot{\sigma}_\mu g'_\mu = -\partial_\nu \theta'_{\mu\nu} \cdot \sigma_\mu ,$$

or, upon deriving the expression for $\dot{\sigma}_\mu$ from (20):

$$-\frac{1}{c^2}\sigma_\alpha \dot{u}_\alpha \cdot u_\beta g'_\beta - \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\theta'_{\mu\nu} u_\alpha g'_\beta = -\partial_\nu \theta'_{\mu\nu} \cdot \sigma_\mu ,$$

so

$$\boxed{\mu_0 \sigma_\alpha \dot{u}_\alpha = \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\theta'_{\mu\nu} u_\alpha g'_\beta - \sigma_\mu \partial_\nu \theta'_{\mu\nu} ,}$$

which provides us with the expression for the scalar product $\sigma_\alpha \dot{u}_\alpha$ and the orthogonality condition for the two vectors σ_α and $\dot{u}_\alpha$:

$$\frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}\theta'_{\mu\nu} u_\alpha g'_\beta = \sigma_\mu \partial_\nu \theta'_{\mu\nu} .$$

If that condition is satisfied then the last angle in our system of vectors u_μ, $\dot{u}_\mu$, p_μ, s_μ will be a right angle, and we will recover the generalized Darboux-Frenet system of axes that we pointed out in the case of pure matter (Chapter III).

We will better comprehend the significance of our formalism by integrating the fundamental equations over the volume of an *infinitesimal* droplet that is cut from the fluid, according to the usual method:

Upon setting:

$$\Gamma_{\mu\nu} = 2 \int_{V_0} \theta'_{<\mu\nu>} d\upsilon_0 ,$$

one will immediately get:

$$G'_\mu U_\nu - G'_\nu U_\mu + \Gamma_{\mu\nu} = \dot{S}_{\mu\nu}$$

for the torque equation, which is an equation whose significance is clear.

The calculation is a little more delicate for the force equation. One gets:

$$\dot{G}_\mu d\tau = -\int_\Omega \partial_\nu \theta'_{\mu\nu} d\omega .$$

We can transform the right-hand side into a hypersurface integral that is taken, on the one hand, over the proper-space endcaps C_1 and C_2, and on the other hand, over the boundary P of the current tube (Appendix A):

$$\dot{G}_\mu d\tau = -\int_{C_1} \theta'_{\mu\nu} d\sigma_\nu - \int_{C_2} \theta'_{\mu\nu} d\sigma_\nu - \int_P \theta'_{\mu\nu} d\sigma_\nu .$$

The end terms are not zero, as they were in the Takabayasi formalism. Upon transforming the two integrals over the ends $d\sigma_\nu$ into $-u_\nu\, d\upsilon_0$ and $+u_\nu\, d\upsilon_0$, it will follow that:

$$\dot{G}_\mu d\tau = \int_{C_1} \theta'_{\mu\nu} u_\nu \, d\upsilon_0 - \int_{C_2} \theta'_{\mu\nu} u_\nu \, d\upsilon_0 - \int_P \theta'_{\mu\nu} d\sigma_\nu \,,$$

so

$$\dot{G}_\mu d\tau = \frac{d}{d\tau} \int_{V_0} \theta'_{\mu\nu} u_\nu \, d\upsilon_0 d\tau - \int_P \theta'_{\mu\nu} d\sigma_\nu \,.$$

One can differentiate under the integral sign, and that will give:

(V.23) $$\dot{G}_\mu d\tau = -\int_{V_0} \partial_\lambda (\theta'_{\mu\nu} u_\nu u_\lambda) \, d\upsilon_0 d\tau - \int_P \theta'_{\mu\nu} d\sigma_\nu \,.$$

In order to interpret the last term, it must be placed into the proper system. One knows that $d\sigma_\nu$ has only spatial components then, and they will be:

$$d\sigma_j^0 = - \, ds_j^0 \, d\tau \,,$$

in which ds_j^0 is the surface element in proper space.

One then introduces a volumetric force density:

$$f_\mu = - \, \partial_\lambda (\theta'_{\mu\nu} u_\nu u_\lambda) \,.$$

The proper space components of f_μ are:

$$f_k^0 = -\partial_4 (\theta'_{\kappa 4} \, ic \cdot ic) = - \, ic \, \frac{\partial}{\partial t} \theta'_{\kappa 4} \,.$$

As one sees, they are related to the proper-space components of our stress tensor.

The proper-time component is:

$$f_k^0 = -\partial_4 (\theta'_{44} \, ic \cdot ic) = 0,$$

the pure time component of $\theta'_{\mu\nu} = \theta_{\mu\nu} + q_\mu \, u_\nu + q_\nu \, u_\mu$ is obviously zero.

The force density f_μ is then a proper-space vector (like the heat current in the old formalism, to which it is obviously related). One will then have:

$$\dot{G}_k^0 = \int_{V_0} t_k^0 \, d\upsilon_0 + \int_{S_0} \theta'^0_{kj} \, ds_j^0 \,.$$

One see that the surface integral represents simply the classical force that is produced by the *surface actions* that are expressed by the stress tensor θ'^0_{kj} (which is identical to θ^0_{kj} in proper space, since the terms $q_k^0 \, u_j^0$ and $q_j^0 \, u_k^0$ are zero). The novelty, when compared to the Takabaysi fluid, is in the term $\int_{V_0} f_k^0 \, d\upsilon_0$, which manifests the existence of *volume* *actions* that are related to the divergence $\partial_\lambda(\theta'_{\mu\nu} u_\mu u_\nu)$. The interactions that are introduced in the form a coupling Lagrangian then translate, in general, into not only the internal stresses that are exerted by the surface forces, but, at the same time, into an *internal field* that acts upon the entire volume of the drop and is expressed by the volumetric force density:

$$f_\mu = -\,\partial_\lambda(\theta'_{\mu\nu} u_\mu u_\nu)\,,$$

which is not situated in proper space, moreover.

Finally, in the proper system, the fourth component of equation (23), namely:

$$\dot{G}_4^0 = -\,ic\;\dot{\mathfrak{M}}_0 = \int_{S_0} \theta'^0_{4j}\, ds_j^0\,,$$

will give us the variation of the total proper mass, since the component t_4^0 is zero. This integral represents the work that is done by the stress force on the surface of the drop.

