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# The Lagrange-Euler equations of mechanics

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## Introduction

Mathematicians have frequently addressed the study of non-holonomic (i.e., non-integrable) constraint equations in mechanics in the last twenty years. First, one deals with demonstrating the presence of such constraint equations and showing how they could be treated in kinetic problems. However, one then asks, above all else, to what extent the fundamental laws of mechanics can still be applied to them. As long as the principle of virtual work, the so-called **Hamilton** principle, and the principle of least action come into question, everything can probably be explained; it would suffice to recall the investigations of **Voss**, **Routh**, **C. Neumann**, **Hertz**, and **Hölder** ([1]).

Moreover, **Lagrange** ([2]) knew of non-holonomic constraint equations, if only in statics. Although he possessed all of the means to also solve problems with non-holonomic constraint equations in form of the transitivity equations and his central equation, he still did not address those problems.

The following two questions demand to be resolved:

1. Can one (at least, under certain assumptions) still apply the **Lagrange** equations when non-holonomic constraint equations appear, and indeed in such a way that in the expression for the *vis viva* $T$, the non-holonomic constraint equations are employed in order to eliminate coordinates whose derivatives alone enter into $T$?

2. Which equations enter in place of the **Lagrange** equations in the general case?

The fact that first question can generally be answered with "no" was known already to **C. Neumann** ([3]); up to now, **Hadamard** ([4]) probably examined it at the most fundamental level. However, there are two reasons why I regard the question as still not having been resolved: First of all, **Hadamard**'s conditions are only also necessary for its confirmation, in general. However, as we will see, we can cast the correct light upon the first question only when we connect it with the second one.

Answers to the second question exist in the literature already. **Voss** treated the general problem in his cited paper, but only with the use of **Lagrange** multipliers ([5]). Special problems were frequently addressed with special methods by various authors. **Appell** ([6]) once more took up the resolution of the question in general; however, the first form that he gave to it was followed through only slightly. An expression entered in

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([1]) **Voss**, "Über die Differentialgleichungen der Mechanik," Math. Ann. **25** (1884).
**Routh**, *Advanced Rigid Dynamics*, 1884, pp. 329, § 445 in the German translation. See also the remark of **F. Klein** in the German translation, pp. 534.
**C. Neumann**, "Grundzüge der analytischen Mechanik, insbesondere der Mechanik starrer Körper," Leipziger Berichte 1887 and 1888.
**Hertz**, "Die Prinzipien der Mechanik," Werke III, Leipzig, 1894.
**Hölder**, "Über die Prinzipien von Hamilton and Maupertuis," Göttinger Nachrichten, 1896, pp. 122.
See also **Richard Greiner**, "Über die Einführung der Bedingung in das Hamiltonsche Prinzip," Dissertation (Teubner, 1901).

([2]) **Lagrange**, *Mécanique analytique*, t. I. Part One, Sect. IV, § 11, no. 13, pp. 77. (Bertrand's third problem).

([3]) In addition to the aforementioned presentation, see **C. Neumann**, "Über die rollende Bewegung eines Körpers auf einer gegebenen Horizontalebene unter dem Einfluß der Schwere," Math. Ann., Bd. 27, and Leipziger Berichte 1885.

([4]) **Hadamard**, "Sur les mouvements de roulement," Mémoire de la société des sciences physiques et naturelles de Bordeaux. (4), t. V, 1895. Reprinted in **Appell** "Les mouvements de roulement en Dynamique," Scientia no. 4, 1899.

([5]) The paper of **Korteweg** "Über eine ziemlich verbreitete unrichtige Behandlungsweise eines Problemes der rollender Bewegung…" Nieuw archief voor wiskunde, 1899, takes essentially the same viewpoint.

([6]) See the previously-cited monograph in the collection Scientia, and no. 23, in particular. One will find **Appell**'s further papers on this topic mentioned in the article by **Voss** in *der Enzyklopädie der Mathematischen Wissenschaften*, IV, 1, no. 38, pp. 82.

place of $\partial T / \partial q_i$ that **Appell** denoted by $R_i$ , but with no further discussion. But what does the $R_i$ mean? **Appell**'s second attempt, in which introduced a new function $S$ of the acceleration in place of the *vis viva*, cannot be methodologically satisfying, despite its esthetic advantages. On the one hand, it required the transformation of the second derivatives of the coordinates, but then the *vis viva* $T$ completely lost its dominant role for him, such that the systems with non-holonomic constraint equations were separated from the systems with holonomic constraint equations by a deep chasm, which does not correspond to the difference between the two problems. Finally, the holonomic constraint equations define only one special case of the non-holonomic ones.

On that basis, the relationship between the general constraint equations and **Lagrange**'s equations should also emerge clearly, such that the latter will emerge as a special case. Such a general form for the equations of motion of mechanics is possible; exhibiting that form should be the first priority of the following examinations. I call those equations the **Lagrange-Euler** equations (see § **6**); they are denoted by (IV) [(IV′), resp.] in the following text. I came across them when I posed the following very general question.

*Which equations will enter into mechanics in place of **Lagrange**'s equations when I introduce any n independent linear couplings* $\omega_1, \ldots, \omega_n$ *of the* $dq_\lambda / dt$ *in place of the n position-dependent parameters (viz., **Lagrangian** coordinates)* $q_1, \ldots, q_n$ *, along with the latter parameters*?

One can find that question being posed before in a paper by **Boltzmann** ([1]), which I learned about only during the publication of this article from a cordial communication by the author. Boltzmann also gave essentially the correct answer already ([2]): The **Lagrange-Euler** equations demand that ([3]). The way that one obtains them led me from the *principle of virtual displacements* to **Lagrange**'s *central equation* (§ **4**) and to the use of the *transitivity equations* (§ **2**) for achieving the desired objective (§ **5**). However, seeing that objective clearly, in general, and in detail was first made possible in light of **Lie**'s ideas: The study of the **Lagrange-Euler** equations led me ever deeper into the theory of groups. Therefore, I have referred to the initially-formal connection between my research and that of **Lie** already in § **3** ([4]).

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([1]) **Boltzmann**, "Über die Frage der Lagrangeschen Gleichungen für nicht-holonome, generalisierte Koordinate," Sitzungsberichte der Wiener Akademie, Bd. CXI, Abt. IIa, Dec. 1902.

([2]) *Loc. cit.*, page 1612, equations (24). However, those equations are still not formed completely. By contrast, they are more general than the Lagrange-Euler equation in regard to the presence of time.

([3]) See the remark on page 15 in regard to the connection between the Lagrange-Euler equations and the completely-general equations that **Lagrange** and **Poisson** derived by transformation.

([4]) I cannot see clearly the extent to which **Lie** himself has thought of applying his ideas to mechanics. I found no applications to mechanics in the work that **Engel** and **Scheffers** produced; indeed, the introduction to the third section of *Transformationsgruppen* that **Lie** himself wrote contained the statement on page VII: "The principles of mechanics have a group-theoretic origin," although the meaning of that was a riddle to this reader. I would almost like to believe that **Lie** was thinking of only the theory of integrating the differential equations of mechanics, the connection between **Jacobi**'s canonical substitution with his contact transformations, and an extension of the theory of geodetic lines. The following changes in the cited location no longer allow me to suspect that. The meaning of the sentence "Kinematics and its laws can, in part, be assigned to some entirely special cases of my general theorems" is especially difficult for me to fathom, due to the "in part." The

The fact that addressing the general question that was posed above was also made possible by a systematic resolution of the first question that was posed on page 2 needs to be explained. §§ **7** and **8** in the present paper deal with the response to that.

The fact that the **Lagrange-Euler** equations had already been posed for some special cases (if one ignores the aforementioned general investigations by **Boltzmann**) should not be amazing. Yet, the *impulse equations* belong to them, as well as the ordinary *Euler equations of the rigid body that rotates about a fixed point*; one will get the former when one replaces $\omega_1$, $\omega_2$, $\omega_3$ with $\pi$, $\kappa$, $\rho$, resp., which are the components of the rotation vector with respect to three axes that are fixed in space, but the latter when one introduces $p$, $q$, $r$, namely, the components of the rotation vector with respect to three axes that are fixed in the body, in place of $\omega_1$, $\omega_2$, $\omega_3$. One has **Lagrange** ([1]) to thank for the derivation of **Euler**'s equation in the general form, which he also obtained in the manner that was outlined above. I see a second major point in the following considerations in an investigation of the systematic positions that the impulse equations and **Euler**'s equation assume in mechanics. I will characterize them group-theoretically and show *inter alia* that **Euler**'s equations exist for any mechanical system in a certain extended sense, but not the impulse equations (§ **9** and § **10**).

To my knowledge, the **Lagrange-Euler** equations have been exhibited in two special cases; however, the link to the group-theoretic viewpoint was still not exhibited. The equations that **C. Neumann** gave in the cited paper in the Annals under no. 45 on page 492 belong to those equations, as well as the equations that **Carvallo** ([2]) presented in his award-winning paper on the rolling of a body on a plane. However, except for the special case of wheels, his equations were not developed as far as those of **Neumann**, and in that regard, they do not imply any progress from Appell's first equations, and go beyond them only in regard to the general concept of the velocity parameters that was employed.

My own general impulse equations (§ **9**) are characterized by the fact that $T$ admits the infinitesimal transformations that correspond to constant infinitesimal, and in fact, virtual, values of the velocity parameters. If we set $d\vartheta_\lambda = \omega_\lambda \cdot dt$ then the variation of $T$ will be performed here in such a way that $d\delta\vartheta_\lambda = 0$. By contrast, one will get the general **Euler** equations (§ **10**) when $T$ admits the transformations above, but under the assumption that the velocity parameters themselves remain unchanged under the variation, so one sets $\delta\omega_\lambda = 0$ – i.e., $\delta d\vartheta_\lambda = 0$. Now, the remarkable relation exists between the two assumptions $d\delta\vartheta_\lambda = 0$ and $\delta d\vartheta_\lambda = 0$ for rigid bodies that they will both say the same thing when one replaces the $\omega$ with $\pi$, $\kappa$, $\rho$ in the first case and $p$, $q$, $r$ in the second. Hence, in § **11**, I shall pose the more general question: *When can one take the $\omega$ to $\omega'$ by a linear transformation with variable coefficients in such a way that the*

papers that **Lie** cited by **Painlevé** (Comptes rendus **114** and **116**), **Staude** (Leipziger Berichte 1892 and 1893), and **Stäckel** (Leipziger Berichte 1893, 1897, Crelle's Journal **107**, Comptes rendus **119**) probably have hardly any points in common with my own investigations. By contrast, let it be emphasized that **Hadamard** had referred to the connection between his studies and **Lie**'s theory of groups in a second brief note: "Sur certains systèmes d'équations aux différentielles totales" (Procès-verbaux des séances de la société des sciences physique et naturelles de Bordeaux, 1894-95, reprinted in the aforementioned volume of Scientia).

([1]) **Lagrange**, *loc. cit.*, t. II, Section IX. Chap. I, § II, no. 22, pp. 208.

([2]) **Carvallo**, "Théorie du mouvement du monocyle et de la bicyclette," Journal de l'École Polytechnique Paris, II Série, Cahiers 5, 6 (1900 and 1901), and in particular, cahier 6, no. 72, pp. 36.

*assumption* $d\delta\vartheta_\lambda = 0$ *will go to* $\delta d\vartheta'_\lambda = 0$*, and the impulse equations will also be transformed into Euler's equations by it?* The answer is very simple and noteworthy: "The infinitesimal transformations that correspond to the $\delta\vartheta_\lambda$ must generate a finite *n*-parameter group. Furthermore, that condition is also sufficient." On top of that, one also gets that the group that belongs to the $\omega'$ is precisely the reciprocal group of the given one (§ **11**).

For rigid bodies, the coefficients of the aforementioned transformation will be the direction cosines of the one axis-cross with respect to the other one. If one asks when the coefficients have that character in general then one will encounter a special class of groups – I call then *rotation groups* – that are characterized by the fact that **Lie**'s composition constants $c_{i,k,s}$ will admit the infinitesimal transformations of a cyclic permutation of their indices for a certain choice of infinitesimal transformation. I shall then call a mechanical system with $n$ degrees of freedom that admits an $n$-parameter rotation group a *rigid body with n degrees of freedom* (§ **12**) ([1]).

Since the "rigid bodies" that belong to the same rotation group, relative to which the mathematical steps are closely related, in the sense that their purely-kinematical equations can be converted into other ones by a point transformation, while their Euler equations will be identical in their form, only one type of "rigid body" will belong to any rotation group.

That justifies the fact that I will deal with the rotation groups only in the last paragraphs. When I use the beautiful investigations of **Killing** and **Cartan** ([2]) as my foundation, I will succeed in dispatching the real rotation groups up to a certain degree; only one complication remains to be overcome for the complex ones, which I will refer to in § **14**. Finally, I will exhibit the only types of "rigid bodies" of less than eight degrees of freedom that can be represented by systems that are composed of independent, ordinary, rigid bodies and points with translational motions.

That is the content of the present paper in brief. I would like to remark that the purely-mechanical part can be understood with no deeper knowledge of group theory, so it will almost suffice for one to know about the concepts of "infinitesimal transformation" and "group." When I require further theorems from group theory at other points, I will cite the two books: **Lie-Engel**, *Theorie der Transformationsgruppen*, in three parts, Leipzig, Teubner, 1888-1893, and **Lie-Scheffers**, *Kontinuierliche Gruppen*, Leipzig, Teubner, 1893, and indeed I shall briefly refer to them by the names of their authors in what follows. Similarly, I shall refer to the monograph of **Klein** and **Sommerfeld**, *Über die Theorie des Kreisels*, Leipzig, Teubner, 1897, which I will frequently cite for the more convenient enlightenment of the reader when I treat the rigid body as an example.

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([1]) Moreover, I remark that despite those statements, I do not abandon the mechanics of three-dimensional space at any point.

([2]) Bibliography on page 50.

## CHAPTER I
## Geometric-kinematical considerations

**§ 1. Introduction of new velocity parameters. –** We consider a mechanical system with $n$ degrees of freedom. Let $\bar{x}$ be the vector that determines the position of an arbitrary system point at any time, and let $q_1, \ldots, q_n$ be $n$ independent **Lagrangian** coordinates that establish the position of the system uniquely, while $a$, $b$, $c$ are three parameters that allow one to characterize the individual points of the system. (One can regard $a$, $b$, $c$ as, say, the three rectangular coordinates of the point at any arbitrary, but well-defined and possible, position of the system.) One then has ([1]):

$$\bar{x} = \bar{x}\,(a, b, c; q_1, \ldots, q_n),$$

along with the velocity vector:

$$\bar{v} \equiv \frac{d\bar{x}}{dt} \equiv \dot{\bar{x}} = \sum_{\lambda=1}^{n} \frac{\partial \bar{x}}{\partial q_\lambda} \dot{q}_\lambda \,, \tag{1}$$

when we denote time by $t$ and the derivative with respect to time by a dot.

We shall defer the introduction of non-holonomic constraint equations until later.

We now think of new velocity parameters $\omega_\rho$ being defined in place of the $\dot{q}_\lambda$ by the linear equations:

$$\dot{q}_\lambda = \sum_\rho \xi_{\rho,\lambda}\, \omega_\rho \,. \tag{A}$$

The $\xi_{\rho,\lambda}$ (like all of the functions that will occur, moreover) shall be regular functions of the $q$ for the motion in the domain that comes under consideration, but otherwise arbitrary up to the condition that the determinant:

$$|\, \xi_{\rho,\lambda} \,|$$

must be non-zero. That restriction obviously says nothing but the fact that the $\omega_\rho$ are in a position to describe the velocity state completely.

We can then solve equations (A) for the $\omega$ and thus obtain:

$$\omega_\rho = \sum_\lambda \pi_{\rho,\lambda}\, \dot{q}_\lambda \,. \tag{A′}$$

The relations:

$$\sum_\rho \xi_{\rho,\lambda}\, \pi_{\rho,\lambda} = \delta_{\lambda,\,\mu} \tag{2}$$

exist between $\xi$ and $\pi$, in which $\delta_{\lambda,\,\mu}$ is equal to zero or 1 according to whether $\lambda$ and $\mu$ are different or equal, resp.

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([1]) We shall overlook system couplings that include time here; therefore, time will not enter explicitly into $\bar{x}$ .

If we introduce the $\omega$ into equation (1) then $\bar{v}$ will take on the form:

$$\bar{v} = \sum_\rho \bar{e}_\rho \, \omega_\rho \, , \tag{3}$$

in which:

$$\bar{e}_\rho = \sum_\lambda \frac{\partial \bar{x}}{\partial q_\lambda} \xi_{\rho,\lambda} \tag{4}$$

or

$$\frac{\partial \bar{x}}{\partial q_\lambda} = \sum_\rho \bar{e}_\rho \cdot \pi_{\rho,\lambda} \, . \tag{4′}$$

For the sake of simplicity, we would like to assume here that the $\xi$ do not depend upon time explicitly; that does not represent an essential assumption, though.

The fact that we have introduced the $\omega$ into equations (A) linearly might be justified on the grounds of need and convenience. In the few cases that have been treated already (see the Introduction), one must always deal with linear equations, and the non-holonomic constraint equations, of which we will speak later on, will always be linear. Whether one can be in complete agreement with **Hertz**'s proof that only linear constraint equations are possible will not be discussed here.

## § 2. – The transitivity equations.

If we let $\delta q_\lambda$ denote a virtual, infinitely-small change in the coordinate $q_\lambda$ then we shall let the virtual displacements $\delta\vartheta_\rho$ be defined by the equations:

$$\delta\vartheta_\rho = \sum_\rho \pi_{\rho,\lambda} \, \delta q_\lambda \, , \tag{A″}$$

which are analogous to (A′), but we will not say that there must be coordinates $\vartheta_\rho$. Nevertheless, in what follows, we will frequently use the notation:

$$\frac{d\vartheta_\rho}{dt} = \omega_\rho ,$$

in place of $\omega_\rho$ .

