"Ueber die reciproken Figuren der graphischen Statik," J. f. d. reine u. angew. Math. **100** (1887), 365-389.

On the reciprocal figures of graphical statics

(By *Guido Hauck*)

Translated by D. H. Delphenich

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§ 1. Introduction.

Rankine published (*) a theorem on the equilibrium of forces in a *polyhedral rod frame*, which said that a system of external forces that act upon the nodes of such a rod framework will be in equilibrium when the forces are proportional to the areas of the faces of a polyhedron whose edges lie in planes that are drawn through a fixed point *O* and perpendicular to the rods of the frame, and are directed perpendicularly to those planes. The stresses in the latter rods will then be proportional to the triangular areas that project the polyhedral edges from *O* and are perpendicular to the rods in question.

Clerk Maxwell next sought to relate that theorem to the reciprocal planar figures of graphical statics by means of the *polar system of the sphere* (**), when he regarded a planar rod framework as the projection of a corresponding spatial rod framework. Since one must impose the restriction on the external forces into this that they must all go through a point – namely, the center of the sphere – which must be moved to infinity, the verification of that promising thought was not achieved in the desired degree of generality. *Maxwell* thus treated the problem of reciprocal force diagram later in a piece-wise analytical way, in which he replaced the sphere with a *paraboloid of rotation* (***). On the other hand, *Cremona* appealed to the *null system* in order to establish the theory of reciprocal figures in graphical statics with its proper synthetic elegance (****).

Granted, the principle that the spatial structure whose projection is considered to be a planar rod framework *can be assigned a concrete meaning as a spatial rod framework* stayed more or less in the background of the last two papers. These original thoughts of *Maxwell*, as well as his subtle search *to make Rankine's theorem subordinate to it by means of the polar system of the sphere*, has a certain charm to it that makes taking up

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(*) See: *Rankine*, "Principle of the equilibrium of polyhedral frames," Phil. Mag. (4) **27** (1864), pp. 92. The theorem was published without proof.

(**) See: *Clerk Maxwell*, "On reciprocal figures and diagrams of forces," *ibidem*, pp. 250. Furthermore: *Clerk Maxwell*, "On the application of the theory of reciprocal polar figures to the construction of diagrams of forces," Engineer **24** (1867), pp. 402.

(***) See: *Clerk Maxwell*, "On reciprocal figures, frames and diagrams of forces," Transactions of the Royal Society of Edinburgh **26** (1870), pp. 1. Unfortunately, this extremely significant treatise does not seem to be sufficiently well-known, which is very regrettable in relation to the *Culmann's* polemic regarding *Maxwell* in the Foreword (pp. x) of the 2nd ed. of Culmann's *Graphische Statik*.

(****) See: *Cremona*, "Le figure reciproche nella statica grafica," Milano, 1872.

that search again seem to be all the more justified as the resolution of various open problems of paramount importance that the theory of reciprocal force diagrams still hopes to establish at this time.

When seen from this viewpoint, it will be shown in the present article in the synthetic way that the *polar system of the sphere*, and in fact, *any second-order surface of rotation* will fulfill the condition in the same way as the null system, in such a way that rod frameworks and force frameworks can *generally* be considered to be the projection of reciprocal polyhedral structures. The proof comes about with the aid of the method of *reciprocal projection* – a type of projection whose presentation already seems to be justified, insofar as the *surface locus projection* (*) that was previously introduced by *Fr. Neumann* for crystallographic purposes proves to be a special case of the stated projection method. Furthermore, the planar rod framework will related to the spatial rod framework, and the connection between Rankine's theorem and the reciprocal planar figures of graphical statics will be made clear. The following simple consideration will define the starting point for this:

If one regards a planar system of n forces that is found to be in equilibrium as the orthogonal parallel projection of a spatial system of n forces then the latter will not necessarily be likewise in equilibrium. It is only requisite that it can be reduced to a single force (a single force-couple, resp.) as a resultant that is perpendicular to the plane of projection. However, one can also add the resultant with the opposite direction to the spatial system, and the planar system of n forces in equilibrium will then be represented as the projection of a spatial system of $n + 1$ forces in equilibrium, one of which is perpendicular to the plane of projection.

§ 2. The reciprocal projection.

The foregoing problem is connected with the theory of the *trilinear affinity* of planar systems in the broader sense.

Three planar figures have a trilinear affinity when they can be exhibited as the *projections of one and the same spatial figure*. One can now apply a manner of projection in place of the usual kind of projection that we call *reciprocal projection* and would like to define as follows:

We imagine that a spatial polar system has been given and determine the projection of each straight line L of the object to be projected in such a way that we draw, not the line L itself, but the line $\mathfrak{L}$ that is conjugate to it in the polar system, from the center of projection O to a plane of projection and make it intersect the plane of projection. Under this kind of projection, a *planar surface* of the object will project to a point that is obtained when one draws a projecting ray from O to the pole of the surface and makes it intersect the plane of projection. The object-points possess no direct projections.

