From: G. Hamel, *Elementare Mechanik: ein Lehrbuch*, Teubner, Leipzig, 1912.

Section Two

Statics

Chapter V

Statics of rigid bodies (theory)

111. Problem statement and definitions. On the basis of the argument that was made in no. 106, in this chapter we will once more seek to find equilibrium conditions that are necessary, as well as sufficient, but restricted this time to rigid bodies as the objects in question; i.e., bodies that always remain congruent to themselves.

Previously (no. 47), we stated equilibrium conditions that were true in full generality, namely, that for any volume element the sum of the forces that are distributed in the space it occupies and across its bounding surface must be null. With this condition, however, we can still accomplish only so much, since the internal stresses of the rigid body are necessarily reaction forces, and are thus unknown, but there are, of course, no deformations in the interior of a rigid body (see no. 58).

We now seek conditions in which only the external forces that act on a rigid body figure: In the following, when we speak casually of forces we will always mean external forces.

However, we will extend the problem of this chapter in the following way:

We refer to two force systems on a material system as "equivalent" when they produce the same state of velocity and acceleration, and we ask:

When are force systems on a rigid body "equivalent" to each other?

If a force system is so numerous that it is as if no force at all were acting, then we also say that the forces cancel out on the rigid body, or preserve its equilibrium, while we shall speak of the equilibrium of the body itself when it constantly remains at rest.

§ 23. The admissible operations and their invariants

112. Case of a finite number of forces. Concept of moment. We next make the assumption that a finite number of finite forces: $\bar{k}_1, \ldots, \bar{k}_n$ act upon our rigid body at the given points $A_1, \ldots, A_n$, which are described by the vectors $\bar{a}_1, \ldots, \bar{a}_n$. Later (no. 113), we will see the meaning of this assumption, which does not strictly correspond to nature itself.

We shall call the line through the point A with the direction of $\bar{k}$ the *line of action* of the force.

We then speak of the following, most plausible, axiom:

When two equal and opposite forces act along the same line of action, one may add or omit them at will.

(One can derive this axiom, for the more specific questions of the equilibrium of the body from the law of adequate grounds [see no. 63], when one adds the axiom that the question of the equivalence alone will decide between the force itself and its point of application. Thus, in this case there is either a distinguished direction along which the body moves or a distinguished axis around which it rotates.)

Thus, we come to the previously-discussed parallelogram law (no. 44): One may combine forces that act at a point into a resultant.

From our new axiom, we immediately infer the *translation theorem*, whose traces emerged in the Middle Ages with Jordanus de Nemore, became more distinct with Benedetti, the forerunner of Galilei, and came into focus with Varignon, who summarized elementary statics in his "projet d'une nouvelle mécanique" in 1687, and in his much-neglected "Nouvelle mécanique" (1725), he made it a foundation of statics:

One may translate a force on a rigid body arbitrarily along its line of action; i.e., two equal forces along the same line of action are equivalent to each other.

Proof: The force $\bar{k}$ acts at the point A. At the point B, which lies on the line of action of $\bar{k}$, one adds a force $\bar{k}_1 = \bar{k}$ and a force $\bar{k}_2 = -\bar{k}$, which, from our recent axiom, we may do. However, from the same axiom, $\bar{k}$ and $\bar{k}_2$ cancel out, which leaves $\bar{k}_1$. $\bar{k}$ is then equivalent to $\bar{k}_1$. Q.E.D.

As admissible operations, we would now like to point out the following ones:

1. Translation of a force along its line of action.
2. The insertion of two equal and opposite forces at the same point of application.
3. Combination of two forces that act at the same point using the parallelogram law.

We call any expression that includes the forces an *invariant* when it remains unchanged under these operations.

We would now like to show that there are two invariants:

a) *The geometric sum of all forces:*

$$\bar{K} = \sum \bar{k}_\nu \, .$$

b) *The geometric sum of the moments of all the forces, relative to any point:*

$$\bar{M} = \sum \overline{a_\nu k_\nu} \, .$$

In this, we therefore understand the moment of a force that acts at A relative to the point O to mean the outer product $\overline{ak}$ of the vectors $\bar{a} = \overline{OA}$ and $\bar{k}$; i.e., a vector that is perpendicular to the plane through O and the line of action of $\bar{k}$ and is directed such that when one looks outward from it the force $\bar{k}$ points to the left (when one brings both $\overline{ak}$ and $\bar{k}$ to O), and whose magnitude is:

$$\overline{ak} = a \cdot k \cdot \sin(\bar{a}, \bar{k}) = h \cdot k,$$

where h is the length of the perpendicular that one can erect from O to the line of action $\bar{k}$. h is also called the "lever arm" of the force $\bar{k}$ relative to O.

That $\bar{K}$ and $\bar{M}$ are invariant is easy to confirm.

For $\bar{K} = \sum \bar{k}_\nu$, the assertion is entirely self-explanatory, since it is clear that each of the terms indeed remain unchanged by operation 1, and a sum remains unchanged when one insertions two equal and opposite terms or when one combines two terms.

The fact that $\bar{M} = \sum \overline{a_\nu k_\nu}$ can be shown as follows: Each term remains unchanged under operation 1, since the plane, sense, and magnitude of a moment remains unchanged. $\bar{M}$ does not change under 2 either, since two equal and opposite $\bar{k}$ with the same $\bar{a}$ have equal and opposite moments. The fact that $\bar{M}$ also is invariant under 3 emerges from the assertion that:

$$\overline{ak_1} + \overline{ak_2} = \overline{a(k_1 + k_2)},$$

in which perhaps $\bar{k}_1$ and $\bar{k}_2$ act at the same point $A(\bar{a})$ and will thus be combined into $\overline{k_1 + k_2}$. This statement is, however, correct, since it is nothing but the distributive law of outer multiplication (see Appendix I, 5).

One can thank Varignon for the latter fundamental theorem, if only in the case where all of the vectors lie in the same plane.

113. Reduction of the general case to the previous one. Axiom group VII. We now consider the case in which an infinite number of spatially or superficially distributed forces $\overline{dk}$ act on the body.

The concept of translating an isolated force is meaningless. We thus express the axiom in the preceding paragraph in connection with the parallelogram law in a somewhat altered form, in such a way that for the case of finite forces the two become identical:

Axiom VII.1: If the lines of action of several forces $\overline{dk}$ go through a point then the forces are all equivalent to a single force whose line of action goes through the same point and whose sum $\mathrm{S}\,\overline{dk}$ is the force in question.

In addition, we make use of the inverse parallelogram law:

One can decompose any force into three components that act on the same point and have given directions, assuming that these directions do not all lie in a plane (see Appendix I, 1).

We would like to show that one can reduce any force system on a rigid body to three forces.

Proof: We choose three points O_1, O_2, O_3 in a plane outside of the rigid body that do not lie in the same line. Any force that acts at a point A can then be decomposed into the directions AO_1, AO_2, AO_3, since these directions certainly do not lie in a plane. The entire force system now consists of three groups: The first one has a line of action through O_1, the second one, through O_2, and the third one, through O_3. From axiom VII.1, we can combine each group into a single force. Q.E.D.