§ 7. The case of a "perfect fluid" drop. – In conclusion, we shall apply our theory to several well-defined cases of interaction. We first consider the case of a *perfect fluid* that obeys an equation of state that generalizes what we said about the classical perfect fluid. We have to replace the mass density $\rho\, m_0$ with the more general form $\mu_0(\rho)$ in the Lagrangian for the spinning "pure matter" fluid:

$$\mathcal{L} = \mu_0(\rho)\, c^2 + ic\,\rho\; a_\mu^4 \partial_\mu S + ic\,\rho h_0\; a_\mu^4 a_\lambda^1 \partial_\mu a_\lambda^2 + \lambda_{\mu\nu}(a_\mu^\xi a_\nu^\xi - \delta_{\mu\nu})\,.$$

Nothing has changed with regard to the "pure matter" fluid, at least, as far as the Belinfante tensor and the internal angular momentum are concerned, which will take the usual form. If one differentiates with respect to ρ then one will get:

$$\dot{S} = -\,\mu_0' c^2 - h_0\, a_\lambda^1\, \dot{a}_\lambda^2\,,$$

which will give a non-zero value to the Lagrangian:

$$\mathcal{L} = (\mu_0 - \rho\mu_0')\, c^2\,.$$

The derivations with respect to the a_μ^ξ must give the same expressions that they do for the pure matter fluid, and one will therefore obtain an antisymmetric expression:

$$\text{(V.24)} \qquad \dot{s}_{\mu\nu} = ic\,\rho\,\left(a_\nu^4 \partial_\mu S - a_\mu^4 \partial_\nu S\right) + ic\,\rho h_0\; a_\lambda^1 (a_\nu^4 \partial_\mu a_\lambda^2 - a_\mu^4 \partial_\nu a_\nu^2)\,.$$

Upon contracting this with a_μ^4 and replacing $\dot{S}$ with its value, one will get:

(V.25) $$ic\,\rho\,\partial_\nu S = -\rho\mu_0' c^2 a_\nu^4 - ic\rho h_0 a_\nu^1 \partial_\nu a_\lambda^2 - s_{\nu\lambda}\dot{a}_\lambda^4 .$$

One can then define the energy-momentum tensor by taking the term that contains the Lagrangian into account:

$$t_{\mu\nu} = -\, a_\nu^4(\rho\mu_0' c^2 a_\mu^4 + ic\rho h_0\, a_\lambda^1 \partial_\mu a_\lambda^2 + s_{\mu\lambda}\dot{a}_\lambda^4) + ic\rho h_0\, a_\nu^4 a_\lambda^1 \partial_\mu a_\lambda^2 - \delta_{\mu\nu}(\mu_0 - \rho\mu_0' c^2),$$

or

$$t_{\mu\nu} = \rho\mu_0'\, u_\mu u_\nu + \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda u_\nu - \delta_{\mu\nu}(\mu_0 - \rho\mu_0')\, c^2 .$$

Upon contracting the usual expression for momentum by u_μ, one will derive from this that:

$$-\, c^2\, g_\mu = \rho\mu_0'\, u_\mu c^2 - s_{\mu\lambda}\dot{u}_\lambda - \mu_0 c^2 u_\mu + \rho\mu_0' c^2 u_\mu ,$$

$$g_\mu = \mu_0\, u_\mu + \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda .$$

There is no heat flux $t_{\mu\nu}\, u_\mu = -\, c^2\mu_0\, u_\mu$, so the stress tensor will reduce to:

$$\begin{aligned}\theta_{\mu\nu} &= t_{\mu\nu} - g_\mu\, u_\nu \\ &= (\rho\mu_0' - \mu_0)\, u_\mu u_\nu + c^2\delta_{\mu\nu}(\rho\mu_0' - \mu_0)\end{aligned}$$

so

$$\theta_{\mu\nu} = c^2(\rho\mu_0' - \mu_0)\,\eta_{\mu\nu} .$$

It is symmetric, situated in proper space, and takes the form of a classical "perfect fluid" stress that corresponds to a *pressure* of:

$$\boxed{\pi = c^2(\rho\mu_0' - \mu_0).}$$

We then have a force density:

$$\varphi_\mu = -\,\partial_\nu\,(\eta_{\mu\nu}\,\pi),$$

but not a torque density, since $\theta_{\mu\nu}$ is symmetric. The torque equation is then simply:

$$\dot{s}_{\mu\nu} = g_\mu\, u_\nu - g_\nu\, u_\mu ,$$

as one easily verifies by replacing $\partial_\nu S$ with its expressions in (25) in equation (24).

The equations that are obtained by contracting (24) with $u_\mu\, s_\nu$, $\varepsilon_{\mu\nu\alpha\beta}\, u_\alpha$, and $\varepsilon_{\mu\nu\alpha\beta}\,\sigma_\nu$ will give, along with the usual relation:

$$g_\mu\,\sigma_\mu = 0,$$

$$\boxed{c^2\dot{\sigma}_\beta = \dot{u}_\alpha\sigma_\alpha \cdot u_\beta,}$$

and:

$$\sigma_0 \dot{\sigma}_0 \, u_\beta = 0,$$

respectively. It then results that, on the one hand, *the norm of the spin is constant:*

$$\boxed{\dot{\sigma}_0 = 0}$$

(which is the case for all fluids with symmetric stresses, by reason of the relation $\sigma_0 \dot{\sigma}_0$ = $\theta_{\mu\nu}\, s_{\mu\nu}$), and on the other hand, $\dot{\sigma}_0$, which is collinear with the current, is orthogonal to the space-time acceleration:

$$\dot{\sigma}_\beta \, \dot{u}_\beta = 0,$$

as well as to the transverse momentum:

$$\dot{\sigma}_\beta \, p_\beta = 0.$$

The force equation:

(V.26) $$\dot{g}_\mu = - \partial_\mu (\eta_{\mu\nu}\, \pi)$$

will give, upon contracting with u_μ, the relation for proper energy:

$$\dot{\mu}_0 c^2 = u_\mu \, \partial_\nu \theta_{\mu\nu},$$

which takes the usual form that it takes in the case of perfect fluids here:

$$\dot{\mu}_0 c^2 = \partial_\nu (\theta_{\mu\nu} u_\mu) - \eta_{\mu\nu}\, \pi \, \partial_\nu u_\mu ,$$

such that, since $\theta_{\mu\nu} u_\mu = 0$ and $\eta_{\mu\nu}\, \partial_\nu u_\mu = \partial_\nu u_\mu$:

$$\boxed{\dot{\mu}_0 c^2 = -\pi \, \partial_\mu u_\mu .}$$

This expression permits us to write the differential equation of the streamlines:

$$\begin{aligned}
\mu_0 \dot{u}_\mu + \frac{1}{c^2} s_{\mu\nu} \ddot{u}_\nu &= - \eta_{\mu\nu}\, \partial_\lambda (\eta_{\nu\lambda}\, \pi) \\
&= - \partial_\lambda (\eta_{\nu\lambda}\, \pi) + \eta_{\mu\nu}\, \pi \, \partial_\lambda \left(\frac{u_\mu u_\nu}{c^2} \right) \\
&= - \eta_{\mu\nu}\, \partial_\lambda \, \pi - \pi \, \partial_\lambda \left(\frac{u_\mu u_\nu}{c^2} \right) + \pi \, \partial_\nu \left(\frac{u_\mu u_\nu}{c^2} \right) + \frac{u_\nu u_\lambda}{c^2} \pi \, \partial_\lambda \left(\frac{u_\mu u_\nu}{c^2} \right) \\
&= - \eta_{\mu\nu}\, \partial_\lambda \, \pi - \pi \, \frac{u_\nu u_\lambda}{c^2} \frac{\pi}{c^2} u_\nu \partial_\lambda u_\mu + \frac{u_\nu u_\lambda}{c^2} \frac{\pi}{c^2} u_\mu \partial_\lambda u_\nu .
\end{aligned}$$

The last term goes to zero, since $u_\nu \partial_\lambda u_\nu = 0$.

The second can be written $-(\pi / c^2)\ \dot{u}_\mu$, so:

$$\frac{1}{c^2} s_{\mu\nu} \ddot{u}_\nu + \left(\mu_0 + \frac{\pi}{c^2}\right) \dot{u}_\mu = - \eta_{\mu\nu} \partial_\lambda \pi .$$

One will then see the pseudo-mass density for perfect fluids appear (Appendix *B*):

$$\mu = \mu_0 + \frac{\pi}{c^2} .$$

On the other hand, since the vectors $s_{\mu\nu} \ddot{u}_\nu$ and $\dot{u}_\mu$ are in proper space, the left-hand side can be projected, without modification, onto proper space:

$$\eta_{\mu\lambda} \left[\frac{1}{c^2} s_{\lambda\nu} \ddot{u}_\nu + \mu \dot{u}_\lambda\right] = - \eta_{\mu\nu} \partial_\lambda \pi,$$

so that finally, we will have the equation:

$$\boxed{\eta_{\mu\lambda} \left[\frac{1}{c^2} s_{\lambda\nu} \ddot{u}_\nu + \mu \dot{u}_\lambda + \partial_\lambda \pi\right] = 0.}$$

This equation, which generalizes Mathisson's equation, says that upon projecting into proper space, the vector:

$$\frac{1}{c^2} s_{\lambda\nu} \ddot{u}_\nu + \mu \dot{u}_\lambda + \partial_\lambda \pi$$

will be zero.

One sees Lichnerowicz's internal force field density appear, and that general expression will make two special cases emerge directly: The perfect fluid without spin, for which:

$$s_{\nu\lambda} = 0 \qquad \text{and} \qquad \dot{u}_\lambda + \frac{\partial_\lambda \pi}{\mu} = 0,$$

and the "pure matter" fluid with spin, for which:

$$\frac{\partial_\lambda \pi}{\mu} = 0 \qquad \text{and} \qquad \frac{1}{c^2} s_{\lambda\nu} \ddot{u}_\nu + \mu \dot{u}_\lambda = 0.$$

Finally, upon contracting (26) with σ_μ:

$$\begin{aligned} \mu \dot{u}_\mu \sigma_\mu &= - \sigma_\mu \partial_\nu (\eta_{\mu\nu} \pi) \\ &= - \partial_\nu (\eta_{\mu\nu} \sigma_\mu \pi) + \eta_{\mu\nu} \pi \partial_\nu \sigma_\mu , \end{aligned}$$

$$= -\partial_\nu(\sigma_\nu\,\pi) + \pi\left(\partial_\mu\sigma_\mu + \frac{u_\mu u_\nu \partial_\nu\sigma_\mu}{c^2}\right)$$

$$= -\sigma_\nu\,\partial_\nu\,\pi + \frac{\pi}{c^2}\dot{\sigma}_\mu u_\mu\,.$$

Ultimately, upon introducing the pseudo-mass density $\mu = \mu_0 + \pi / c^2$ once more, we will have:

$$\boxed{\mu\left(\dot{u}_\nu + \frac{\partial_\nu\pi}{\mu}\right) = 0.}$$

One sees the vector $\dot{u}_\nu + \dfrac{\partial_\nu\pi}{\mu}$ appear, which will be zero in the case of the perfect fluid without spin, and which will reduce to $\dot{u}_\nu$ in the case of the "pure matter" fluid with spin. In the general case, it is that vector, and not $\dot{u}_\nu$, to which the spin will be orthogonal.

We can apply these formulas to the "perfect gas with spin" by generalizing the case of the "classical perfect gas." The equation of state $\mu_0 = C\rho^{k+1}$ corresponds to the pressure $\pi = c^2\,k\,C\rho^{k+1}$, and it will result from this that one can compute:

The pseudo-mass density: $\quad \mu = \mu_0 + \dfrac{\pi}{c^2} = \dfrac{k+1}{k}\dfrac{\pi}{c^2}.$

The gradient: $\quad \partial_\lambda\pi = (k+1)\,\pi\,\partial_\lambda \log\rho.$

The force field: $\quad k_\lambda = \dfrac{\partial_\nu\pi}{\mu} = k\,c^2\,\partial_\lambda \log\rho\,.$

One then gets the variation of the pseudo-mass density:

$$\dot{\mu} = \frac{1}{c^2}u_\nu\,\partial_\lambda\,\pi = k_\mu\,u_\nu\,\partial_\lambda \log\rho$$

from that of the mass density:

$$\dot{\mu}_0 = -\frac{1}{c^2}\,\pi\,\partial_\nu\,u_\nu\,.$$

The equation of the streamlines is:

$$\eta_{\mu\nu\lambda}\left[\frac{1}{c^2}s_{\lambda\nu}\ddot{u}_\lambda + \mu(\dot{u}_\lambda + kc^2\partial_\lambda \log\rho)\right] = 0.$$

