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On virtual displacements in mechanics

By

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Introduction.

When Lagrange elevated the principle of virtual work, which had already been expressed in a certain degree of generality by Johann Bernoulli, to the status of the fundamental tool for all mechanics, he was so deeply involved with isoperimetric considerations that – in any case, in his analytical arguments – he identified virtual displacements with the variations of the isoperimetric problem, and thus implicitly employed the relation that is true for any coordinate q and necessary for the calculus of variations, namely:

$$d\delta q - \delta dq = 0$$

for mechanics, as well.

Since then, it has become a dogma that $d\delta q - \delta dq = 0$ must be the characteristic feature of a true (holonomic) coordinate, while:

$$d\delta\vartheta - \delta d\vartheta \neq 0$$

characterizes the non-holonomic (generalized) coordinate ϑ.

Now, the coupling of mechanics with the calculus of variations has become so fruitful (Lagrange, Hamilton, Jacobi) that one cannot deny that the aforementioned way of looking at virtual displacements is one-sided and its mechanical interpretation does not correspond completely, so it actually stands in the way of the connection to broader relations that lie outside the field of view of the calculus of variations. Moreover, the noteworthy opinion that is so extensive in the literature that the essence of Lagrangian mechanics is stuck in the so-called variational principles seems to me to have its roots in this dogma.

In the following, it shall be shown that any assumption about $d\delta q - \delta dq$ *is completely unnecessary for mechanics and that the singling out of non-holonomic coordinates* ϑ *lies only in the relationship between* $d\delta\vartheta - \delta d\vartheta$ *and in the event that the* q *are true (holonomic) coordinates.*

This theorem is, in itself, self-explanatory, since the principle of virtual work does not include the δdq at all. However, if one regards the equations of motion of mechanics as not only the result of a more or less arbitrary and coincidental conversion of d'Alembert's principle, while the emphasis lies in the meaning that the concepts of energy and impulse thus take on, then the stated law admits further development – i.e., an explicit verification following a path along which one stands to encounter the aforementioned concepts, as well as the expression $d\delta q - \delta dq$. Now, there is an equation that seems to me to unite essentially almost all of these paths, and that is the *general central equation*. For that reason, in what follows we would like to base the exposition of virtual displacements itself on this fundamental equation, in order to arrive at the general equations of motion from it, independently of any assumption on $d\delta\vartheta - \delta d\vartheta$.

However, from the standpoint that we arrive at, we then catch a glimpse of the relationship between mechanics and the calculus of variations, from which we will especially see how the clarity that was first attained in recent times concerning the so-called Hamilton principle is completely present when one starts for *non-holonomic equations of motion* with the central equation.

Finally, we consider the relationship between mechanics and the theory of Lie groups. Indeed, I have already presented the affinity between these two domains in my Habilitationsschrift: "Die Lagrange-Eulerschen Gleichungen der Mechanik" [*]), and I will also add nothing new to its individual results here, while our more general way of looking at virtual displacements will give us an, in part, new and, as I also believe, clearer representation of the nature of things. In particular, it has now been possible for me to be led to the proof of the theorem of the simultaneous existence of Euler equations and impulse equations in a purely abstract way. A purely group-theoretic theorem that I had only briefly mentioned in my Habilitationsschrift thus now serves as a welcome bridge.

§ 1.

Definition of virtual displacements for holonomic and non-holonomic coordinates.

We first consider a free point with a well-defined motion in space, such that its position vector:

$$\bar{x} = \bar{\varphi}(t)$$

can be known for all values of time t in a certain interval, and therefore also its velocity:

$$\bar{v} = \frac{d\bar{x}}{dt} = \dot{\bar{\varphi}}(t)$$

at every location along its path.

We now associate the points at each moment with a virtual displacement $\delta\bar{x}$ by setting:

$$\delta\bar{x} = \bar{w}(t),$$

[*]) Also in the Zeitschrift f. Math. u. Phys. Bd 50, Heft 1, 1904. In the sequel, this will always be referred to as "L. E. Gl."

where $\overline{w}$ is a continuous and continuously differentiable, but otherwise arbitrary, function of time. If we write:

$$\overline{x}' = \overline{x} + \delta\overline{x} = \overline{\varphi}(t) + \overline{w}(t)$$

then every point of the path is assigned to a neighboring point.

Now, for the considerations of the calculus of variations, it is necessary and sufficient that the totality of these neighboring points can be regarded as a new, varied path, for which $d\overline{x}' / dt$ is then also defined naturally as:

$$\frac{d\overline{x}'}{dt} = \dot{\overline{\varphi}} + \dot{\overline{w}},$$

such that the variation $\delta d\overline{x} = d\overline{x}' - d\overline{x}$ is given by:

$$\delta d\overline{x} = \dot{\overline{w}}(t)dt = d\delta\overline{x}\,.$$

With this way of looking at things, there thus exists the following relation:

$$\delta d\overline{x} - d\delta\overline{x} = 0$$

However, this assumption is in no way necessary; if we regard $d\overline{x}$ and $\delta\overline{x}$ as two independent displacements then certainly $d\delta\overline{x}$ is known from $\delta\overline{x}$, but not $\delta d\overline{x}$, since $d\overline{x}$ is indeed first defined for the path itself.