However, when position-determining coordinates $\vartheta_\rho$ do exist, the known equations from the calculus of variations:

$$d \, \delta\vartheta_\rho = \delta \, d \, \vartheta_\rho$$

will then exist, as well. However, when that is not the case, some other equations will enter in place of these equations that we, with **Heun** ([1]), would like to call the

([1]) **Heun**, "Die Bedeutung des d'Alembertschen Prinzips für starre Systeme und Gelenkmechanismen," Archiv der Mathematik und Physik III, 2, pp. 300.

*Lagrangian transitivity equation*, since **Lagrange** ([1]) exhibited them already for the case of a rigid body that rotates around a fixed point. (One employs three $\omega$ here, namely, the three components of the Eulerian rotation vector. Furthermore, **Kirchhoff** [in his *Mechanik*, pp. 59, equations 9] and **C. Neumann** [in the cited paper] have also derived the transitivity equations for that case.)

In our general case, it follows from (A″) that:

$$d\,\delta\vartheta_\rho = \sum_\lambda \pi_{\rho,\lambda}\, d\,\delta q_\lambda + \sum_{\lambda,\sigma} \frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} dq_\sigma \delta q_\lambda\,;$$

by contrast, (A′) implies that:

$$\delta\, d\vartheta_\rho = \sum_\lambda \pi_{\rho,\lambda}\, d\,\delta q_\lambda + \sum_{\lambda,\sigma} \frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} \delta q_\sigma dq_\lambda\,.$$

If we subtract these two equations and observe that:

$$d\,\delta q_\lambda - \delta\, dq_\lambda = 0$$

then that will yield:

$$d\delta\vartheta_\rho - \delta\, d\vartheta_\rho = \sum_{\lambda,\sigma}\left(\frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial \pi_{\rho,\sigma}}{\partial q_\lambda}\right) dq_\sigma \delta q_\lambda$$

when we switch the $\lambda$ and $\sigma$ in the last sum in the second equation. If we now introduce $d\vartheta$ and $\delta\vartheta$ in place of $dq_\sigma$ and $\delta q_\lambda$ using equations (A) then we will get:

(I) $$d\delta\vartheta_\rho - \delta\, d\vartheta_\rho = \sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\, \delta\vartheta_\mu\, d\vartheta_\nu\,,$$

in which we have set:

(5) $$\beta_{\mu,\nu,\rho} = \sum_{\sigma,\lambda}\left(\frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial \pi_{\rho,\sigma}}{\partial q_\lambda}\right) \xi_{\mu,\lambda}\, \xi_{\nu,\rho}\,.$$

*Equations (I) are the desired transitivity equations;* they express the difference $d\delta\vartheta_\rho - \delta\, d\vartheta_\rho$ as a bilinear function of the $d\vartheta$ and $\delta\vartheta$; the coefficients $\beta$ are connected with the $\pi$ (the $\xi$, resp.) by the relations (5).

It is only when the $\beta$ are zero that the $\vartheta$ will be actual coordinates, and the $\beta$ will be zero only when:

$$\frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial \pi_{\rho,\sigma}}{\partial q_\lambda} = 0,$$

so when the $\vartheta$ are actual coordinates. We can then solve equations (5) and get:

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([1]) **Lagrange**, *Mécanique analytique*, t. II, Part Two, Section IX, Chap. I, § **1**, pp. 200.

(5′) $$\frac{\partial \pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial \pi_{\rho,\sigma}}{\partial q_\lambda} = \sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\, \pi_{\mu,\nu}\, \pi_{\nu,\rho} \,.$$

## § 3. Connection with Lie's theory of groups

It would be useful to discuss the formal connection with **Lie**'s theory of groups here.

We can regard equations (A) and the more general one for virtual displacements:

(A‴) $$\delta q_\lambda = \sum_\rho \xi_{\rho,\lambda}\, \delta\vartheta_\rho$$

as the equations for infinitesimal transformations. The $\delta\vartheta_\rho$ are the infinitesimal changes in the parameters, so they can be regarded as constant here; the $\xi_{\rho,\lambda}$ are the quantities that **Lie** likewise denoted by $\xi_{\rho,\lambda}$.

That concept corresponds to the *symbol of the* $\rho^{th}$ *infinitesimal transformation:*

(B) $$X_\rho f = \sum_\lambda \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda} \,.$$

We have still not said whether or not these $n$ generate a *finite group*. However, two possible displacements – i.e., ones that are compatible with the constraints of the given system – will probably again yield another one, and as a result, the $n$ partial differential equations $X_\rho f = 0$ must define a complete system; i.e., relations of the form:

$$(X_\rho, X_\sigma) = \sum_\tau \gamma_{\rho,\sigma,\tau} X_\tau \,,$$

in which $(X_\rho, X_\sigma)$ means the **Jacobi** bracket symbol. That is obvious here, moreover, since the $(X_\rho, X_\sigma)$, which are linear in the $\partial f / \partial q_\lambda$ ($\lambda = 1, 2, \ldots, n$) must naturally also be capable of being expressed linearly in terms of the $n$ completely-independent $X_\tau$.

I now assert that the $\gamma$ are nothing but the $\beta$ that were introduced above (§ **2**). Namely, one has:

$$q_\lambda + dq_\lambda = q_\lambda + \sum_\nu X_\nu \, d\vartheta_\nu \,,$$

in which the operation $X_\nu$ shall refer to $f = q_\lambda$, as in what follows. Furthermore:

$$q_\lambda + dq_\lambda + \delta q_\lambda + \delta dq_\lambda = q_\lambda + \sum_\nu X_\nu \, d\vartheta_\nu + \sum_\mu X_\mu \, \delta\vartheta_\mu + \sum_{\nu,\mu} X_\mu X_\nu \, d\vartheta_\nu \, \delta\vartheta_\mu + \sum_\nu X_\nu \, \delta d\vartheta_\nu \,,$$

whereas:

$$q_\lambda + \delta q_\lambda + dq_\lambda + d\delta q_\lambda = q_\lambda + \sum_\mu X_\mu \, \delta\vartheta_\mu + \sum_\nu X_\nu \, d\vartheta_\nu + \sum_{\nu,\mu} X_\nu X_\mu \, d\vartheta_\nu \, \delta\vartheta_\mu + \sum_\mu X_\mu \, d\delta\vartheta_\mu \,.$$

Since the left-hand sides are equal to each other (since $d\delta q = \delta dq$, by the conventions of the calculus of variations), the right-hand sides must also be equal. When one introduces the **Jacobi** symbol:

$$(X_\nu , X_\mu) = X_\nu X_\mu - X_\mu X_\nu ,$$

and subtracts the first equation from the second one, one will get:

$$0 = \sum_{\nu,\mu} (X_\nu , X_\mu)\, d\vartheta_\nu\, \delta\vartheta_\mu + \sum_\nu X_\nu\, (d\delta\vartheta_\nu - \delta d\vartheta_\nu)\,,$$

or when one employs the formula $(X_\nu , X_\mu)$ and inverts the summation indices in the last sum:

$$0 = \sum_{\nu,\mu,\rho} \gamma_{\nu,\mu,\rho}\, d\vartheta_\nu\, \delta\vartheta_\mu X_\rho + \sum_\rho (d\delta\vartheta_\rho - \delta d\vartheta_\rho) X_\rho\ .$$

However, $X_\rho (q_\lambda) = \xi_{\rho,\,\lambda}$ ; it then follows that:

$$\begin{aligned} d\,\delta\vartheta_\rho - \delta\, d\,\vartheta_\rho &= - \sum \gamma_{\nu,\mu,\rho}\, d\vartheta_\nu\, \delta\vartheta_\mu \\ &= \sum \gamma_{\mu,\nu,\rho}\, d\vartheta_\nu\, \delta\vartheta_\mu\ , \end{aligned}$$

i.e., due to equations I (§ **2**):

$$\gamma_{\mu,\,\nu,\,\rho} = \beta_{\mu,\,\nu,\,\rho}$$

and

(II) $$(X_\rho , X_\sigma) = \sum_\tau \beta_{\rho,\sigma,\tau} X_\tau\ .$$

*The coefficients $\beta_{\rho,\,\sigma,\,\tau}$ that appear in equations (II) for the definition of the bracket symbol are precisely the same coefficients that we learned about in the transitivity equations.*

Now, according to **Lie** ([1]), the necessary and sufficient condition for the transformations (B) to generate a finite, $n$-parameter group is that the $\beta$ must be constants. (Second Fundamental Theorem) We then have the theorem:

*The coefficients $\beta$ in the Lagrangian transitivity equations are constant when and only when the associated infinitesimal transformations (B) of the system generate an $n$-parameter group.*

In that case, we would like to set:

$$\beta_{\mu,\,\nu,\,\rho} = c_{\mu,\,\nu,\,\rho}\,,$$

as **Lie** did. It must be further remarked that:

([1]) **Lie-Engel** I, Chap. 9.

*If we integrate the ordinary differential equations:*

$$\frac{d\bar{x}}{dt} = \sum_\lambda \bar{e}_\lambda \, \omega_\lambda$$

*for constant, but arbitrary, $\omega$ then we will get any possible position from the initial position $\bar{x}_0$; $\omega_1 t$, …, $\omega_n t$ are the so-called canonical parameters of the finite equations for the group that are obtained by integration* (i.e., the isomorphic group in $\bar{x}$).

Now, I shall further assert that the transitivity equations are most closely connected with the equations that define the infinitesimal transformations of the *adjoint group* ([1]).

Namely, the symbol of those infinitesimal transformations ([2]) is:

$$E_\nu f = \sum_{\mu,\kappa} c_{\mu,\nu,\kappa} \omega_\mu \frac{\partial f}{\partial \omega_\kappa},$$

so for $f = \omega_\kappa$, one will have:

$$\delta_\nu \, \omega_\kappa = \sum_\mu c_{\mu,\nu,\kappa} \omega_\mu \, \delta\vartheta_\nu$$

and

$$\delta\omega_\kappa \equiv \sum_\nu \delta_\nu \omega_\kappa = \sum_{\nu,\mu} c_{\mu,\nu,\kappa} \omega_\mu \delta\vartheta_\nu .$$

However, those are our transitivity equations, except that we have set $d\delta\vartheta_\kappa = 0$, which indeed also corresponds to the assumption of constant $\delta\vartheta$ that is necessary here. That assumption is not made in the general transitivity equations, and those equations will still remain true when the $\beta$ are not constant, in addition.

However, if the $\beta$ are constant then **Lie**'s two *characteristic relations* ([3]) will naturally be fulfilled:

$$(6') \qquad \left\{ \begin{array}{c} c_{\rho,\sigma,\tau} + c_{\sigma,\rho,\tau} = 0, \\ \sum_\sigma (c_{\iota,\kappa,\sigma} c_{\sigma,\lambda,\tau} + c_{\kappa,\lambda,\sigma} c_{\sigma,\iota,\tau} + c_{\lambda,\iota,\sigma} c_{\sigma,\kappa,\tau}) = 0, \end{array} \right.$$

the first of which is explained immediately by (5), while the second one follows from the **Jacobi** identity:

$$((X_\iota \, X_\kappa) \, X_\lambda) + ((X_\kappa \, X_\lambda) \, X_\iota) + ((X_\lambda \, X_\iota) \, X_\kappa) \equiv 0.$$

It would be useful to derive the general equations that correspond to these.

The fact that the equation:

$$(6a) \qquad \beta_{\rho,\sigma,\tau} + \beta_{\sigma,\rho,\tau} = 0$$

---

([1]) Naturally, the actual basis for understanding the equation $\gamma_{\mu,\,\nu,\,\rho} = \beta_{\mu,\,\nu,\,\rho}$ is connected with this. One can derive the meaning of the adjoint group directly from the transitivity equations I (§ **2**).

([2]) **Lie-Engel** I, pp. 275. Theorem 48. **Lie-Scheffers**, pp. 464, equation (19). The quantities that were denoted by *e* there correspond to our $\omega$.

([3]) **Lie-Engel** I, pp. 170. Theorem 27.

remains valid is clear.

Furthermore, the **Jacobi** identity:

$$((X_\iota X_\kappa) X_\lambda) + ((X_\kappa X_\lambda) X_\iota) + ((X_\lambda X_\iota) X_\kappa) \equiv 0$$

remains valid; i.e.:

$$\left(\sum_\sigma \beta_{\iota,\kappa,\sigma} X_\sigma, X_\lambda\right) + \left(\sum_\sigma \beta_{\kappa,\lambda,\sigma} X_\sigma, X_\iota\right) + \left(\sum_\sigma \beta_{\lambda,\iota,\sigma} X_\sigma, X_\kappa\right) = 0$$

or

$$\sum_{\sigma,\tau} (\beta_{\iota,\kappa,\sigma}\beta_{\sigma,\lambda,\tau} + \beta_{\kappa,\lambda,\sigma}\beta_{\sigma,\iota,\tau} + \beta_{\lambda,\iota,\sigma}\beta_{\sigma,\kappa,\tau}) X_\tau$$
$$- \sum_{\sigma,\tau}\left(\frac{\partial \beta_{\iota,\kappa,\sigma}}{\partial q_\rho}\xi_{\lambda,\rho} + \frac{\partial \beta_{\kappa,\lambda,\sigma}}{\partial q_\rho}\xi_{\iota,\rho} + \frac{\partial \beta_{\lambda,\iota,\sigma}}{\partial q_\rho}\xi_{\kappa,\rho}\right) X_\sigma = 0,$$

and since the $X_\sigma$ are completely independent, it follows from this that:

$$\text{(6b)}\qquad \sum_\sigma (\beta_{\iota,\kappa,\sigma}\beta_{\sigma,\lambda,\tau} + \beta_{\kappa,\lambda,\sigma}\beta_{\sigma,\iota,\tau} + \beta_{\lambda,\iota,\sigma}\beta_{\sigma,\kappa,\tau}) = \sum_{\sigma,\tau}\left(\frac{\partial \beta_{\iota,\kappa,\sigma}}{\partial q_\rho}\xi_{\lambda,\rho} + \frac{\partial \beta_{\kappa,\lambda,\sigma}}{\partial q_\rho}\xi_{\iota,\rho} + \frac{\partial \beta_{\lambda,\iota,\sigma}}{\partial q_\rho}\xi_{\kappa,\rho}\right).$$

That is the second equation, which replaces **Lie**'s equation (6′).

## Chapter II. Kinetic considerations

### § 4. Lagrange's central equation

At the summit of our actual mechanical considerations, we place the fundamental equation that we would like to refer to as ***Lagrange****'s central equation.* It defined the actual core of the so-called ***Hamilton*** *principle*, but it is more encompassing, since it also includes the principle of varied action. Since it seems to be less known, permit me to derive it here briefly.

Let $m$ ($a$, $b$, $c$) be the mass of a system point and let:

$$\text{(C)}\qquad T = \tfrac{1}{2}\mathrm{S}\, m\dot{\bar{x}}^2$$

be its kinetic energy. (We shall let $\mathrm{S}$ denote the summation over the points of the system, in which we leave open the issue of whether we are dealing with a sum over a discrete set of points or an integral). If we denote the inner (i.e., scalar) product of two vectors $\bar{x}$ and $\bar{y}$ by $\bar{x}\cdot\bar{y}$ then the identity:

$$\frac{d}{dt}\left[\mathrm{S}\, m(\delta\bar{x}\cdot\dot{\bar{x}})\right] = \mathrm{S}\, m(\delta\bar{x}\cdot\ddot{\bar{x}}) + \delta T$$

will be true for all virtual $\delta\bar{x}$.

If we now make **d'Alembert**'s Ansatz:

$$m\ddot{\bar{x}} = \bar{K} + \bar{R}, \tag{D}$$

in which $\bar{K}$ means the impressed elementary force that acts upon the point, but $\bar{R}$ means the reaction force, then from the *principle of virtual work*, we will have:

$$\mathrm{S}\,\bar{R}\cdot\delta\bar{x} = 0$$

for all displacements that are compatible with the constraints of the system; i.e., for all $\delta q_\lambda$, and so for all $\delta\vartheta_\lambda$, as well.

If we then define *n system forces* $Q_\lambda$ by the identity in the $\delta\vartheta_\lambda$:

$$\sum_{\lambda=1}^{n} Q_\lambda\,\delta\vartheta_\lambda = \mathrm{S}\,\bar{K}\cdot\delta\bar{x} \tag{E}$$

then we can convert our previous identity into the following equation:

$$\frac{d}{dt}\left[\mathrm{S}\,m\left(\dot{\bar{x}}\cdot\delta\bar{x}\right)\right] = \mathrm{S}\,Q_\lambda\,\delta\vartheta_\lambda + \delta T,$$

which is true for all $\delta\vartheta_\lambda$. Finally, if we define *n impulse components* $J_\lambda$ by the identities:

$$\sum_{\lambda} J_\lambda\,\delta\vartheta_\lambda = \mathrm{S}\,m\dot{\bar{x}}\cdot\delta\bar{x}, \tag{E′}$$

which corresponds to (E), then our equation will read:

$$\frac{d}{dt}\left(\sum_{\lambda} J_\lambda\,\delta\vartheta_\lambda\right) = \sum Q_\lambda\,\delta\vartheta_\lambda + \delta T, \tag{III}$$

and that is ***Lagrange**'s central equation* ([1]); its meaning can be expressed as follows:

*The total variation of the virtual work of the impulse is equal to the virtual work done by the impressed forces plus the virtual variation of the kinetic energy.*

---

([1]) **Lagrange**, *loc. cit..*, t. I. Part Two, Sect. IV. Nos. 1 to 5, especially no. 3, pp. 283. Strictly speaking, **Lagrange** described only the identity that was written down at the beginning of this paragraph, and since the principle of least action was at the foreground in his era, Lagrange strived to employ that principle, above all. (See also t. I, Part Two, Sect. III, § VI, and Sect. IV, no. 6. He went over to the integral principle directly at those two locations.) Furthermore, the term "central equation" was used already by **Heun** in his *Vorlesungen über Mechanik.*

That equation encompasses the whole volume of forces that act over time in kinetics; the so-called Hamilton principle will follow from it by integration when one assumes that the $\delta q_\lambda$ are equal to zero at the limits of the integral.