The fact that the sums (integrals):

$$\bar{K} = \mathrm{S}\,\overline{dk}\,, \qquad \bar{M} = \mathrm{S}\,\overline{a\,dk}$$

are invariant under the operations allowed by axiom VII.1 and the parallelogram law should be clear.

From the following examinations of this chapter (no. 116, 124), it will then emerge that these are also the only invariants.

To axiom VII.1, we add the following plausible axioms, which are put to the test in daily life:

Axiom VII.2: A rigid body is certainly not in equilibrium when a single non-zero force acts upon it.

Axiom VII.3: A rigid body is certainly not in equilibrium when a so-called "force-couple" acts upon it – i.e., two equal and opposite non-zero forces that do not have the same line of action – or when *one* isolated force and *one* force-couple act upon it.

The fundamental concept of force-couple goes back to Poinsot ("Statique," 1803).

Axiom VII.4: A rigid body is certainly in equilibrium when no forces at all act upon it.

(If we were to assume the general center of mass theorem [see no. 52 or 107] then axiom VII.2 would naturally be an immediate consequence of it; likewise, VII.3 would be a consequence of the moment theorem [see no. 108 and 114]. Above all, only axioms VII are necessary for a self-contained presentation of statics.

114. Definition of the moment for a different reference point. The moment of a force-couple. If we define the sum of the moments for a new reference point O' then we obtain no essentially new invariants.

Thus, let $\overline{O'A} = \bar{y}$, $\overline{O'O} = \bar{s}$ so that:

$$\bar{a} = \bar{y} - \bar{s}\,,$$

and it follows that the new moment is:

$$\bar{M}' = \mathrm{S}\,\overline{y\,dk} = \mathrm{S}\,\overline{a\,dk} + s\mathrm{S}\,\overline{dk}$$

or:

$$\bar{M}' = \bar{M} + \overline{sK}\,.$$

One thus obtains the moment relative to the new reference point when one adds a moment to the old moment that one obtains when one lets the resultant $\bar{K}$ *act upon the old reference point O and forms its moment relative to the new one.*

From this, it immediately follows that the moment of a force-couple – we understand this to mean the sum of the moments of the two forces of the pair – is independent of the choice of reference point. One then has $\bar{K} = 0$. If we then choose the reference point to be the point of application of one of the two forces then we recognize that the moment of a force-couple is perpendicular to its plane in such a way that when one looks along the moment vector the force exhibits a sense of rotation as an arrow and that the magnitude of the moment is equal to the product of the magnitudes of the forces and lever arm of the couple; i.e., the normal distance between the two lines of action.

§ 24. Combination of forces in the plane

115. Combination of two forces. Let a finite number of finite forces be given – from no. 113, we can indeed restrict ourselves to this case – which lie in a plane and act upon a rigid body. We likewise choose the reference point *O* of the moment to be in that plane. Thus, all moments are perpendicular to this plane and differ from each other only by their magnitudes and senses. For that reason, it suffices here to regard the moments as scalars that can be positive or negative. If we imagine an *x*, *y* coordinate system as being defined in the plane then we would like to choose the direction of the *z*-axis to be the positive direction for the moment; we let the *x*, *y*, *z* system be a right-handed system – i.e., the *x*-axis points to the right, while the *y*-axis points to its left, so the *z*-axis points upwards. A moment is positive then when the force vector, as seen from above – i.e., along the *z*-axis – points to the left relative to the reference point *O*.

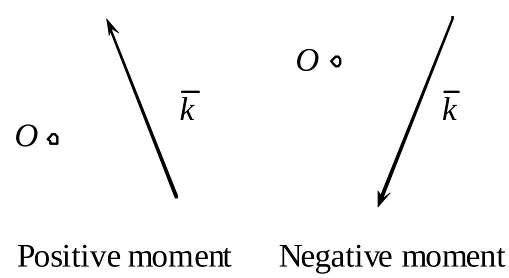

Fig. 67

Now, if the lines of actions through the given forces intersect at a point *S* then we can, from axiom VII.1, replace the two forces with a single resultant.

If two forces are parallel and point in the same direction then one can likewise replace them with a single resultant.

Proof: Let $\bar{p}$ and $\bar{q}$ be parallel and point in the same direction. At two points *A* and *B* of the two lines of action, one inserts two opposite and equal forces $\bar{s}$ and $-\bar{s}$ with the same line of action, which we may do. Now, one combines $\bar{p}$ and $\bar{s}$ into a resultant $\bar{u}$, and combines $\bar{q}$ and $-\bar{s}$ into a resultant $\bar{v}$.

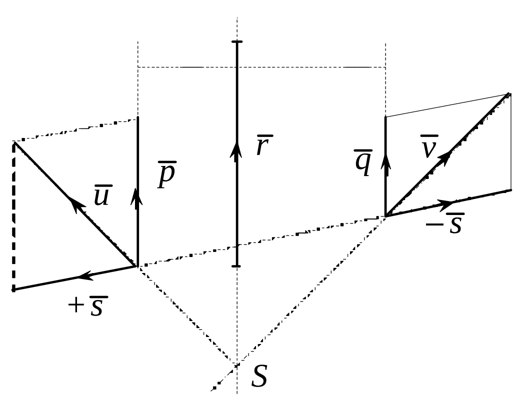

Fig. 68

The lines of action of $\bar{u}$ and $\bar{v}$ intersect, however, at a point *S* and can therefore be combined into a resultant $\bar{r}$.

From the theorems that $\sum \bar{k}$ and $\sum \overline{ak}$ are invariant, one may now determine the direction, magnitude, and position of $\bar{r}$ with nothing further.

One must have:

$$\bar{r} = \bar{p} + \bar{q}\,;$$

i.e., $\bar{r}$ has the same direction as the given forces, and its magnitude is equal to the ordinary sum:

$$r = p + q,$$

Furthermore, one must have:

$$\overline{xr} = \overline{ap} + \overline{bq}$$

relative to an arbitrary point *O*, where $\bar{x}$, $\bar{a}$, $\bar{b}$ are the vectors to the points of application of $\bar{r}$, $\bar{p}$, $\bar{q}$.

If we choose *O* to be on the chosen line of action of $\bar{r}$ then one has:

$$O = \overline{ap} + \overline{bq}\,.$$

From this, it next follows that: $\bar{r}$ lies between $\bar{p}$ and $\bar{q}$, since only then can $\bar{p}$ and $\bar{q}$ have equal and opposite moments. Furthermore: If *a* and *b* are the distances from the line of action of $\bar{r}$ from $\bar{p}$ and $\bar{q}$, then it follows that:

$$pa = qb;$$

i.e., $\bar{r}$ divides the distance between $\bar{p}$ and $\bar{q}$ into inverse ratios of forces.

With that, this case is resolved.

One can likewise reduce two oppositely-directed, unequal forces to a single force.

The proof is analogous to the previous one. Moreover, the forces $\bar{u}$, $\bar{v}$ now intersect at a point *S*, while $\bar{u}$, $\bar{v}$ are rotated around $\bar{p}$, $\bar{q}$ with the same sense. However, the angle between $\bar{p}$ and $\bar{u}$ will be less the angle between $\bar{q}$ and $\bar{v}$, because we have assumed that $p > q$.