The angle between the spin and acceleration is given by:

$$(\dot{u}_\nu + k\, c^2\, \partial_\nu \log \rho)\ \sigma_\nu = 0.$$

§ 8. The case of an "angular momentum-vorticity" interaction. – Now, consider a more complicated case in which the interaction brings the proper angular momentum into play. We consider an energy term of the form $\frac{1}{2} s_{\mu\nu}\,(\partial_\mu u_\nu - \partial_\nu u_\mu)$ – or simply $s_{\mu\nu}\,\partial_\mu u_\nu$ – which is a term that we encountered in the representative fluid for the Dirac equation, along with the ones that depended upon the variable *A*. If we express that energy in the proper system then it will follow that since s^0_{k4} and s^0_{4k} are zero:

$$\tfrac{1}{2}\, s^0_{jk}\,(\partial_j u_k - \partial_k u_j)^0 = \tfrac{1}{2}\, s^0_{jk}\,(\partial_j\, \upsilon_k)^0 - (\partial_k\, \upsilon_j)^0]\ .$$

One sees the "vorticity" tensor of classical hydrodynamics figure in the bracket, which must not be confused with the Lichnerowicz's relativistic vorticity (Appendix *B*), which seems difficult to express in the case of the fluid with spin. If one expresses the spatial tensor s^0_{jk} as a function of spin (Chapter III) then one will get:

$$\tfrac{1}{2}\ \varepsilon_{ijk}\ \sigma^0_i\,(\partial_j\, \upsilon_k)^0 - (\partial_k\, \upsilon_j)^0]\ ,$$

or simply $\sigma^0_i\, \varepsilon_{ijk}\,(\partial_j\, \upsilon_k)^0$.

The Lagrangian that was introduced, which will be a maximum when σ^0_i is parallel to the vector $\varepsilon_{ijk}\,(\partial_j\, \upsilon_k)^0$, thus expresses an interaction that tends to make the spin parallel to the dual of the vorticity. It was such an interaction that Vigier, in an article that is currently going to print, recently brought into play in the case of the wave function of a neutrino. As a function of the vierbein a^ξ_μ, it takes the form:

$$ic\ \rho h_0\ (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu)\,\partial_\mu a^4_\nu\ .$$

The Lagrangian of the fluid is then:

$$\boxed{\mathcal{L} = \rho \mathfrak{M}_0 c^2 + ic\,\rho a^4_\mu \partial_\mu S + ic\,\rho h_0\ a^4_\mu a^1_\lambda\ \partial_\mu a^2_\lambda + ic\,\rho h_0 (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu) + \lambda_{\mu\nu}(a^\xi_\mu a^\xi_\nu - \delta_{\mu\nu}).}$$

In order to obtain the internal angular momentum, one forms the Belinfante tensor by means of the derivatives:

$$\frac{\partial \mathcal{L}}{\partial a^2_{\alpha,\lambda}} = ic\ \rho h_0\ a^4_\lambda a^1_\alpha\,,$$

$$\frac{\partial \mathcal{L}}{\partial a^4_{\alpha,\lambda}} = ic\ \rho h_0\ (a^1_\lambda a^2_\alpha - a^1_\alpha a^2_\lambda)\,.$$

Upon using the infinitesimal Lorentz transformation for the vectors, namely:

$$\mathfrak{L}^{\alpha\beta}_{\mu\nu} = \tfrac{1}{2}\delta^{\alpha\beta}_{\mu\nu},$$

one will then have that:

$$\begin{aligned} f_{\mu\nu\lambda} &= \tfrac{1}{2} ic\,\rho h_0\, a^4_\lambda (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu) + \tfrac{1}{2} ic\,\rho h_0\, a^1_\lambda a^2_\alpha\, \delta^{\alpha\beta}_{\mu\nu} a^4_\beta - \tfrac{1}{2} ic\,\rho h_0\, a^2_\lambda a^1_\alpha\, \delta^{\alpha\beta}_{\mu\nu} a^4_\beta \\ &= \tfrac{1}{2}\rho h_0\,[u_\lambda (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu) + a^1_\lambda (a^2_\mu u_\nu - a^2_\nu u_\mu) - a^2_\lambda (a^1_\mu u_\nu - a^1_\nu u_\mu)], \\ &= \tfrac{1}{2}\rho h_0\,[u_\lambda (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu) + u_\nu (a^1_\lambda a^2_\mu - a^2_\lambda a^1_\mu) - u_\nu (a^1_\lambda a^2_\mu - a^2_\lambda a^1_\nu)]. \end{aligned}$$

Upon introducing the expression for $s_{\mu\nu}$, this will become:

$$f_{\mu\nu\lambda} = \tfrac{1}{2}(s_{\mu\nu}\, u_\lambda + s_{\lambda\mu}\, u_\nu - s_{\lambda\nu}\, u_\mu).$$

The internal angular momentum is given by:

$$\tfrac{1}{2} f_{\mu\nu\lambda}\, u_\lambda = s_{\mu\nu}\ .$$

It is found to be simply the density that relates to the *angular momentum of the particles* (which is vacuously unnecessary in the general case, where the stresses can intervene in the expression for the hydrodynamical angular momentum). It is then orthogonal to the current (which is no longer necessarily the case for any fluid that is composed of particles that each possess this property). We remark that the Belinfante tensor does not reduce to $\tfrac{1}{2}\ s_{\mu\nu}\ u_\lambda$, as it does for the "pure matter" fluid. We will then have to perform a gauge transformation that is expressed by:

$$\varphi_{\mu\nu\lambda} = \tfrac{1}{2}(s_{\mu\nu}\, u_\lambda - s_{\lambda\nu}\, u_\mu),$$

$$\Phi_{\mu\nu\lambda} = s_{\lambda\nu}\ u\ .$$

This is the transformation that we have encountered in the Dirac case.

The Euler-Lagrange equations that are derived from the Lagrangian are:

For the variable ρ:

(V.27)
$$\mathfrak{M}_0\, c^2 + \dot{S} + h_0 a^1_\lambda \dot{a}^2_\lambda + \frac{1}{\rho} s_{\mu\nu} \partial_\mu u_\nu = 0.$$

It results from this that $\mathcal{L} = 0$.