From now on, we thus make no assumptions about:

$$d\delta\overline{x} - \delta d\overline{x}\,;$$

i.e., we let the definition of $d\overline{x}$ *be completely free outside of the path that we have in mind.*

Obviously, we can apply the same argument to any variable q that we follow as a function of time t.

Moreover, we fix our attention on a mechanical system of n degrees of freedom, so for any of its points and for all time we can set:

$$\overline{x} = \overline{\varphi}(a, b, c; q_1, \cdots, q_n)\,,$$

where a, b, c mean three parameters that are independent of the point that serve to characterize the individual points of the system; let q_1, …, q_n be the time-varying Lagrange coordinates. Time t does not enter into $\overline{\varphi}$ explicitly, so the system is *scleronomic*, according to a terminology of Boltzmann. This assumption temporarily serves to simplify things. (For the non-scleronomic case, see the conclusion of § 3.)

We will temporarily omit possible non-holonomic condition equations; we will speak about them briefly in § 3.

Since the n parameters or coordinates q are, moreover, essential, there thus exists no identity relation of the form:

$$\sum \overline{l}_\lambda \psi_\lambda = 0$$

in the a, b, c, where we have set:

$$\overline{l}_\lambda = \frac{\partial \overline{x}}{\partial q_\lambda},$$

and where the ψ_λ are independent of the a, b, c.

With these assumptions, we define virtual displacements $\delta \overline{x}$ by:

$$\delta \overline{x} = \sum_{\lambda=1}^{n} \overline{l}_\lambda \, \delta q_\lambda ,$$

and in a similar way, virtual variations for any function χ of the q by:

$$\delta \overline{x} = \sum_{\lambda=1}^{n} \frac{\partial \chi}{\partial q_\lambda} \delta q_\lambda ,$$

where the statements that were made at the start of this paragraph can be made for the δq. In particular, corresponding to the *character of the δ-operation as a differentiation*, it follows from:

$$d\overline{x} = \sum_\lambda \overline{l}_\lambda dq_\lambda$$

that

$$(1) \qquad \delta d\overline{x} = \sum_\lambda \frac{\partial \overline{l}_\lambda}{\partial q_\rho} dq_\lambda \delta q_\rho + \sum_\lambda \overline{l}_\lambda \, \delta dq_\lambda \quad {}^{*}),$$

in which we would, however, like to have the definition of δdq_λ close at hand.

With this arbitrariness, it is, however, important to prove the following theorem:

Any assumption about $d\delta q - \delta dq$ for any independent system of parameters for the q implies a decision regarding the corresponding expressions for any other independent system of parameters.

Obviously, it suffices to show the mutual dependency between $d\delta q - \delta dq$ and $d\delta\overline{x} - \delta d\overline{x}$:

Along with equation (1), there exists the following one:

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*) The fact that we focus on the differential character of the δ-operation here – in other words, that we also do not leave the Lagrangian space of motion with the (infinitely small) variation of $d\overline{x}$ – is also ultimately arbitrary and unnecessary. Thus, we would like to accept this assumption here.

(2) $$d\delta\overline{x} = \sum_{\lambda,\rho}\frac{\partial\overline{l}_\rho}{\partial q_\lambda}dq_\lambda\delta q_\rho + \sum_\lambda \overline{l}_\lambda d\delta q_\lambda\,,$$

and here because one has:

$$\frac{\partial\overline{l}_\rho}{\partial q_\lambda} = \frac{\partial\overline{l}_\lambda}{\partial q_\rho} = \frac{\partial^2\overline{\varphi}}{\partial q_\rho\partial q_\lambda}\,,$$

by subtracting (1) and (2), this yields:

(3) $$d\delta\overline{x} - \delta d\overline{x} = \sum_\lambda \overline{l}_\lambda(d\delta q_\lambda - \delta dq_\lambda)\,.$$

The dependency of $d\delta\overline{x} - \delta d\overline{x}$ on $d\delta q_\lambda - \delta dq_\lambda$ then comes to light; however, the converse is also true. If one then thinks of equation (3) as being true for all *a*, *b*, *c* and observes that the $d\delta q - \delta dq$ are independent of the *a*, *b*, *c* then all of the $d\delta q - \delta dq$ must be expressible in terms of the components of a suitably-chosen $d\delta\overline{x} - \delta d\overline{x}$. A determinant of degree *n* that comes into consideration in the determination of the *n* quantities $d\delta q - \delta dq$ must then be non-zero, because no relation of the form:

$$\sum \overline{l}_\lambda \psi_\lambda = 0$$

shall exist for all *a*, *b*, *c*, and with that our assertion is proved. *In particular, all* $d\delta q - \delta dq$ *must then vanish when* $d\delta\overline{x} - \delta d\overline{x}$ *is continuously zero.* Otherwise, one cannot naturally specify $d\delta\overline{x} - \delta d\overline{x}$ arbitrarily for all *a*, *b*, *c*, but only for a total of *n* independent displacements; the other components of the $d\delta\overline{x} - \delta d\overline{x}$ are then chosen such that equations (3) remain possible for all *a*, *b*, *c*.

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We would now like to also extend the definition of virtual displacements to *non-holonomic* velocity parameters.