## § 5. The Lagrange-Euler equations

We now introduce our substitution:

$$\dot{\bar{x}} = \sum_\lambda \bar{e}_\lambda \, \omega_\lambda \tag{3}$$

into equation (III). The kinetic energy will then become:

$$T = \tfrac{1}{2} \mathrm{S}\, m \sum_\lambda \bar{e}_\lambda \, \omega_\lambda \cdot \sum_\mu \bar{e}_\mu \, \omega_\mu \,, \text{ or}$$

(7)

$$T = \tfrac{1}{2} \sum_\lambda \omega_\lambda \omega_\mu \cdot \mathrm{S}\, m \bar{e}_\lambda \cdot \bar{e}_\mu \,.$$

Naturally, $T$ is a quadratic form for the parameters $\omega_\lambda$, which determine the velocity.

According to (3) (§ **1**), it also follows immediately from (E′) (§ **4**) that:

$$J_\lambda = \mathrm{S}\, m \sum_\mu \bar{e}_\mu \, \omega_\mu \, \bar{e}_\lambda \,, \qquad \text{or from (7)}$$

(IVa)

$$J_\lambda = \frac{\partial T}{\partial \omega_\lambda} \,.$$

*The $\lambda^{th}$ impulse component is equal to the partial derivative of the kinetic energy with respect to the $\lambda^{th}$ velocity parameter.*

As a result, we can also write (III) as:

$$\sum_\lambda \frac{dJ_\lambda}{dt} \delta\vartheta_\lambda + \sum_\lambda J_\lambda \frac{d}{dt} \delta\vartheta_\lambda = \sum_\lambda Q_\lambda \delta\vartheta_\lambda + \sum_\lambda J_\lambda \delta\omega_\lambda + \sum_\lambda \frac{\partial T}{\partial q_\lambda} \delta q_\lambda \,.$$

If we now observe the transitivity equation:

$$\frac{d}{dt} \delta\vartheta_\lambda - \delta\omega_\lambda = \sum_{\mu,\nu} \beta_{\mu,\nu,\lambda} \, \delta\vartheta_\lambda \omega_\nu \,,$$

as well as equations (A‴) (§ **3**), and imagine that the equation above must be valid for all $\delta\vartheta_\lambda$ then we will get:

(IV′ b) $$\frac{dJ_\lambda}{dt} + \sum_{\mu,\rho} \beta_{\mu,\nu,\lambda}\, \omega_\mu J_\rho - \sum_\rho \frac{\partial T}{\partial q_\rho} \xi_{\lambda,\rho} = Q_\lambda .$$

With that, we have already found the general equations of mechanics that we seek. If we introduce the symbolic differential quotients:

(F) $$\left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = \sum_\rho \frac{\partial T}{\partial q_\rho} \xi_{\lambda,\rho} ,$$

which are defined by analogy to actual differential quotients, then we can also write:

(IVb) $$\frac{dJ_\lambda}{dt} + \sum_{\mu,\rho} \beta_{\mu,\nu,\lambda}\, \omega_\mu J_\rho - \left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = Q_\lambda .$$

We would like to refer to these equations as the **Lagrange-Euler equations of mechanics** ([1]).

With that, we have discovered the *Theorem:*

*If we introduce parameters $\omega_1, \dots, \omega_n$ into the consideration of a mechanical system of n degrees of freedom that determine the velocity as independent, linear functions of the $\dot{q}_1, \dots, \dot{q}_n$, along with the coordinates $q_1, \dots, q_n$ that determine position then the more general equation IV will enter in place of the* ***Lagrange*** *equations. In them: T means the kinetic energy, $J_\lambda$ means the impulse components that belong to the $\omega_\lambda$ by way of (E′), and the $\beta$ are the coefficients of the transitivity equations:*

$$d\delta\vartheta_\lambda - \delta d\vartheta_\lambda = \sum_{\mu,\nu} \beta_{\mu,\nu,\lambda}\, \delta\vartheta_\mu\, d\vartheta_\nu \qquad \left(\omega_\nu = \frac{d\vartheta_\nu}{dt}\right).$$

$\left(\frac{\partial T}{\partial \vartheta_\lambda}\right)$ *means nothing but* $\sum_\rho \frac{\partial T}{\partial q_\rho} \xi_{\lambda,\rho}$ *and will go to* $\frac{\partial T}{\partial \vartheta_\lambda}$ *when all $J_\lambda$ are actual coordinates that determine position. In that case, all $\beta$ will also be zero, since the $\beta$ are determined in terms of the $\xi$ by way of equations (5) (page 8) and (2) (page 6).*

([1]) If one expresses $T$ in terms of the $q_\lambda$ and $J_\lambda$ in equations (IV) then, after some other minor alterations, one will get equations that represent a special case of the **Lagrange-Poisson**-transformed ones. (**Lagrange**, t. I, pp. 315; see also **Cauchy**, "Report on the recent progress of theoretical dynamics." Works III, no. 195) They express $dJ_\lambda / dt$ and $dq_\lambda / dt$ linearly in terms of $\partial T / \partial q$ and $\partial T / \partial J$ with coefficients that amount to the Poisson bracket symbols for the special substitutions $q_\lambda = q_\lambda$ and $J_\lambda = \sum_\tau \xi_{\lambda,\tau}\, p_\tau$ (in which $p_\tau = \partial T / \partial \dot{q}_\tau$).

It should be stressed that if one wishes to exhibit these equations then one needs to known only the expression for the kinetic energy $T$ and the quantities $\xi_{\lambda,\rho}$ , which establish the connection between the $\dot{q}_\lambda$ and the $\omega_\lambda$ .

### § 6. Discussion of the Lagrange-Euler equations. Example of rigid body.

I would like to justify the fact that I have referred to equations IV as the **Lagrange-Euler** equations of mechanics as follows:

First, they subsume the **Lagrange** equations; indeed, one needs only to assume that the $\vartheta$ are coordinates in order to get immediately:

$$J_\lambda = \frac{\partial T}{\partial \omega_\lambda},$$
$$\frac{dJ_\lambda}{dt} - \frac{\partial T}{\partial \vartheta_\lambda} = Q_\lambda ,$$

i.e., the Lagrangian equations. However, in addition, we also recall that the first and last terms on the left-hand side suggest the structure of the Lagrange equation in its general form, and finally, one has Lagrange to thank for their derivation.

In order to justify the name **Euler** and at the same time to give an example, permit me to apply the entire theory to the rigid bodies that rotate about a fixed point.

If we take the Euler angles $\vartheta$, $\varphi$, $\psi$ to be the position-determining coordinates, and let $p$, $q$, $r$ be the components of the rotation vector relative to the axes through the fixed point in the body that take the form of the principal axes of inertia then $p$, $q$, $r$ will be linear couplings of the $\dot{\vartheta}$, $\dot{\varphi}$, $\dot{\psi}$ of the kind that we assumed in equations (A). Namely, one has (1):

$$\dot{\varphi} = r - \cot\vartheta\,(p \sin\varphi + q \cos\varphi),$$

$$\dot{\psi} = \frac{1}{\sin\vartheta}(p \sin\varphi + q \cos\varphi),$$

$$\dot{\vartheta} = p \cos\varphi + q \sin\varphi.$$

The kinetic energy assumes the following form in the $p$, $q$, $r$:

$$T = \tfrac{1}{2}(A\,p^2 + B\,q^2 + Cr^2),$$

in which $A$, $B$, $C$ are the (constant) principle moments of inertia (2). Our $J$ will now become:

---

(1) **Klein-Sommerfeld**, pp. 45, equations (7) and (9).

(2) *Ibidem*, pp. 100, equation 13.

$$J_1 = \frac{\partial T}{\partial p} = Ap, \qquad J_2 = \frac{\partial T}{\partial q} = Bq, \qquad J_3 = \frac{\partial T}{\partial r} = Cr,$$

and thus precisely the ones that **Klein** and **Sommerfeld** called the impulse components $L$, $M$, $N$ ([1]). Now, in order to derive the transitivity equations, we construct the symbols for the three infinitesimal transformations using the equations above:

$$X_p = \frac{\partial f}{\partial \varphi}(-\cot\vartheta \sin\varphi) + \frac{\partial f}{\partial \varphi}\frac{\sin\varphi}{\sin\vartheta} + \frac{\partial f}{\partial \vartheta}\cos\varphi,$$

$$X_q = \frac{\partial f}{\partial \varphi}(-\cot\vartheta \cos\varphi) + \frac{\partial f}{\partial \psi}\frac{\cos\varphi}{\sin\vartheta} - \frac{\partial f}{\partial \vartheta}\sin\varphi,$$

$$X_r = \frac{\partial f}{\partial \varphi}.$$

If we calculate the bracket expression $(X_p, X_q)$ from this then we will get precisely $\partial f / \partial \varphi$; i.e., $X_r$, etc. The relations then exist:

$$(X_p, X_q) = X_r, \qquad (X_q, X_r) = X_p, \qquad (X_r, X_p) = X_q.$$

(Let it be expressly remarked that these relations are true only when $p$, $q$, $r$ are referred to axes that are fixed in the body; for $\pi$, $\kappa$, $\rho$, which are the components of the same vector with respect to three axes that are fixed in space, the signs will be just the opposite.)

Hence, all $\beta$ are constant (which is certainly not surprising), and indeed:

$$\begin{array}{lll} \beta_{1,2,1} = 0, & \beta_{1,2,2} = 0, & \beta_{1,2,3} = 1, \\ \beta_{2,3,1} = 1, & \beta_{2,3,2} = 0, & \beta_{2,3,2} = 0, \\ \beta_{3,1,1} = 0, & \beta_{3,1,2} = 1, & \beta_{3,1,3} = 0. \end{array}$$

Finally, since the $\left(\frac{\partial T}{\partial \vartheta_\lambda}\right)$ are all zero, our equations (IV b) will read:

$$\frac{dJ_1}{dt} + \omega_2 J_3 - \omega_3 J_2 = Q_1,$$

$$\frac{dJ_2}{dt} + \omega_3 J_1 - \omega_1 J_3 = Q_2,$$

$$\frac{dJ_3}{dt} + \omega_1 J_2 - \omega_2 J_1 = Q_3,$$

$$(\omega_1 = p, \ \omega_2 = q, \ \omega_3 = r),$$

([1]) **Lagrange** had no special term for them, but the facts were already completely present for him.

and these are precisely the equations that, e.g., **Klein-Sommerfeld**, pp. 141 (3′) gave, up to notations; they are the **Euler equations** for the motion of a rigid body about a fixed point. Moreover, as was mentioned already in the introduction, **Lagrange** had already derived these equations in their general form, and in essentially the same way (*loc. cit.*, t. II, Section IX, Chap. I, § II, no. 22).

However, it emerges clearly from that example that the terms in equations (IV b) that are in the second position have entirely the same type as the terms that enter into the Euler equations for the derivatives of the impulse components, and that justifies the name "Lagrange-Euler equations." Furthermore, in the later considerations, it will be precisely **Euler**'s particular vectorial notation that will suggest itself more strongly in conjunction with **Lie**'s ideas.

## § 7. Non-holonomic constraint equations

The main advantage of the **Lagrange-Euler** equations consists of the fact that they allow one to treat any non-holonomic constraint equations that might appear in a systematic way that is similar to what **Lagrange** applied to holonomic constraints.

We assume that $\nu < n$ linear, independent, and generally non-integrable constraint equations that do not include time explicitly are given for the $\delta q$. *By a suitable choice of the* $\omega$, *we can likewise arrange that the constraint equations read simply:*

(G) $$\delta\vartheta_{n-\nu+1} = 0, \quad \delta\vartheta_{n-\nu+2} = 0, \ldots, \delta\vartheta_n = 0.$$

Naturally, the $n$ equations:

(V a) $$\omega_{n-\nu+1} = 0, \ldots, \omega_n = 0$$

will be correspondingly fulfilled throughout the motion.

Nothing major has changed from our previous investigations. We now have $n - \nu$ arbitrary, independent displacements $\delta\vartheta_1, \ldots, \delta\vartheta_{n-\nu}$. The law of virtual displacements will be true for the new reactions that appear in equations (V a) when we consider only (G), and we will then get the equations of motion when we append the first $n - \nu$ of equations (IV b) to equations (V a), in which we observe that the $Q_1, .., Q_{n-\nu}$ are to be calculated from only the impressed external force using formulas (E) (§ **4**). When we take equations (V a) into account, we will get the $n - \nu$ equations:

(V b) $$\frac{dJ_\lambda}{dt} + \sum_{\substack{\mu=1,2,\ldots,n-\nu \\ \rho=1,\ldots,n}} \beta_{\lambda,\mu,\rho}\,\omega_\mu J_\rho - \left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = Q_\lambda$$

$$(\lambda = 1, 2, \ldots, n - \nu).$$

Equations (V a), (V b), (IV a), and (A) will then suffice completely to determine the motion. We then have the theorem:

*If $\nu$ generally non-holonomic constraint equations are given for a mechanical problem with n degrees of freedom then one introduces linear combinations of the $\dot{q}$ by way of equations (A) (page 6) in such a way that the constraint equations assume the form* $\delta\vartheta_{n-\nu+1} = 0$, $\delta\vartheta_n = 0$. *One calculates the forces $Q_\lambda$ in terms of only the impressed forces using (E) (page 13). Equations (Va), (Vb), (IVa), and (A) then determine motion of the system completely.* However, let it be further remarked if one is to calculate the *vis viva* then one generally known the terms that are linear in $\omega_{n-\nu+1}$ , …, $\omega_n$ , since one requires the $J_{n-\nu+1}$ , …, $J_n$ in equations (Vb).

Now, the most important case is naturally the one in which one takes $\omega_1$ , …, $\omega_{n-\nu}$ to be simply $\dot{q}_1$, …, $\dot{q}_{n-\nu}$ – i.e., velocities of $n - \nu$ suitably-chosen coordinates.

Naturally, one will then have that all of the:

$$\beta_{\lambda,\mu,\rho} = 0$$

for the $\rho \leq n - \nu$. Now, one will have:

$$d\delta\vartheta_\rho = \delta d\vartheta_\rho$$

for such a $\rho$. Equations (V b) then assume the form:

$$\text{(V b}'\text{)} \qquad \frac{dJ_\lambda}{dt} + \sum_{\substack{\mu=1,2,\ldots,n-\nu \\ \rho=n-\nu+1,\ldots,n}} \beta_{\lambda,\mu,\rho}\dot{q}_\mu J_\rho - \left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = Q_\lambda$$

$$(\lambda = 1, 2, \ldots, n - \nu).$$

The expression that appears to the left of $dJ_\lambda / dt$ is precisely the same as the one that **Appel** denoted by – $R_\lambda$ . (See Introduction) Naturally, in the present case, one also has:

$$\xi_{\lambda,\lambda} = 1 \qquad \text{and} \qquad \xi_{\rho,\lambda} = 0 \qquad \text{for all } \rho \neq \lambda$$

for $\lambda \leq n - \nu$. [See equations (A)]

## § 8. When can one employ the Lagrangian equations and the "illegitimate form" of the *vis viva* for non-holonomic constraint equations?

We shall now address the question that many authors ([1]) have treated in recent times, when we simply put the **Lagrangian** equations in place of equations (V b′) and in that way, at the same time, the "illegitimate form" ([2]) of the *vis viva* that arises by setting $\omega_{n-\nu+1}$, …, $\omega_n$ in $T$ can be employed, as long as one assumes that the coordinates $q_{n-\nu+1}$, …, $q_n$ do not enter into the abbreviated $T$ anywhere.

From the last remark of the previous paragraphs, one will have:

([1]) See the literature that was cited in the Introduction.
([2]) An expression that **C. Neumann** used. See "Beiträge zur analytischen Mechanik," Leipziger Berichte, 1899, page 437; also **Voss**, *Enzyklopädie der mathematischen Wissenschaften IV*, 1, no. 38.

$$\left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = \frac{\partial T}{\partial q_\lambda} + \sum_{\rho=n-\nu+1}^{n} \frac{\partial T}{\partial q_\rho} \xi_{\lambda,\rho}$$

for $\lambda \leq n - \nu$. Now when the illegitimate $T$ is free of $q_{n-\nu+1}, \ldots, q_n$, if we employ (Va) ([1]) already then what will remain is:

$$\left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = \frac{\partial T}{\partial q_\lambda} \qquad (\lambda = 1, 2, \ldots, n-\nu).$$

Therefore, everything comes down to investigating when ([2]):

$$\sum_{\substack{\rho > n-\nu \\ \mu \leq n-\nu}} \beta_{\lambda,\mu,\rho} \omega_\mu J_\rho = 0,$$

and indeed identically in all $\omega_\mu$ for $\mu \leq n - \nu$.

Now, upon consideration of (V a):

$$J_\rho = \sum_{\kappa \leq n-\nu} \frac{\partial^2 T}{\partial \omega_\rho \, \partial \omega_\kappa} \omega_\kappa .$$

Hence, one must have:

$$\sum_{\rho > n-\nu} \beta_{\lambda,\mu,\rho} \frac{\partial^2 T}{\partial \omega_\rho \, \partial \omega_\kappa} = 0 \tag{8}$$

for all $\lambda$, $\mu$, $\kappa = 1, 2, \ldots, n - \nu$.

These are the necessary and sufficient conditions for one to be able to employ simply the **Lagrangian** equations for $q_1, \ldots, q_n$, which arise from the abbreviated $T$.

We need to discuss those conditions. If $\nu = 1$, so only *one* non-holonomic condition is present, things will be very simple. Namely, either all $\dfrac{\partial^2 T}{\partial \omega_\rho \, \partial \omega_\kappa}$ $(\kappa < n)$ must vanish (i.e., the total kinetic energy $T$ is composed of the part that is free of $\omega_\mu$ and a term in $\omega_n^2$) or all $\beta_{\lambda,\mu,n}$ must be zero (i.e., the infinitesimal transformations that belong to the $q_1, \ldots, q_n$ commute with each other), and the constraint equation will be integrable.

