"Über nichtholonome Systeme," Math. Ann. **92** (1924), pp. 33-41.

On non-holonomic systems

by

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The stimulus for this work was given in two notes of Ivan Tzénoff: "Sur les equations du movement des systèmes matériels non holonomes," which appeared in 1920 in Liouville's Journal [1]) and then in 1924 in these Annals [2]). Tzénoff derived equations of motion that represent a combination of the equations of Lagrange and Appell. The calculation can be simplified considerably so that one recognizes the result immediately; this shall be done in § 1.

Moreover, Tzénoff compared his equations with the ones that Woronetz derived in § 5 his 1911 work [3]) "Über die Bewegung eines starren Körpers, der ohne Gleitung auf einer beliebigen Fläche rollt" and which agree in content with the equations of motion that I gave in my 1903 Habilitationsschrift (also in the Zeitschrift für Math. und Physik [4]), in 1904): "Die Lagrange-Eulerschen Gleichungen der Mechanik," as well as in the Annals paper [5]) "Über die virtuellen Verschiebungen in der Mechanik," except that my equations are considerably more far-reaching, in that they admit the use of arbitrary non-holonomic velocity parameters. However, my equations are clearly the same in form as the Lagrange ones, which one cannot say about those of Woronetz. Since my papers are obviously little known – Tzénoff did not mention them – in § 2, I would like to briefly show how the equations of Woronetz are included as special cases of my own ones.

Thirdly, at the conclusion of his paper, Tzénoff mentioned a remarkable theorem of Woronetz on Hamilton's principle (§ 7 of the cited paper). However, not only Woronetz, but also Tzénoff, proved its validity only to the extent that it agreed with their own equations of motion, while one can also immediately recognize the theorem – practically without calculation – from Hamilton's principle, when correctly understood, if one employs the results of my work; this shall be carried out in § 3 of this Note. I avail myself of the notation of my textbook on "Elementare Mechanik" (Teubner, Leipzig, 1912, 2nd ed., 1922).

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[1]) Journal de Math. pures et appliquées (8) **3**.
[2]) Math. Annalen **91**.
[3]) Math. Annalen **70**.
[4]) Zeitschr. f. Math. u. Physik **50**.
[5]) Math. Annalen **59**.

§ 1.

The equation of Tzénoff

Suppose we have a rheonomic system with a finite number of degrees of freedom; i.e., the position vector of any point of the system may be represented by:

(1) $$\overline{r} = \overline{r}\,(q_1, q_2, \ldots, q_n, q_{n+1}, \ldots, q_{n+k}; t),$$

such that the kinetic energy E becomes a quadratic function of $\dot{q}$ whose coefficients can that depend upon the q and the time t.

Furthermore, the following conditions, which are generally non-holonomic, may be given:

(2) $$\dot{q}_{n+m} = \sum_{s=1}^{n} a_{m,s}\dot{q}_s + a_m \qquad (m = 1, 2, \ldots, k),$$

which correspond to the following conditions for the virtual displacements:

(2′) $$\delta q_{n+m} = \sum_s a_{m,s}\delta q_s .$$

From (2), one has:

(3) $$\ddot{q}_{n+m} = \sum_s a_{m,s}\ddot{q}_s + \ldots$$

and thus:

(4) $$\frac{\partial \ddot{q}_{n+m}}{\partial \ddot{q}_s} = a_{m,s} .$$

From (1), it follows in a well-known way that the velocity of a system point is:

$$\overline{v} = \frac{d\overline{r}}{dt} = \sum_{i=1}^{n} \frac{\partial \overline{r}}{\partial q_i}\dot{q}_i + \sum_{s=1}^{k} \frac{\partial \overline{r}}{\partial q_{n+s}}\dot{q}_{n+s} + \frac{\partial \overline{r}}{\partial t}$$

and for the acceleration:

$$\overline{w} = \frac{d^2\overline{r}}{dt^2} = \sum_{i=1}^{n} \frac{\partial \overline{r}}{\partial q_i}\ddot{q}_i + \sum_{s=1}^{k} \frac{\partial \overline{r}}{\partial q_{n+s}}\ddot{q}_{n+s} + \cdots$$

and thus:

