"Das Hamiltonsche Prinzip bei nichtholonomen Systemen," Math. Ann. **111** (1935), pp. 94-97.

Hamilton's principle for non-holonomic systems

by

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In the paper "Le principe de Hamilton et l'holonomisme," in Prace mit.-fiz. **38**, Kerner has proved the theorem:

Holonomity is the necessary and sufficient condition for Hamilton's principle to be true,

or, more precisely:

Holonomity is the necessary and sufficient condition for the correct equations of mechanics to be equivalent to the equations that are obtained from the Lagrangian method of variational calculus with supplementary conditions with the help of parameters.

It is good to add that:

The former equations are completely equivalent to the latter ones, and not, say, obtained by specializing the additional parameter.

Otherwise, the theorem can be false.

Geometrically, the following fact is fundamental: As is well-known, non-holonomic conditions do not reduce the dimension of the space of motion. On the contrary, this is not true for the direction of motion. Thus, the neighboring paths that one must consider in the calculus of variations are in a space of higher dimension than the ones that one arrives at by supplementary displacements. With that, it can happen that these displacements involve a choice of paths in the calculus of variations, which likewise, along with the starting paths, are themselves the correct paths of mechanics, while the theorem of Kerner extends the identity and thus the lowering of the dimension count.

In the following, it will be shown:

1. The Hamilton principle is always correct for a correct formulation, which is an old, but less mentioned theorem.

2. The theorem of Kerner may be proved by my method using brief, effortless calculations (see: "Die Lagrange-Eulerschen Gleichungen der Mechanik," in Zeitschr. f.

Math. u. Phys. **50**, "Über die virtuellen Verschiebungen in der Mechanik," Math. Annalen **70**, and "Über nichtholonomen Systemen," Math. Annalen **92**).

3. It can be false, when one does not observe the additional remark above.

§ 1

Hamilton's principle

From d'Alembert's principle, in Lagrangian form:

$$S\, dm\, \overline{w}\, \delta\overline{r} = S\, d\overline{k}\, \delta\overline{r}$$

(S means one sums over the system, $\overline{r}$ is the position vector, dm, the mass element, $\overline{w}$, the acceleration vector and $d\overline{k}$ is the vector of applied force), what follows, with the always supplementary assumption that:

$$d\, \delta\overline{r} - \delta\, d\overline{r} = 0,$$

is the central Lagrangian equation:

$$\frac{d}{dt} S\, dm\, \overline{v}\, \delta\overline{r} - \delta(E - U) = 0,$$

where $\overline{v}$ refers to the velocity, E, to the kinetic energy, and U, to the potential energy that are assumed to be present. It yields, by integrating over the time interval from t_1 to t_2, at whose ends the virtual displacements shall be zero:

$$\int_{t_1}^{t_2} \delta L\, dt = 0 \qquad\qquad (L \equiv E - U),$$

hence, Hamilton's principle. The variations must therefore be regarded as supplementary displacements here, from which, the neighboring paths that one arrives at will not be supplementary to the rule. For the sake of simplicity, we take the system to be scleronomic.

Let the Lagrangian coordinates be $q_1, q_2, \ldots, q_n$. In place of the $\dot{q}$, we think of there being n linearly independent couplings introduced between them:

$$\frac{d\vartheta_i}{dt} \equiv \omega_i = \sum_{s=1}^{n} b_{i,s}\dot{q}_s \qquad \text{or, when solved:} \qquad \dot{q}_s = \sum_{i=1}^{n} c_{s,i}\omega_i\,,$$

which we can do in such a way that the non-holonomic condition equations become:

$$\omega_{k+1} = 0, \qquad \omega_{k+2} = 0, \ldots, \omega_n = 0 \qquad\qquad k < n.$$

The commutation equations may be:

$$d\,\delta\vartheta_\sigma - \delta\,d\,\vartheta_\sigma = \sum_{i,s}\beta_{i,s,\sigma}\delta\vartheta_i\,d\vartheta_s\,.$$

They, together with Hamilton's principle, immediately give the correct equations of motion for $k < n$ (since $\delta\vartheta_\sigma = 0$ for $\sigma > k$)