There will then be a resultant $\bar{r}$. From the fact that:

$$\bar{r} = \bar{p} + \bar{q}\,,$$

it follows, as before, that:

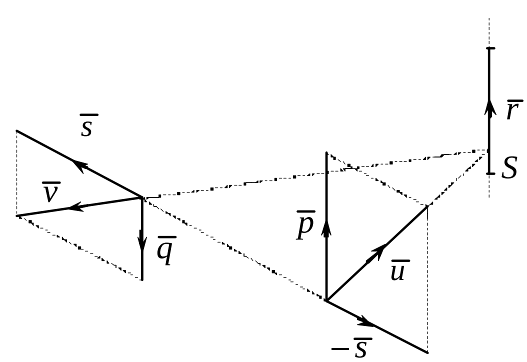

Fig. 69

The resultant has the same direction as the greater of the two forces ($\bar{p}$) and its magnitude is equal to the difference of the given forces:

$$r = p - q.$$

If we again choose the reference point for the moment to be on $\bar{r}$ then this gives:

$$O = \overline{ap} + \overline{bq} .$$

From this, one recognizes that $\bar{r}$ must lie outside of $\bar{p}$ and $\bar{q}$, and that one must have:

$$ap = bq$$

moreover, if a and b mean the distances between r and p (q, resp.).

From this, it ultimately follows that since $p > q$, one must have $b > a$; i.e., the resultant lies on the side of the greater of the two forces.

What still remains is the case of equal and opposite forces – i.e., a force-couple.

One cannot reduce a force-couple to a single force.

If this were possible, with $\bar{r}$ the resultant, then one would have $\bar{r} = \bar{p} + (-\bar{p}) = 0$.

On the other hand, one would have:

$$\overline{xr} = \overline{ap} + \overline{b(-p)} = \overline{(a-b)p} = \bar{M} ,$$

where $\bar{M}$ is the non-vanishing moment of the force-couple. Thus, $\bar{r}$ must be non-zero.

One can interpret this result by saying: A force-couple is equivalent to a force of magnitude zero ($\bar{r} = 0$) that acts on the infinitely distant line ($\bar{x} = \infty$). However, one gains nothing in practice by this way of looking at things.

116. Combination of arbitrarily many forces. The equilibrium conditions. When arbitrarily many forces $\bar{k}'$, $\bar{k}''$, …, $\bar{k}^{(n)}$ are given with the points of application $\bar{a}'$, $\bar{a}''$, …, $\bar{a}^{(n)}$, we can always combine them in pairs at most one force or force-couple remains; thus, among three forces, at most two of them can be equal and oppositely directed.

Therefore, one can reduce a force system into the plane to a single force or a single force-couple,

which are also free to be zero.

However, one can determine this result directly from our two invariants:

$$\bar{K} = \sum \bar{k} \qquad \text{and} \qquad \bar{M} = \sum \overline{ak} .$$

a) If:

$$\bar{K} \neq 0,$$

then the result is a single force. If the result is a force-couple, or if the force system were equivalent to zero, then one would need to have $\bar{K} = 0$.

$\bar{K}$ then already gives the magnitude and direction of the resultant.

In order to find its line of action – the so-called "central axis" of the force system – we then make use of the fact that the total moment is zero. Let $\bar{x}$ be the vector from the reference point O to a point on the central axis, so one must have:

$$\overline{xK} = \overline{M}\,.$$

This is the equation of the central axis (see Appendix II, 2). Let h be the distance from the central axis to O, so the equation that was presented in the statements at the beginning of no. 115 can also be written as:

$$\pm\, h \cdot K = M.$$

The $\pm$ sign is determined from that of M. This sign and the magnitude of h determine the position of the central axis completely, since its direction is indeed given by $\overline{K}$ (cf., Fig. 67).

b) If $\overline{K} = 0$, but $\overline{M} \neq 0$ then only one force-couple can be the result of the reduction, since the other cases are excluded. M is the moment of this force-couple.

c) If $\overline{K} = 0$ and $\overline{M} = 0$ then the force system is equivalent to zero, and it brings equilibrium to the rigid body. It must then be that the force system can be reduced to two equal and opposite forces in the same line of action, which then cancel each other.

$\overline{K} = \sum \overline{k} = 0$ *and* $\overline{M} = \sum \overline{ak} = 0$ *are then, in any case, sufficient equilibrium conditions for rigid bodies; that they are also necessary follows from axioms* VII.2 *and* VII.3.

If a force acts on the rigid body then from axiom VII.4 the body is in equilibrium; i.e., it remains at rest if was once at rest.

It suffices for us to state the condition $\overline{M} = 0$ for a point: If, following no. 114, for a new reference point one has $\overline{M}' = \overline{M} + \overline{sK}$ then $\overline{M}'$ is itself zero when $\overline{M}$ and $\overline{K}$ are zero.

In the special case of three forces acting on the rigid body, when at least two of them ($\overline{k}'$ and $\overline{k}''$) intersect at a point S, the equilibrium condition can be expressed in a particularly intuitive way as:

$$\sum \overline{k} = \overline{k}' + \overline{k}'' + \overline{k}''' = 0,$$

which says that the forces define a closed triangle when they are placed end-to-end. The condition:

$$\sum \overline{ak} = 0$$

says that when we make S the reference point:

$$\overline{a'''k'''} = 0;$$

i.e., $\overline{k}'''$ likewise goes through S:

The lines of action of three forces must go through a point.

The combination of force-couples is included in our foregoing considerations. Two force-couples are then nothing but four forces of a special type.

One immediately recognizes that:

Two force-couples in the same plane again yield a force-couple whose moment is equal to the sum of the moments.

Two force-couples with equal and opposite moment cancel each other.

From this, it ultimately follows that:

Two force-couples of equal moments are equivalent to each other.

In general:

Two force-couples of equal moments and equal resultants are equivalent to each other.

Proof: Let S_1 be one and let S_2 be the other system. One constructs the opposite system S_2' to S_2 in which one has rotated all of the forces to the opposite direction. Thus, S_2 and S_2' cancel each other out, and one also adds both of them to S_1 . Moreover, S_1 and S_2' also cancel, since the sum of their forces and the sum of their moments is, by assumption, zero. What then remains is S_2 . Q.E.D.

Thus, it likewise proved, at least for the plane, that:

$\bar{K}$ and $\bar{M}$ are the only invariants.

Force systems with equal $\bar{K}$ and $\bar{M}$ are then equivalent in all cases.

117. Analytical formulation of the result. Along the right-angled axes x, y, $\bar{k}$ has the components k_x, k_y , $\bar{K}$ has the components K_x, K_y, and $\bar{a}$ has the components a, b. Let the reference point O likewise be the origin of the coordinate system.

Then from:

$$\bar{K} = \sum \bar{k}$$

one derives the two equations:

$$K_x = \sum k_x \,,$$
$$K_y = \sum k_y \,,$$

and since $\overline{ak}$ has the components:

$$0,\ 0,\ ak_y - bk_x \,,$$

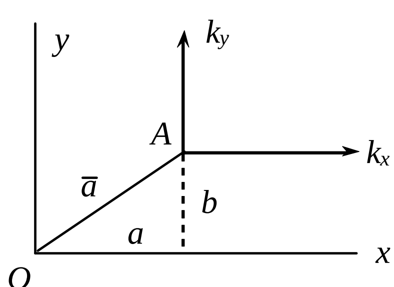

Fig. 70.