We will obtain the same result when more than one constraint equation is present. We will then proceed as follows:

Instead of $\omega_{n-\nu+1}, \ldots, \omega_n$, we can also introduce any sort of new linear and independent coupling of them. Hence, we set:

$$\omega_\sigma = \sum_\rho \alpha_{\sigma,\rho} \, \omega'_\rho \qquad (\rho, \sigma = n - \nu + 1, \ldots, n),$$

---

([1]) Which is permitted here.

([2]) The theorem of **Korteweg** (*loc. cit.*) that one can employ the illegitimate form of $T$ for infinitely-small motions follows immediately here from the form of the terms in question.

in which one must have only $|\alpha_{\sigma,\rho}| \neq 0$. If we introduce this substitution into $T$ then we will get:

$$\frac{\partial^2 T}{\partial\omega_\rho\,\partial\omega_\kappa} = \sum_{\sigma>n-\nu}\frac{\partial^2 T}{\partial\omega_\sigma\,\partial\omega_\kappa}\alpha_{\sigma,\rho} \qquad \begin{pmatrix}\rho = n-\nu+1,\dots,n\\ \kappa = 1,2,\dots,n-\nu\end{pmatrix}.$$

We now construct a matrix of $\nu$ (or less) columns and $(n - \nu)^2$ rows from the $\beta_{\lambda,\mu,\sigma}$, in which we associated the various $\sigma$ with the columns and the various $\mu$ and $\lambda$ with rows. However, we would like to drop the columns with $\sigma$ for which all $\frac{\partial^2 T}{\partial\omega_\kappa\,\partial\omega_\rho}$ are zero ($\kappa \leq n - \nu$). Let $\varepsilon$ be the rank of that matrix then (i.e., let the highest non-vanishing determinant that it contains have degree $\varepsilon$), and let that be true precisely for the last rows and columns $\rho$, $\sigma = n, n - 1, \dots, n - \varepsilon + 1$, which we can assume with no loss of generality. We then set:

$$\alpha_{\sigma,\rho} = \beta_{\lambda,\mu,\sigma}$$

for all $\sigma > n - \nu$ and $\rho = n, n - 1, \dots, n - \varepsilon + 1$, in which we associate each $\rho$ with a certain pair of values for $\lambda$, $\mu$, namely, one for which the $\varepsilon$-term determinant does not vanish. We set all other $\alpha_{\sigma,\rho}$ equal to zero, with the exception of:

$$\alpha_{\rho,\rho} = 1$$

for

$$\rho = n - \varepsilon, n - \varepsilon + 1, \dots, n - \nu + 1.$$

The determinant of the $\alpha_{\rho,\sigma}$ that are thus determined does not vanish, while for $\rho = n, n - 1, \dots, n - \varepsilon + 1$ and $\kappa \leq n - \nu$:

$$\frac{\partial^2 T}{\partial\omega_\rho\,\partial\omega_\kappa} = \sum_{\sigma}\frac{\partial^2 T}{\partial\omega_\sigma\,\partial\omega_\kappa}\beta_{\mu,\lambda,\sigma},$$

and that must now be zero, from equations (8). With those substitutions, we have then succeeded in eliminating terms from $\omega_\kappa\,\omega_\rho$ from $T$ with $\kappa = 1, 2, \dots, n - \nu$ and $\rho = n, n - 1, \dots, n - \varepsilon + 1$ [in the event that (8) is fulfilled], while the coefficients of $\omega_\kappa\,\omega_\rho$ with $\rho = n - \varepsilon, n - \varepsilon - 1, \dots, n - \nu + 1$ remain unchanged, and therefore vanish if that is what they did before.

If we determine the $\beta$ that are associated with the recently-introduced $\omega$, and if they are still not all zero, then we can apply the procedure above; one can still make the coefficients of $T$ that are endowed with $\omega_\kappa \cdot \omega_\rho$ ($\rho > n - \nu$, $\kappa \leq n - \nu$) vanish. That process must terminate, since the number of those coefficients is finite – i.e., either all coefficients of $\omega_\kappa\,\omega_\rho$ can be made to vanish, or when that is not the case, all $\beta_{\mu.\lambda,\rho}$ for which not all coefficients of $\omega_\kappa\,\omega_\rho$ vanish will vanish. With that, we have reached our goal. We have then arrived at the following theorem:

*If we are allowed to pose the* ***Lagrangian*** *equations for the first* $n - \nu$ *coordinates that comes from the abbreviated – i.e., illegitimate – T then the r constraint equations must be brought into a form:*

$$\omega_{n-\nu+1} = 0, \ldots, \omega_{n-\nu+\tau} = 0; \qquad \omega_{n-\nu+\tau+1} = 0, \ldots, \omega_n = 0$$

*by linear combinations such that terms with* $\omega_\kappa \cdot \omega_{n-\nu+\rho}$ *do not enter into the expression for the vis viva, while for* $\lambda, \mu = 1, 2, \ldots, n - \nu$:

$$\beta_{\lambda,\mu,n-\nu+\tau+\sigma} = 0 \qquad (\sigma = 1, 2, \ldots, \nu - \tau),$$

*such that the brackets* $(X_\lambda, X_\mu)$ *will be composed from only* $X_{n-\nu+1}, .., X_{n-\nu+\tau}$. $\tau$ can have all values from 0 to $\nu$ in this. In particular, if $\tau = 0$ then the first $n - \nu$ infinitesimal transformations must commute with each other, so the infinitesimal motions that are actually possible will generate a group. By contrast, if $\tau = \nu$ then the energy must be composed of two separate terms: The part that is free of $\omega_{n-\nu+1}, \ldots, \omega_n$ and a part that includes only those non-holonomic velocity parameters.

The conditions are also sufficient. *In the case* $\tau = 0$, *the equations* $\omega_{n-\nu+1} = 0, \ldots, \omega_n = 0$ *must be capable of being brought into an integrable form* such that in reality no non-holonomic constraint equations are present, moreover. Since the displacements that are compatible with the $\nu$ **Pfaffian** equations $d\vartheta_{n-\nu+1} = 0, \ldots, d\vartheta_n = 0$ will be represented by $\delta q_1, \ldots, \delta q_{n-\nu}$, but from the above they will generate an $n - \nu$-parameter group in the variables $q_1, \ldots, q_n$, precisely $\nu$ finite constraint equations will exist, namely, the $n$ equations that arise by eliminating the $n - \nu$ parameters from the equations of the finite group. However, with that, we have once more arrived at the result that **Hadamard** stated in no. **8** of the treatise that was cited in the introduction. However, he had to add the words "*en générale*"; his *en générale* corresponds to our $\tau = 0$; i.e., to the case in which there is *no* $\rho > n - \nu$ for which all coefficients of $\omega_\kappa \omega_\rho$ ($\kappa \le n - \nu$) in $T$ will vanish.

If we now ask, more generally, not whether we can employ *all* constraint equations that we get from the outset by elimination from $T$, but *which* of them that we can employ, then that will yield an equation as a criterion that is entirely analogous to (8), except that the summation is extended over only the $\rho$ that come into question. We will come to the following theorem by essentially the same argument that we presented before:

*We form all* $(X_\lambda, X_\mu)$ *for* $\lambda, \mu = 1, 2, \ldots, n - \nu$ *and select the independent ones from among them. If there are* $\omega_{n-\nu+1}, \ldots, \omega_n$ *for which the associated infinitesimal transformations are independent of those brackets then we can employ those independent* $\omega_\rho$ *for the elimination from T.* Generally speaking, $\nu - (n - \nu)(n - \nu - 1) / 2$ such $\omega$ will go away.

That is precisely **Hadamard**'s result ([1]) that he stated in the conclusion of no. **7** in the first-mentioned paper, except that in general this result only implies the necessary

([1]) **Hadamard** has also expressed it in this group-theoretic form in his second note: "Sur certains systèmes d'équations."

condition: *We can, in fact, also drop an* $\omega_\rho$ *from T only when the associated infinitesimal transformation indeed results from the definition of the* $(X_\lambda, X_\mu)$, *but* $\omega_\rho$ *itself does not appear in T in conjunction with* $\omega_1, \omega_2, \ldots, \omega_{n-\nu}$, *and with that additional restriction, the condition will also be general and necessary.*

However, things will become clearest here when we pose the problem somewhat more generally and renounce the demand that the first $n - \nu$ $\omega$ must be precisely the $\dot{q}$, and ask:

*Let* $\nu$ *constraint equations be given. Which of those equations can one employ for the presentation of the* $n - \nu$ *Lagrange-Euler equations* (Vb) *for the elimination from T?*

The path to the solution of this problem remains entirely the old one, so it results from equations (8), and the *answer* then reads:

*If one has brought the constraint equations into the form* $\omega_{n-\nu+1} = 0, \ldots, \omega_n = 0$ *then one can set equal to zero, in addition to such independent combinations of those* $\omega$ *that do not enter into T in conjunction with the first* $n - \nu$ $\omega$, *also the ones* ***before*** *exhibiting the first* $n - \nu$ ***Lagrange-Euler*** *equations in T that are linearly-independent of those combinations of the* $\omega$ *that belong to the infinitesimal transformations that result from the* $(X_\rho, X_\sigma)$ $(\rho, \sigma = 1, 2, \ldots, n - \nu)$. *Hence if the partial differential equations* $X_\rho f = 0$ $(\rho = 1, \ldots, n - \nu)$ *then define, in particular, a complete system, or – what amounts to the same thing – two possible* [1] *infinitesimal displacements again produce a possible infinitesimal displacement, then one can set all* $\omega_{n-\nu+1}, \ldots, \omega_n$ *equal to zero in T from the outset.*

## Example of the two-wheeled wagon

As an example, we consider the motion of a two-wheeled wagon on a horizontal plane. The wheels might be perpendicular to it and fixed along an axle of length $2l$; we let their diameters be $2r$, while we would like to ignore their thicknesses. Let the axle of the wagon be permanently linked with the wagon. If we assume a rectangular coordinate system $x$, $y$, $z$ such that the $z$-axis falls along the vertical then the center of the wagon axis might have the coordinates $x_0$, $y_0$, and $r$.

Let $\vartheta$ be the angle that the direction that points from the left-hand wheel to the right-hand one makes with the $x$-axis. We still require two more coordinates then in order to establish the position of the wheels. For that purpose, we employ in each wheel the angle $\varphi_1$ (left) [$\varphi_2$ (right), resp.] that a certain radius vector in each wheel subtends with the forward-pointing horizontal, and indeed we would like to measure the angle in the sense that forward, below, backward, above follow in sequence, such that $\varphi_1$ and $\varphi_2$ will increase when the wagon rolls forward.

We would like to ignore any rotation of the wagon about the horizontal axis, *such that the five coordinates* $x_0$, $y_0$, $\vartheta$, $\varphi_1$, $\varphi_2$ *will suffice to determine the position of the system completely.*

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[1] i.e., compatible with the constraint equations.

Now let $a$, $b$, $c$ be the coordinates of a point of the wagon relative to the system that is fixed in the wagon and is constructed from the axle, the shaft that is used for towing it, and a perpendicular to the latter two axes, so for a point of the wagon itself one will have:

$$x = x_0 + a \cos \vartheta - b \sin \vartheta,$$

$$y = y_0 + a \sin \vartheta + b \cos \vartheta,$$

$$z = c + r,$$

so

$$\dot{x} = \dot{x}_0 - (a \sin \vartheta + b \cos \vartheta)\, \dot{\vartheta},$$

$$\dot{y} = \dot{y}_0 - (a \cos \vartheta - b \sin \vartheta)\, \dot{\vartheta},$$
$$\dot{z} = 0.$$

However, if we establish a point in the wheels by the polar coordinates $\rho_1$, $\alpha_1$ ($\rho_2$ and $\alpha_2$, resp.), in which $\alpha_1$ and $\alpha_2$ might be measured from the directions that are fixed in the wheels in the same sense as $\varphi_1$ and $\varphi_2$, then we will have:

$$x = x_0 - l \cos \vartheta - \rho_1 \cos (\varphi_1 + \alpha_1) \sin \vartheta,$$

$$y = y_0 - l \sin \vartheta + \rho_1 \cos (\varphi_1 + \alpha_1) \cos \vartheta,$$

$$z = r - \rho_1 \sin (\varphi_1 + \alpha_1)$$

for a point on the *left* wheel, so:

$$\dot{x} = \dot{x}_0 + [l \sin \vartheta - \rho_1 \cos (\varphi_1 + \alpha_1) \cos \vartheta]\, \dot{\vartheta} + \rho_1 \sin (\varphi_1 + \alpha_1) \sin \vartheta \cdot \dot{\varphi}_1,$$

$$\dot{y} = \dot{y}_0 - [l \cos \vartheta + \rho_1 \cos (\varphi_1 + \alpha_1) \sin \vartheta]\, \dot{\vartheta} - \rho_1 \sin (\varphi_1 + \alpha_1) \cos \vartheta \cdot \dot{\varphi}_1,$$

$$\dot{z} = - \rho_1 \cos (\varphi_1 + \alpha_1)\, \dot{\varphi}_1,$$

and we will have:

$$x = x_0 + l \cos \vartheta - \rho_2 \cos (\varphi_2 + \alpha_2) \sin \vartheta,$$

$$y = y_0 + l \sin \vartheta + \rho_2 \cos (\varphi_2 + \alpha_2) \cos \vartheta,$$

$$z = r - \rho_2 \sin (\varphi_2 + \alpha_2)$$

for a point on the *right* wheel, so:

$$\dot{x} = \dot{x}_0 + [- l \sin \vartheta - \rho_2 \cos (\varphi_2 + \alpha_2) \cos \vartheta]\, \dot{\vartheta} + \rho_2 \sin (\varphi_2 + \alpha_2) \sin \vartheta \cdot \dot{\varphi}_2,$$

$$\dot{y} = \dot{y}_0 - [-l\cos\vartheta + \rho_2\cos(\varphi_2 + \alpha_2)\sin\vartheta]\,\dot{\vartheta} - \rho_2\sin(\varphi_2 + \alpha_2)\cos\vartheta\cdot\dot{\varphi}_2,$$

$$\dot{z} = -\rho_1\cos(\varphi_2 + \alpha_2)\,\dot{\varphi}_2.$$

Now since the lowest point of each wheel must be at rest during the rolling motion, and since $\varphi_1 + \alpha_1 = \pi / 2$, $\varphi_2 + \alpha_2 = \pi / 2$, $\rho = r$ for those points, we will get the four conditions:

$$0 = \dot{x}_0 + l\sin\vartheta\,\dot{\vartheta} + r\sin\vartheta\,\dot{\varphi}_1,$$
$$0 = \dot{y}_0 - l\cos\vartheta\,\dot{\vartheta} - r\cos\vartheta\,\dot{\varphi}_1,$$
$$0 = \dot{x}_0 - l\sin\vartheta\,\dot{\vartheta} + r\sin\vartheta\,\dot{\varphi}_2,$$
$$0 = \dot{y}_0 + l\cos\vartheta\,\dot{\vartheta} - r\cos\vartheta\,\dot{\varphi}_2.$$

(The two $\dot{z}$ will each be zero.)

However, these four equations are not independent of each other; one can derive the following *three independent constraint equations* from them:

$$\omega_3 \equiv \dot{\varphi}_1 + \frac{1}{r}(\omega_1 + l\dot{\vartheta}) = 0,$$
$$\omega_4 \equiv \dot{\varphi}_2 + \frac{1}{r}(\omega_1 - l\dot{\vartheta}) = 0,$$
$$\omega_5 \equiv \dot{x}_0\cos\vartheta + \dot{y}_0\sin\vartheta = 0.$$

In them, one has set:

$$\omega_1 \equiv \dot{x}_0\sin\vartheta - \dot{y}_0\cos\vartheta;$$

$\omega_1$ means the component of the velocity in the direction of the shaft, so it is the velocity of the actual forward motion.

We would now like to introduce:

$$\omega_1, \qquad \dot{\vartheta} = \omega_2, \qquad \omega_3, \qquad \omega_4, \qquad \omega_5$$

as the five independent velocity parameters; the constraint equations then read simply:

$$\omega_3 = 0, \qquad \omega_4 = 0, \qquad \omega_5 = 0.$$

The *transitivity equations* in this case are especially easy to exhibit; one gets:

$$\beta_{5,2,1} = 1, \qquad \beta_{2,5,1} = -1, \qquad \text{all other } \beta_{\kappa,\lambda,1} = 0,$$

$$\beta_{\kappa,\lambda,2} = 0,$$

$$\beta_{\kappa,\lambda,3} = \beta_{\kappa,\lambda,4} = \frac{1}{r}\beta_{\kappa,\lambda,1},$$

$$\beta_{1,2,5} = -1, \qquad \beta_{2,1,5} = 1, \qquad \text{all other } \beta_{\kappa,\lambda,5} = 0.$$

Now since all $\beta_{\kappa,\lambda,5}$ and $\beta_{\kappa,\lambda,4}$ for which the $\kappa$ and $\lambda$ have the values 1 or 2 are zero, one can employ the conditions $\omega_3 = 0$ and $\omega_4 = 0$ directly for the purpose of exhibiting $T$, but not $\omega_5 = 0$, since $\beta_{1,2,5}$ is not zero.

If we now let $M$ denote the mass of the wagon alone, and let $a^*$ and $b^*$ denote its center of mass coordinates, while $k$ is the radius of inertia relative to a vertical through the center of the axle then the *vis viva* for the wagon, but without its wheels, will be:

$$T_w = \frac{M}{2}[\dot{x}_0^2 + \dot{y}_0^2 + k^2\dot{\vartheta}^2 - 2\dot{x}_0\dot{\vartheta}(a^*\sin\vartheta + b^*\cos\vartheta) + 2\dot{y}_0\dot{\vartheta}(a^*\cos\vartheta - b^*\sin\vartheta)]$$

$$= \frac{M}{2}[\omega_1^2 + k^2\omega_2^2 - 2a^*\omega_1\,\omega_2 - 2b^*\omega_2\,\omega_5]\,.$$

we have already dropped $\omega_5^2$.