We think of the velocity state of the system as being depicted by new velocity parameters ω, in place of the *n* quantities dq_λ / dt, which are introduced by means of the linear equations:

(4) $$\dot{q}_\lambda = \sum_{\rho=1}^{n}\xi_{\rho,\lambda}\omega_\rho \ldots$$

Let the determinant $|\,\xi_{\rho,\lambda}\,|$ be non-zero in the domain that comes under consideration for the motion, such that we can also solve equations (4) for the ω:

(4′) $$\omega_\rho = \sum_{\lambda=1}^{n}\pi_{\rho,\lambda}\dot{q}_\lambda\,.$$

Now, despite the fact that the ω_ρ are not generally total derivatives with respect to time, we would still like to write symbolically:

$$\omega_\rho = \frac{d\vartheta_\rho}{dt}$$

and refer to ϑ_ρ as a *non-holonomic (generalized) coordinate.*

We also define n virtual displacements that correspond to equations (4′) by way of:

$$\delta\vartheta_\rho = \sum_{\lambda=1}^{n} \pi_{\rho,\lambda}\,\delta q_\lambda\,. \tag{4″}$$

When expressed in terms of the $d\vartheta$ ($\delta\vartheta$, resp.), now let:

$$d\overline{x} = \sum_\lambda \overline{l}_\lambda\, d\vartheta_\lambda\,,$$

$$\delta\overline{x} = \sum_\lambda \overline{l}_\lambda\, \delta\vartheta_\lambda\,,$$

where we have, however, set:

$$\overline{l}_\lambda = \sum \frac{\partial\overline{x}}{\partial q_\rho}\,\xi_{\lambda,\rho}\,, \tag{5}$$

moreover, or, by analogy, it can be written symbolically as a true differential quotient:

$$\overline{l}_\lambda = \left(\frac{\partial\overline{x}}{\partial\vartheta_\lambda}\right).$$

The expressions $d\delta\vartheta - \delta d\vartheta$ again relate to the $d\delta q - \delta dq$ as a complete dependency. Namely, from equations (4″) and (4′), one has:

$$d\delta\vartheta_\rho - \delta d\vartheta_\rho = \sum_{\lambda=1}^{n} \pi_{\rho,\lambda}(d\delta q_\lambda - \delta dq_\lambda) + \sum_{\sigma,\lambda}\left(\frac{\partial\pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial\pi_{\rho,\sigma}}{\partial q_\lambda}\right)\delta q_\lambda\, dq_\sigma\,,$$

or

$$d\delta\vartheta_\rho - \delta d\vartheta_\rho = \sum_\lambda \pi_{\rho,\lambda}(d\delta q_\lambda - \delta dq_\lambda) + \sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\,\delta\vartheta_\mu\, d\vartheta_\nu\,, \tag{I}$$

in which we have set, to abbreviate:

$$\beta_{\mu,\nu,\rho} = \sum_{\sigma,\lambda}\left(\frac{\partial\pi_{\rho,\lambda}}{\partial q_\sigma} - \frac{\partial\pi_{\rho,\sigma}}{\partial q_\lambda}\right)\xi_{\mu,\lambda}\,\xi_{\nu,\sigma}\,.$$

One easily recognizes that the $\beta_{\mu,\nu,\rho}$ vanish when and only when the ϑ are true (i.e., holonomic) coordinates.

We would like to call equations (I) the "*transitivity*" – or "*transition*" – *equations of the first form.* We obtain the second form when we solve (I) for $d\delta q_\lambda - \delta dq_\lambda$:

$$d\delta q_\lambda - \delta dq_\lambda = \sum_\rho \xi_{\rho,\lambda}(d\delta q_\lambda - \delta dq_\lambda) + \sum_\rho \xi_{\rho,\lambda} \sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\delta\vartheta_\mu d\vartheta_\nu\,,$$

and substitute this into the equation:

$$d\delta\overline{x} - \delta d\overline{x} = \sum_\lambda \frac{\partial\overline{x}}{\partial q_\lambda}(d\delta q_\lambda - \delta dq_\lambda)\,,$$

in which we observe equations (5). We then obtain:

$$\text{(I}'\text{)} \qquad d\delta\overline{x} - \delta d\overline{x} = \sum_\rho \overline{l}_\rho(d\delta\vartheta_\rho - \delta d\vartheta_\rho) + \sum_\rho \overline{l}_\rho \sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\delta\vartheta_\mu d\vartheta_\nu\,,$$

which is *the transition equation of the second form.*

In our more general way of looking at virtual displacements, non-holonomic coordinates ϑ are therefore not characterized by $d\delta\vartheta - \delta d\vartheta$ being non-zero, but are, moreover, characterized by the form of the relation (I′) [(I), resp.], *in which $d\delta\vartheta - \delta d\vartheta$ stands for $d\delta\overline{x} - \delta d\overline{x}$ ($d\delta q - \delta dq$, resp.), if the q are true coordinates.*

§ 2.

The general central equation and the Lagrange-Euler equations.

In order to now show that mechanics is independent of any assumption about $d\delta\overline{x} - \delta d\overline{x}$ that goes beyond the previous statements, we next derive *the general central equation*, which defines the nucleus of all kinetics.