(see Appendix II, 1), the equation:

$$\bar{M} = \sum \overline{ak}$$

becomes the one equation:

$$M = \sum (ak_y - bk_x) \,.$$

The validity of this formula also follows immediately from the illustration (Fig. 70).

As the equilibrium conditions, one thus has:

$$\sum k_x = 0,$$
$$\sum k_y = 0,$$

and

$$\sum (ak_y - bk_x) = 0.$$

This equation once again leads one to the first two equations that were discussed in Section I.

If $\bar{K} \neq 0$ then since one must have:

$$\overline{xK} = \bar{M}$$

the equation of the central axis reads:

$$yK_x - xK_y = M;$$

x, y are then the running coordinates of the central axis.

Problem 58. Reduce the forces whose magnitudes are 10, 12, 7, 13 kg, whose lines of action intersect an axis at angles of 30°, 90°, 45°, 120° at four points, which are at distances of 3, 2, and 4 m from each other.

§ 26. Combination of forces in space

124. Reduction to one force and one force-couple. Let there be given a series of forces $\bar{k}_1, \ldots, \bar{k}_n$ with the points of application $A_1, \ldots, A_n$. We choose a reference point O, and the vectors OA may be called $\bar{a}$: $\bar{a}_1, \ldots, \bar{a}_n$.

To each force $\bar{k}_\nu$, we now add a force at O:

$$\bar{k}'_\nu = \bar{k}_\nu$$

and a force:

$$\bar{k}''_\nu = -\bar{k}_\nu,$$

which we may do.

We add up the forces $\bar{k}'_\nu$ at O to form a resultant that will be applied to O and whose magnitude and direction is determined by:

$$\bar{K} = \sum_{\nu=1}^{n} \bar{k}_\nu .$$

$\bar{k}_\nu$ and $\bar{k}''_\nu = -\bar{k}_\nu$ define a force-couple of moment $\overline{a_\nu k_\nu}$. We thus confront the problem of combining force-couples whose planes intersect – so they all have the point O in common.

We only need to show that we can combine two such couples into one: The fact that it then has a moment that is equal to the geometric sum of the moments of the given couples follows from the invariance of this sum under the allowable operations.

In order for us to now show that one can combine two force-couples with intersecting planes, we proceed as follows: We choose a line segment AB in the line of intersection of both planes as it pleases us. We can then replace each force-couple with one for which A, B are the points of application. Indeed, we need only for there to be two such forces in each plane that are applied at A and B and preserve the sense and magnitudes of the moments, which is always possible. Now, however, we have four forces in all, of which two of them are applied at A and two of them are applied at B. The fact that we can reduce them to just one is clear. Moreover, one sees immediately that a force-couple again emerges since this is already true from the fact that the sum of the forces still remains zero.

We can thus, in fact, combine force-couples and obtain the result:

A spatial system of forces on a rigid body may be reduced to a single force and force-couple. The single force is equal to the geometric sum of the given forces:

$$\bar{K} = \sum \bar{k}_\nu$$

and is applied at an arbitrarily chosen point O; the force-couple has a moment that is equal to the geometric sum of the moments of the given forces:

$$\bar{M} = \sum \overline{a_\nu k_\nu}\,.$$

For infinitely many, infinitely small, continuously distributed forces $d\bar{k}$, one naturally has:

$$\bar{K} = \mathrm{S}\; d\bar{k}\,,$$
$$\bar{M} = \mathrm{S}\; \overline{a\,dk}$$

(cf., no. 113).

Naturally, the equilibrium conditions read precisely as they do in a plane, as a result of Axiom VII, 2, 3, 4:

$$\bar{K} \equiv \mathrm{S}\; d\bar{k} = 0 \qquad \text{and} \qquad \bar{M} \equiv \mathrm{S}\; \overline{a\,dk} = 0.$$

The fact that two systems of forces are equivalent when and only when the sums of the forces $\bar{K}$ and the sums of the moments $\bar{M}$ agree is proved in precisely the same way as the corresponding theorem for systems of forces in the plane (see no. 116).

Thus, $\bar{K}$ and $\bar{M}$ are also the only invariants.

125. The force screw (dyname). Our result still depends on the choice of the reference point O. If we choose another point O' and let $\overline{OO'} = \bar{s}$ then $\bar{K}$ remains unchanged. However, from no. 114, the moment relative to O' is now $\bar{M}'$, where:

$$\bar{M}' = \bar{M} - \overline{sK}\,.$$

The additional component $-\overline{sK}$ is now always perpendicular to $\bar{K}$, but, by a suitable choice of $\bar{s}$ – i.e., O' – all of the other components preserve their directions and magnitudes.

If one thus decomposes $\bar{M}$ and $\bar{M}'$, the moments at each point, into a component parallel to $\bar{K}$ and another one perpendicular to $\bar{K}$ then one cannot change the former, while the latter is arbitrary. One can thus make it equal to zero by a suitable choice of $\bar{s}$ (O', resp.); i.e., one can arrange for the moment to takes on the same direction as the resulting force $\bar{K}$.

If we call the combination of a force and a force-couple whose plane is perpendicular to the force – so its moment vector lies in the same line as the force – a *force screw* or *dyname* then we can say:

One can always reduce a force system on a rigid body to a force screw; thus, one can arrange for one to have:

$$\bar{M} = p\bar{K}\,.$$

The number p for a line segment, which can be positive or negative, is called the parameter of the force screw.

Thus, in general, one cannot make the moment in space vanish, so the force system does not reduce to a single force. The facts that the component of $\bar{M}$ that is parallel to $\bar{K}$ indeed cannot change, and that it is not always zero from then on, can be recognized from the fact that one can just as well regard a force screw as a force system as fundamental.

The points for which $\bar{M}$ takes on the same direction as $\bar{K}$ lie on a line: the so-called *central axis* of the force system.

If we let O be a point for which one thus has $\bar{M} = p\bar{K}$ then the supplementary term $\overline{sK}$ that appears for a choice of another point O' and which is perpendicular to $\bar{K}$ – and therefore to $\bar{M}$ – is zero when and only when $\bar{s}$ has the same direction as $\bar{K}$. The central axis then has the same direction as $\bar{K}$.

If one lets $\bar{M}$ and $\bar{K}$ be given for any point O then p may be easily computed, and the equation for the central axis in the Plückerian form can be presented (see Appendix II, 2).

If we let $\bar{x} = \overline{OX}$ and let X be a point of the central axis then for X the moment is:

$$\bar{M}' = \bar{M} - \overline{xK}.$$

However, we shall have $\bar{M}' = p\bar{K}$.

Thus, we have:

$$\overline{xK} = \bar{M} - \overline{pK}.$$

If we abbreviate $\bar{M} - \overline{pK}$ by $\bar{c}$ then:

$$\overline{xK} = \bar{c}$$

is already the equation for the central axis, and $\bar{K}$, $\bar{c}$ are the Plücker vectors. One must still fulfill:

$$\bar{c} \cdot \bar{K} = 0,$$

i.e.:

$$\left(\bar{M} - \overline{pK}\right) \cdot \bar{K} = 0$$

or:

$$\bar{M} \cdot \bar{K} - p\,K^2 = 0,$$

from which, one obtains:

$$p = \frac{\bar{M} \cdot \bar{K}}{K^2} = \frac{M_x K_x + M_y K_y + M_z K_z}{K_x^2 + K_y^2 + K_z^2}$$

when we introduce the orthogonal components.