If we then denote the mass of each wheel by $m$ and assume that the center of mass of each wheel lies in the hub and that the wheel possesses rotational symmetry about the hub then the *vis viva* of the left wheel will be:

$$T_l = \frac{m}{2}[\dot{x}_0^2 + \dot{y}_0^2 + (l^2 + 2k_0^2)\,\dot{\vartheta}^2 - k_0^2\dot{\varphi}_1^2 + 2l\,\dot{x}_0\sin\vartheta\cdot\dot{\vartheta} - 2l\,\dot{y}_0\cos\vartheta\cdot\dot{\vartheta}],$$

in which $k_0$ denotes the polar radius of inertia. (By assumption, $2mk_0^2$ will then be the moment of inertia around an axis in the plane of the wheel that goes through the center of mass.)

For the right wheel, one will have:

$$T_r = \frac{m}{2}[\dot{x}_0^2 + \dot{y}_0^2 + (l^2 + 2k_0^2)\,\dot{\vartheta}^2 + k_0^2\dot{\varphi}_2^2 - 2l\,\dot{x}_0\sin\vartheta\cdot\dot{\vartheta} + 2l\,\dot{y}_0\cos\vartheta\cdot\dot{\vartheta}]\,.$$

If we add $T_l$ and $T_r$ then the last two terms will cancel, and after one introduces the $\omega$, while considering the facts that $\omega_3 = \omega_4 = 0$, what will remain will be:

$$T_l + T_r = \frac{1}{2}\left[\omega_1^2\left(2m + 2m\frac{k_0^2}{r^2}\right) + \omega_2^2\,2m\left(l^2 + 2k_0^2 + k_0^2\frac{l^2}{r^2}\right)\right].$$

*With that, the total vis viva* (i.e., when one sets $\omega_3$ and $\omega_4$ equal to zero and drops the term with $\omega_5^2$) *will be:*

$$T = \frac{1}{2}\left\{\omega_1^2\left[M + 2m\left(1 + \frac{k_0^2}{r^2}\right)\right] + \omega_2^2\left[Mk^2 + 2m\left(l^2 + 2k_0^2 + \frac{k_0^2\,l^2}{r^2}\right)\right] - 2Ma^*\omega_1\,\omega_2 - 2b^*\omega_2\,\omega_5\right\}.$$

$T$ no longer contains any coordinates explicitly. *Therefore, the equations of motion read simply:*

$$\frac{dJ_1}{dt} - \omega_2\, J_5 = Q_1\,,$$
$$\frac{dJ_2}{dt} + \omega_1\, J_5 = Q_2\,,$$

and therefore one will have:

$$J_1 = \left(\frac{\partial T}{\partial \omega_1}\right)_{\omega_5=0} = \omega_1\left[M + 2m\left(1 + \frac{k_0^2}{r^2}\right)\right] - \omega_3\, M\, a^*,$$
$$J_2 = \left(\frac{\partial T}{\partial \omega_2}\right)_{\omega_5=0} = \omega_2\left[Mk^2 + 2m\left(l^2 + 2k_0^2 + \frac{k_0^2\, l_2}{r^2}\right)\right] - \omega_1\, M\, a^*,$$

$$J_5 = \left(\frac{\partial T}{\partial \omega_5}\right)_{\omega_5=0} = \omega_2\, M\, b^*,$$

which are then linear couplings of $\omega_1$ and $\omega_2$ with constant coefficients.

$Q_1$ means the force that pulls in the direction of the shaft, while $Q_2$ means the rotational moment relative to the center of the wagon axle. That might suffice for the purposes of the present study. If one would actually like to examine the motion of such a wagon then one would now have to enter into the much-more-difficult part of the problem, namely, the study of the force systems for any sort of forward motion of the wagon.

## § 9. The impulse equations

We shall now turn to some other questions, and indeed we will give free rein to those ideas that come out of the consideration of the motion of the rigid body almost intrinsically.

It is entirely clear that the expression for the *vis viva* of the rigid body admits the group of its own motions; i.e., the rotations.

We ask, more generally:

*When does our expression for the vis viva T admit an infinitesimal transformation* $X_\rho f$ ? From our definition of $X_\rho f$, and in the sense of group theory, we would like to regard the $\delta\vartheta$ as constants here, such that one can set:

$$d\ \delta_\rho\, \vartheta_\lambda = 0\ .$$

The answer to this question is not hard to find. If $T$ is to admit the $\rho^{\text{th}}$ infinitesimal transformation, and indeed the extended point-transformation, then one must have:

$$X_\rho\, T = 0;$$

i.e.:

$$\sum_\sigma \frac{\partial T}{\partial q_\sigma}\xi_{\rho,\sigma} + \sum_\sigma \frac{\partial T}{\partial \omega_\sigma}\delta_\rho \omega_\sigma = 0.$$

[from (B), page 9]

However, from [(I), page 8]:

$$\delta_\rho\, \omega_\sigma = -\sum_\mu \beta_{\rho,\mu,\sigma}\omega_\mu\, \delta\vartheta_\rho .$$

Hence, one must have:

$$\sum_\sigma \frac{\partial T}{\partial q_\sigma}\xi_{\rho,\sigma} - \sum_{\sigma,\mu} \frac{\partial T}{\partial \omega_\sigma}\beta_{\rho,\mu,\sigma}\omega_\mu = 0. \tag{9}$$

However, if we imagine that, from [(F), page 15], the first sum will be nothing but $\left(\frac{\partial T}{\partial \vartheta_\rho}\right)$, while the second one means precisely $\sum_{\sigma,\mu}\beta_{\rho,\mu,\sigma}\omega_\mu J_\sigma$, then we will see that all terms up to the first one on the left in the $\rho^{\text{th}}$ equation (IVb) will drop out, and since all of the conclusions can be inverted, we will have the theorem:

*The assumption that the expression for the vis viva will admit the $\rho^{th}$ infinitesimal transformation $X_\rho$ under the assumption that $d\,\delta_\rho\,\vartheta_\lambda = 0$ has the consequence that the* ***Lagrange-Euler*** *equation reads simply:*

$$\frac{dJ_\rho}{dt} = Q_\rho , \tag{VI}$$

*and conversely: If it so happens that this equation assumes such a simple form then $T$ must admit the $\rho^{th}$ infinitesimal transformation.* The analytic expression for $T$ to admit that transformation is equation (9).

We would like to refer to equations (VI) as *the impulse equations.*

Incidentally, that yields yet another theorem:

*One can also write the general* ***Lagrange-Euler*** *equations thus:*

$$\frac{dJ_\rho}{dt} - X_\rho T = Q_\rho , \tag{IVc}$$

*in which $X_\rho$ is the symbol of the $\rho^{th}$ infinitesimal extended point-transformation.*

In this form, our equations come to light more clearly as generalizations of the **Lagrangian** equation in a different way. $X_\rho T$ is intrinsically connected with $\partial T / \partial \vartheta_\rho$ conceptually, and indeed it also goes over to it when the $\vartheta_\rho$ are actual coordinates.

**Example:** If we consider a rigid body that rotates around a fixed point and introduce the projections $\pi$, $\kappa$; $\rho$ ([1]) of the rotation vector onto three orthogonal axes that are fixed in space as the parameters $\omega$ then, as we remarked before, $T$ will admit the group of those motions, and therefore the equations of motion will read simply:

$$\frac{dJ_1}{dt} = Q_1 , \qquad \frac{dJ_2}{dt} = Q_2 , \qquad \frac{dJ_3}{dt} = Q_3 ,$$

in which one has ([2]):

$$J_1 = \frac{\partial T}{\partial \pi}, \qquad J_2 = \frac{\partial T}{\partial \kappa}, \qquad J_3 = \frac{\partial T}{\partial \rho}.$$

We shall return to the general case.

*If $T$ admits precisely $\nu \leq n$ of the infinitesimal transformations (B) then $\nu$ equations that are analogous to the area theorems will exist in the case of force-free motion, namely:*

$$J_\lambda = \text{const.}$$

Furthermore, one has the theorem:

*If we know any $\nu$ infinitesimal transformations:*

$$X_\rho f = \sum_\lambda \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda} \qquad (\lambda = 1, 2, \ldots, \nu)$$

*the $T$ admits for a mechanical system with $n$ degrees of freedom then we make the substitutions (A), for a suitable choice of the other $\xi_{\rho,\kappa}$. The first $n$ equations of motion then read simply:*

$$\frac{dJ_\rho}{dt} = Q_\rho .$$

*In particular, if the motion is force-free then we will know $\nu$ first integrals directly, namely,* $J_\rho$ = const.

It would be consistent with the **Thomson-Helmholtz** view of things to refer to the $\omega_\rho$ for which $X_\rho\, T = 0$ and $Q_\rho = 0$ (so $J_\rho$ = const.) as *cyclic velocity parameters;* they then define the natural generalization of the cyclic coordinates (their derivatives with respect to time, resp.).

Let it be stated expressly that not every mechanical system possesses impulse equations.

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([1]) This is the same notation as in **Klein-Sommerfeld**, pp. 45, equations (8).

([2]) *Ibidem*, pp. 115, Theorem IIa.

## § 10. Euler's equations

We shall take our basis to be our ongoing assumption that just $d\delta\vartheta_\rho$ shall be zero under infinitesimal transformation (variation), which is entirely arbitrary, if also closely related to group-theoretic concepts. Any other assumption about $d\delta\vartheta_\rho$ is likewise permitted. For example, we can then consider a variation for which, in particular:

$$d\delta\vartheta_\rho = \sum \beta_{\lambda,\mu,\rho} d\vartheta_\mu \, \delta\vartheta_\lambda \, ,$$

such that one will have directly that:

$$\delta d\, \vartheta_\rho = 0.$$

What does that mean?

Only that we leave the velocity parameters unchanged under the variation. However, we once more consider the rigid body and now introduce the components $p$, $q$, $r$ of the rotation vector along three orthogonal axes that are fixed in the body to be $\omega_1$, $\omega_2$, $\omega_3$, resp. The assumption that $\delta p = 0$, $\delta q = 0$, $\delta r = 0$ will then mean that we generate the varied motion directly by an infinitely-small, constant rotation around an axis that is fixed in space; then and only then do the velocity components not change relative to the axes that are fixed in the body. (Analytically-speaking: The position of the body at every time is fixed by $p$, $q$, $r$, up to a rotation of the coordinate system.) In this example, we see how the assumption that $d\delta\vartheta_1 = 0$, $d\delta\vartheta_2 = 0$, $d\delta\vartheta_3 = 0$ (the $\delta\vartheta$ are referred to axes that are fixed in space) can be closely related to the assumption that $\delta d\vartheta_1 = 0$, $\delta d\vartheta_2 = 0$, $\delta d\vartheta_3 = 0$ (the $\delta\vartheta$ are referred to axes that are fixed in the body). A deeper study of this remarkable relationship in the general case shall follow later; for now, that remark might suffice to justify the consideration of the assumption:

$$\delta_\lambda \, d\vartheta_\lambda = 0.$$

We once more ask: *When does $T$ admit the $\rho^{th}$ infinitesimal transformation under this new assumption?*

Obviously, one must then have:

$$\sum_\sigma \frac{\partial T}{\partial q_\sigma} \xi_{\rho,\sigma} = 0, \tag{10}$$

or

$$\left(\frac{\partial T}{\partial \vartheta_\rho}\right) = 0,$$

and the $\rho^{th}$ **Lagrange-Euler** equation now reads:

$$\frac{dJ_\rho}{dt} + \sum \beta_{\lambda,\mu,\rho} \omega_\mu \, J_\lambda = Q_\rho \, . \tag{VII}$$

We would like to call such an equation an **Euler equation** (in the broad sense).

We have then arrived at the theorem:

*The assumption that the expression for the vis viva T admits the* $\rho^{th}$ *infinitesimal transformation under the assumption that* $\delta_\rho\, d\vartheta_\lambda = 0$ *(for all* $\lambda$*) has the consequence that the simple* ***Euler*** *equation (VII) will enter in place of the* $\rho^{th}$ ***Lagrange-Euler*** *equation. Conversely: If one succeeds in giving the* $\rho^{th}$ *equation that form then T must admit the* $\rho^{th}$ *infinitesimal transformation under the assumption that* $\delta_\lambda\, d\vartheta_\lambda = 0$.

If we assume that $T$ admits *all n* infinitesimal transformations then we will naturally get nothing but **Euler**'s equations. However, it also follows from equations (10), which must now be true for all $\rho$, that:

$$\frac{\partial T}{\partial q_\sigma} = 0 \qquad \text{for all } \sigma;$$

i.e., $T$ must have constant coefficients.

**Theorem:** *A mechanical system will move in accordance with Euler's equation (VII) if and only if T has constant coefficients after one introduces the* $\omega$*, so if and only if T admits all infinitesimal transformations (B) under the assumption that* $\delta\omega_\lambda = 0$.

*In particular, if the n infinitesimal transformations generate a group, so when all* $\beta$ *are constant, then the* ***Euler****ian equations (VII) will contain nothing but the* $\omega$ *as variables on the left-hand side.*

In this case, one would probably be advised to refer to equations (VII) as ***Euler****'s equation in the narrow sense.*

Since the rigid body obviously fulfills all of the stated conditions when one introduces the projections $p$, $q$, $r$ of the rotation vector onto axes that are fixed in the body, its equations of motion must have precisely the form (VII), in which $J_1 = \partial T / \partial p$, $J_2 = \partial T / \partial q$, $J_3 = \partial T / \partial r$, and $T$ means a quadratic form in the $p$, $q$, $r$ with constant coefficients. One then needs only to determine the special form of the $\beta$, as was done above on page 17 in order to be able to write down the ordinary **Euler** equations for rigid bodies directly on the basis of this argument and our general theorems.

For the general, it should be remarked that: Since one can always introduce velocity parameters such that $T$ constant coefficients (since one can in fact bring any definite quadratic form $T$ into the form $\sum_{i=1}^{n} \omega_i^2$ by a real linear substitution), there will be **Euler** equations in the broad sense for any system. However, the $\beta$ will naturally not be constant.

*Any system of n degrees of freedom then possesses* ***Euler*** *equations in the broad sense, but not also in the narrow sense.*

## § 11. The relationship between the assumptions $d\delta\vartheta = 0$ and $\delta d\vartheta = 0$

We now take up the question of the relationship between the two assumptions $d\delta\vartheta = 0$ and $\delta d\vartheta = 0$ that we already brushed upon briefly above.

Since one can go from the $p$, $q$, $r$ to the $\pi$, $\kappa$, $\rho$ by a rotation of the coordinate system – i.e., by a linear transformation (with variable coefficients) – then we shall now pose the following general question:

*If we introduce n new velocity parameters* $\omega'_1, \ldots, \omega'_n$ *in place of* $\omega_1, \ldots, \omega_n$ *by way of the equations:*

$$\omega_\lambda = \sum_\kappa \varepsilon_{\lambda,\kappa}\omega'_\kappa , \tag{H}$$

*in which the determinant* $|\varepsilon_{\lambda,\kappa}|$ *should not vanish identically then can we arrange for the assumption that* $d\delta\vartheta_\lambda = 0$ *to go to* $\delta d\vartheta'_\lambda = 0$ *by a suitable choice of the* $\varepsilon$ (which depend upon the $q$)?

(Henceforth, we shall denote all quantities that refer to the $\omega'$ with primes.)

It follows from the equation:

$$\delta\vartheta_\lambda = \sum_\kappa \varepsilon_{\lambda,\kappa}\delta\vartheta'_\kappa ,$$

which corresponds to (H), that:

$$\left\{\begin{array}{l} d\delta\vartheta_\lambda = \sum_\kappa d\varepsilon_{\lambda,\kappa}\delta\vartheta'_\kappa + \sum_\kappa \varepsilon_{\lambda,\kappa}d\delta\vartheta'_\kappa, \\ \text{and from (H) itself :} \\ \delta d\vartheta_\lambda = \sum_\kappa \delta\varepsilon_{\lambda,\kappa}d\vartheta'_\kappa + \sum_\kappa \varepsilon_{\lambda,\kappa}\delta d\vartheta'_\kappa. \end{array}\right. \tag{11}$$

The first set of these equations can be employed to calculate $d\delta\vartheta'_\kappa$ when one considers that $d\delta\vartheta_\lambda = 0$; however, when one considers the transitivity equations on the left of the second set and the assumption that $\delta d\vartheta'_\kappa = 0$ on the right, it will follow that:

$$-\sum_{\mu,\nu} \beta_{\mu,\nu,\lambda} d\vartheta_\nu\, \delta\vartheta_\mu = \sum_\kappa \delta\vartheta_{\lambda,\kappa}\, d\vartheta'_\kappa .$$

If one now once more sets:

$$d\vartheta'_\kappa = \sum_\lambda E_{\kappa,\lambda}\, d\vartheta_\lambda$$

using (H), in which:

$$\sum_\lambda E_{\kappa,\lambda}\, \varepsilon_{\lambda,\mu} = \delta_{\kappa,\mu} ,$$

then equations (11) must be true for all $d\vartheta$ and $\delta\vartheta$. On the same basis, when one substitutes:

$$\delta\varepsilon_{\lambda,\kappa} = \sum_{\iota}\frac{\partial\varepsilon_{\lambda,\kappa}}{\partial q_\iota}\delta q_\iota = \sum_{\iota}\frac{\partial\varepsilon_{\lambda,\kappa}}{\partial q_\iota}\xi_{\nu,\iota}\,\delta\vartheta_\nu$$

it will follow further that:

$$-\beta_{\nu,\mu,\lambda} = \sum_{\kappa,\iota}\frac{\partial\varepsilon_{\lambda,\kappa}}{\partial q_\iota}\xi_{\nu,\iota}\,E_{\kappa,\mu}\,.$$

If one solves these equations for the derivatives of $\varepsilon_{\lambda,\kappa}$ then one will get:

$$\text{(12)}\qquad \frac{\partial\varepsilon_{\lambda,\kappa}}{\partial q_\iota} = -\sum_{\nu,\mu}\beta_{\nu,\mu,\lambda}\pi_{\nu,\iota}\,\varepsilon_{\mu,\kappa}\,.$$

*These are the constraint equations for the* $\varepsilon_{\lambda,\kappa}$. The assumption that $d\delta\vartheta_\lambda = 0$ will go to the assumption that $\delta d\vartheta'_\lambda = 0$ under the transformation (H) if and only if they are fulfilled.