If one allows these reciprocal projections, along with the usual projections, then of the three planar figures that represent the projections of one and the same spatial figure, either all three of them will always be equivalent or two of them will always be equivalent, while the third one will be inequivalent.

(*) See *Fr. Neumann, Beiträge zur Krystallonomie,*" Berlin and Poznan, 1823.

Two inequivalent projections can also be regarded as the images (in the usual sense) of two reciprocal spatial figures.

The reciprocal manner of projection simplifies essentially when one gives the center of projection O a special position in regard to the polar system that the projection process is based upon. Indeed, in what follows, we would like to always couple the concept of *reciprocal projection* with the assumption that the *center of projection O lies at the center of the polar system*.

Thus, if one has any straight line L and lays the *reciprocally-projected plane* from O to its conjugate $\mathfrak{L}$ then that plane will be identical with the *diametral plane that is conjugate* to the direction L. Moreover, if one has a planar surface and draws the *reciprocally-projected ray* from the pole of O then it will represent the *diameter that is conjugate* to the position of plane. We thus have the following definition:

1. *The reciprocal projection of a straight line is obtained when one determines the diametral plane that conjugate to its direction (as the reciprocally-projected plane) and makes it intersect the plane of projection.*

The reciprocal projection of a planar surface is obtained when one determines the diameter that is conjugate to its position (as the reciprocally-projected ray) and makes it intersect the plane of projection.

This definition immediately yields the following theorem *as the fundamental property of reciprocal projections:*

2. *The reciprocal projections of parallel lines fall upon the same line. The reciprocal projections of parallel planes fall upon the same point.*

Furthermore, if one has a number of straight lines that lie in the same plane or are parallel to it then their conjugate diameters will all go through the diameter that is conjugate to that plane. The theorem follows from this that:

3. *The reciprocal projections of straight lines that lie in the same plane or are parallel to the same plane all intersect at a point that represents the reciprocal projection of that plane.*

On the other hand, if one has several planes that go through the same straight line or are parallel to it then their conjugate diameters will all lie in the diametral plane that is conjugate to that straight line. It will then follow that:

4. *The reciprocal projections of all planar surfaces that go through the same axis or are parallel to that axis lie along a straight line that represents the reciprocal projection of that axis.*

If one takes the polar system that is at the basis of the projection process to be *the polar system of a sphere* then one will obtain a special case of reciprocal projection that behaves similarly with respect to the general case as orthogonal projection does with respect to skew parallel projection, and which proves to be identical with a process of projection that *Fr. Neumann* already defined and exhibited for the purposes of crystallography; for that reason, we would like to refer to this special case briefly as *Neumann projection* in the sequel. Namely, since any diametral plane is perpendicular to the line-direction that is conjugate to it in the polar system of the plane, and any diameter is perpendicular to the plane position that it conjugate to it that will specialize our definition (Theorem 1) above for the polar system of the sphere to the following theorem, which coincides with *Neumann's* definition:

5. *Neumann's projection of a straight line is obtained when one lays a plane through the center of projection O that is perpendicular to the line and makes it intersect the plane of projection.*

The Neumann projection of a planar surface is obtained when one draws a ray through the center of projection O that is perpendicular to the surface and makes it intersect the plane of projection.

Neumann called the projection of a surface its *surface location*, and the projection of an edge direction – or "zone axis" – the corresponding *zone line*, which is a terminology that we would also like to carry over to the general reciprocal projection. The simplicity by which the Neumann method of projection gives one a picture of the zone connection of the surfaces of a crystal is demanded by Theorem 4 above, which would read, in Neumann's terminology: *The surface locations of all surfaces that belong to the same zone will lie along a straight line, namely, the zone line in question*.

§ 3.

Relationship between orthogonal projection, reciprocal projection relative to a second-order surface of rotation, and reciprocal projection relative to a null system.

If one simultaneously draws the Neumann projection $\mathfrak{l}$ and the orthogonal parallel projection l of a straight line L onto the same plane of projection $\mathfrak{P}$ then, from a known theorem, since $\mathfrak{l}$ represents the trace of plane that is perpendicular to L, l must be perpendicular to $\mathfrak{l}$. For this, one requires a special relationship between the Neumann projection and the orthogonal parallel projection of a polyhedral structure onto the same plane of projection. It is illustrated in more detail in Fig. 1, which represents the Neumann projection (dashed line) and, at the same time, the orthogonal parallel projection (plain line) of a polyhedron (*):

(*) The projected polyhedron is the iso-angular, semi-regular polyhedron whose 26 faces consist of 6 regular octangles (hexahedral faces), 8 regular hexangles (octahedral faces), and 12 squares (garnet faces – *Granatoëderflächen*). The plane of projection is parallel to a hexahedral face, while the center of projection O is taken along the perpendicular to the plane of projection that goes through the center of the polyhedron. The surface locations of the hexahedral, octahedral, garnet faces are indicated by $\mathfrak{h}$, $\mathfrak{o}$, $\mathfrak{g}$, resp. The surfaces that are perpendicular to the plane projection project orthogonally as straight lines; the associated surface locations (viz., two points $\mathfrak{h}$ and two point $\mathfrak{g}$) lie at infinity.