We start from the identity [*]):

$$\frac{d}{dt}(\dot{\overline{x}}\cdot\delta\overline{x}) \equiv \ddot{\overline{x}}\cdot\delta\overline{x} + \delta\left(\tfrac{1}{2}\dot{\overline{x}}^2\right) + \dot{\overline{x}}\left(\frac{d}{dt}\delta\overline{x} - \delta\frac{d\overline{x}}{dt}\right),$$

in which, following Newton, an overhead dot denotes the derivative with respect to time, and $\overline{x}\cdot d\overline{x}$ denotes the inner – or work – product of the vectors $\overline{x}$ and $d\overline{x}$, etc. This equations immediately brings the concepts of virtual work of velocity, virtual work of acceleration, and kinetic energy to the foreground.

[*]) Under the assumption that $d\delta\overline{x} - \delta d\overline{x} = 0$ of Lagrange, t. I, sec. Part, Sect. IV, no. 3.

If we now let $m(a, b, c)$ be the mass of a system point, and we understand the operation S to mean summation – or possibly integration – over all a, b, c then it follows from the identity above, when we then introduce the *kinetic energy of the system:*

$$T = \tfrac{1}{2} Sm\dot{\bar{x}}^2 ,$$

that

$$\frac{d}{dt}\left\{Sm\dot{\bar{x}} \cdot \delta\bar{x}\right\} = Sm\ddot{\bar{x}} \cdot \delta\bar{x} + \delta T + Sm\dot{\bar{x}} \cdot \left(\frac{d\,\delta\bar{x}}{dt} - \delta\frac{d\bar{x}}{dt}\right).$$

Now since, from d'Alembert's principle, for all δq_λ, and therefore also for all $\delta\vartheta_\lambda$, one has:

$$Sm\ddot{\bar{x}} \cdot \delta\bar{x} = S\bar{K} \cdot \delta\bar{x} ,$$

where $\bar{K}$ means the force that is applied to the system point, when we define *n system forces* Q_λ by the equation that exists for all $\delta\vartheta_\lambda$:

$$\sum_{\lambda=1}^{n} Q_\lambda \delta\vartheta_\lambda = S\bar{K} \cdot \delta\bar{x} ,$$

and *n impulse components* ϑ_λ by the analogous equation:

$$\sum_{\lambda=1}^{n} J_\lambda \delta\vartheta_\lambda = Sm\dot{\bar{x}} \cdot \delta\bar{x} ,$$

the identity above goes to the following equation, which is valid for all $\delta\vartheta_\lambda$:

$$\frac{d}{dt}\left(\sum_\lambda J_\lambda \delta\vartheta_\lambda\right) = \sum_\lambda Q_\lambda \delta\vartheta_\lambda + \delta T + Sm\dot{\bar{x}} \cdot \left(\frac{d\,\delta\bar{x}}{dt} - \delta\frac{d\bar{x}}{dt}\right).$$

In order to convert the last term, we now appeal to the transition equation in the second form (I′), and since one easily finds that:

$$J_\lambda = Sm\dot{\bar{x}} \cdot \bar{l}_\lambda = \frac{\partial T}{\partial \omega_\lambda} ,$$

we obtain:

$$\text{(II)} \qquad \frac{d}{dt}\left(\sum_\lambda J_\lambda \delta\vartheta_\lambda\right) = \sum_\lambda Q_\lambda \delta\vartheta_\lambda + \delta T + \sum_\lambda J_\lambda \left(\frac{d\,\delta\bar{x}}{dt} - \delta\frac{d\bar{x}}{dt}\right) - \sum_\lambda J_\lambda \sum_{\mu,\nu} \beta_{\mu,\nu,\lambda} \delta\vartheta_\mu \frac{d\vartheta_\nu}{dt} .$$

This is the *general central equation;* it is distinguished from Lagrange's central equation [*]) by the appearance of the last two sums, which collectively vanish when we make the assumption that $d\delta\overline{x} - \delta d\overline{x} = 0$.

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We now derive the general equations of motion from the central equation. If one performs the differentiation with respect to time in (II) and considers that one has:

$$\delta T = \sum_{\lambda} J_{\lambda} \frac{\delta d\vartheta_{\lambda}}{dt} + \sum_{\lambda} \left(\frac{\partial T}{\partial \vartheta_{\lambda}} \right) \delta\vartheta_{\lambda} ,$$

where once more once sets:

$$\left(\frac{\partial T}{\partial \vartheta_{\lambda}} \right) = \sum_{\rho} \frac{\partial T}{\partial q_{\rho}} \xi_{\lambda,\rho} ,$$

by analogy with a true differential quotient, then all of the terms in $d\delta\vartheta$ and $\delta d\vartheta$ drop out, and since the central equation is true identically in all $\delta\vartheta$, this yields the equations:

$$\text{(IIIa)} \qquad \frac{dJ_{\lambda}}{dt} + \sum_{\mu,\nu} \beta_{\lambda,\mu,\nu} \omega_{\mu} J_{\nu} - \left(\frac{\partial T}{\partial \vartheta_{\lambda}} \right) = Q_{\lambda} ,$$

which, together with:

$$\text{(IIIb)} \qquad J_{\lambda} = \frac{\partial T}{\partial \omega_{\lambda}} ,$$

define the *"Lagrange-Euler," – i.e., the general kinetic – equations of mechanics.* [**])

They were derived with no assumptions on $d\delta\overline{x} - \delta d\overline{x}$ *, and with that, the assertion is proved that any assumption on* $d\delta\overline{x} - \delta d\overline{x}$ *that arises from the general differential nature of the d-operation is superfluous for mechanics in itself.*

If the ϑ are true coordinates then the β vanish, and we have the ordinary Lagrangian equations in (III).