We shall now address the problem of exhibiting the integrability equations for (12). If we differentiate these equations with respect to $q_\tau$ and then present the same equations when we switch $\iota$ and $\tau$ then since the left-hand sides of these two equations agree, the right-hand sides must also be equal. When we combine both sides simultaneously, we will then get:

$$\sum_{\nu,\mu}\beta_{\nu,\mu,\lambda}\varepsilon_{\mu,\kappa}\left(\frac{\partial\pi_{\nu,\iota}}{\partial q_\tau} - \frac{\partial\pi_{\nu,\tau}}{\partial q_\iota}\right) + \sum_{\nu,\mu}\left(\frac{\partial\beta_{\nu,\mu,\lambda}}{\partial q_\tau}\pi_{\nu,\iota} - \frac{\partial\beta_{\nu,\mu,\lambda}}{\partial q_\iota}\pi_{\nu,\tau}\right)\varepsilon_{\mu,\kappa}$$

$$+\sum_{\nu,\mu}\beta_{\nu,\mu,\lambda}\left(\frac{\partial\varepsilon_{\mu,\kappa}}{\partial q_\tau}\pi_{\nu,\iota} - \frac{\partial\varepsilon_{\mu,\kappa}}{\partial q_\iota}\pi_{\nu,\tau}\right) = 0.$$

If we substitute equations (5′) (page 9, § **2**), as well as equations (12) in this then we will get:

If we switch $\rho$ and $\mu$ in the last sum then since the equations are true for all $\kappa$, we can also write:

$$\sum_{\nu,\rho,\sigma}\beta_{\nu,\mu,\lambda}\beta_{\sigma,\rho,\nu}\,\pi_{\rho,\iota}\,\pi_{\sigma,\iota} - \sum\beta_{\nu,\rho,\lambda}\beta_{\sigma,\mu,\nu}\,\pi_{\sigma,\tau}\,\pi_{\nu,\iota} + \sum\beta_{\nu,\mu,\lambda}\beta_{\sigma,\mu,\rho}\,\pi_{\sigma,\iota}\,\pi_{\nu,\tau}$$

$$= \sum_{\nu}\left(\frac{\partial\beta_{\nu,\mu,\lambda}}{\partial q_\iota}\pi_{\nu,\tau} - \frac{\partial\beta_{\nu,\mu,\lambda}}{\partial q_\tau}\pi_{\nu,\iota}\right).$$

We shall now steer clear of applying formula (6b) (page 12, § **3**). We can write the foregoing formula with permutation of the summation indices as:

$$\sum_{\rho,\sigma}\pi_{\rho,\iota}\,\pi_{\sigma,\iota}\sum_{\nu}(\beta_{\rho,\sigma,\nu}\beta_{\nu,\mu,\lambda} + \beta_{\mu,\rho,\nu}\beta_{\nu,\sigma,\lambda} + \beta_{\sigma,\mu,\nu}\beta_{\nu,\rho,\lambda})$$

$$= \sum_{\rho} \pi_{\rho,\tau} \frac{\partial \beta_{\mu,\rho,\lambda}}{\partial q_\iota} + \sum_{\sigma} \pi_{\sigma,\iota} \frac{\partial \beta_{\sigma,\mu,\lambda}}{\partial q_\tau}.$$

It follows upon solving for the sum over $\nu$ that:

$$\sum_{\nu} (\beta_{\rho,\sigma,\nu} \beta_{\nu,\mu,\lambda} + \beta_{\mu,\rho,\nu} \beta_{\nu,\sigma,\lambda} + \beta_{\sigma,\mu,\nu} \beta_{\nu,\rho,\lambda})$$
$$= \sum_{\iota} \xi_{\sigma,\iota} \frac{\partial \beta_{\mu,\rho,\lambda}}{\partial q_\iota} + \sum_{\sigma} \xi_{\rho,\tau} \frac{\partial \beta_{\sigma,\mu,\lambda}}{\partial q_\tau}.$$

If we switch the summation symbols $\tau$ and $\iota$ in the last sum and compare the formula, thus-altered, with formula (6b) then we will see directly that we can also simply write it as:

$$\sum_{\iota} \xi_{\mu,\iota} \frac{\partial \beta_{\rho,\sigma,\lambda}}{\partial q_\iota} = 0.$$

Since this should be true for all $\mu$, it will follow that the necessary integrability condition for equations (12) is that:

(VIII) $$\frac{\partial \beta_{\rho,\sigma,\lambda}}{\partial q_\iota} = 0$$

for all $\rho, \sigma, \lambda, \iota$; i.e., all $\beta$ must be constant.

Now it follows from the general theory of systems of partial differential equations like (12) ([1]) that the condition that was found is also sufficient for (12) to possess solutions. Indeed, the general solution of (12) contains precisely $n^2$ constants, say the values of $\varepsilon_{\lambda,\mu}$ for a well-defined system of values of the $q$. Furthermore, each $\varepsilon_{\lambda,\mu}$ depends upon only $n$ of those constants, since the system (12) splits into $n$ separate independent systems that are represented by the various values of the index $\kappa$.

We then have the following Theorem:

*It is possible to convert the condition* $d\delta\vartheta = 0$ *into the condition* $\delta d\vartheta = 0$ *by a transformation of the form* (H) *iff the n infinitesimal transformations* $X_\rho = \sum_{\lambda} \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda}$ *generate an n-parameter group. The transformation coefficients* $\varepsilon$ *are determined completely by the partial differential equations* (12), *up to a well-defined system of initial values for the q.*

In particular, for rigid bodies, the $\pi, \kappa, \rho$ will transform into the $p, q, r$ in precisely the same way that the axis-cross that is fixed in space transforms into the axis-cross that is fixed in the body. Therefore, the $\varepsilon_{\lambda,\kappa}$ will be the nine direction cosines that the system that is fixed in the body define with the one that is fixed in space for rigid bodies.

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([1]) See, e.g., **Lie-Engel**, v. I, pp. 179, Theorem I.

How do the other quantities transform under the substitution (H), namely, the $\beta$ (for which we will now write $c$), the $\bar{e}$, the $x$, the $J$, and the $Q$? It follows from the first row of equations (11) that:

$$\begin{aligned} d\delta\vartheta'_\kappa &= -\sum_{\mu,\lambda} d\varepsilon_{\lambda,\mu}\,\delta\vartheta'_\mu\,E_{\kappa,\lambda} \\ &= -\sum_{\mu,\lambda,\nu,\iota} \frac{\partial\varepsilon_{\lambda,\mu}}{\partial q_\iota}\xi_{\nu,\iota}\,d\vartheta_\nu\,\delta\vartheta'_\mu\,E_{\kappa,\lambda} \\ &= -\sum_{\mu,\lambda,\nu,\rho,\iota} \frac{\partial\varepsilon_{\lambda,\mu}}{\partial q_\iota}\xi_{\nu,\iota}\,\varepsilon_{\nu,\rho}E_{\kappa,\lambda}d\vartheta'_\rho\,\delta\vartheta'_\mu \;. \end{aligned}$$

Hence, when we switch the indices $\rho$ and $\mu$, we will have:

$$c'_{\rho,\mu,\kappa} = -\sum_{\lambda,\nu,\iota} \frac{\partial\varepsilon_{\lambda,\rho}}{\partial q_\iota}\xi_{\nu,\iota}\,\varepsilon_{\lambda,\mu}E_{\kappa,\lambda}\,.$$

If we then substitute the value of $\partial\varepsilon_{\lambda,\rho}/\partial q_\iota$ from (12) in this then we will get:

$$\begin{aligned} c'_{\rho,\mu,\kappa} &= \sum_{\sigma,\gamma,\lambda} c_{\alpha,\gamma,\lambda}\sum_{\iota,\nu}\pi_{\alpha,\iota}\,\xi_{\nu,\iota}\,\varepsilon_{\gamma,\rho}\varepsilon_{\nu,\mu}E_{\kappa,\lambda} \\ &= \sum_{\alpha,\gamma,\lambda} c_{\alpha,\gamma,\lambda}\varepsilon_{\alpha,\mu}\varepsilon_{\gamma,\rho}E_{\kappa,\lambda}\,. \end{aligned}$$

or finally, when written more clearly:

(13) $$c'_{\rho,\mu,\kappa} = -\sum_{\alpha,\gamma,\lambda} c_{\alpha,\gamma,\lambda}\varepsilon_{\alpha,\rho}\varepsilon_{\gamma,\mu}E_{\kappa,\lambda}\,.$$

Naturally, the $c'$ will also be constant, since no significant difference exists between the variations $d$ and $\delta$.

In order to find the transformation of the $\bar{e}$, we start from the equation:

$$d\bar{x} = \sum \bar{e}_\lambda\,d\vartheta_\lambda = \sum \bar{e}'_\lambda\,d\vartheta'_\lambda = \sum \bar{e}'_\lambda\,E_{\lambda,\kappa}\,d\vartheta_\kappa\,.$$

Hence:

$$\bar{e}_\lambda = \sum_\kappa E_{\kappa,\lambda}\,\bar{e}'_\kappa\,.$$

It similarly follows from:

$$dq_\lambda = \sum_\rho \xi_{\rho,\lambda}\,d\vartheta_\rho = \sum_\rho \xi'_{\rho,\lambda}\,d\vartheta'_\rho$$

that

(14′) $$\varepsilon_{\rho,\lambda} = \sum_\kappa E_{\kappa,\rho}\,\xi'_{\kappa,\lambda}\,,$$

and in precisely the same way:

(14″) $$J_\lambda = \sum_\kappa E_{\kappa,\lambda}\,J'_\kappa\,,$$

$$(14''') \qquad Q_\lambda = \sum_\kappa E_{\kappa,\lambda}\, Q'_\kappa \,.$$

---

Now, if the *vis viva T* admits all the infinitesimal transformations $\sum_\rho \xi_{\lambda,\rho}\frac{\partial f}{\partial q_\rho}$, in the sense that one takes $d\delta\vartheta = 0$, then naturally $T$ will also admit the transformations $\sum_\rho \xi'_{\lambda,\rho}\frac{\partial f}{\partial q_\rho}$, and it is obvious that one must now set $\delta d\vartheta'_\rho = 0$. With that, we have the Theorem:

*If the equations of motion of a system can be brought into the form of impulse equations:*

$$\frac{dJ_\lambda}{dt} = Q_\lambda\,, \qquad \textit{in which} \qquad J_\lambda = \frac{\partial T}{\partial \omega_\lambda}\,,$$

*then one can introduce, under* ***one*** *condition, new velocity parameters $\omega'$ by the substitutions* (H) *in such a way that the new equations of motion will read:*

$$\frac{dJ'_\lambda}{dt} + \sum c'_{\lambda,\mu,\nu}\,\omega'_\mu\, J'_\nu = Q'_\lambda\,,$$

*in which one now has $J'_\lambda = \partial T / \partial\omega'_\lambda$, and the coefficients of the form T that is quadratic in the $\omega'_\lambda$ are constant. The single condition reads: One must have that $\beta = c$, and therefore also $\beta' = c'$, must be constant; i.e., the infinitesimal transformations* (B) (page 9, § **3**) *must generate an n-parameter group.*

Briefly stated:

*If there are* ***Euler*** *equations in the narrow sense then there will also be impulse equations; however, the converse it true only when the transformations* (B) *generate a group.*

There are a few things that can be said about the $\varepsilon_{\lambda,\kappa}$, namely, the coefficients in (H).

If we denote the initial values of the $\varepsilon_{\lambda,\kappa}$ by $\varepsilon^{(0)}_{\lambda,\kappa}$ then the integral equations of (12):

$$\varepsilon_{\lambda,\kappa} = \varphi_{\lambda,\kappa}(q_1, \ldots, q_n\,;\ \varepsilon^{(0)}_{1,\kappa}, \ldots, \varepsilon^{(0)}_{n,\kappa})$$

will define the finite equations of a group. If we set:

$$\sum_\mu c_{\mu,\nu,\lambda}\,\varepsilon_{\mu,\kappa} = \xi_{\nu,\lambda,\kappa}$$

then equations (12) will read:

$$\frac{\partial \varepsilon_{\lambda,\kappa}}{\partial q_\iota} = \sum_\nu \pi_{\nu,\iota}\, \xi_{\nu,\lambda,\kappa}\,.$$

Since the functions $\varphi_{\lambda,\kappa}$ satisfy these differential equations, from **Lie**'s ([1]) first fundamental theorem, the $n$ equations

$$\varepsilon_{\lambda,\kappa} = \varphi_{\lambda,\kappa}$$

(for $\lambda = 1, 2, \ldots, n$, but a well-defined, if also arbitrary, $\kappa$) will, in fact, represent a group; $q_1, \ldots, q_n$ play the role of the parameters.

The equations:

$$\varepsilon_{\lambda,\kappa} = \varphi_{\lambda,\kappa}(q_1, \ldots, q_n\,;\, \varepsilon^{(0)}_{1,\kappa}, \ldots, \varepsilon^{(0)}_{n,\kappa}),$$
$$\varepsilon'_{\lambda,\kappa} = \varphi_{\lambda,\kappa}(q'_1, \ldots, q'_n\,;\, \varepsilon_{1,\kappa}, \ldots, \varepsilon_{n,\kappa})$$

will then imply the equations:

$$\varepsilon'_{\lambda,\kappa} = \varphi_{\lambda,\kappa}(q''_1, \ldots, q''_n\,;\, \varepsilon^{(0)}_{1,\kappa}, \ldots, \varepsilon^{(0)}_{n,\kappa}),$$

in which $q''_1, \ldots, q''_n$ are functions of only $q_1, \ldots, q_n$ and $q'_1, \ldots, q'_n$.

However, the infinitesimal transformations of that group:

$$\sum_\lambda \xi_{\mu,\lambda,\kappa} \frac{\partial f}{\partial \varepsilon_{\lambda,\kappa}} = \sum_{\lambda,\nu} c_{\mu,\lambda,\kappa} \frac{\partial f}{\partial \varepsilon_{\lambda,\kappa}}$$

are nothing but the infinitesimal transformations of the adjoint group ([2]) that belongs to the group $X_\rho = \sum_\lambda \xi_{\rho,\lambda} \dfrac{\partial f}{\partial q_\lambda}$.

We then have the theorem:

*The arbitrary values of the q that are associated with the $\varepsilon$ by* (12) *emerge from the arbitrarily-chosen initial values of the $\varepsilon$ by an application of the adjoint group to the given group* (B)*, in which the coordinates $q_1, \ldots, q_n$ play the role of parameters. However, the parameter group of that new group is once more the old group.*

In order to prove the last statement, we write (12) as:

$$\xi_{\mu,\lambda,\kappa} = \sum_\iota \xi_{\mu,\iota} \frac{\partial \varepsilon_{\lambda,\kappa}}{\partial q_\iota}\,.$$

However, from a theorem that **Lie** proved ([1]), it will follow from this equation that:

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([1]) **Lie-Scheffers**, pp. 376; **Lie-Engel** III, pp. 563.
([2]) **Lie-Scheffers**, pp. 464, formula (19).

$$\sum_\iota \xi_{\mu,\iota} \frac{\partial f}{\partial q_\iota};$$

i.e., the symbol of the $\mu^{\text{th}}$ infinitesimal transformation of the original group (B), is simultaneously the symbol of the $\mu^{\text{th}}$ infinitesimal transformation of the parameter group.

We would like to derive yet another remarkable relationship. Namely, I assert that *the given group* (B) *and the reciprocal group* ([2]) *that corresponds to the* $\xi'_{\rho,\lambda}$ [see equation (14′)] *are simply-transitive groups.*

Namely, if we set:

$$X'_\rho = \sum_\lambda \xi'_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda},$$

analogous to (B), then I assert that we will have:

$$(X_\rho,\ X'_\sigma) = 0$$

for every pair of values $\rho$, $\sigma$, and since our groups are simply-transitive, from the assumptions that:

$$|\ \xi_{\rho,\lambda}\ | \neq 0 \qquad \text{and} \qquad |\ \varepsilon_{\rho,\lambda}\ | \neq 0,$$

the first statement will follow from our new one by **Lie**'s theorems in Chapter 20 of *Transformationsgruppen.*

However, our new assertion is very easy to prove. Since:

$$(X_\rho\ X'_\sigma) = \sum_{\lambda,\mu} \left( \frac{\partial \xi'_{\sigma,\lambda}}{\partial q_\mu} \xi_{\rho,\mu} - \frac{\partial \xi_{\sigma,\lambda}}{\partial q_\mu} \xi'_{\rho,\mu} \right) \frac{\partial f}{\partial q_\lambda},$$

and from (14′):

$$\xi'_{\kappa,\lambda} = \sum_\rho \xi_{\rho,\lambda} \varepsilon_{\rho,\kappa},$$

the statement will follow from a simple calculation when we observe equations (12) for the $\varepsilon_{\rho,\kappa}$ and equations (5) for the $c_{\lambda,\mu,\nu}$ (page 8).

Since one can determine the $\varepsilon_{\rho,\kappa}$ uniquely from the given $\xi'_{\kappa,\lambda}$ and $\xi_{\rho,\kappa}$ using equations (14), that will imply the following purely-group-theoretic theorem:

*If we have two classes of n infinitesimal transformations in n variables:*

$$X_\rho = \sum_\lambda \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda} \qquad \text{and} \qquad X'_\sigma = \sum_\lambda \xi'_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda},$$

---

([1]) **Lie-Scheffers**, v. I, pp. 407, Theorem 72.
([2]) **Lie-Scheffers**, v. I, pp. 380, Theorem 68.

*and none of the determinants* | $\xi_{\rho,\kappa}$ | *and* $\left|\xi'_{\kappa,\lambda}\right|$ *vanish identically then the transformations of the one class can be switched with all transformations of the other class only when the transformations of each class generate an n-parameter group. Naturally, the groups will then be simply-transitive and reciprocal to each other.*

Since I will not need this theorem, I can probably suppress its proof; however, all of the pieces that are necessary for that proof are contained in the considerations of these paragraphs.