(5) $$\frac{\partial \overline{w}}{\partial \ddot{q}_i} = \frac{\partial \overline{r}}{\partial q_i} \qquad (i = 1, 2, \ldots, n + k).$$

Moreover, it follows from (1) that the virtual displacements are:

(6) $$\delta\overline{r} = \sum_{i=1}^{n} \frac{\partial \overline{r}}{\partial q_i}\delta q_i + \sum_{s=1}^{k} \frac{\partial \overline{r}}{\partial q_{n+s}}\delta q_{n+s},$$

and thus, from Lagrange's principle (the combination of the principle of virtual work with d'Alembert's principle):

$$S\,dm\,\bar{w}\cdot\delta\bar{r} = S\,d\bar{k}\cdot\delta\bar{r}\,,$$

upon substituting (6) one here:

(7) $$\sum_{i=1}^{n} Q_i\,\delta q_i + \sum_{s=1}^{k} Q_{n+s}\,\delta q_{n+s} = \sum_{i=1}^{n} K_i\,\delta q_i + \sum_{s=1}^{k} K_{n+s}\,\delta q_{n+s}\,,$$

where the Q's are the Lagrangian acceleration components $S\,dm\,\bar{w}\cdot\partial\bar{r}/\partial q$, and the K's are the force components $S\,d\bar{k}\cdot\partial\bar{r}/\partial q$. ($S$ means the summation over the system.)

If one now substitutes (2′) in (7) and observes that the δq_i are now completely arbitrary then one obtains the *equations of motion in the raw form (Rohform):*

(8) $$Q_i + \sum_{m=1}^{k} a_{m.i}\,Q_{n+m} = K_i + \sum_{m=1}^{k} a_{m.i}\,K_{n+m} \equiv K_i'.$$

Now, it is known that:

(9) $$Q_i = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i}.$$

However, from (5), we also have:

(10) $$S\,dm\,\bar{w}\,\frac{\partial\bar{r}}{\partial q} = S\,dm\,\bar{w}\,\frac{\partial\bar{w}}{\partial\ddot{q}_i} = \frac{\partial W}{\partial\ddot{q}_i}\,,$$

where W is the well-known Appell acceleration function $1/2\,S\,dm\,\bar{w}^2$.

If, in (8), we use the expressions (9) for the Q_i and the expressions (10) for the Q_{n+m} then we obtain:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i} + \sum_{m=1}^{k} a_{m.i}\,\frac{\partial W}{\partial \ddot{q}_{n+m}} = K_i'\,,$$

or, from (4):

(I) $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i} + \frac{\partial W'}{\partial \ddot{q}_i} = K_i' \qquad (i = 1, 2, \ldots, n),$$

where the prime on W' shall imply that the differentiation with respect to the q_i shall be assumed to involve only the terms that correspond to the $\ddot{q}_{n+m}$ (from (3)).

These are the equations of Tzénoff in the second form.

Its first form, however, can also be obtained immediately:

If one lets E'' denote the expression for E when one has replaced the $\dot{q}_{n+m}$ by (2) then one has:

(11) $$\frac{\partial E''}{\partial \dot{q}_i} = \frac{\partial E}{\partial \dot{q}_i} + \sum_m \frac{\partial E}{\partial \dot{q}_{n+m}}\frac{\partial \dot{q}_{n+m}}{\partial \dot{q}_i} = \frac{\partial E}{\partial \dot{q}_i} + \frac{\partial E'}{\partial \dot{q}_i}\,,$$

when the prime on the E' shall imply the same thing here as it did for W': that one differentiates only with respect to the variables that appear in the $\dot{q}_{n+m}$ (from (2)).

