"Ueber die Beziehung des Nullsystems und linearen Strahlencomplexes zum Polarsystem des Rotationsparaboloids," Zeit. f. Math. u. Phys. **31** (1886), 362-368.

On the relationship between null systems and linear ray complexes and the polar systems of paraboloids of rotation

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Along with Table V, Fig. 5a and 5b (†).

The very close relationship that exists between a *null system* and the *polar system of a paraboloid of rotation* seems to have not been commented upon, up to now. However, it can be arrived at very simply [entirely apart from grapho-statical considerations (*)] as soon as one considers the null system to be two reciprocal spatial systems and poses the question of which peculiarities of the null system come at the cost of a specialization of the reciprocal transformation and which of them come at the cost of a specialization of position.

As examples of the former properties, which we would like to call *intrinsic properties*, we can immediately recognize:

1. The existence of a central axis.
2. The fact that the pole of the plane at infinity lies at infinity.

Both properties also characterize the *polar system of the paraboloid of rotation* intrinsically. Therefore, this leads one to suspect that the null system can exhibit only one other reciprocal position of the aforementioned two reciprocal systems Σ and Σ' that are united by the polar system of a paraboloid of rotation. The suspicion is justified insofar as, indeed, it is not Σ and Σ' themselves that can always be brought into such a mutual position in which they will constitute of a null system, but, in fact, Σ and a system Σ'' that is symmetric with Σ'.

(†) Translator's note: These accompanying figures were not available to me.

(*) Cf., on this subject, my paper: "Ueber die reciproken Figuren der graphischen Statik," in volume 100 of the Journal f. d. reine u. angew. Math.

This shall be done more rigorously in what follows, where the most essential properties of null systems (*) will be derived from those of the polar system of a paraboloid of rotation in the most elementary way.

§ 1.

Let the axis of the paraboloid of rotation be $\mathfrak{A}$, let its vertex be S, and let the tangential plane at the vertex be σ'; let the parameter of the meridian parabola be equal to $2p$. Let Σ be the spatial system that is considered to be the *original system*. Let Σ' be the system – we call it the *polar system* – that is defined by the polar planes, polars, and poles of the points, lines, and planes, resp., of Σ relative to the paraboloid of rotation. Finally, Σ'' means the *mirror image* of the system Σ' relative to the plane σ' as its plane of reflection.

Now, if P is any point of the system Σ then one will obtain its polar plane π' as follows (cf., Tab. V, Fig. 5a): Lay the meridian plane through the point P and the axis $\mathfrak{A}$, which will serve as the reference plane, and denote the meridian parabola of all of its vertex tangents in it by $\mathfrak{s}'$. One then draws a parallel to $\mathfrak{A}$ through P that cuts the meridian parabola at X, extends PX to M', and draws the line $\mathfrak{p}'$ through M' that is parallel to the tangent to the parabola at the point X. $\mathfrak{p}'$ is then the polar of the point P relative to the meridian parabola. Finally, if one lays the plane π' through $\mathfrak{p}'$ that is perpendicular to the meridian parabola then π' will be the polar plane of the point P relative to the paraboloid of rotation.

The line $\mathfrak{p}'$ cuts the tangent $\mathfrak{s}'$ to the vertex at U' and the axis $\mathfrak{A}$ at r'. The tangent to the parabola at X cuts the axis at t. One drops the perpendiculars Pp and Xx to the axis from the points P and X, respectively. One will then have $pS = Sr'$ (since $px = Px = XM' = tr'$ and $xS = St$). The point p is then the *mirror image* of r' relative to the plane σ'. It follows from this that the line $U'p$ is the mirror image $\mathfrak{p}''$ of the line $\mathfrak{p}'$ and that the plane that is laid through $\mathfrak{p}''$ perpendicular to the meridian plane will represent the mirror image π'' of the plane π', moreover.

One now rotates the plane π around the axis $\mathfrak{A}$ through an angle of 90°, so it will describe the surface of a cone of rotation whose vertex is p. At the end of the rotation, *it will go through the line pP*. It will be divided into an upper and a lower sheet (**) by the line pP. The upper sheet will lie in front of the meridian plane or behind it according to whether the rotation around the axis results to the *left* or the *right*, resp. (i.e., as seen from above, and opposite to the hands of a clock or in the same sense, resp.). The first position will be denoted by π''_1 and the second one by π''_2. Conversely, if we were to consider the

(*) On this subject, one confers: *Möbius:* "Ueber eine besondere Art dualer Verhältnisse zwischen Figuren im Raume," Crelle J., X, pp. 317. Furthermore: Reye, *Die Geometrie der Lage,* 2nd ed., 1882, Part II, pp. 69.