### § 12. The "rotation group" and the "rigid body with *n* degrees of freedom"

As we see from equations (H), (14), (14′), (14″), (14‴), so the $\bar{e}$, $J$, $\xi$, $Q$ will transform differently from the $\omega$. There would be a certain interest to examining the case in which the $\bar{e}$, $J$, $\xi$, $Q$ transform precisely like the $\omega$ for rigid bodies.

However, one sees immediately that one only needs to have:

$$E_{\kappa,\lambda} = \varepsilon_{\kappa,\lambda} \tag{J}$$

then.

However, the equations:

$$\sum_\lambda E_{\kappa,\lambda}\varepsilon_{\lambda,\mu} = \delta_{\kappa,\mu}$$

then imply that:

$$\begin{cases} \sum_\lambda \varepsilon_{\lambda,\kappa}^2 = 1, \\ \sum_\lambda \varepsilon_{\lambda,\kappa}\varepsilon_{\lambda,\mu} = 0, \quad \text{when } \kappa \neq \mu. \end{cases} \tag{15}$$

However, since the assumption (J) implies that the sub-determinant of each element in the determinant | $\varepsilon_{\lambda,\kappa}$ | ([1]) is equal to that element, it will also follow that:

$$\begin{cases} \sum_\lambda \varepsilon_{\lambda,\kappa}^2 = 1, \\ \sum_\kappa \varepsilon_{\lambda,\kappa}\varepsilon_{\mu,\kappa} = 0, \quad \text{when } \lambda \neq \mu. \end{cases} \tag{15′}$$

The quantities $\varepsilon_{\lambda,\kappa}$ the have the character of direction cosines in an *n*-dimensional space; as is easy to see, $n(n-1)/2$ of them are independent, so the substitution (H) will be orthogonal under the assumption (J).

We then have the theorem:

*If the substitution (H) is orthogonal then the* $\omega$ *will transform precisely like the* $J$, $\bar{e}$, $\xi$, *and* $Q$.

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([1]) Which will have the value ± 1, as a result of (15); we would like to choose + 1.

We would like to call the totality of $n$ quantities that transform like the $\omega$ a *vector relative to the system* ([1]). We shall then also speak briefly of the "velocity vector" $\omega$, the "impulse vector" $J$, and the "force vector" $Q$. However, in order to avoid misunderstanding, we would like to always put those terms in quotation marks.

Now how does the assumption (J) relate to the differential equations (12) of the previous paragraph?

In place of (12), one can also write:

$$c_{\mu,\nu,\lambda} = \sum_{\pi,\iota} \frac{\partial \varepsilon_{\lambda,\kappa}}{\partial q_\iota^2}\, \xi_{\nu,\iota}\, E_{\kappa,\mu} ,$$

so from (J):

$$c_{\mu,\nu,\lambda} = \sum_{\pi,\iota} \frac{\partial \varepsilon_{\lambda,\kappa}}{\partial q_\iota^2}\, \varepsilon_{\mu,\kappa}\, \xi_{\nu,\iota} .$$

However, from (15′), that is the same thing as:

$$-\sum_{\pi,\iota} \frac{\partial \varepsilon_{\mu,\kappa}}{\partial q_\iota^2}\, \varepsilon_{\lambda,\kappa}\, \xi_{\nu,\iota} , \qquad \text{i.e., as} - c_{\lambda,\nu,\mu} .$$

Thus, in any event, a necessary condition for (J) to be possible is that:

$$c_{\mu,\nu,\lambda} + c_{\lambda,\nu,\mu} = 0.$$

If one combines this with:

$$c_{\mu,\nu,\lambda} + c_{\nu,\mu,\lambda} = 0$$

then it will follow that:

$$\text{(IX)} \qquad \left\{ \begin{array}{cccccc} & c_{\mu,\nu,\lambda} & = & c_{\nu,\lambda,\mu} & = & c_{\lambda,\mu,\nu} \\ = & -c_{\lambda,\nu,\mu} & = & -c_{\nu,\mu,\lambda} & = & -c_{\mu,\lambda,\nu} . \end{array} \right.$$

Conversely, if (IX) is fulfilled then it will follow from equations (12) when one multiplies by $\varepsilon_{\lambda,\kappa}$ and sums over $\lambda$ that:

$$\tfrac{1}{2} \frac{\partial}{\partial q_\iota} \sum_\lambda \varepsilon_{\lambda,\mu}^2 = \sum_{\mu,\nu,\lambda} c_{\mu,\nu,\lambda} \pi_{\nu,\iota} \varepsilon_{\mu,\kappa} \varepsilon_{\lambda,\kappa} ,$$

and that is zero, since every term on the right also occurs with its opposite. It likewise follows that:

$$\sum_\lambda \left( \varepsilon_{\lambda,\kappa} \frac{\partial \varepsilon_{\lambda,\tau}}{\partial q_\iota} + \varepsilon_{\lambda,\tau} \frac{\partial \varepsilon_{\lambda,\kappa}}{\partial q_\iota} \right) = \sum_{\mu,\nu,\lambda} c_{\mu,\nu,\lambda} \pi_{\nu,\iota} (\varepsilon_{\mu,\kappa} \varepsilon_{\lambda,\tau} + \varepsilon_{\mu,\tau} \varepsilon_{\lambda,\kappa}) = 0.$$

Therefore, one will have:

([1]) We are then using this term in a somewhat more restricted sense that **Hertz** used.

$$\sum_{\lambda} \varepsilon_{\lambda,\kappa} \cdot \varepsilon_{\lambda,\tau} = \text{const.}$$

for all $\kappa$; $\tau$, and when one chooses the initial values $\varepsilon^{(0)}_{\lambda,\kappa}$ of the $\varepsilon$ corresponding to equations (15), one will find that the $\varepsilon$ always fulfill equations (15), and therefore (J), as well. We then have the theorem:

*The substitution* (H) *that takes* $d\delta\vartheta = 0$ *to* $\delta d\vartheta = 0$ *can be orthogonal if and only if the constants of the group* (B) *fulfill equations* (IX); *i.e., when they admit a cyclic permutation of their indices.*

**Definitions:** We would like to call such a group a "rotation group" and a mechanical system with $n$ degrees of freedom whose *vis viva* $T$ admits such an $n$-parameter rotation group [whether the assumption is $d\delta\vartheta = 0$ or $\delta d\vartheta = 0$ makes no difference ([1])] will be a "rigid body with $n$ degrees of freedom." Here again, we would like to always apply quotation marks.

We would like to refer to an orthogonal substitution briefly as a "rotation" in what follows. We would also like to say that the "velocity vector" $\omega$ ["impulse vector" $J$, resp.] are referred to the components $\omega'$ ($J'$, resp.) ($\delta\omega' = 0$) in "a coordinate system that is fixed in the 'rigid body,'" while the components $\omega$ [$J$, resp.] ($d\delta\vartheta = 0$) are referred to a "coordinate system that is fixed in space."

With all of those preliminaries, we now have the following Theorem:

*A "rigid body with n degrees of freedom" always moves in accordance with the* ***Euler*** *equations in the narrow sense:*

$$\frac{dJ'_\lambda}{dt} + \sum_{\mu,\nu} c'_{\lambda,\mu,\nu} \omega'_\mu J'_\nu = Q'_\lambda ,$$

*in which* $J'_\lambda = \partial T / \partial \omega'_\lambda$ *are the components of the "impulse vector,"* $\omega'_\lambda$ *are the components of the "velocity vector," and* $Q'_\lambda$ *are the components of the "force vector" with respect to a "coordinate system that is fixed in the body." The vis viva T is a quadratic form of the* $\omega'$ *with constant coefficients, while the* $c'$ *are* ***Lie****'s characteristic constants for the motion of the "body," and they admit a cyclic permutation of their indices. All vectors will be transformed in the same way under a "rotation"; in particular, there is a "rotation" with* $n$ ($n$ – 1) / 2 *arbitrary constants that takes the assumption* $\delta d\vartheta'_\lambda = 0$ (one sets $d\vartheta'_\lambda / dt = \omega'_\lambda$) *to* $d\delta\vartheta_\lambda = 0$; *i.e., the coordinate system that is fixed in the body to one of the* $\infty^{n(n-1)/2}$ *"coordinate systems that are fixed in space."*

*However, the "rigid body" satisfies the n impulse equations:*

$$\frac{dJ_\lambda}{dt} = Q_\lambda ,$$

([1]) Since the group that belongs to the $c'$ is also a rotation group, as one easily sees from (13).

*relative to such a "coordinate system that is fixed in space," in which one again has* $J_\lambda = \partial T / \partial \omega_\lambda$ . *In particular, for a force-free motion there will exist n first integrals:*

$$J_\lambda = \text{const.},$$

*so the "impulse vector" will remain constant in space.*

## Chapter III. Group-theoretic considerations

### § 13. The group of all rotations in *n*-dimensional space

Obviously, the "rigid body with $n$ degrees of freedom" is essentially characterized kinematically by its group. One can always bring the $T$ for the "rigid bodies" into the form $\sum_i \omega_i^2$ , i.e., by a linear transformation of the $\omega$ with *constant* coefficients, without changing the group. The **Euler** equations then contain nothing besides the $c$ that would be kinematically characteristic. Equations (A) indeed define the group then, and since our group is naturally simply-transitive, groups that belong to the same $c$, which then have the same composition, will also be similar ([1]); i.e., all equations (A) that belong to the same $c$ can be converted into each other by a point-transformation.

The constants $c$ then determine the "body" kinematically. By contrast, many "rigid bodies" will belong to one group, since the group will remain unchanged under a linear substitution of the $\omega$ with constant coefficients, but the $T$ will take the form $\sum \omega_i^2$ that was assumed above only for all orthogonal substitutions of the $\omega$, as we shall soon see. Therefore, the theorem that the group characterizes "rigid bodies" will remain true only when we count all "rigid bodies" as having the same type when they indeed belong to the same group, but whose $T$ includes arbitrary constant coefficients ([2]). In this sense, e.g., all ordinary rigid bodies that can rotate about a fixed point can be assigned the same type. With that convention, we can say:

*If we would like to learn about all types of "rigid bodies" in a mathematical-kinematical way then it will suffice to exhibit all rotation groups.*

Let it be next remarked that the term "rotation" for an orthogonal substitution is also justified only insofar as the expressions $\sum \omega_\lambda'^2$ , $\sum \omega_\lambda''^2$ , and $\sum \omega_\lambda' \omega_\lambda''$ are invariants for two "vectors" $\omega_1'$ , …, $\omega_n'$ and $\omega_1''$ , …, $\omega_n''$ , and therefore, the distance between their end-points $\sqrt{\sum (\omega_\lambda' - \omega_\lambda'')^2}$ , as well. In that way, the orthogonal substitutions are conversely singled out from the other linear ones uniquely, and indeed it will already suffice to

([1]) **Lie-Engel** I, pp. 340, Theorem 64. **Lie-Scheffers**, pp. 435, Theorem 30.

([2]) i.e., instead of posing the diversity of Euler equations in the manifold of the same $c$ that define the group, while we keep the form $T = \sum \omega_i^2$ , we now find it preferable to pose them on the manifold of constants in $T$, in order to be able to demand that the $c$ should be free.

demand the invariance of $\sum \omega_\lambda^2$. Therefore, all rotations again define a group, namely, the *group of all rotations in n-dimensional space.*

We would like to prove the following theorem about that group:

*The group of all rotations of the n-dimensional space and the totality of its subgroups is characterized by the fact that the $n^2$ coefficients of each of its infinitesimal transformations* ([1]):

$$\delta_\lambda \omega_\rho = \sum_{\mu=1,\ldots,n} a_{\lambda,\mu,\rho} \omega_\mu \delta\tau_\lambda \qquad (\rho = 1, 2, \ldots, n)$$

*define a skew determinant of degree n, and indeed for every λ that is present.* That then means that one has:

$$a_{\lambda,\mu,\rho} = - a_{\lambda,\rho,\mu}$$

and

$$a_{\lambda,\mu,\mu} = 0.$$

The proof reads:

It must follow from the invariance of $\sum \omega_\rho^2$ that:

$$\sum_\rho \omega_\rho \delta_\lambda \tau_\rho = 0,$$

so

$$\sum_{\mu,\rho} a_{\lambda,\mu,\rho}\, \omega_\rho \omega_\mu = 0,$$

or

$$\sum_{\rho,\mu\le\rho} \omega_\rho \omega_\mu (a_{\lambda,\mu,\rho} + a_{\lambda,\rho,\mu}) = 0$$

for all $\omega$.

The assertion that was made then follows from that. The proof can also be inverted, and therefore the given condition is *also sufficient.*

It follows from this theorem, for example, that the adjoint group to a rotation group is an $n$-parameter subgroup of the group of all rotations, and similarly for the group of $\varepsilon$ that was mentioned on page 37, when the conditions (IX) (page 40) are fulfilled.

In what follows, we would like to refer to the cone:

$$\omega_1^2 + \omega_2^2 + \ldots + \omega_n^2 = 0,$$

which naturally goes to itself under all "rotations," as the *absolute cone.*

We can then prove the following theorem about the group of all rotations, which will be important for us:

---

([1]) The fact that $a$ is constant, so the infinitesimal transformations are linear functions of the $\omega$, is self-explanatory, since the finite transformation equations are indeed linear and homogeneous.

*One can take any linear structure through the origin in the space of $\omega$ to another one of equal dimension by a rotation when neither of the two linear structures contacts the absolute cone.*

We next prove that one can take the $(n - 1)$-dimensional plane:

$$\sum_\nu \lambda_\nu \, \omega_\nu = 0$$

to $\omega_n = 0$ when one does not have $\sum \lambda_\nu^2 = 0$, so when the plane does not contact the absolute cone.

If we then rotate things by setting:

$$\omega_\nu = \sum_\kappa h_{\nu,\kappa} \, \omega'_\kappa ,$$

in which:

$$\sum_\mu h_{\mu,\mu}^2 = 1,$$

$$\sum_\kappa h_{\nu,\kappa} h_{\mu,\kappa} = 0 \qquad \text{for } \nu \neq \mu,$$

and should $\sum \lambda_\nu \omega_\nu$ then go to $\omega'_n \cdot$ const. then we would need to have:

$$\sum_\nu \lambda_\nu h_{\nu,\kappa} = 0 \qquad \text{for } \kappa = 1, 2, \ldots, n - 1,$$

while $\sum_\nu \lambda_\nu h_{\nu,n}$ cannot be zero.

We now proceed as follows:

We choose any system of solutions of:

$$\sum_\nu \lambda_\nu h_{\nu,1} = 0$$

that is not identically zero. Upon multiplying all $h_{\nu,1}$ by the same factor, we can now arrange that:

$$\sum h_{\nu,1}^2 = 1.$$

If we were to then have $\sum h_{\nu,1}^2 = 0$ for all systems of solutions of the equation $\sum \lambda_\nu \, h_{\nu,1} = 0$ then we would need to have $h_{\nu,1} = \tau_0 \, \lambda_\nu$ ([1]), and therefore $\sum \lambda_\nu^2 = 0$, as well, which should be excluded. ($\tau_0 = 0$ is also impossible, as well as $\tau_0 = \infty$.) We now take a non-vanishing solution of the equations:

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([1]) since for all $dh_{\nu,1}$ that satisfy $\sum \lambda_\nu \, dh_{\nu,\lambda} = 0$, one must also have $\sum h_{\nu,1} \, dh_{\nu,1} = 0$.

$$\sum_\nu \lambda_\nu\, h_{\nu,2} = 0 \quad \text{and} \qquad \sum h_{\nu,1}\, h_{\nu,2} = 0.$$

Once more, one cannot have $\sum h_{\nu,2}^2 = 0$, since otherwise one would need to have:

$$h_{\nu,2} = \tau_1\, h_{\nu,1} + \tau_0\, \lambda_\nu .$$

It follows from this upon multiplying by $\lambda_\nu$ and summing that $\tau_0 = 0$. Multiplication by $\lambda_\nu$ and summing over $n$ will also give $\tau_1 = 0$ them so all $h_{\nu,2}$ would be zero, which should not happen, though.

One can then also succeed in having:

$$\sum h_{\nu,2}^2 = 1.$$

We then proceed.

At the penultimate step, we must solve the equations:

$$\sum \lambda_\nu\, h_{\nu,n-1} = 0, \qquad \sum h_{\nu,1}\, h_{\nu,n-1} = 0, \qquad \ldots, \qquad \sum h_{\nu,n-2}\, h_{\nu,n-1} = 0.$$

These are $n - 1$ linear homogeneous equations in $n$ unknowns, so they possess a non-zero solution. If one also had that $\sum_\nu h_{\nu,n-1}^2 = 0$ for this solution then it would follow that:

$$h_{\nu,n-1} = \tau_0\, \lambda_\nu + \tau_1\, h_{\nu,1} + \ldots + \tau_{n-2}\, h_{\nu,n-2} .$$

Just as before, one must once more conclude from this that one has $\tau_0 = 0$, $\tau_1 = 0$, …, $\tau_{n-2} = 0$ in succession, so the $\tau$ would all be zero, which cannot be. One can then also arrange that:

$$\sum h_{\nu,n-1}^2 = 1.$$

It ultimately remains for us to solve:

$$\sum_\nu h_{\nu,1}\, h_{\nu,n} = 0, \ldots, \qquad \sum_\nu h_{\nu,n-1}\, h_{\nu,n} = 0.$$

These are once more $n - 1$ homogeneous equations, so they have a non-zero solution, and we also do not have $\sum h_{\nu,n}^2 = 0$, since otherwise it would have to follow that:

$$h_{\nu,n} = \tau_1\, h_{\nu,1} + \ldots + \tau_{n-1}\, h_{\nu,n-1} ,$$

from which we could once more have to conclude the vanishing of all $\tau$. However, all of the required equations for the $h$ are satisfied with that; all that remains is to show that $\sum h_{\nu,n} \lambda_\nu$ *does not* vanish.