If we substitute the value of p into the expression for $\bar{c}$ then we obtain:

$$\bar{c} = \bar{M} - \frac{\bar{M} \cdot \bar{K}}{K^2} \cdot \bar{K} = \frac{1}{K^2}\left(K^2 \bar{M} - (\bar{M} \cdot \bar{K})K\right).$$

However, from the development formula (see Appendix), this is:

$$\bar{c} = \frac{1}{K^2}\overline{K(MK)} = -\frac{1}{K^2}\overline{(MK)K}.$$

The equation for the central axis thus reads definitively:

$$\overline{xK} = \frac{1}{K^2}\overline{(KM)K} \,,$$

so one of its points is given by:

$$\bar{x}_0 = \frac{1}{K^2}\overline{KM} \,.$$

If this is the base point for the perpendicular from O to the central axis then one indeed has $\bar{x}_0$ perpendicular $\bar{K}$.

The parametric equation of the central axis reads:

$$\bar{x} = \bar{x}_0 + \lambda \cdot \bar{K}$$

where λ runs through all values from $-\infty$ to $+\infty$.

126. Analytical formulation of the result. Since $\overline{ak}$ has the perpendicular components:

$$a_y k_z - a_z k_y \,, \quad a_z k_y - a_y k_z \,, \quad a_x k_y - a_y k_x \,,$$

(see Appendix II, 1), one then has:

$$K_x = \sum k_x \,, \quad K_y = \sum k_y \,, \quad K_z = \sum k_z \,,$$

$$M_x = \sum (a_y k_z - a_z k_y) \,, \qquad M_y = \sum (a_z k_x - a_x k_z) \,, \qquad M_z = \sum (a_x k_y - a_y k_x) \,,$$

and the equilibrium conditions for the rigid bodies read:

$$\sum k_x = 0, \qquad \sum k_y = 0, \qquad \sum k_z = 0,$$

$$\sum (a_y k_z - a_z k_y) = 0, \qquad \sum (a_z k_x - a_x k_z) = 0, \qquad \sum (a_x k_y - a_y k_x) = 0.$$

Problem 60: Compute the parameter and determine the central axis for the force system that consists of three forces with the orthogonal components 0, 15 kg, 0; 10 kg, 0, 0; 0, 0, 21 kg, when the coordinates of the point of application are 0, 0, 10m; 0, 0, 0, and 5m, 0, 0.

127. The moment relative to an axis. The expression:

$$a_x k_y - a_y k_x \,,$$

or a sum over these quantities when they originate from various forces, admits a double interpretation: Firstly, it gives the components of the moment vector relative to a point O of the z-axis, along this axis, but then it also means a moment that lies on the z-axis that we obtain from the given force system when we set k and z equal to zero. Thus, the components of the moment along the x and y axes will be zero, although the component along the z-axis remains unchanged.

However, setting k_z and z equal to zero means nothing more than projecting the force (forces, resp.) onto the xy-plane.

In an analogous way, we can take the moment of any force system relative to any axis:

We understand the moment of a force system relative to a line to mean the moment that we obtain when we project the force system onto a plane perpendicular to the line and take the moment of this projection relative to the point of intersection O of the line with the plane.

Since we can make any line in space the z-axis of a right-angled coordinate system, if we please, it then follows in general that:

The moment relative to a line is equal to the projection of that moment vector onto the line that one has constructed relative to a point of the line.

If one has given the line a definite sense of direction then one can regard the moment relative to it as a scalar, which is positive or negative, depending upon whether it does or does not agree with the sense of the line.

If h is the shortest distance of the force from the line then h is also the shortest distance of the aforementioned projection of the force one the point of intersection O. Moreover, if α is the angle that line subtends with the line of action of the force k then the projection of the force k on the plane is perpendicular to the line $k \sin \alpha$ and its moment relative to the line is the absolute magnitude of:

$$k \sin \alpha \cdot h.$$

Except for the trivial case of $k = 0$, *the moment of a force relative to a line is therefore zero if either the force is parallel to the line or it intersects the line.*

128. The equilibrium conditions, as expressed by the annulling of the moments. The question arises: Can one replace the equilibrium conditions for rigid bodies in such a way that one expresses the annulling of several moments?

If the moment vector for two points O and O' vanishes, so $\bar{M} = 0$ and $\bar{M}' = 0$, then it follows from:

$$\bar{M}' = \bar{M} - \overline{sK},$$

that:

$$\overline{sK} = 0;$$

i.e., if a $\bar{K}$ is possibly present then it can only lie on the connecting line OO'.

If one knows, perhaps, from the outset that this is impossible – e.g., all $d\bar{k}$ are vertical, but OO' is horizontal – then one must have that $\bar{M} = 0$ and $\bar{M}' = 0$. In general, however, there will be a third moment $\bar{M}''$, relative to a point O'', which must be set equal to zero, where O'' cannot lie on the line $O'O$. That certainly suffices, since $\bar{K}$ cannot be simultaneously lie on three different lines OO', $O'O''$, $O''O$.

For there to truly be equilibrium, it thus suffices to set equal to zero the moment relative to three points that do not lie in a line.

Does one also encounter moments relative to lines?

Since every equation that expresses that the moment relative to a line is equal to zero implies a scalar equation, the conditions:

$$\bar{K} = 0 \qquad \text{and} \qquad \bar{M} = 0$$

however, represent two vectorial – i.e., six scalar – equations, one can expect that in general the annulling of the moments for six lines will suffice to guarantee equilibrium.

For a special type of six lines this certainly suffices: Three lines g_1, g_2, g_3 may define a triangle $OO'O''$, while the other three h, h', h'' can go through a point O (O', O'', resp.), but not lie in the plane of $OO'O''$. Then, because the moment for g_1, g_2, h'' vanishes – these three may go through O'' – so does the moment vector $\bar{M}''$ for O'' vanish when its orthogonal projections onto three lines that do not lie in a plane vanish. This is true for any of the three points $OO'O''$, so, from the above, there actually exists equilibrium.

In the next paragraphs we will examine the exceptional cases in which one may not conclude equilibrium from the vanishing of the moments for six lines.

Problem 61: A table with three legs – whose lower ends A, B, C we would like to regard as points – is supported by a horizontal surface. Aside from the support reactions N_1, N_2, N_3 that govern them, and are directed vertically upwards, let the resulting force G that acts on the table be directed downwards and let its line of action intersect the triangle A, B, C at a point S whose distances r_1, r, r_3 are given by the three sides of the triangle. In addition, let the height of the triangle be given.

Compute the support reactions N_1, N_2, N_3. Why can equilibrium exist only if S lies in the interior of the triangle – i.e., when r_1, r, r_3 are positive?

Use the equilibrium condition that was given at the end of this section, and take the sides of the triangle and the verticals through the vertices as the moment axes.