(**) If one recalls Fig. 5a (Tab. V) then the branch of the axis $\mathfrak{A}$ that falls inside of the paraboloid might be referred to as the *upper* one and the one that falls outside of it, the *lower* one.

plane π_1'' or π_2'' as belonging to the original system Σ and correspondingly denote them by ρ_1 or ρ_2, resp., then the pole R_1' of ρ_1 would lie on the perpendicular to the meridian plane through r' at a distance $p'R_1' = pP$ in front of the meridian plane. As a result, its mirror image R_1'' would lie in front on the perpendicular through p at the aforementioned distance and would then coincide with P after a *left rotation* around the axis by 90° (cf., Tab. V, Fig. 5b, which presents the projection onto a plane that is perpendicular to the axis). The mirror image R_2'' of the polar R_2' of ρ_2 would lie at an equal distance behind the meridian plane and, correspondingly, its coincidence with P would be obtained by a *right rotation* around 90°.

If one performs the aforementioned operation with all points P and planes ρ of the system Σ then the system Σ' that is defined by the polar planes π' and poles R' – and consequently, its mirror image Σ'' – will be reciprocal to Σ, and after its left or right rotation around the axis $\mathfrak{A}$ through 90°, in either case, Σ'' will be situated in an *involutory* position with respect to Σ such that every plane π'' will go through the point P that corresponds to it and any point R will lie in the plane ρ that corresponds to it. That is:

The system Σ and the system Σ'' that is rotated through 90° from it to the left or right collectively define a null system.

From a different viewpoint, this can also be stated: If two polyhedra correspond to each other reciprocally relative to the polar system of a paraboloid of rotation then one of then can always be brought into a position with respect to the mirror image of the other one such that each of them is circumscribed by the other one.

§ 2.

Let the distance from the point P to the axis $\mathfrak{A}$ be equal to e, and let the angle that the plane π' makes with the axis $\mathfrak{A}$ be α'. Now (cf., Tab. V, Fig. 5a), if the normal to the parabola that is drawn through the point X cuts the axis at n then the sub-normal xn will be a constant that equals one-half the parameter p. Furthermore, one will have $Xx = c$ and the angle $xXn = \alpha'$ in the rectangular triangle Xxn. As a result, one will have:

$$e \tan \alpha' = p. \tag{1}$$

Since the mirror image π'' of the plane π' makes the same angle α' with the axis $\mathfrak{A}$ as π', and since that angle will not change under the rotation of the plane π'' around the axis, relation (1) will be true for the null system in the same way that it is true for the polar system of the paraboloid of rotation. The axis angle α' of the polar plane is determined directly from the distance e from the axis to the pole by means of that in both systems, and conversely. The polar plane to any pole can then be given – and conversely – when, in addition, one also observes that:

In the null system, the polar plane goes through the perpendicular Pp that goes from the pole P to the axis. By contrast, in the polar system of the paraboloid of rotation, it will go through the mirror image $R'\,r'$ of that perpendicular that has been rotated through 90°.

If one refers to the quantity p that related to the null system as the *constant of the null system* then one can, moreover, state the theorem:

The polar system of a null system can be considered to be the mirror image, rotated around the axis by 90°, *of the polar system of a paraboloid of rotation whose parameter is twice as large as the constant of the null system*.

§ 3.

Let $\mathfrak{l}$ and $\mathfrak{l}'$ be two conjugate straight lines in the polar system of the paraboloid of rotation. Let the shortest distance from the line $\mathfrak{l}$ to the axis $\mathfrak{A}$ be equal to e, and let the angle that its direction makes with the axis be equal to α. Let the distance from the axis to the line $\mathfrak{l}'$ be equal to e', and let its angle with the axis be equal to α'.

The meridian plane that is parallel to $\mathfrak{l}'$ will serve as the reference plane (cf., Tab. V, Fig. 5a). One draws the plane π' perpendicular to the meridian plane, which will intersect the meridian plane along $\mathfrak{p}'$. $\mathfrak{p}'$ will then be parallel to $\mathfrak{l}'$ and will represent its projection onto the meridian plane.