For the variable a^1_μ:

$$ic\,\rho h_0\, a^4_\lambda\, \partial_\lambda a^3_\mu + ic\,\rho h_0\, a^2_\alpha\, (\partial_\mu a^4_\alpha - \partial_\mu a^4_\alpha) + 2\lambda_{\mu\alpha} a^1_\alpha = 0.$$

For the variable a^2_μ:

$$\partial_\lambda (ic\,\rho h_0\, a^4_\lambda a^1_\mu) = -\,ic\,\rho h_0\; a^1_\alpha(\partial_\mu a^4_\alpha - \partial_\alpha a^4_\mu) + 2\lambda_{\mu\alpha} a^2_\alpha = 0.$$

For the variable a^3_μ:

$$2\lambda_{\mu\alpha} a^3_\alpha = 0.$$

For the variable a^4_μ:

$$ic\rho\,\partial_\mu S + ic\,\rho h_0\, a^1_\lambda \partial_\mu a^2_\lambda + 2\lambda_{\mu\alpha} a^4_\alpha = \partial_\lambda[ic\,\rho h_0\,(a^1_\lambda a^2_\mu - a^2_\lambda a^1_\mu)]\,.$$

These equations transform into:

$$0 = \rho h_0\, \dot a^2_\mu + \rho h_0\, a^2_\mu(\partial_\mu u_\alpha - \partial_\alpha u_\mu) + 2\lambda_{\mu\alpha} a^1_\alpha\,,$$

$$0 = -\rho h_0\, \dot a^1_\mu - \rho \dot h_0\, a^1_\mu - \rho h_0\, a^1_\mu(\partial_\mu u_\alpha - \partial_\alpha u_\mu) + 2\lambda_{\mu\alpha} a^2_\alpha\,,$$

$$0 = 2\lambda_{\mu\alpha} a^3_\alpha\,,$$

$$0 = ic\rho\,\partial_\mu S + ic\,\rho h_0\, a^1_\mu \partial_\mu a^2_\lambda - ic\,\partial_\lambda s_{\lambda\mu} + 2\lambda_{\mu\alpha} a^4_\alpha\,.$$

Upon multiplying these equations by a^1_ν, a^2_ν, a^3_ν, a^4_ν, adding them, and taking the antisymmetric part in μ and ν of the result (which will eliminate the $\lambda_{\mu\nu}$), one will get:

$$\begin{aligned}
&\rho h_0(\dot a^2_\mu a^1_\nu - \dot a^1_\mu a^2_\nu) - \rho \dot h_0 a^1_\mu a^2_\nu - \rho h_0(\dot a^2_\nu a^1_\mu - \dot a^1_\nu a^2_\mu) + \rho \dot h_0 a^1_\nu a^2_\nu + \rho u_\nu \partial_\mu S - \rho u_\mu \partial_o S \\
&+ \rho h_0\, a^1_\lambda u_\nu \partial_\mu a^2_\lambda - \rho h_0\, a^1_\lambda u_\mu \partial_\nu a^2_\lambda + \rho h_0[a^2_\alpha a^1_\nu(\partial_\mu u_\alpha - \partial_\alpha u_\mu) - a^2_\alpha a^1_\mu(\partial_\nu u_\alpha - \partial_\alpha u_\nu) \\
&- a^1_\alpha a^2_\nu(\partial_\mu u_\alpha - \partial_\alpha u_\mu) + a^1_\alpha a^2_\mu(\partial_\nu u_\alpha - \partial_\alpha u_\nu)] - u_\nu \partial_\lambda s_{\lambda\mu} + u_\mu \partial_\lambda s_{\lambda\nu} = 0.
\end{aligned}$$

Upon remarking that the first four terms represent the derivative $-\dot s_{\mu\nu}$, and upon contracting with u_μ, this will give:

$$\begin{aligned}
-\,\dot s_{\mu\nu} u_\mu + \rho u_\nu \dot S + c^2 \rho\,\partial_\nu S + \rho h_0(u_\nu a^1_\lambda \dot a^2_\lambda + c^2 a^1_\lambda \partial_\nu a^2_\lambda) + \rho h_0(\dot u_\alpha a^2_\alpha a^1_\nu - \dot u_\alpha a^1_\alpha a^2_\nu) \\
- u_\nu u_\mu\, \partial_\lambda s_{\lambda\mu} - c^2\, \partial_\lambda s_{\lambda\mu} = 0.
\end{aligned}$$

Upon replacing $\dot S$ with its value in (27), one will have:

$$\begin{aligned}
-\rho\,(\mathfrak{M}_0 c^2 u_\nu + h_0 a^1_\lambda \dot a^2_\lambda u_\nu + \tfrac{1}{2}\, s_{\alpha\beta}\,\partial_\alpha u_\beta u_\nu - c^2\,\partial_\nu S) \\
+ \rho h_0 a^1_\lambda(u_\nu \dot a^2_\lambda + c^2 \partial_\nu a^2_\lambda) - c^2 \partial_\nu s_{\lambda\mu}\left(\delta_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) = 0.
\end{aligned}$$

Hence, the expression for the gradient $\partial_\nu S$ will be:

(V.28) $$\rho\,\partial_\nu S = \frac{1}{c^2} s_{\alpha\beta}\,\partial_\nu u_\beta \cdot u_\nu + \eta_{\beta\nu}\,\partial_\alpha s_{\alpha\beta} + \rho\,\mathfrak{M}_0\,u_\nu - \rho h_0\;a^1_\lambda\,\partial_\nu a^2_\lambda\,.$$

We can then construct the canonical energy-momentum tensor. One has:

$$\frac{\partial\mathcal{L}}{\partial S_{,\nu}}\partial_\nu S = \rho\,u_\nu\,\partial_\mu S\,,$$

so, from (28), one will get:

$$\frac{1}{c^2} s_{\alpha\beta}\,\partial_\alpha u_\beta\,u_\mu\,u_\nu + \eta_{\beta\mu}\,\partial_\alpha s_{\alpha\beta} u_\nu + \rho\,\mathfrak{M}_0\,u_\mu\,u_\nu - \rho h_0\,a^1_\lambda u_\nu\,\partial_\mu a^2_\lambda \qquad (?)$$

$$\frac{\partial\mathcal{L}}{\partial a^2_{\lambda,\nu}}\partial_\mu a^2_\lambda = \rho h_0\,u_\nu a^1_\lambda\,\partial_\mu a^2_\lambda\,,$$

which cancels the last term in the preceding expression:

$$\frac{\partial\mathcal{L}}{\partial u_{\lambda,\nu}}\partial_\mu u_\lambda = \rho h_0 (a^1_\nu a^2_\lambda - a^2_\nu a^1_\lambda)\,\partial_\mu u_\lambda = s_{\nu\lambda}\,\partial_\mu u_\lambda\,,$$

so, finally:

$$t_{\mu\nu} = \rho\,\mathfrak{M}_0\,u_\mu\,u_\nu + s_{\nu\lambda}\,\partial_\mu u_\lambda + \partial_\lambda s_{\lambda\mu}\,u_\nu\,.$$