§ 27. The null system [1])

129. Reduction to two forces. We know that in general one cannot reduce a force system to one force (no. 124), but to three forces (no. 113). Can one reduce the system to two forces?

For the case of merely a force-couple, the answer is in the affirmative, but trivial. Thus, we exclude this case ($\bar{K} = 0$, $p = \infty$) for the time being. Likewise, we omit the case $p = 0$ – i.e., $\bar{M} = 0$ – since one can indeed reduce the system to one force.

In the general case, one can, however, likewise replace the system with two forces. Thus, let the force system at a point O be given by $\bar{M}$ and $\bar{K}$, so one can indeed represent the force-couple $\bar{M}$ by two forces and $\bar{q}$ and $-\bar{q}$, of which $-\bar{q}$ acts on O. One can then combine $-\bar{q}$ and $\bar{K}$ into one force $\bar{r}$ at O and thus has two forces $\bar{q}$ and $\bar{r}$ for its resultant.

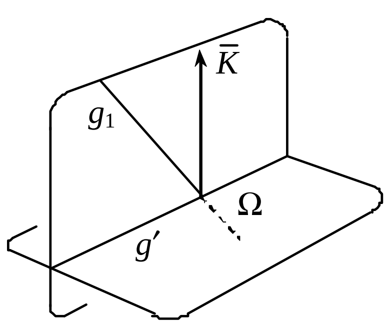

Fig. 79

Thus, one can still arbitrarily choose the magnitude and direction of $-\bar{g}$ at O when it is in the plane of the force-couple. This has the consequence that one can, at the very least, prescribe the line of action of $\bar{r}$ arbitrarily.

[1]) These paragraphs can be omitted by the beginner.

Namely, let the line of action g_1 be given, so one chooses a point O on it and thinks of the associated plane of the force-couple – which is perpendicular to $\bar{M}$ – as being distinguished. One lays a plane through $\bar{K}$ and the given line of action g_1 of $\bar{r}$; this plane intersects the plane of the force-couple in a line g'. One can, in general, lay a force $-\bar{q}$ along this line in such a way that the resultant of $-\bar{q}$ and $\bar{K}$ has the prescribed direction: Indeed, one needs only to decompose $\bar{K}$ into two components along g_1 and g', which is always possible when g_1 and g' do not coincide. $\bar{r}$ is thus completely determined by g': It is equal to $\bar{K}$ (and thus $\bar{q} = 0$) when g' coincides with the direction of $\bar{K}$, and it is infinite (and thus $\bar{q}$, as well) when g' lies in the plane of the force-couple. The second force $\bar{q}$ is then completely determined, along with its line of action. Then, in direction and magnitude it must certainly be given by the fact that $-\bar{q} = \bar{r} - \bar{K}$ and its position is given by the fact that $\bar{q}$ and $-\bar{q}$ must give the moment $\bar{M}$.

One can therefore always reduce a force system to two forces: In general, one can choose a line of action at will, so the other one is uniquely determined, as well as the magnitudes of the forces.

The lines of space are therefore pair-wise associated in general relative to a force screw; one calls these pairs *conjugate*. Their relationship – namely, that one can reduce the force system to two forces that lie in it – is in the nature of this statement.

However, there are two exceptions:

1. The lines that are parallel to the central axis – i.e., they have the direction $\bar{K}$. $\bar{q}$ will then be null, and thus, the lever arm of the force-couple will be infinite. All of these lines are thus associated with infinitely distant lines as conjugates. This exception may be regarded as having been dealt with by these remarks.

2. The lines that are perpendicular to the direction of the associated moment at a point: For them, $\bar{q}$ will be infinite, and in order for $\bar{M}$ to be finite, the conjugate line must coincide with it.

These lines, which one also calls *null lines,* are therefore conjugate to themselves.

They are called null lines, because they have the characteristic property that for them the moment of the force system vanishes. Thus, from no. 127, it follows that the moment vector is perpendicular to them.

130. Null points and null planes. From the preceding remarks, the null lines that go through a point define a plane, namely, the plane of force-couples that belong to the point. One also calls this plane the *null plane* of the point and the point is the *null point* of the plane.

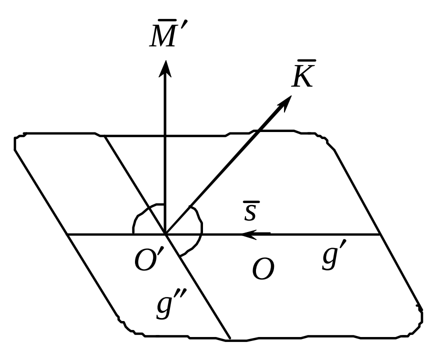

Fig. 80

That there is one and only one null plane through each point is clear from the discussion above. However, there is also, conversely, one and only one null point for each plane.

Namely, let the planes $\bar{K}$ and $\bar{M}'$ be determined for a point O', so one can go from the line g' in the plane that is perpendicular to $\bar{M}'$, and is therefore a null line, to another point O. The associated $\bar{M}$ then differs from $\bar{M}'$ by a component $\overline{sK}$ that is perpendicular to g' and lies in a line g'' that is perpendicular to $\bar{K}$. Everywhere else, the magnitude of $\overline{sK}$ varies freely with $\bar{s} = \overline{OO'}$. One can therefore choose O in such a way that $\bar{M}$ is perpendicular to the plane; indeed, one needs only to decompose $\bar{M}'$ into a component along g'' and one that is perpendicular to the plane (which is possible, since g'', $\bar{M}'$, and the perpendicular to the plane are all perpendicular to g', and thus lie in a plane), and the former one can be made to vanish by the addition of $\overline{sK}$.

O is then the desired null point.

Only for a family of planes does the null point lie at infinity – namely, for those planes that include the direction $\bar{K}$. Thus, g'' is indeed perpendicular to the plane and $\overline{sK}$; hence, $\bar{s}$ must also be infinite.

Conjugate lines, null lines, null planes, and null points now have the following relationship to each other:

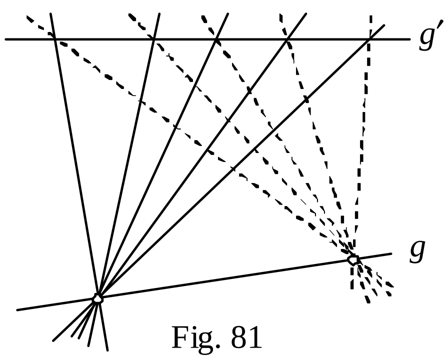

Fig. 81

1. Each line that cuts two conjugate lines is a null line.

Thus, since one can reduce the force system to two forces in the conjugate lines and they cut the aforementioned line the moment vanishes for them.

2. The null plane of each point of a line g includes the line g' that is conjugate to it.

Thus, all lines through the point that cut the conjugate line are, from 1., null lines; however, they also define a plane, namely, the null plane of the point.

3. The null point of each plane through a line g' lies on the line g that is conjugate to it.

Thus, from 1., all null lines of this plane indeed go through the other line.

The null lines thus determine the opposite association of conjugate lines. One now calls the totality of null lines, along with the opposite association of points to planes and lines to each other that is given by it, a *null system*.