[*]) Essentially in Lagrange, t. I, sec. Part. Sect. IV, no. 7.

[**]) "L. E. Gl.," § 5, (IV). Moreover, these equations have already been derived by V. Volterra ("Sopra una classe di equazioni dinamiche," Atti di Torino XXXIII, 1898) and P. Woronetz ("On the equations of motion for non-holonomic systems," [Russian], Math. Sbornik, Moscow, t. XXII, 1901).

§ 3.

Virtual displacements and variations.

(Hamilton's principle and the principle of varied action.)

One will naturally employ the complete freedom to be able to specify $d\delta\vartheta - \delta d\vartheta$ for any holonomic or non-holonomic coordinate system in order to, e.g., arrive at a simplification of the derivation of the kinetic equations in individual cases, as well as when one proceeds from d'Alembert's principle to one's objective along another path that does not involve the central equation.

I would not like to go into the methodological advantages of our general conception here, since this viewpoint will be developed in concrete cases in the soon-to-appear *Lehrbuch der Kinematik* of K. Heun. By contrast, allow me to point out what a freer standpoint one now has for connecting mechanical relationships with other domains of mathematics; e.g., the calculus of variations and the theory of Lie groups.

In order to stick to known things, I would first like to briefly sketch out the relationship to the calculus of variations. This is then the place to make the assumption that:

$$d\delta\overline{x} - \delta d\overline{x} = 0,$$

since this equation, as we already remarked, is characteristic of the calculus of variations.

The central equation then reads:

$$\frac{d}{dt}\sum_\lambda J_\lambda \delta\vartheta_\lambda = \sum_\lambda Q_\lambda \delta\vartheta_\lambda + \delta T,$$

which we would like to distinguish from the general one as the Lagrangian equation.

If we set the virtual work done by the applied force in this equal to:

$$\sum_\lambda Q_\lambda \delta\vartheta_\lambda = \delta A$$

then it follows by integrating between the limit points 0 and 1 that:

$$\text{(IV)} \qquad \sum_\lambda J_\lambda \delta\vartheta_\lambda \Big|_0^1 = \int_0^1 (\delta A + \delta T)\, dt .$$

If we take $\delta\vartheta = 0$ at the limits then the so-called Hamilton principle follows, which takes the following form for the case in which a force function V exists:

$$\delta \int (V + T)\, dt = 0.$$

However, if we think of $\int_0^1 (V+T)\,dt$ as a function of the limit coordinates $q^{(0)}$ and $q^{(1)}$, when one uses the equations of motion in integral form, and sets this equal to:

$$S(q_1^{(0)},\ldots,q_n^{(0)};q_1^{(1)},\ldots,q_n^{(1)})$$

then equation (IV) yields *Hamilton's principle of varied action*, when it is converted into the following form for non-holonomic coordinates:

$$J_\lambda^{(1)} = \left(\frac{\partial S}{\partial \vartheta_\lambda^{(1)}}\right); \qquad J_\lambda^{(0)} = -\left(\frac{\partial S}{\partial \vartheta_\lambda^{(0)}}\right).$$

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Now, if our system of n degrees of freedom is then subject to $\nu < n$ generally non-holonomic condition equations in which time does not enter explicitly, then one bring these condition equations into the form:

$$\omega_{n-\nu+1} = 0, \qquad \ldots, \qquad \omega_n = 0,$$

in which one chooses the ω suitably.

The variations are therefore now restricted by the equations:

$$\delta\vartheta_{n-\nu+1} = 0, \ldots, \delta\vartheta_n = 0,$$

so, from the principle of virtual work, the virtual displacements must satisfy the equations of condition; nothing changes from the previous argument for the remaining ones, except that of the n Lagrange-Euler equations, now only the first $n - \nu$ remain, from which, the ν equations arise [*]):

$$\omega_{n-\nu+1} = 0, \qquad \ldots, \qquad \omega_n = 0.$$

In this representation, it now quite clear from here on how one is to understand Hamilton's principle for non-holonomic condition equations. As before, it must be:

$$\delta\int (V+T)\,dt = 0$$

in which, however, according to the principle of virtual displacements, the variations are restricted by the conditions:

$$\delta\vartheta_{n-\nu+1} = 0, \ldots, \delta\vartheta_n = 0,$$

from which the equations:

$$d\delta\vartheta_{n-\nu+1} = 0, \ldots, d\delta\vartheta_n = 0$$

follow.

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[*]) "L. E. Gl." § 7.