If one lays the plane λ through the line $P'M'$ perpendicular to the meridian plane then this will represent the conjugate diametral plane of the line $\mathfrak{p}'$, and consequently, also the line $\mathfrak{l}'$ that is parallel to $\mathfrak{p}'$. Thus, the polar $\mathfrak{l}$ of $\mathfrak{l}'$ must then lie in it. However, since the diametral plane that goes through $\mathfrak{l}'$ is parallel to the meridian plane, and consequently, perpendicular to the diametral plane λ, one will then have the theorem:

The diametral planes that are laid through any two conjugate straight lines will be perpendicular to each other in the polar system of the paraboloid of rotation.

It will then follow further from this, when one rotates the mirror image $\mathfrak{l}''$ of $\mathfrak{l}'$ (which lies in the aforementioned diametral plane) around the axis $\mathfrak{A}$ by 90°, that:

The diametral planes that are laid through any two conjugate straight lines will be parallel to each other in the null system.

The shortest distance from the line $\mathfrak{l}'$ to the axis $\mathfrak{A}$ is perpendicular to the meridian plane, so let its foot on the axis be r', and let its foot on $\mathfrak{l}'$ be R'. One will then have $r'R' = e'$. Moreover, the angle that the line $\mathfrak{p}'$, which is parallel to $\mathfrak{l}'$, makes with the axis is equal to α'. It will then be identical with the angle between the axis and the plane π'.

Since $\mathfrak{l}'$ lies in the plane π', its conjugate $\mathfrak{l}$ must go through the pole P of π'. Moreover, since $\mathfrak{l}$ lies in the diametral plane that is perpendicular to the meridian plane, it will project onto the meridian plane along PM', and Pp will represent the shortest distance e between and $\mathfrak{l}$ and $\mathfrak{A}$. Finally, since $\mathfrak{l}'$ goes through the point R', $\mathfrak{l}$ must lie in the polar plane ρ of R'. From § 1, this will go through pP; its upper sheet will lie in front of the meridian plane or behind it, according to whether the point R' lies in front of it or behind it, respectively. Correspondingly, the upper branch of the line $\mathfrak{l}$ will also lie in front of the meridian plane or behind it according to whether the line $\mathfrak{l}'$ lies in front of it or behind it, resp. The angle that the plane ρ makes with the axis $\mathfrak{A}$, and thus, also with the meridian plane, will be measured by the angle that is defined by $\mathfrak{l}$ and PM' and will be, correspondingly, equal to the angle α between the axis and the line $\mathfrak{l}$.

Since it is then proved that the distances e and e' from the two conjugate lines $\mathfrak{l}$ and $\mathfrak{l}'$ to the axis, and the angles α and α' that they make with the axis are identical with the distances from the axis to the two points P and R', resp., and the angles between the axis and the two polar planes ρ and π', resp., the relation:

$$e \tan \alpha' = e' \tan \alpha = p \tag{2}$$

will follow immediately from the theorem in § 2.

If one rotates the mirror image $\mathfrak{l}'$ of $\mathfrak{l}$ (which possesses the aforementioned distance e' to the axis and angle α' with the axis) around the axis through 90° then either the distance to the axis or the angle with the axis will change. The relation (2) is thus true for the null system, as well as for the polar system of the paraboloid of rotation. The distance to the axis and the angle with the axis of one of two conjugate lines will be determined directly from the distance to the axis and the angle with the axis of the other one by means of it. One make the following remark in regard to their mutual position: The mirror image R'' of the point R' lies on the line that is perpendicular to the meridian plane at p, and will thus coincide with the line pP after a rotation around the axis by 90°. Therefore:

The distances from two conjugate lines to the axis will fall on the aforementioned line in the null system. By contrast, in the polar system of the paraboloid of rotation they will fall on lines, one of which will represent the mirror image of the other one, rotated through 90°.

§ 4.

If the relation:

$$e \tan \alpha = p \tag{3}$$

exists between the distance e to the axis and the angle α between the axis and a straight line $\mathfrak{l}$ then it will follow from relation (2) of the previous paragraph that the distance e

from the conjugate line $\mathfrak{l}'$ to the axis and the angle α' between it and the axis will possess the aforementioned values:

$$e' = e, \quad \alpha' = \alpha.$$

It will follow that conversely: If two conjugate straight lines $\mathfrak{l}$ and $\mathfrak{l}'$ have the same distance to the axis then they will also have the same angles with the axis, and conversely, and relation (3) will then exist between the two.