Likewise, one has:

$$\frac{\partial E''}{\partial q_i} = \frac{\partial E}{\partial q_i} + \sum_m \frac{\partial E}{\partial \dot{q}_{n+m}} \frac{\partial \dot{q}_{n+m}}{\partial q_i} = \frac{\partial E}{\partial q_i} + \frac{\partial E'}{\partial q_i}. \tag{12}$$

If one solves (11) and (12) for $\partial E / \partial \dot{q}_i$ ($\partial E / \partial q_i$, resp.) and substitutes in (I) then one obtains the *first form of Tzénoff:*

$$\frac{d}{dt}\left(\frac{\partial E''}{\partial \dot{q}_i}\right) - \frac{\partial E''}{\partial q_i} + \frac{\partial E'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial E'}{\partial \dot{q}_i}\right) + \frac{\partial W'}{\partial q_i} = K_i'. \tag{II}$$

§ 2.

The Lagrange-Euler equations

In my papers cited at the beginning, I proved the following (one also compares the presentation in Heuns: Lehrbuch der Mechanik **1**, Kinematik, Göschen, 1906):

Let $\bar{r}$ be a function of only $q_1, q_2, \ldots, q_n$, so E is a function of only the q and $\dot{q}$. One introduces new velocity quantities as independent linear functions of the $\dot{q}$:

$$\omega_i = \sum_{s=1}^{n} b_{i,s} \dot{q}_s \qquad \text{or, when solved:} \qquad \dot{q}_s = \sum_{i=1}^{n} c_{s,i} \omega_i . \tag{13}$$

Correspondingly, the virtual displacements are:

$$\delta\vartheta_i = \sum_{s=1}^{n} b_{i,s} \delta q_s \qquad \text{or, when solved:} \qquad \delta q_s = \sum_{i=1}^{n} c_{s,i} \delta\vartheta_i . \tag{14}$$

From these two definitions, under the additional – but not actually necessary – assumption that:

$$\delta\, dq_i = d\, \delta q_i , \tag{15}$$

which is equivalent to:

$$\delta d\bar{r} = d\delta\bar{r} ,$$

there follow the *commutation equations:*

$$d\, \delta\vartheta_m - \delta\, dJ_m = \sum_{i,s} \beta_{i,s,m} \delta\vartheta_i\, d\vartheta_s , \tag{16}$$

where $d\vartheta = \omega\, dt$ and we have set:

(17) $$\beta_{i,s,m} = \sum_{h,l}\left(\frac{\partial b_{m,l}}{\partial q_h} - \frac{\partial b_{m,h}}{\partial q_l}\right)c_{l,i}\,c_{h,s}\,.$$

It is useful, but not necessary, that the $\delta\vartheta_i$ be regarded as constants, and the second equations (14), as infinitesimal transformations, so the second equations (13), together with the commutation equations (16), give the extended point transformations $\delta\, dq_s$.

From the Lagrangian principle, one will then have:

(18) $$\sum_i Q_i\,\delta\vartheta_i = \sum_i K_i\,\delta\vartheta_i\,,$$

where the right-hand side is the virtual work of the applied force $S\,d\bar{k}\,\delta\bar{r}$, while:

(19) $$Q_i = \frac{dJ_i}{dt} - X_i\,E.$$

Thus, the impulse quantity J_i refers to the derivative of E with respect to ω_i , while $X_i\,E$ is the extended point transformation of E associated with $\delta\vartheta_i$. When written out completely, one has:

(19′) $$Q_i = \frac{d}{dt}\left(\frac{\partial E}{\partial\omega_i}\right) + \sum_{s,m}\beta_{i,s,m}\,\omega_s\,\frac{\partial E}{\partial\omega_m} - \left(\frac{\partial E}{\partial\vartheta_i}\right).$$

Here, $\partial E\,/\,\partial\omega_i$ is an abbreviation for $\sum_s (\partial E/\partial q_s)\,c_{s,i}$. If the J_i are regarded as arbitrary coordinates then the β are null, and we have the well-known Lagrangian expression (9) for the Q_i in the (19′).