Thus, for non-holonomic condition equations, Hamilton's principle no longer has anything whatsoever to do with the problem of the calculus of variations of finding a minimum to:

$$\int (V + T)\, dt = 0$$

under the conditions:

$$\omega_{n-\nu+1} = 0, \quad \ldots, \quad \omega_n = 0.$$

Then, for this problem the variations are subject to only the restrictions:

$$\delta d\vartheta_{n-\nu+1} = 0, \ldots, \delta d\vartheta_n = 0,$$

which are generally completely different from:

$$d\delta\vartheta_{n-\nu+1} = 0, \ldots, d\delta\vartheta_n = 0.$$

I might perhaps emphasize that one has, above all, Lagrange himself to thank for the possibility of arriving at the foregoing facts so quickly and clearly, since he gave us the central equations in their essence. (*loc. cit.*)

Let us make yet another remark: If the system is non-scleronomic then the time does not enter into $\bar{x}$ and $\xi_{\rho,\lambda}$ explicitly, and if the equations that link the ϑ and the q are inhomogeneous, moreover, then one introduces an $(n + 1)^{\text{th}}$ coordinate:

$$q_{n+1} = \vartheta_{n+1} = t,$$

in place of t, which also makes the equations between q and ϑ homogeneous, and one now works with the $n + 1$ coordinates in precisely the same way as one did with the n coordinates up to now. To the condition equations that are possibly present, one adds:

$$\delta\vartheta_{n+1} = 0.$$

In the remaining n, ($n - \nu$, resp.) Lagrange-Euler equations, one then again sets $q_{n+1} = t$ everywhere.

The validity of this remark follows from the fact that time is not be varied in the principle of virtual work.

§ 4.

Virtual displacements and infinitesimal transformations.

(Euler equations and impulse equations.)

In order to exhibit the relationship to the theory of Lie groups, a multiple way of looking at $\delta\vartheta$ and $\delta d\vartheta$ is recommended.

We would now like to always regard the $\delta\vartheta$ as constant (except for in the last investigation that we make).

This is closely related to regarding the equations:

$$\delta q_\lambda = \sum_\rho \xi_{\rho,\lambda} \delta\vartheta_\rho$$

as infinitesimal transformations. The $\delta\vartheta_\rho$ are infinitesimal constants and the $\xi_{\rho,\lambda}$ are the quantities that Lie likewise pointed out.

This concept corresponds, as a symbol, to the ρ^{th} infinitesimal transformation:

$$X_\rho f = \sum_\rho \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda}.$$

We can now recognize the role that the β in the transition equations play in this way of looking at things when we also assume that:

$$\delta d\vartheta = 0,$$

hence, that the ω remain unchanged.

The transition equations of the first form then, in fact, read:

$$d\delta q_\lambda - \delta dq_\lambda = -\sum_{\mu,\nu,\rho} \xi_{\rho,\lambda}\beta_{\mu,\nu,\rho}\delta\vartheta_\mu d\vartheta_\nu ,$$

or, when we set all $\delta\vartheta$ and $d\vartheta$ equal to zero up to $\delta\vartheta_\mu = 1$ and $d\vartheta_\nu = 1$, in the easily-understood notation:

$$d\delta q_\lambda - \delta dq_\lambda = -\sum_\rho \xi_{\rho,\lambda}\beta_{\mu,\nu,\rho} = \sum_\rho \xi_{\rho,\lambda}\beta_{\nu,\mu,\rho} ,$$

– since one has $\beta_{\mu,\nu,\rho} = -\beta_{\nu,\mu,\rho}$ – i.e., when we, following Jacobi:

$$X_\nu X_\mu - X_\mu X_\nu = (X_\nu, X_\mu),$$

set:

$$(X_\nu, X_\mu) = \sum_\rho \beta_{\nu,\mu,\rho} X_\rho .$$

The $\beta_{\nu,\mu,\rho}$ are precisely the coefficients with whose aid the Jacobi symbols (X_ν, X_μ) are expressed in terms of the X_ρ linearly. In particular, if the n infinitesimal transformations generate an n-parameter group with the Lie composition constants $c_{\nu,\mu,\rho}$ then:

$$\beta_{\nu,\mu,\rho} = c_{\nu,\mu,\rho} ;$$

the coefficients in the transition equations are then constant. [*])

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[*]) "L. E. Gl." § 3, (II). There, the proof is achieved directly; also the emphasis was more on the applicability of the transition equations to the equations that define the infinitesimal transformations of the adjoint group.

If we make the assumption that $\delta d\vartheta = 0$, for the moment, then δT is now nothing but the infinitesimal transformation of T with ω held constant, or one has:

$$\left(\frac{\partial T}{\partial \vartheta_\lambda}\right) = X_\lambda T$$

for constant ω.

Therefore, when T admits all infinitesimal transformations for constant ω, as one easily recognizes from the general central equation (II), the equations of motion read simply:

$$\frac{dJ_\lambda}{dt} + \sum_{\nu,\rho} \beta_{\lambda,\nu,\rho}\omega_\nu J_\rho = Q_\rho .$$

I have referred to these equations as the *Euler equations in the broad sense.* [**])

The J_λ are thus linear functions of the ω_λ with constant coefficients, because T will be just such a quadratic form of the ω; if the $\beta = c$ are still constant then the left-hand (i.e., kinematic) sides of the equations still include only the ω as variables. (*Euler equations in the restricted sense*) Any free system possesses Euler equations in the broad sense; these equations are thus just as general as the Lagrange equations.

The assumption:

$$d\delta q - \delta dq = 0$$

also has an interpretation in terms of group theory. Namely, by means of it, one obtains from:

$$\delta q_\lambda = \sum_\rho \xi_{\rho,\lambda}\delta\vartheta_\rho$$

the extended point transformation:

$$\delta dq_\lambda = d\delta q_\lambda = \sum_{\rho,\sigma,\mu} \frac{\partial \xi_{\rho,\sigma}}{\partial q_\sigma}\xi_{\mu,\sigma}\delta\vartheta_\rho d\vartheta_\mu ,$$

and the transition equations of the first form now yield the infinitesimal changes in the ω, insofar as these are determined by the coupling of the dq / dt and the infinitesimal changes that we just carried out. However, from equations (I):

(6) $$\delta\omega_\rho = -\sum_{\mu,\nu} \beta_{\mu,\nu,\rho}\omega_\nu\delta\vartheta_\mu$$

becomes the variation of ω_ρ under the extended point transformation.