131. Relation of the null system to the force screw. For the sake of convenience in expression, we now call the direction of the central axis that is given by $\bar{K}$ "vertical upwards," and the directions perpendicular to it "horizontal," so it is now clear that all horizontal lines that cut the central axis are null lines. For points of the central axis, the moment then indeed lies in it.

If we now choose a point A outside of the central axis at a distance a from it and make the base point O of this perpendicular be the reference point then the moment at A is:

$$\bar{M}' = \bar{M} - \overline{aK},$$

if one has $\bar{a} = \overline{OA}$.

$-\overline{aK}$ is horizontal, and indeed point to the left, when one looks from O to A; the magnitude of this supplement is aK. Thus, $\bar{M}'$ will deviate from the vertical to the left or right – but always perpendicular to $\bar{a}$ – according to whether $\bar{M}$ points upwards to downwards – i.e., whether it is positive or negative.

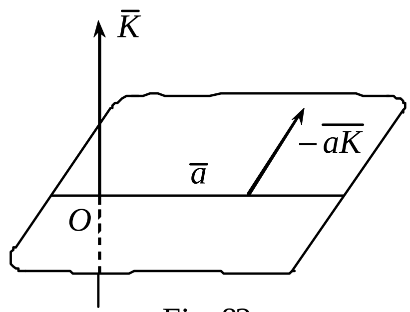

Fig. 82

If we measure the angle α by which $\bar{M}'$ deviates from the normal to the null plane at A to the left as being positive then, when one includes the sign, one has:

$$\tan \alpha = \frac{aK}{M} = \frac{a}{p}.$$

If one then removes the point A from the central axis then the direction of the associated null plane is increasingly steep; for $a = \infty$ one will have $\alpha = \pi / 2$. We also already know that the planes that are parallel to the central axis have infinitely distant null points.

One also sees from this that the horizontal null lines cut the central axis.

Furthermore, one sees that the null system admits any twist around the central axis; i.e., if one displaces the entire system along the central axis or rotates about it then it goes into itself.

Now, how do the conjugate lines relate to each other?

Let g be chosen such that $\bar{a} = \overline{OA}$ is its shortest distance from the central axis.

The conjugate line g' of the previous section must then lie in the null plane that belongs to A.

Above all, g will then cut the line OA – let B be its point of intersection – and since one shall have:

$$\bar{K} = \bar{q} + \bar{r}$$

– this puts $\bar{q}$ in g, $\bar{r}$ in g' – and $\bar{K}$, as well as $\bar{q}$, is perpendicular to OA, one must then also have that g' is perpendicular to OA:

The two conjugate lines, together with the central axis, thus cut the same line AOB perpendicularly.

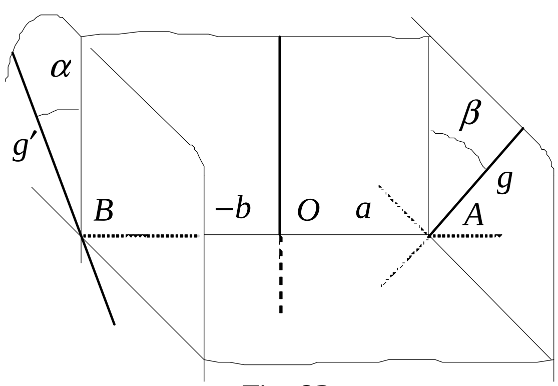

Fig. 83

Let OB be set equal to b: It will be positive when it coincides in direction with OA, and otherwise, it is negative.

Now, the normal to the null plane at A deviates from the vertical by an angle α, for which one has:

$$\tan a = \frac{a}{p}$$

– α is defined to be positive when the deviation in the direction $\bar{a}$ is seen as pointing to the left – and as a result, the null line g' deviates by an angle of $\pi/2 - \alpha$.

On the other hand, g lies in the null plane of B. If one thus has:

$$\tan b = \frac{b}{p}$$

– for positive and negative b – then g deviates from the vertical by $\pi / 2 - \beta$ to the right.

Along with the line g, now let a and $\pi / 2 - \beta$ be given – i.e., a and b – so from the previous formulas one computes distance b to the conjugate line g', including the signs, from:

$$b = p \tan \beta,$$

and its inclination $\pi / 2 - \alpha$ from:

$$\tan \alpha = \frac{a}{p}.$$

If the two conjugate curves are perpendicular to each other then one speaks of a *force cross* (Kraftkreuz), and one must have $\tan \alpha \cdot \tan \beta = 1$; i.e.:

$$ab = p^2.$$

One easily recognizes from these formulas that the null lines are conjugate to themselves. One must then have $\alpha = \beta$, and it then follows that $a = b$.

One sees that the null system depends only upon the central axis and the parameter p, but not on the magnitude of K.

132. The null system as a linear complex. If we translate our problem into the language of vector calculus then it reads: One shall determine all possible vectors $\bar{q}$, $\bar{r}$ and $\bar{a}$, $\bar{b}$ such that:

$$\bar{q} + \bar{r} = \bar{K},$$
$$\overline{aq} + \overline{br} = \bar{M}.$$

$\bar{a}$, $\bar{b}$ are the vectors from the reference point O to the points A (B, resp.) of the two conjugate lines.

One solves the equations for the Plücker vectors (see Appendix II, 2) $\bar{q}$ and $\overline{aq}$ of one line:

$$\left.\begin{aligned} \bar{q} &= \bar{K} - \bar{r}, \\ \overline{aq} &= \bar{M} - \overline{br}. \end{aligned}\right\} \tag{1}$$

One then only has that the inner product vanishes, so one must have:

$$(\bar{K} - \bar{r}) \cdot (\bar{M} - \overline{br}) = 0$$

or:

$$\bar{K} \cdot \bar{M} = \bar{r} \cdot \bar{M} + \bar{K} \cdot \overline{br}\,.$$

If we set:

$$\bar{r} = r \cdot \bar{\eta}\,,$$

where $\bar{\eta}$ is a unit vector in the second line, then it follows that:

$$r = \frac{\bar{K} \cdot \bar{M}}{\bar{\eta} \cdot \bar{M} + \bar{K} \cdot \overline{b\eta}}\,. \tag{2}$$

This result shows the older one: One can, in general, choose the one line of action – i.e., its Plücker vectors $\bar{\eta}$ and $\overline{b\eta}$ – arbitrarily; the other line and the magnitudes of the forces $\bar{q}$ and $\bar{r}$ are uniquely determined (from equations (1) and (2)).

As exceptions, we mention only the cases:

1. $\bar{q} = 0$ – i.e., $\bar{K} = \bar{r}$, but $\overline{aq} = \bar{M} - \overline{br} \neq 0$, such that $\bar{a} = \infty$.

2. $r = \infty$, so one also has $q = \infty$. This comes about when the denominator in (2) vanishes; i.e.:

$$\bar{\eta} \cdot \bar{M} + \bar{K} \cdot \overline{b\eta} = 0. \tag{I}$$

One easily sees that (I) is now actually the equation of the null line. Then, from the commutation theorem (see Appendix I, 6) the left-hand side of (I) is equal to:

$$\bar{\eta} \cdot (\bar{M} - \overline{bK}) = \bar{\eta} \cdot \bar{M}'\,,$$

where $\bar{M}'$ means the moment vector for the point *B*.