In order to rejoin the Weyssenhoff formalism, we must perform the gauge transformation $t'_{\mu\nu} = t_{\mu\nu} - \Phi_{\mu\nu\lambda}$:

$$\begin{aligned} t'_{\mu\nu} &= \rho\,\mathfrak{M}_0\,u_\mu\,u_\nu + s_{\nu\lambda}\,\partial_\mu u_\lambda + \partial_\lambda s_{\lambda\mu}\,u_\nu - \partial_\lambda (s_{\nu\lambda}\,u_\mu) \\ &= \rho\,\mathfrak{M}_0\,u_\mu\,u_\nu + s_{\nu\lambda}\,(\partial_\mu u_\lambda - \partial_\lambda u_\mu) + u_\mu\,\partial_\lambda s_{\lambda\nu} + u_\nu\,\partial_\lambda s_{\lambda\mu}\,. \end{aligned}$$

One easily decompose this tensor along the current, which will give us:

– The momentum (we suppress the prime):

$$g_\mu = \rho\,\mathfrak{M}_0\,u_\mu + \partial_\lambda s_{\lambda\mu} - \frac{1}{c^2}\partial_\lambda s_{\lambda\nu}\,u_\nu\,u_\mu\,.$$

– The non-kinetic part:

$$\tau_{\mu\nu} = s_{\nu\lambda}\,(\partial_\mu u_\lambda - \partial_\lambda u_\mu) + u_\mu\,\partial_\lambda s_{\lambda\nu} + \frac{1}{c^2}\partial_\lambda s_{\lambda\alpha} u_\alpha\,u_\nu\,u_\mu = s_{\nu\lambda}\,(\partial_\mu u_\lambda - \partial_\lambda u_\mu) + \eta_{\nu\alpha}\partial_\lambda s_{\lambda\alpha} u_\mu\,.$$

The latter tensor can be decomposed according to the Takabayasi formalism into:

– The heat current:

$$q_\nu = -\frac{1}{c^2} s_{\nu\lambda}\dot{u}_\lambda + \eta_{\nu\alpha}\partial_\lambda s_{\lambda\alpha},$$

– and a proper-space tensor:

$$\tau_{\mu\nu} = s_{\nu\lambda}(\partial_\mu u_\lambda - \partial_\lambda u_\mu) + \frac{1}{c^2} u_\mu s_{\nu\lambda}\dot{u}_\lambda = s_{\nu\lambda}(\eta_{\mu\alpha}\partial_\alpha u_\lambda - \partial_\lambda u_\mu),$$

so, upon subtracting the zero term $(1 / c^2)\, u_\alpha u_\mu \partial_\lambda u_\alpha = 0$ we will get:

$$\boxed{\theta_{\mu\nu} = \eta_{\mu\alpha} s_{\nu\lambda}(\partial_\alpha u_\lambda - \partial_\lambda u_\alpha).}$$

This stress tensor expresses precisely a coupling between the proper angular momentum, which is in proper space, and the projection of the vorticity tensor into proper space, which is not contained in proper space, in general.

We can make an important remark in regard to this expression: If one contracts $\theta_{\mu\nu}$ with the vorticity:

$$\theta_{\mu\nu}(\partial_\mu u_\nu - \partial_\nu u_\mu) = s_{\nu\lambda}\,\eta_{\mu\alpha}(\partial_\alpha u_\lambda - \partial_\lambda u_\alpha)(\partial_\mu u_\nu - \partial_\nu u_\mu)$$

then it will follow, upon taking the properties of $\eta_{\mu\alpha}$ into account, that:

$$\begin{aligned}\theta_{\mu\nu}&(\partial_\mu u_\nu - \partial_\nu u_\mu)\\ &= s_{\nu\lambda}\,\eta_{\mu\alpha}\partial_\alpha u_\lambda\,\partial_\mu u_\nu - s_{\nu\lambda}\partial_\alpha u_\lambda\,\partial_\nu u_\alpha - s_{\nu\lambda}\partial_\lambda u_\alpha\,\partial_\mu u_\nu + s_{\nu\lambda}\partial_\lambda u_\mu\,\partial_\nu u_\mu .\end{aligned}$$

The last term is zero, by antisymmetry. The second and third ones will be cancelled after one changes the dummy indices. What will then remain is:

$$\begin{aligned}\theta_{\mu\nu}(\partial_\mu u_\nu - \partial_\nu u_\mu) &= s_{\nu\lambda}\left(\delta_{\alpha\beta} + \frac{u_\mu u_\alpha}{c^2}\right)\partial_\alpha u_\lambda\,\partial_\mu u_\nu\\ &= s_{\nu\lambda}\partial_\mu u_\lambda\,\partial_\mu u_\nu + \frac{1}{c^2} s_{\nu\lambda}\dot{u}_\lambda\dot{u}_\nu .\end{aligned}$$

Both terms are zero, by antisymmetry; one then has:

$$\theta_{\mu\nu}(\partial_\mu u_\nu - \partial_\nu u_\mu) = 0.$$

We know that $\theta_{\mu\nu}$ is not the true stress tensor. We can immediately write:

(V.29) $$\boxed{\theta_{<\mu\nu>}\partial_\mu u_\nu = 0,}$$

which is a relation that persists for the tensor $\theta'_{\mu\nu}$, which has the same antisymmetric part as $\theta_{\mu\nu}$.