If that were in fact the case then since the equations $\sum h_{\nu,n}\, h_{\nu,\kappa} = 0$ ($\kappa < n$) are fulfilled, a linear relation $\tau_0\, \lambda_\nu + \tau_1\, h_{\nu,1} + \ldots + \tau_{n-1}\, h_{\nu,n-1} = 0$ would have to exist, which would likewise be impossible, from the previous argument and the assumption about the $h$.

One can then satisfy all conditions for the $h$.

With that, it has been shown that, in fact, any plane $\sum \lambda_\nu \omega_\nu = 0$ can be brought into the form $\omega_n = 0$ by a rotation when one does not have $\sum \lambda_\nu^2 = 0$; i.e., when the plane does not contact the absolute cone.

Now, in order show something similar for any lower-dimensional [say $(n - \nu)$-dimensional] linear manifold, we lay a manifold of the next-higher dimension through it that does not contact the absolute cone either (which is always possible), etc., and finally, an $(n - 1)$-dimensional plane. We then rotate it into $\omega_n = 0$. In that plane, we we rotate the next-lower-dimensional manifold into $\omega_{n-1} = 0$, and proceed in that way until we have ultimately brought the given $n - n$-dimensional manifold into the form $\omega_n = 0$, $\omega_{n-1} = 0$, … $\omega_{n-\nu+1} = 0$.

The stated theorem is proved with that.

## § 14. The composition of real rotation groups.

With those preliminaries, we turn to the study of rotation groups.

First of all, the group of the rigid body is a rotation group, as would emerge from the formulas on page 16.

However, there are only two three-parameter rotation groups, to begin with. Since the $c$ with two equal indices must always vanish, what will then remain is just $c_{1,2,3}$ . If $c_{1,2,3}$ is also zero then all $c$ will be zero, and the group will consist of nothing but commuting transformations, and would be similar to the group of translations. However, when $c_{1,2,3}$ is not zero, one can arrange that $c_{1,2,3} = 1$ by a suitable choice of the $\omega$, in such a way that the group is then similar to the group of rotations in three-dimensional space ([1]).

From the remarks at the beginning of the previous paragraphs, we can also say that:

*There are essentially two "rigid bodies with three degrees of freedom": the rigid bodies that rotate about a fixed point and the freely-moving point.*

It is a complicating fact that the characteristic distinction (IX) (page 40) of the rotation group does not emerge in all of its representations. Namely, in place of the $\omega$, one can introduce linear couplings $\omega'$ with *constant* coefficients:

([1]) Compare **Lie-Scheffers**, pp. 571. Of the seven types of groups that were given there, we can then convert only the first and last types into rotation groups. In regard to I, see the form I′ on page 568.

(K)
$$\begin{cases} \omega_\kappa = \sum_\lambda h_{\kappa,\lambda}\omega'_\lambda, \\ \text{or} \\ \omega'_\lambda = \sum_\kappa H_{\lambda,\kappa}\omega_\kappa, \end{cases}$$

in such a way that the relations (IX) are no longer fulfilled for the $c'$ that belong to the $\omega'$.

In fact, it follows from (K), in analogy to (14′), that:

$$\xi_{\rho,\lambda} = \sum_\kappa H_{\kappa,\rho}\xi'_{\kappa,\lambda},$$

and from that:

$$X_\rho = \sum_\kappa H_{\kappa,\rho}X'_\kappa,$$

so

$$X'_\kappa = \sum_\lambda h_{\lambda,\kappa}X_\lambda .$$

That then implies the formulas ([1]):

$$\sum_\lambda h_{\tau,\lambda}\, c'_{\mu,\nu,\lambda} = \sum_{\kappa,\sigma} c_{\sigma,\kappa,\tau} h_{\sigma,\mu} h_{\kappa,\nu}$$

or also:

(16)
$$c'_{\mu,\nu,\lambda} = \sum_{\sigma,\kappa,\tau} c_{\sigma,\kappa,\tau} h_{\sigma,\mu} h_{\kappa,\nu} H_{\lambda,\tau} .$$

Now, when the $c_{\sigma,\kappa,\tau}$ satisfy equations (IX) then one will have:

$$c'_{\mu,\nu,\lambda} + c'_{\lambda,\nu,\mu} = 0$$

only when one has:

$$\sum_{\sigma,\kappa,\tau} c_{\sigma,\kappa,\tau} h_{\kappa,\nu}(h_{\sigma,\mu}H_{\lambda,\tau} + h_{\sigma,\lambda}H_{\mu,\tau}) = 0$$

for all $\lambda$, $\mu$, $\nu$.

Since the determinant of the $h$ is naturally non-zero, it will follow that:

$$\sum_{\sigma,\tau} c_{\sigma,\kappa,\tau}(h_{\sigma,\mu}H_{\lambda,\tau} + h_{\sigma,\lambda}H_{\mu,\tau}) = 0,$$

or, when one considers (IX):

$$\sum_{\tau,\sigma<\tau} c_{\sigma,\kappa,\tau}[h_{\sigma,\mu}H_{\lambda,\tau} + h_{\sigma,\lambda}H_{\mu,\tau} - h_{\tau,\mu}H_{\lambda,\sigma} - h_{\tau,\lambda}H_{\mu,\sigma}] = 0.$$

([1]) See **Lie-Engel** I, pp. 290, formula (3).

In general, that condition will not be fulfilled, but it will probably be always fulfilled when $h_{\sigma,\lambda} = H_{\sigma,\lambda}$ ; i.e., when the substitution (K) is orthogonal. We then have the theorem:

*The condition* (IX) *for the rotation keeps its form under an orthogonal substitution of the* $\omega$ *with constant coefficients,* but not for all linear substitutions, in general.

We would now like to prove *a characteristic property of the adjoint group of a rotation group.* The $\nu^{\text{th}}$ infinitesimal transformation of that adjoint group reads:

$$\delta_\nu \omega_\lambda = \sum_\mu c_{\mu,\nu,\lambda} \omega_\mu \delta\vartheta_\nu ,$$

as has been stated several times. Now, when equations (IX) are fulfilled, the determinant:

$$|\, c_{\mu,\nu,\lambda} \,| \qquad \begin{pmatrix} \mu = 1, 2, \ldots, n \\ \lambda = 1, 2, \ldots, n \end{pmatrix}$$

will be skew for all $\nu$.

Therefore, from the theorem that was proved on page 43, *the adjoint group to a rotation group will be an n-parameter subgroup of the group of all rotations in n-dimensional space.*

That theorem, together with the aforementioned one, makes it possible for us to now carry out a more precise study of the rotation groups.

To that end, with **Lie**, we appeal to the lemma that we can represent every infinitesimal transformation of our group by a ray $\omega_1 : \omega_2 : \ldots : \omega_n$ in $n$-dimensional space of the $\omega$. Any $\nu$-parameter subgroup will then correspond to an $\nu$-dimensional linear manifold through the origin that remains unchanged under all transformations of the adjoint group, whose representative points lie in it ([1]). If the subgroup is invariant in the group then the planar manifold of *all* transformations of the adjoint group will remain unchanged ([2]).

We now make the assumption that *the rotation group is real.*

From now on, we shall assume that it possesses a real invariant subgroup.

One can introduce new $\omega$ by a substitution (K) (page 47), and indeed by a real rotation, such that the planar manifold that corresponds to the subgroup will go to the manifold $\omega_n = 0$, $\omega_{n-1} = 0$, …, $\omega_{\nu+1} = 0$. (See the Theorem on page 43.) In that way, one can arrange that the $\nu$ infinitesimal transformations of the invariant subgroup will be $X_1$, …, $X_\nu$ , precisely, since otherwise the characteristic property (IX) of the $c_{\lambda,\mu,\nu}$ would be lost (from the Theorem above). Now, since the $X_1$, …, $X_\nu$ initially define a group by themselves, any $c$ will be zero when it has two indices from the sequence 1, 2, …, $\nu$ and one index from the sequence $\nu$ + 1, …, $n$. Since that subgroup is invariant, moreover, and therefore only linear couplings of $X_1$, …, $X_\nu$ will arise when one forms the brackets $(X_\kappa, X_{\nu+1})$ ($\kappa$ = 1, 2, …, $\nu$), all $c$ with one index from the sequence 1, 2, …, $\nu$ and two

([1]) **Lie-Scheffers**, Chap. 18, §§ **1** and **3**, especially pp. 478.
([2]) *Ibidem*, pp. 485.

indices from the sequence $\nu + 1, \ldots, n$ will also vanish. However, as a result of that, the transformations $X_{\nu+1}, \ldots, X_n$ themselves will define, not just another group, but in fact an invariant subgroup of the entire group. In addition, any element of the first group will commute with every element of the second one, since every $c$ does indeed vanish when it includes one index from the sequence $1, 2, \ldots, \nu$ and one from the sequence $\nu + 1, \ldots, n$, regardless of what the third index might be.

Now, if one or both of the subgroups again contain an invariant subgroup in their own right then we proceed as follows: Ultimately, we must naturally reach the goal that the entire group consists of nothing but invariant subgroups that all commute with each other, and are real and simple in the reals, and thus no longer contain any more invariant subgroups. With that, we have the theorem:

*Any real rotation group consists of a sequence of real rotation groups* $X_1, \ldots, X_{\nu_1}; X_{\nu_1+1}, \ldots, X_{\nu_2}; \ldots, X_{\nu_\kappa+1}, \ldots, X_n$ *that all commute with each other and are simple in the reals in each of them.*

If we would now like to learn more about these simple-in-the-reals subgroups then, as we will see, we will require a brief consideration of the homogeneous entire invariants of degree two of the associated adjoint group.

We know one such invariant, namely:

$$\omega_1^2 + \omega_2^2 + \ldots + \omega_n^2 .$$

The adjoint group is indeed a subgroup of the group of all rotations.

I now assert that *a real, simple in the reals, rotation group can have no other quadratic homogeneous invariants of its adjoint group.*

Namely, if such a thing does exist then one can bring it into the form ([1]):

$$A_1\,\omega_1^2 + \ldots + A_n\,\omega_n^2$$

by a real rotation. However, if that should be an invariant then one would need to have:

$$\sum_\kappa A_\kappa \omega_\kappa \delta_\lambda \omega_\tau = 0$$

for all $\omega$ and all $\lambda$, i.e.:

$$\sum_{\kappa,\tau} A_\kappa \omega_\kappa c_{\tau,\lambda,\kappa}\,\omega_\tau = 0,$$

so one will also need to have:

$$A_\kappa c_{\tau,\lambda,\kappa} + A_\tau c_{\kappa,\lambda,\tau} = 0$$

for all $\kappa, \tau, \lambda$.

From (IX), one can also write the condition as:

$$c_{\tau,\lambda,\kappa}\,(A_\kappa - A_\tau) = 0.$$

([1]) See **Baltzer**: *Determinanten*, page 187 *et seq.* (5th ed., Hirzel, Leipzig, 1881).

Hence, either all $A$ are equal to each other (and that will give only the old invariants) or one can sort the $A$ into classes whose individual members are equal to each other. However, from the constraint equation that was written down above, all $c_{\tau,\lambda,\kappa}$ must then vanish when only two of their indices belong to different classes. However, that means: Corresponding to the decomposition of $A$, the given rotation group also decomposes into invariant subgroups, and indeed into real subgroups, but that should not happen. Hence, there is in fact only one quadratic invariant:

$$\omega_1^2 + \ldots + \omega_n^2 .$$

However, from the studies of **Killing** ([1]) and **Cartan** ([2]), the adjoint group to *any* group will possess an invariant of degree two, and indeed one gets that invariant as follows: One forms the "characteristic determinant":

$$\Delta(\alpha) = \begin{vmatrix} \sum_\rho c_{\rho,1,1}\omega_\rho - \alpha & \sum_\rho c_{\rho,2,1}\omega_\rho & \cdots & \sum_\rho c_{\rho,n,1}\omega_\rho \\ \sum_\rho c_{\rho,1,2}\omega_\rho & \sum_\rho c_{\rho,2,2}\omega_\rho - \alpha & \cdots & \sum_\rho c_{\rho,n,2}\omega_\rho \\ \cdots & \cdots & \cdots & \cdots \\ \sum_\rho c_{\rho,1,n}\omega_\rho & \sum_\rho c_{\rho,2,n}\omega_\rho & \cdots & \sum_\rho c_{\rho,n,n}\omega_\rho - \alpha \end{vmatrix},$$

and develops this in powers of $\alpha$. The coefficient $\chi_2(\omega)$ of $a^{n-2}$ is then a quadratic form in the $\omega$ and at the same time an invariant of the adjoint group.

We assume that this form is regular; i.e., that its discriminant is non-zero. We can then arrange that this form will assume the form:

$$\omega_1^2 + \omega_2^2 + \ldots + \omega_\mu^2$$

by a suitable linear substitution (K) of the $\omega$ that is possibly complex.

However, the arbitrary group above must go to a generally *not*-necessarily real subgroup under this transformation. From the theorem on page 43, the coefficients $c_{\tau,\lambda,\kappa}$ must define a skew determinant for all $\lambda$; i.e., one must have:

$$c_{\tau,\lambda,\kappa} = - c_{\kappa,\lambda,\tau} ,$$

and that is indeed characteristic of the rotation group. With that, we have arrived at the new Theorem:

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([1]) **Killing**, "Die Zusammensetzung der stetigen, endlichen Transformationsgruppen," Math. Ann. **31**, **33**, **34**, **36** (especially **31**, § 2)

([2]) **E. Cartan**, "Über die einfachen Transformationsgruppen," Leipziger Berichte, 1893 and "Sur la structure des groupes de transformations finis and continus," Thèse, Paris, 1894.

*Any group whose **Killing** invariant $\chi_2$ ($\omega$) is a quadratic form with non-vanishing discriminant can be brought into the form of a rotation group – i.e., a group whose c possess cyclically-permuting indices – possibly by a complex linear transformation of the* $\omega$ (naturally, with constant coefficients).

Now **Cartan** has shown that $\chi_2$ ($\omega$) is actually regular for simple groups (Leipziger Berichte, pp. 401, Theorem I), and likewise for semi-simple groups (Thèse, Chapter IV). It then follows from this that:

*Any simple or semi-simple group can be brought into the form of a rotation group.*

Now let that result be mentioned only in passing. In order to further investigate the rotation groups that are simple in the reals, we need the theorem that was contained in Theorems I and IV in Chapter four of **Cartan**'s Thèse: Any group for which $\chi_2$ ($\omega$) is a regular form is either simple or composed of simple, invariant, and mutually-commuting subgroups; i.e., it is either simple or semi-simple.

Now, our real simple rotation group certainly possesses a quadratic form with non-vanishing discriminant as an invariant of its adjoint group, namely, $\omega_1^2 + \ldots + \omega_n^2$. Since, from the theorem that was proved on page 49, there can be no other quadratic invariant, the **Killing** invariant $\chi_2$ ($\omega$) must have the form const. $(\omega_1^2 + \ldots + \omega_n^2)$.

Now, the constant cannot be zero, since $\chi_2$ ($\omega$) will have the form:

$$\sum_{\rho,\sigma}\left(\sum_{\kappa}\omega_\kappa c_{\kappa,\rho,\sigma}\right)^2$$

in the case of a rotation group (see **Killing**, Annalen, **31**, pp. 261). However, that expression can be identically zero for real *c* only when all *c* are zero.

Therefore, the theorem of Cartan that was cited above will find an application to the real, simple-in-the-reals, rotation groups: They can be decomposed into a sequence of invariant, mutually-commuting, simple groups that will naturally be no longer real (when the real simple group was not already simple in the complexes, as well).

If we now accept the theorem above that was already assumed to begin with then that will imply the following Theorem:

*Any real rotation group can be decomposed into a series of groups that are invariant, mutually-commuting, simple, but in general, no longer real, and which can also once more take the form of rotation groups. One can then convert any simple group into a rotation group, but not generally in the real field.*

Since **Killing** and **Cartan** have set down all types of simple groups, that will resolve the problem of the examination of real rotation groups, up to a certain degree. However, there would still be some value to the question of investigating, e.g., whether there are also *real* simple groups for each type of simple groups, which is a question that probably cannot be always answered in the affirmative.

I have still not been able to carry out an investigation of complex rotation groups from the standpoint that was assumed for real groups. Indeed, in general, the arguments proceed analogously, but there is one special case that leads to difficulties: namely, the assumption that a rotation group appears whose only invariant subgroups are the ones that represent linear manifolds that contact the absolute cone. One can no longer give those subgroups in the form $X_1, \ldots, X_\nu$ , without changing the characteristic (IX) of rotation groups. We shall then suppress an investigation of complex rotation groups here.

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One special result might be mentioned that would follow from an application of the results that are contained here to the arguments at the beginning of the third chapter:

According to **Killing** and **Cartan**, there are not simple groups below $n = 8$, except for the one-parameter groups of translations and three-parameter groups of rotations.

*Therefore, the only types of "rigid bodies" with less than eight degrees of freedom are the following ones:*

1. The material point with one, two, or three translational motions.
2. The rigid body that rotates about a fixed point.
3. The rigid body that can rotate about a fixed point a material point that is independent of it and possesses one, two, or three translational motions.
4. Two mutually-independent rigid bodies that each rotates about a fixed point.
5. Two mutually-independent points with one, two, or three translational motions.

$$\left.\begin{array}{l}6.\\7.\\8.\end{array}\right\}\quad\begin{array}{l}\text{One of the last three cases, along with}\\\text{an independent material point}\\\text{with one translational motion.}\end{array}$$

That should be understood to mean: The system of equations of a "rigid body" of, e.g., seven degrees of freedom can be brought into a form such that some of the equations read simply:

$$\frac{dJ_k}{dt} = Q_k ;$$

the others can be summarized into classes of three each, such that each class assumes precisely the form of the ordinary Euler equations (see page 17). However, all of the $J$ can be linear combinations of the seven $\omega$ (with constant coefficients).

The following remark is true in general:

*A decomposition of the group indeed corresponds to a formal decomposition of the equations, but since the impulse components J can be linear combinations of* ***all*** *$\omega$, one cannot speak of a decomposition of the "rigid body," but only of a decomposition of its type.* (See the discussion of the concept of type in the beginning of the third chapter.)

Karlsruhe, in October 1903.

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