(I) $$\frac{dp_i}{dt} + \sum_{s,m}\beta_{i,s,m}\omega_s p_m - \sum_s \frac{\partial L}{\partial q_s}c_{s,i} = 0, \qquad i = 1, 2, \ldots, k.$$

The $p_m = \partial L / \partial\omega_h$ are the impulses. On the contrary, the problem of the calculus of variations:

$$\delta\int L\,dt = 0,$$

with the supplementary conditions $\omega_{k+1} = 0$, $\omega_{k+2} = 0$, …, $\omega_n = 0$, gives:

$$\int \delta(L + \sum_{k+1}^{n}\lambda_\nu\omega_\nu)\,dt = 0$$

or:

$$\int\left[\sum_m (p_m + \lambda_m)\,\delta\omega_m + \sum_\nu \frac{\partial L}{\partial q_\nu}\delta q_\nu\right]dt = 0,$$

where $\lambda_1 = 0$, $\lambda_2 = 0$, …, $\lambda_k = 0$, and the other λ are the Lagrangian parameters. As before, it now follows that the $\delta\vartheta_i$ are all to be treated as arbitrary:

(II) $$\frac{d}{dt}(p_i + \lambda_i) + \sum_{s,m}\beta_{i,s,m}\omega_s(p_m + \lambda_m) - \sum_s \frac{\partial L}{\partial q_s}c_{s,i} = 0,$$

which is now true for all i.

§ 2.

Proof of Kerner's theorem

Should (I) and (II) both be correct, it then follows from a comparison of the first k equations that:

$$\sum_{s,m}\beta_{i,s,m}\omega_s\lambda_m = 0 \qquad \text{for} \qquad \begin{matrix} i = 1, 2, \cdots, k, \\ s = 1, 2, \cdots, k, \end{matrix}$$

or, since the values of ω_s can be given freely at any location:

(III) $$\sum_{m=k+1,k+2,\cdots,n} \beta_{i,s,m}\lambda_m = 0 \qquad s \text{ and } i = 1, 2, 3, \ldots, k.$$

Thus, the λ give the differential equations:

$$\frac{d}{dt}(p_i + \lambda_i) + \sum_{s,m} \beta_{i,s,m}\omega_s(p_m + \lambda_m) - \sum_s \frac{\partial L}{\partial q_s} c_{s,i} = 0, \qquad i = k+1, k+2, \ldots, n.$$

Now, should a complete identity between both systems of equations exist in the sense described in the introduction, then the λ, which can be chosen freely at any location, are subject to no finite restrictions like (III), such that one must have:

$$\beta_{i,s,m} = 0 \qquad \text{for } s \text{ and } i = 1, 2, \ldots, k \qquad \text{and} \qquad m = k+1, k+2, \ldots, n.$$

However, the commutation equations for the last ϑ then take the form:

$$d\,\delta\vartheta_\sigma - \delta\, d\vartheta_\sigma = \sum_{i,s=k+1,k+2,\ldots,n} \beta_{i,s,\sigma}\delta\vartheta_\sigma d\vartheta_s, \qquad s = k+1, k+2, \ldots, n.$$

From a theorem of Frobenius (Crelle **82**, pp. 267, see also the Enzyklopädie der math. W. II, A.5, 15, pp. 319, rem. 90), this is, however, sufficient for us to conclude the integrability of our equations of condition. Kerner's theorem is thus proved.

§ 3.

A counter-example

The following example shall show that the theorem is false when one does not require the complete identity.

Let:

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \omega^2),$$

with the particular non-holonomic condition:

$$\omega \equiv \dot{q}_3 + q_1\dot{q}_3 = 0.$$

Because, from a well-known theorem (see my first paper, pp. 25) one can set $\omega = 0$ from the outset, the correct equations of motion read:

$$\ddot{q}_1 = 0, \qquad \ddot{q}_2 = 0, \qquad \omega = 0.$$

On the contrary, the equations of the variational problem read:

$$\ddot{q}_1 - \lambda \dot{q}_2 = 0, \quad \frac{d}{dt}(\dot{q}_2 + \lambda q_1) = 0, \qquad \frac{d}{dt}\lambda = 0, \qquad \omega = 0,$$

such that λ = const. If one now adds the restriction that $\lambda = 0$ then one obtains the correct equations of motion.

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