We can now perform the transformation that was described in our study of the general case, and which makes the heat current disappear. One then sets:

$$\begin{aligned} g'_\mu &= g_\mu - q_\mu \\ &= \rho\,\mathfrak{M}_0\, u_\mu - \frac{1}{c^2}\partial_\lambda s_{\lambda\nu} u_\nu u_\mu + \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda - \frac{1}{c^2}\partial_\lambda s_{\lambda\alpha} u_\alpha u_\mu \\ &= \rho\,\mathfrak{M}_0\, u_\mu - \frac{2}{c^2}\partial_\lambda s_{\lambda\nu} u_\mu u_\nu + \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda \end{aligned}$$

so:

$$\boxed{g'_\mu = \left[\rho\mathfrak{M}_0 + \frac{1}{c^2} s_{\lambda\alpha}(\partial_\lambda u_\alpha - \partial_\alpha u_\lambda) + \frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda,\right]}$$

which is a relation that brings the proper mass density into consideration:

$$\boxed{\mu_0 = \rho\mathfrak{M}_0 + \frac{1}{c^2} s_{\lambda\alpha}(\partial_\lambda u_\alpha - \partial_\alpha u_\pi)}$$

and the transverse momentum, with its usual expression:

$$p'_\mu = -\frac{1}{c^2} s_{\mu\lambda}\dot{u}_\lambda .$$

One can easily write the torque equation, but it is more interesting to write the three equations to which it reduces by going to the general case directly. One then has, along with the usual equation:

$$\boxed{g'_\mu \sigma_\mu = 0,}$$

that:

$$\begin{aligned} \theta_{\mu\nu}\, s_{\mu\nu} &= -\, s_{\mu\nu}\, s_{\nu\lambda}\, (\eta_{\nu\alpha}\,\partial_\alpha\, u_\lambda - \partial_\lambda\, u_\mu) \\ &= (\sigma_\mu\, \sigma_\lambda - \sigma_0^2\, \eta_{\mu\lambda})\, (\eta_{\nu\alpha}\,\partial_\alpha\, u_\lambda - \partial_\lambda\, u_\mu), \end{aligned}$$

so, upon taking into account that:

$$\eta_{\mu\lambda}\, \sigma_\mu = \sigma_\alpha \qquad \text{and} \qquad \eta_{\mu\lambda}\,\partial_\alpha\, u_\lambda = \partial_\mu\, u_\lambda\,,$$

one gets:

$$\theta_{\mu\nu}\, s_{\mu\nu} = \sigma_\alpha\, \sigma_\lambda\, \partial_\alpha\, u_\lambda - \sigma_\mu\, \sigma_\lambda\, \partial_\lambda\, u_\mu - \sigma_0^2\;\; \eta_{\lambda\alpha}\, \partial_\alpha\, u_\lambda + \sigma_0^2\;\; \eta_{\mu\lambda}\, \partial_\lambda\, u_\mu = 0.$$

It then results that:

$$\sigma_0 \dot{\sigma}_0 = 0;$$

i.e., *the norm of the spin is constant.*

Finally, the equation that gives the spin precession can be written:

$$ic\,\varepsilon_{\mu\nu\alpha\beta}\,s_{\nu\lambda}\,u_\alpha\,(\eta_{\mu\rho}\,\partial_\rho\,u_\lambda - \partial_\lambda\,u_\mu) = c^2\;\eta_{\alpha\beta}\,\dot{\sigma}_\alpha\,.$$

The expression $ic\,\varepsilon_{\mu\nu\alpha\beta}\,s_{\nu\lambda}\,u_\alpha$ transforms into:

$$\delta^{\alpha\beta\mu}_{\gamma\rho\lambda}u_\gamma\sigma_\rho u_\alpha \equiv u_\lambda\,(u_\beta\,\sigma_\mu - u_\mu\,\sigma_\beta) + c^2\,(\delta_{\lambda\beta}\,\sigma_\mu - \delta_{\lambda\mu}\,\sigma_\beta) \equiv c^2\,(\eta_{\lambda\beta}\,\sigma_\mu - \eta_{\lambda\mu}\,\sigma_\beta)\,.$$

In the left-hand side, one then has:

$$\begin{aligned} &c^2\,(\eta_{\lambda\beta}\,\sigma_\mu - \eta_{\lambda\mu}\,\sigma_\beta)\,(\eta_{\mu\alpha}\partial_\alpha\,u_\lambda - \partial_\lambda\,u_\mu) \\ &\qquad \equiv c^2\,(\eta_{\lambda\beta}\,\sigma_\mu\partial_\alpha\,u_\lambda - \eta_{\alpha\lambda}\,\sigma_\beta\partial_\alpha\,u_\lambda - \eta_{\lambda\beta}\,\sigma_\mu\partial_\lambda\,u_\mu + \sigma_\beta\partial_\lambda\,u_\lambda) \\ &\qquad \equiv c^2\;\eta_{\lambda\beta}\,(\sigma_\alpha\partial_\alpha\,u_\lambda - \sigma_\mu\partial_\lambda\,u_\mu). \end{aligned}$$

so:

$$\eta_{\lambda\beta}\,(\sigma_\alpha\partial_\alpha\,u_\lambda - \sigma_\mu\partial_\lambda\,u_\mu - \dot{s}_\lambda) = 0,$$

which one can develop into:

$$\dot{\sigma}_\mu + \frac{1}{c^2}u_\lambda u_\beta\dot{\sigma}_\lambda = \sigma_\alpha\partial_\alpha\,u_\beta - \sigma_\mu\partial_\beta\,u_\mu - \frac{1}{c^2}u_\beta\sigma_\mu\dot{u}_\mu\,.$$

One sees that the last terms on both sides of the equation cancel, which finally gives:

$$\boxed{\dot{\sigma}_\beta = \sigma_\alpha(\partial_\alpha u_\beta - \partial_\beta u_\alpha).}$$

We can also apply relation (29) to the torque equation. Upon contracting it with $\partial_\mu u_\nu$, one will get:

$$\dot{s}_{\mu\nu}\partial_\mu u_\nu = (g'_\mu u_\nu - g'_\nu u_\mu)\,\partial_\mu u_\nu - 2\,\theta_{<\mu\nu>}\,\partial_\mu\,u_\nu = -\,g'_\nu\dot{u}_\nu = 0.$$

One will then have the important relation:

(V.30) $$\boxed{\dot{s}_{\mu\nu}\partial_\mu u_\nu = 0.}$$

We must finally express the corrected stress as:

$$\begin{aligned} \theta'_{\mu\nu} &= \theta_{\mu\nu} + q_\mu\,u_\nu + q_\nu\,u_\mu \\ &= s_{\nu\lambda}\,(\eta_{\mu\alpha}\partial_\alpha\,u_\lambda - \partial_\lambda\,u_\mu) - \frac{1}{c^2}s_{\mu\lambda}\dot{u}_\lambda u_\nu - \frac{1}{c^2}s_{\nu\lambda}\dot{u}_\lambda u_\mu + \eta_{\mu\alpha}\,u_\nu\,\partial_\lambda\,s_{\lambda\alpha} + \eta_{\nu\alpha}\,u_\mu\,\partial_\lambda\,s_{\lambda\alpha}, \end{aligned}$$

which one can easily put into the form:

$$\boxed{\theta'_{\mu\nu} = s_{\nu\lambda}(\partial_\mu u_\lambda - \partial_\lambda u_\mu) - \frac{1}{c^2}s_{\mu\lambda}\dot{u}_\lambda u_\nu + \partial_\lambda s_{\lambda\alpha}(\eta_{\mu\alpha}u_\nu + \eta_{\nu\alpha}u_\mu).}$$

One can calculate the force density:

$$
\begin{aligned}
-\varphi'_\mu &= \partial_\nu \theta'_{\mu\nu} \\
&= \partial_\nu s_{\nu\lambda} (\partial_\mu u_\lambda - \partial_\lambda u_\mu) + s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda + \partial_\nu \partial_\lambda s_{\lambda\alpha} (\eta_{\mu\alpha} u_\nu + \eta_{\nu\alpha} u_\mu) \\
&\qquad + \partial_\lambda s_{\lambda\alpha} (\eta_{\mu\alpha} \partial_\nu u_\nu + \eta_{\nu\alpha} \partial_\nu u_\mu) + \partial_\lambda s_{\lambda\alpha} \\
&\qquad + \left[u_\nu \partial_\nu \left(\frac{u_\mu u_\alpha}{c^2} \right) + u_\mu \partial_\nu \left(\frac{u_\nu u_\alpha}{c^2} \right) \right] - \frac{1}{c^2} \dot{s}_{\nu\lambda} \dot{u}_\lambda - \frac{1}{c^2} s_{\nu\lambda} \ddot{u}_\lambda .
\end{aligned}
$$

The relation that gives the variation of the mass density then gives:

$$
\begin{aligned}
\dot{\mu}_0 c^2 &= u_\mu \partial_\nu \theta'_{\mu\nu} \\
&= \partial_\nu s_{\nu\lambda} \dot{u}_\lambda + s_{\nu\lambda} u_\mu \partial_\nu \partial_\mu u_\lambda - c^2 \eta_{\nu\alpha} \partial_\nu \partial_\lambda s_{\lambda\alpha} - \partial_\lambda s_{\lambda\alpha} \dot{u}_\alpha - \partial_\lambda s_{\lambda\alpha} u_\alpha \partial_\nu u_\nu \\
&= u_\mu (s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda - \partial_\mu \partial_\nu s_{\nu\lambda} u_\lambda) - \partial_\nu s_{\nu\lambda} (\dot{u}_\lambda + u_\lambda \partial_\alpha u_\alpha).
\end{aligned}
$$

In order to transform the first term in this, differentiate the relation:

$$s_{\nu\lambda} u_\lambda = 0$$

by μ, and then by ν, and one will get:

$$s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda + \partial_\nu s_{\nu\lambda} \partial_\mu u_\lambda + \partial_\mu s_{\nu\lambda} \partial_\nu u_\lambda = - s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda .$$

Thus:

$$
\begin{aligned}
\dot{\mu}_0 c^2 &= -\partial_\nu s_{\nu\lambda} \dot{u}_\lambda - \partial_\nu s_{\nu\lambda} u_\lambda \partial_\alpha u_\alpha + u_\mu (2 s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda + \partial_\nu s_{\nu\lambda} \partial_\mu u_\lambda + \partial_\mu s_{\nu\lambda} \partial_\nu u_\lambda) \\
&= - \partial_\nu s_{\nu\lambda} u_\lambda \partial_\alpha u_\alpha + u_\mu \partial_\mu s_{\nu\lambda} \partial_\nu u_\lambda + 2 u_\mu s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda .
\end{aligned}
$$

Since one knows that:

$$\dot{s}_{\nu\lambda} \partial_\nu u_\lambda \equiv \partial_\mu (u_\mu s_{\nu\lambda}) \partial_\nu u_\lambda = 0,$$

it will follow that:

$$
\begin{aligned}
\dot{\mu}_0 c^2 &= - \partial_\nu s_{\nu\lambda} u_\lambda \partial_\mu u_\mu - s_{\nu\lambda} \partial_\nu u_\lambda \partial_\mu u_\mu + 2 u_\mu s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda \\
&= - \partial_\nu (s_{\nu\lambda} u_\lambda) \partial_\mu u_\mu + 2 u_\mu s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda .
\end{aligned}
$$

The first term goes to zero:

$$\boxed{\dot{\mu}_0 c^2 = 2 u_\mu s_{\nu\lambda} \partial_\nu \partial_\mu u_\lambda = s_{\nu\lambda} \frac{d}{d\tau} (\partial_\nu u_\lambda - \partial_\lambda u_\nu).}$$

One should compare this relation with the one that is obtained by differentiating the expression for $\mu_0 c^2$, namely:

$$\dot{\mu}_0 c^2 = \rho \dot{M}_0 c^2 + \frac{d}{d\tau}[s_{\nu\lambda}(\partial_\nu u_\lambda - \partial_\lambda u_\nu)],$$

or, upon taking (30) into account:

$$\dot{\mu}_0 c^2 = \rho \dot{M}_0 c^2 + s_{\nu\lambda}\frac{d}{d\tau}(\partial_\nu u_\lambda - \partial_\lambda u_\nu).$$

One then sees that this says simply that the proper mass $\mathfrak{M}_0$ that corresponds to the pure matter fluid, in particular, is *constant.* In other words, the proper mass that relates to the *kinetic* energy of the internal motion of the drop is constant in the course of motion. The only thing that varies (by being a function of the work that is done by the stresses) is the proper mass that is related to the *potential* energy of the drop in rotation in the stress field. Its variation will depend uniquely upon the variations of the local vorticity, which is coupled to the internal angular momentum. We remark that since s_{4k}^0 and s_{k4}^0 are zero in the proper system, the vorticity that enters into consideration will be, in fact, the proper-space tensor of non-relativitistic hydrodynamics, and not Lichnerowicz's relativistic vorticity.