Now, if the q_i are all independent, and thus the $\delta\vartheta$ are all arbitrary, *so we thus have a holonomic, scleronomic system, although it is found to be good (es aber für gut finden) to introduce the non-holonomic velocity quantities* ω, *then, from* (18), *the equations of motion read* [6]):

(III) $$Q_i = K_i \qquad (i = 1, 2, \ldots, n),$$

where the Q_i are to be determined from (19) ((19′), resp.), and one has:

$$K_i = \sum_s c_{s,i}\,S\,d\bar{k}\,\frac{\partial\bar{r}}{\partial q_s} = S\,d\bar{k}\left(\frac{\partial\bar{r}}{\partial\vartheta_i}\right).$$

However, *if the system is non-holonomic and rheonomic* then we can arrange that:

(20) $$q_n = \vartheta_n = t, \qquad \text{and then} \qquad \dot{q}_n = \omega_n = 1,$$

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[6]) Equations (III), with (19′), were already known to Volterra in 1989: "Sopra una classe di equazioni dinamichi," Atti di Torino **33**, as well as Woronetz, loc. cit., eq. (25) § 8.

and correspondingly:

(20′) $$\delta q_n = \delta \vartheta_n = 0,$$

and, moreover, that the non-holonomic condition equations assume the form:

(21) $$\omega_{k+h} = 0 \qquad (h = 1, 2, \ldots, n-k-1),$$

and correspondingly the virtual displacements vanish:

(21′) $$\delta\vartheta_{k+h} = 0, \qquad (h = 1, 2, \ldots, n-k-1),$$

while the first k $\delta\vartheta$ remain free.

As a result, only the first k of the equations of motion (III) persist in the non-holonomic conditions (21), and must then still substitute (20) and (21) in them, such that the summation symbol s in (19′) only takes on the values 1 to k and n. Furthermore, one sees: If an ϑ_i is a true coordinate then, from (16), the β with the last index i are null, and the corresponding terms in (19′) can be neglected.

These are the equations that I gave for non-holonomic systems and called the Lagrange-Euler equations.

In to obtain the equations of Woronetz, we only need to write down the result for the case described in § 1.

We set:

(22) $$\omega_i = \dot{q}_i \qquad \text{and} \qquad \delta\vartheta_i = \delta q_i \qquad \text{for} \qquad i = 1, 2, \ldots, n,$$

such that the β's with a final index that is less than or equal to n are null.

We further set:

(23) $$\omega_{n+m} = \dot{q}_{n+m} - \sum_{s=1}^{k} a_{m,s}\dot{q}_s - a_m \dot{q}_{n+k+1} = 0 \qquad \text{for} \qquad m = 1, \ldots, k,$$

and correspondingly:

(23′) $$\delta\vartheta_{n+m} = \delta q_{n+m} - \sum_{s=1}^{k} a_{m,s}\delta q_s - a_m\, \delta q_{n+k+1} = 0 \qquad \text{for} \qquad m = 1, \ldots, k.$$

Finally, we set:

(24) $$\omega_{n+k+1} = \dot{q}_{n+k+1} = 1$$

and:

(24′) $$\delta\vartheta_{n+k+1} = \delta q_{n+k+1} = 0,$$

such that the βs with the last index $n + k + 1$ are null.

(V) $$\frac{dJ_i}{dt} + \sum_{\substack{s=1,2,\cdots,m \\ m=1,2,\cdots,k}} \beta_{i,s,n+m}\omega_s J_{n+m} + \sum_{m=1,2,\cdots,k} \beta_{i,n+m+1,n+m} J_{n+m} - \left(\frac{\partial E}{\partial \vartheta_i}\right) = K_i \qquad (i = 1, 2, \ldots, n).$$

Therefore, for $i = 1, \ldots, n$, J_i is equal to $\partial E / \partial \omega_i$.

Because, however, equations (22), (23), and (24) may be used in all cases *after* differentiation – although here *before* differentiation, as well – one also has for $i = 1, \ldots, n$:

$$J_i = \frac{\partial E''}{\partial \dot{q}_i}.$$

On the other hand:

$$J_{n+m} = \frac{\partial E}{\partial \omega_{n+m}} = \frac{\partial E}{\partial \dot{q}_{n+m}},$$

and here one may first make use of the condition equations (23) and (24) *after* differentiation.