However, from the central equation in Lagrangian form:

[**]) "L. E. Gl." § 10, (VII).

$$\frac{d}{dt}\sum_{\lambda} J_{\lambda}\delta\vartheta_{\lambda} = \sum_{\lambda} Q_{\lambda}\delta\vartheta_{\lambda} + \delta T,$$

it follows immediately, since the $\delta\vartheta_{\lambda}$ are constant, and since δT is now to be regarded in the sense of extended point transformations, that:

(IIIa′) *)
$$\frac{dJ_{\lambda}}{dt} - X_{\lambda}T = Q_{\lambda} ,$$

where $X_{\lambda}T$ means the λ^{th} infinitesimal, extended point transformation of T.

Naturally, this equation is entirely identical with the corresponding Lagrange-Euler equation; we have only obtained a new conception of it.

In particular, if T admits the λ^{th} infinitesimal, extended, point transformation such that one has:

$$X_{\lambda}T = 0,$$

where now the ω_p that correspond to equations (6) are to be varied, *then the λ^{th} equation of motion reads simply:*

$$\frac{dJ_{\lambda}}{dt} = Q_{\lambda} .$$

I have referred to these equations as *impulse equations*. *)

* *
*

In order to now show the relationship between the impulse equations and the Eulerian equations in a manner that is similar to the situation for a rigid body that rotates around a point, we ask ourselves:

When can one convert the impulse equations:

$$\frac{dJ_{\lambda}}{dt} = Q_{\lambda} ; \qquad J_{\lambda} = \frac{\partial T}{\partial \omega_{\lambda}}$$

(assuming that they represent the equations of motion of the system), by the introduction of new, independent ω':

(7)
$$\omega'_{\kappa} = \sum_{\lambda=1}^{n} E_{\kappa,\lambda}\omega_{\lambda} ,$$

into the Euler equations:

$$\frac{dJ'_{\lambda}}{dt} + \sum_{\mu,\nu} \beta'_{\lambda,\mu,\nu}\omega'_{\mu}J'_{\nu} = Q'_{\lambda} \ ?$$

*) “L. E. Gl.” § 9.
*) “L. E. Gl.” § 9.

(From now on, we provide everything that relates to the ω' with primes.)

If that is to be possible then under the substitution (7) $\delta T = 0$, in which, from (6) one sets:

$$\delta\omega_\lambda = \sum_{\mu,\nu} \beta_{\mu,\nu,\lambda}\,\omega_\mu\,\delta\vartheta_\nu\ ,$$

must go to $\delta' T = 0$, where one now has to take:

$$\delta\omega'_\lambda = 0, \qquad \text{i.e.,} \qquad \delta d\vartheta'_\lambda = 0.$$

However, in order for us to be able to set $\delta T = \delta' T$ in general, moreover, we must regard the variations δ and δ' in the same way, and therefore, perhaps as extended point transformations, in both cases. Contrary to the previous assumption, we would thus now like to assume for the consideration of the Euler equation that:

$$d\delta q - \delta dq = 0,$$

in which clearly $d\delta\vartheta'_\lambda = 0$ is impossible. Moreover, one must now assume *):

$$\frac{d\delta\vartheta'_\lambda}{dt} = -\sum_{\mu,\nu} \beta'_{\mu,\nu,\lambda}\,\omega'_\mu\,\delta\vartheta'_\nu\ .$$

On the contrary, for the ω, one still has:

$$d\delta\vartheta = 0.$$

One shall have $\delta' T = 0$ for ω' held constant whenever one sets $\delta\omega' = 0$; then however, since one has $\delta' T = \delta T = 0$, also for the same kind of variation of ω', the relation:

(7)
$$\omega'_\kappa = \sum_\lambda E_{\kappa\lambda}\,\omega_\lambda$$

corresponds to this. As a result of this relation (7), it must then follow that:

$$\sum_{\lambda=1}^{n} \frac{\partial T}{\partial \omega'_\lambda}\,\delta\omega'_\lambda = \sum_{\lambda=1}^{n} J'_\lambda\,\delta\omega'_\lambda = 0,$$

or, since this naturally shall be true for all J', one must also have as a result of (7) itself that:

$$\delta\omega' = 0.$$

*) These equations are to be regarded as differential equations for the $\delta\vartheta$. At a location along the path, one can choose the $\delta\vartheta$ arbitrarily, but then they are determined for the entire path. The fact that the $\delta\vartheta$ are not constant here does not contradict the conception of the δ-operation as an infinitesimal transformation; taking the $\delta\vartheta$ to be constant is convenient for many purposes.

Thus, we now have to answer the following question:

When can one take the assumption that $d\delta\vartheta = 0$ to the assumption that $\delta d\vartheta' = 0$ by a substitution:

$$\omega'_\kappa = \sum_\lambda E_{\kappa\lambda}\omega_\lambda \text{ ?} \tag{7}$$

(Thus, the determinant $| E_{\kappa,\lambda} |$ shall naturally be non-zero.)