Two conjugate lines of this special kind have a position in the null system such that their distances to the axis (which, from § 3, will lie in the same straight line) will either coincide or lie on different sides of the axis. In order to examine this behavior more closely, we imagine, in connection with the considerations of the previous paragraphs (cf., Tab. V, Fig. 5a), two straight lines $\mathfrak{l}_1'$ and $\mathfrak{l}_2'$, both of which are parallel to $\mathfrak{p}'$ and project onto $\mathfrak{p}'$, with $\mathfrak{l}_1'$, in front of the meridian plane and $\mathfrak{l}_2'$, behind it, but with equal distances to the axes $r'R_1' = r'R_2'$. Their conjugate lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$ will then lie in the plane λ that is perpendicular to the meridian plane and both of them will project onto PM'. They will both go through P and lie symmetrically to the meridian plane: The upper branch of $\mathfrak{l}_1$ will be in front of the meridian plane, while the upper branch of $\mathfrak{l}_2$ will be behind it; their common distance to the axis will be pP. The distances e to the axis and the angles α with the axis for $\mathfrak{l}_1$ and $\mathfrak{l}_2$ might satisfy the relation (3). The distances to the axis and the angles with the axis of $\mathfrak{l}_1'$ and $\mathfrak{l}_2'$ will then have the aforementioned values: viz., $r'R_1' = r'R_2' = pP = e$, and $\alpha' = \alpha$.

Let the mirror images of R_1' and R_2' be R_1'' and R_2'', resp. They will lie on the perpendicular to the meridian plane that goes through p with R_1'' in front of it and R_2'' behind it, resp., and at equal distances $pR_1'' = pR_2'' = pP$ (cf., Tab. V, Fig. 5b). One imagines Pp as being extended to Q. Let the mirror images of $\mathfrak{l}_1'$ and $\mathfrak{l}_2'$ be $\mathfrak{l}_1''$ and $\mathfrak{l}_2''$, resp. In Fig. 5b (Tab. V), the projections of the upper branches for $\mathfrak{l}_1''$ and $\mathfrak{l}_2''$, as well as $\mathfrak{l}_1$ and $\mathfrak{l}_2$, onto the plane that is perpendicular to the axis are marked by p. The upper branches of $\mathfrak{l}_1''$ and $\mathfrak{l}_2''$ are both inclined to the left, the upper branch of $\mathfrak{l}_1$ is inclined in front of it, and $\mathfrak{l}_2$ is inclined behind it. Since the axis angles are, without exception, equal, one can regard these four lines as surface lines of a skew hyperboloid of rotation, whose midpoint is p and whose throat radius is e. Indeed, $\mathfrak{l}_1$ and $\mathfrak{l}_1''$ belong to the one pair of surface lines, while $\mathfrak{l}_2$ and $\mathfrak{l}_2''$ belong to the other family.

If one then rotates the system Σ'' around the axis $\mathfrak{A}$ *towards the left* (i.e., opposite to the hands of a clock) by 90° then R_1'' will coincide with P, and $\mathfrak{l}_1''$ will coincide with $\mathfrak{l}_1$, while R_2'' will arrive at Q, and the upper branch of $\mathfrak{l}_2''$ will come to lie in front of the meridian plane, such that $\mathfrak{l}_2''$ and $\mathfrak{l}_2$ will cross. By contrast, if one rotates it *towards the right* (i.e., in the sense of clock hands) then R_2'' will coincide with P, and $\mathfrak{l}_2''$ with $\mathfrak{l}_2$, while

R_1'' will arrive at Q and the upper branch of $\mathfrak{l}_1''$ will come to lie behind the meridian plane, such that $\mathfrak{l}_1''$ and $\mathfrak{l}_1$ will cross.

The totality of the coincident lines – and thus all of the lines $\mathfrak{l}_1$ under left rotation and all of the lines $\mathfrak{l}_2$ under right rotation – will define a *linear ray complex.* If one imagines two helices that both have $\mathfrak{A}$ for their axis, and one of them is the line $\mathfrak{l}_1$, while the other one has $\mathfrak{l}_2$ for its tangent then the former will be *right-wound, while the latter will be left-wound.* The character of the two ray complexes will also be determined in that way. The two null systems, one of which arises from the system Σ'' by left rotation through 90°, while the other of which arises by right rotation, will thus differ from each other in such a way that:

The ray complex that is defined by the double lines (guiding rays) by left rotation will be right-wound, while the one that is defined by right rotation will be left-wound.

At the same time, it will follow from this that:

A left-wound null system can be converted into a right-wound one by rotating its polar system around the axis by 90°.