If one then computes β from (23) and (23′) *then one has in* (V) *precisely the equations of Woronetz.*

If one compares the various forms of the equations of motion with each other then the form of (19):

$$\frac{dJ}{dt} - X\,E = K,$$

along with the form of the Appell equation:

$$\frac{\partial W}{\partial \ddot{q}} = K,$$

is certainly is the clearest. The question of which calculation is the simplest will depend entirely upon the particular circumstances. Often, the raw form (8), with the use of (9), is the simplest. In each case, however, one can criticize the forms of Appell and Tzénoff for the fact that they require the function *W*, while one must get by with *E* and the condition equations.

§ 3.

The theorem of Woronetz

The central Lagrangian equation:

$$S\,dm\,\overline{w}\cdot\delta\overline{r} \equiv \frac{d}{dt} S\,dm\,\overline{v}\cdot\delta\overline{r} - \delta E = \delta A$$

immediately yields by integration:

$$S\,dm\,\overline{w}\cdot\delta\overline{r}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (\delta E + \delta A)\,dt\,,$$

and if one does not displace the ends of the time interval then one obtains Hamilton's principle:

(25) $$\int (\delta E + \delta A)\, dt = 0.$$

Since the validity of the central Lagrangian equation is entirely connected with the assumption that:

$$d\,\delta\overline{r} - \delta\, d\overline{r} = 0$$

(see my second paper for this), one also has:

(26) $$d\,\delta q = \delta\, dq$$

for all q with no other assumptions, and for that reason one has, if:

$$\omega_i = 0$$

is a non-holonomic condition then one must indeed take:

$$\delta\vartheta_i = 0,$$

but *not:*

$$\delta\omega_i = 0,$$

since otherwise from the commutation equations the present conditions would not be fulfilled in general. As a result, one may also not vary the non-holonomic conditions before applying Hamilton's principle, but only afterwards.

Now, however, if we again make use of the assumptions of § 2 – in particular, equations (20) to (21′) – then one has:

$$(\delta E)_{\omega_{k+h}=0,\omega_n=1} = \sum_{i=1}^{n}\left(\frac{\partial E}{\partial q_i}\right)_{\omega_{k+h}=0,\omega_n=1}\delta q_i + \sum_{i=1}^{n}\left(\frac{\partial E}{\partial \omega_i}\right)_{\omega_{k+h}=0,\omega_n=1}\delta\omega_i$$

$$+ \sum_{i=1}^{n-k-1}\left(\frac{\partial E}{\partial \omega_{k+l}}\right)_{\omega_{k+h}=0,\omega_n=1}\delta\omega_{k+l} + \left(\frac{\partial E}{\partial \omega_i}\right)_{\omega_{k+h}=0,\omega_n=1}\delta\omega_n .$$

In the first two terms, one may also set $\omega_{k+h} = 0$ and $\omega_n = 1$ *before* performing the differentiation, such that the first two terms together yield $\delta E''$; i.e., the variation of E while giving consideration to the non-holonomic condition equations. The last term drops out since one has $\delta\omega_n = \delta\, dq_n / dt = d\,\delta q_n / dt = 0$.

What then remains is:

$$\delta E = \delta E'' + \sum_{l=1,\cdots,n-k-1} J_{k-l}\,\delta\omega_{k+l} ,$$

and Hamilton's principle (25) assumes the form:

$$\text{(VI)} \qquad \int (\delta E'' + \sum_{l=1,2,\cdots,n-k-1} J_{k+l}\delta\omega_{k+l} + \delta A)\, dt = 0,$$

where (26) is to be observed for all q.

This is the theorem of Woronetz:

"In Hamilton's principle, one may make use of the non-holonomic conditions in the kinetic energy before performing the variation, if one corrects this mistake by adding the variation of the left-hand sides of the non-holonomic condition equation $\omega_{k+l} = 0$*, each multiplied by the associated impulse, under the integral."*

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