The answer reads:

The infinitesimal transformations:

$$X_\rho f = \sum_\lambda \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda}$$

must generate an n-parameter group.

For the proof of this assertion, it is useful to know that:

$$\delta q_\lambda = \sum_\rho \xi_{\rho,\lambda} \delta\vartheta_\rho$$

and

$$dq_\lambda = \sum_\rho \xi'_{\rho,\lambda} d\vartheta_\rho$$

can be regarded as two independent classes of n infinitesimal transformations; from the assumption that:

$$d\delta\vartheta = 0 \qquad \text{and} \qquad \delta d\vartheta' = 0,$$

the infinitesimal parameters $\delta\vartheta$ and $d\vartheta'$ are independent of each other.

The assumption:

$$d\delta q - \delta dq = 0$$

then means nothing but the fact that these two classes of transformations commute with each other, so one has:

$$(X_\rho, X'_\rho) = 0$$

for all ρ, σ. I can then indeed think of the operation d as being performed with the help of the $\delta\vartheta'$, while the operation δ is performed with the help of the $\delta\vartheta$.

Conversely, we have two classes of infinitesimal transformations for which the determinants $| \xi_{\rho,\lambda} |$ and $\left|\xi'_{\rho,\lambda}\right|$ are non-zero, so one can always find unique quantities $E_{k,\lambda}$ such that the relation (7) is true for all $\dot{q}_\rho$ identically. If all of the (X_ρ, X'_ρ) then vanish and we assume that $d\delta q - \delta dq = 0$ then under this substitution $d\delta\vartheta = 0$ goes to $\delta d\vartheta' = 0$. The proof of this converse is so elementary that I do not need to go into it.

We thus have only the following theorem to prove, which carries a purely group-theoretic character:

If we have two classes of n infinitesimal transformations in n variables:

$$X_\rho = \sum_\lambda \xi_{\rho,\lambda} \frac{\partial f}{\partial q_\lambda} \qquad \text{and} \qquad X'_\sigma = \sum_\lambda \xi'_{\sigma,\lambda} \frac{\partial f}{\partial q_\lambda},$$

and neither of the determinants $|\ \xi_{\rho\lambda}\ |$ *and* $\left|\xi'_{\rho\lambda}\right|$ *vanish identically then the transformations of the one class commute with all of the transformations of the other class only when the transformations of each class generate an n-parameter group. Naturally, the groups are then simply transitive and mutually reciprocal.* [*])

Proof: Assume that at least one of the two classes does not define a group, so perhaps (X_ρ, X_σ) is not expressible in terms of the X_λ with constant coefficients. It then follows from the Jacobi relation:

$$((X_\rho X_\sigma)X'_\tau) + ((X_\sigma X'_\tau)X_\rho) + ((X'_\tau X_\rho)X_\sigma) = 0$$

that one also has:

$$(X_\rho X_\sigma)X'_\tau) = 0;$$

i.e., that the new infinitesimal transformation (X_ρ, X_σ) likewise commutes with all X'_τ.

If the X_ρ then generate no n-parameter group then there are more than n independent transformations that commute with all X'_τ.

I will now show that there can be at most n such transformations, with which, the proof of the theorem is complete.

In the n-dimensional space of the q, we consider a point P_1 inside the domain in which the determinant $\left|\xi'_{\kappa,\lambda}\right|$ is non-zero, so one can, in any case, determine n constants $e'_1, \ldots, e'_n$ such that a one-parameter group that is defined for a transformation T' by the differential equations:

$$\frac{dq_\lambda}{dt} = \sum_\rho \xi'_{\rho,\lambda} e'_\rho$$

from P_1 to an arbitrary point P_2 in a sufficiently small neighborhood of P_1. This is possible because the determinant of $\xi'_{\rho,\lambda}$ is non-zero.

When I now take any transformation T that commutes with T', and through which P_1 goes to Q_1 and P_2 goes to Q_2, then $TT' = T'T$ says nothing but the fact that Q_2 arises from Q_1 by the same transformation T' as P_2 does from P_1 .

If we then consider all T that take P_1 to Q_1 then this also takes any point P_2 to a completely well-defined point Q_2 that is independent of the special choice of T, namely, the Q_2 that comes from Q_1 by means of the T' that converts P_1 into P_2 . Therefore, all of these transformations T are identical, so there is only one transformation T that takes P_1 to Q_1 , and since Q_1 is determined by n coordinates there are also n transformations that commute with all n transformations T'.

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[*]) "L. E. Gl." § 11, final theorem.

With that, it is then proved that the X_ρ generate a *n*-parameter group, and naturally also the X'_τ.

Now since, conversely, to any group there also exists a reciprocal group that commutes with it, we have proved the theorem:

Simultaneous Euler equations and impulse equations can exist when and only when the coefficients β of the transition equations are constant; i.e., when the n infinitesimal transformations that correspond to virtual displacements generate an n-parameter group.

The rigid body that rotates around a fixed point will serve as a simple example; the group in question is the three-parameter group of rotations.

With that, I have developed the relationship between mechanics and the theory of Lie groups to the extent that seems necessary for the purposes of the foregoing paper. For the further details, I might perhaps refer to my Habilitationsschrift.

Karlsruhe, in February 1904.