$\bar{\eta} \cdot \bar{M}'$ is, however, the moment relative to the second line, and equation (I) says that this shall vanish. This was, however, the characteristic property of the null line.

If we let $\bar{c}$ represent the second Plücker vector $\overline{b\eta}$, to abbreviate, then (I) reads:

$$\bar{\eta} \cdot \bar{M} + \bar{K} \cdot \bar{c} = 0,$$

or, in perpendicular components:

$$\eta_x M_x + \eta_y M_y + \eta_z M_z + c_x K_x + c_y K_y + c_z K_z = 0. \tag{I$'$}$$

That is, however, the most general linear homogeneous equation that can be given between the six Plücker coordinates $\eta_x, \ldots, c_z$. One now calls any manifold of ∞^2 lines – there are ∞^4 lines in space, in all (see Appendix II, 2) – that is given by a homogeneous equation between the six Plücker coordinates a *complex*. If the equation is linear then one will be speaking of a *linear complex*.

Our considerations show *that a system of null lines and a linear complex are identical concepts*.

If one makes the direction of the central axis that is given by $\bar{K}$ be the *z*-axis then, from (I′), one has:

$$p\ \eta_z + c_z = 0. \tag{I″}$$

If a is the shortest distance from the null line to the central axis, α, the angle that it subtends with the central axis then one has $\eta_z = \cos\alpha$; $\bar{c}$ is perpendicular to $\bar{a}$, has the magnitude a, and defines an angle $\pm(\pi / 2 - \alpha)$ with the z-axis, so one has $c_z = \pm a \sin\alpha$, and thus, from (I″), one has:

$$p \cos a \pm a \sin a = 0,$$

$$\tan\alpha = \pm \frac{p}{a}. \tag{I‴}$$

The sign is always given by that of p; a and $\tan\alpha$ are always positive, by definition.

From the formulas (1), (2), and (I) of this section, one can easily obtain all of the previous results. This may be left to the reader.

Problem 62: Show that the tetrahedron that is defined by the two conjugate forces $\bar{r}$ and $\bar{q}$ has a volume that is independent of the choice of line of action. (One confers Appendix II, 1).

133. Resolving the exceptional cases of no. 128. Decomposition of a force system into six lines. We showed in no. 128 that equilibrium is insured, in general, when the moments relative to six lines vanish. However, from the results of the previous section that implies the existence of six equations for $\bar{M}$ and $\bar{K}$ of the form (I′), namely:

$$\eta_x^{(\nu)} M_x + \eta_y^{(\nu)} M_y + \eta_z^{(\nu)} M_z + c_x^{(\nu)} K_x + c_y^{(\nu)} K_y + c_z^{(\nu)} K_z = 0 \qquad (\nu = 1, 2, \ldots, 6).$$

From this, one can now, in fact, always conclude with the vanishing of $\bar{K}$ and $\bar{M}$ when the determinant of six rows and six columns:

$$\begin{vmatrix} \eta_x^{(1)} & \eta_y^{(1)} & \cdots & c_z^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_x^{(6)} & \eta_y^{(6)} & \cdots & c_z^{(6)} \end{vmatrix}$$

is non-zero.

However, if it zero then the above six equations can very well come about – i.e., the six moments vanish, or else equilibrium would not reign.

What does the vanishing of the determinant now mean? (That it does not vanish identically follows from the example that was given in no. 118.)

There is then a force system for which the moments relative to the six lines vanish; i.e., these lines define six lines of a null system.

From the vanishing of the moments for six lines, one can conclude that equilibrium prevails when and only when these six lines do no belong to a null system.

From our considerations, there then follows the purely geometrical theorem:

One can always lay a linear complex through five lines, but not six, in general.

Thus, to five lines, one can always determine a sixth one such that the aforementioned determinant vanishes, but for six arbitrary lines the determinant does not vanish.

One can give the vanishing of the determinant yet another meaning. If the determinant is null then the system of equations:

$$\sum \eta_x^{(\nu)} \lambda_\nu = 0,$$
$$\sum \eta_y^{(\nu)} \lambda_\nu = 0,$$
$$\ldots$$
$$\sum c_z^{(\nu)} \lambda_\nu = 0$$

also has a non-zero solution.

If we interpret λ_ν as the forces that lie along the six lines:

$$\bar{\eta}^{(\nu)} \lambda^{(\nu)} = \bar{k}_\nu$$

then our equations read:

$$\sum \bar{k}_\nu = 0,$$
$$\sum \overline{b_\nu k_\nu} = 0$$

(since one indeed has $\bar{c}^{(\nu)} \cdot \lambda_\nu = \bar{\eta}_x^{(\nu)} \bar{b}_\nu \cdot \lambda_\nu = - \overline{b_\nu k_\nu}$), i.e., the forces $\bar{k}_\nu$ bring about equilibrium since their sum, as well as the sum of their moments, vanishes.

One can then lay six forces that are not all zero and bring about equilibrium along six lines when and only when the six lines belong to a null system (Möbius).

However, when the six lines do not belong to a null system, one can decompose an arbitrary force system along them; i.e., reduce it to six forces that lie along the lines.

One then needs only to solve the linear equations:

$$\sum \bar{\eta}^{(\nu)} \lambda^{(\nu)} = \bar{K},$$
$$\sum \bar{c}^{(\nu)} \lambda^{(\nu)} = -\bar{M},$$

and that can be done, since the determinant of the coefficients of the left-hand side does not vanish.

This theorem finds its application in the theory of spatial frameworks.

134. History and literature. The theorems and concepts that were presented here originate with Möbius, for the most part, who was an outstandingly good geometer, mechanician, and astronomer from the start of the Nineteenth Century. His statics is still very much worth reading. Yet another approach to statics from this era must be mentioned, which likewise placed the geometrical viewpoint very much in the foreground: that of Minding. Later, the geometers were much occupied with the complex theory. There is also a special theory of force screws that was worked out in detail: Let the names of Plücker, Klein, and Ball (Theory of Screws, German by Budde) be mentioned. One often summarizes the theories that were put forth in §§ 26 and 27, along with their extensions, under the name of the "geometry of forces;" the outstanding work of Study "Geometrie der Dynamen" begins on this basis. Among the elementary textbooks, let us mention: Föppl, Technische Mechanik, Bd. II; Timerding, Geometrie der Kräfte, Marcolongo-Timerding, Theoretische Mechanik, Bd. I. One finds a purely analytical presentation in Heuns Kinematik. (Later, we will run into the same things again in the kinematics of rigid bodies; see § 46, no. 263.) A textbook that especially emphasizes the geometric side of mechanics is that of Schell: Theorie der Bewegung und der Kräfte. Let the textbook of Webster: "The dynamics of particles and of rigid, elastic, and fluid bodies" also be pointed out. Naturally, one also finds a presentation of null systems and associated things in the great works on mechanics, such as Appell: Traité de mécanique (3 vols.) and Routh: A Treatise on Analytical Statics (2 vols.). As an overview reference, one can confer article 2 of volume IV (Mechanik) of the Enzyklopedie der mathematische Wissenschaften: Timerding, Geometrische Grundlegung der Mechanik eines starren Körpers.