Any straight line in one figure of projection corresponds to a line in the other one that is perpendicular to it, and indeed, in such a way that all lines of the one figure that meet at a point will correspond to just as many lines in the other figure that define a closed polygon, and conversely.

(In Fig. 1, this arrangement is drawn in such a way that the vertices of the one figure will lie completely inside the corresponding polygon in the other figure, and conversely.)

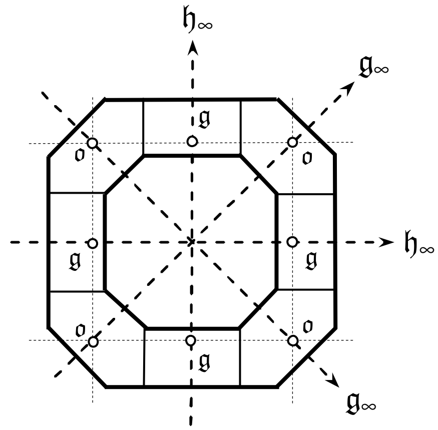

Figure 1.

However, this relationship does not restrict to just Neumann projection, but is also valid in the same way for reciprocal projection relative to the polar system of any second-order surface of rotation when the plane of projection is aligned perpendicular to the axis of rotation. This is easily obtained from the following consideration:

One imagines the polar system of a second-order surface of rotation whose center is O and whose axis of rotation is perpendicular to the plane of projection $\mathfrak{P}$, and then the polar system of a sphere whose center O' lies on the axis of rotation. If one then draws the reciprocal projections of any spatial object relative to both systems then the rays and planes that are reciprocally-projected from O and O' will define a sheaf of diameters and diametral planes relative to the polar systems that are conjugate to those planes and line-directions; as a result, these two sheaves will be collinear with each other. Now, in the polar system of the surface of rotation, as in that of the sphere, any diametral plane that goes through the axis of rotation will be perpendicular to its conjugate diameter. Any such diametral plane will then be conjugate to the same line-direction in both systems. The two collinear sheaves O and O' will thus have the corresponding pencils of planes that cut the rotational axis in common, and will thus be found to be in perspective position (*). It follows from the omnidirectional symmetry that its perspective intersection must be perpendicular to the axis of rotation, and thus, parallel to the plane of projection. If one displaces the sphere parallel to itself, while one lets its center O' move along the axis of rotation, then one will alter the perspective intersection. However, since it remains consistently parallel, it must coincide with the plane of projection $\mathfrak{P}$ for some

(*) Cf., *H. Schroeter,* "Theorie der Oberflächen zweiter Ordnung," § 45.

particular position of O'. If we fix that position then the perspective intersection will, moreover, represent the reciprocal projection onto the plane of projection, relative to the polar system of the sphere, as well as the polar system of the surface of rotation. Conversely, each figure that is drawn in the plane of projection – e.g., like the dashed part of Fig. 1 – can be regarded as the reciprocal projection of a spatial object relative to the one surface, as well as the other one. A theorem that is proved for Neumann projection is therefore true in the same way for the reciprocal projection relative to the surface of rotation.

For the special case of the *paraboloid of rotation*, the center or the center of projection O will lie at infinity. The reciprocally-projected rays and planes will then be parallel to the axis of rotation or *orthogonal to the plane of projection.*

A *null system* can be related to the polar system of a sphere whose center is chosen along the central axis of the null system in the same way that it is related to the polar system of a second-order surface of rotation. Since the center O of the null system lies at infinity along the direction of the central axis, the reciprocally-projected rays and planes will define a sheaf of parallels to the central axis. It will have a collinear relationship to the reciprocally-projected sheaf O' of the polar system of the sphere. The diametral planes that are conjugate to the directions that are perpendicular to the central and go through the central axis are, however, not perpendicular in the null system now, as they are for the sphere, but *parallel* to the directions considered and will thus be perpendicular to the corresponding diametral planes of the sphere. The pencil of planes that it defines can thus be made to coincide with the corresponding pencil of planes of the sphere when one rotates one of the two sheaves around the central axes through an angle of 90°. The two sheaves will then lie perspectively, so the perspective intersection that is perpendicular to the central axis will represent the reciprocal projection of the object, as far as its form is concerned, relative to the null system, as well as the polar system of the sphere. Moreover, since the reciprocal projection relative to the sphere can be identified with the one relative to a second-order surface of rotation, it will then follow that (*):

The reciprocal projection of an arbitrary spatial object relative to a second-order surface rotation onto a plane of projection that is perpendicular to the axis of rotation can always be considered to be the reciprocal projection relative to a null system that is coaxial with the surface of rotation that has been rotated through 90°.

It is further true that: If one has the reciprocal projections of a straight line onto a plane that is perpendicular the axis relative to a second-order surface of rotation, as well as relative to a null system, then the *orthogonal parallel projection* of the line onto that plane of projection will be perpendicular to the former and parallel to the latter.

(*) The relationship between the null system and the polar system of a *paraboloid of rotation* proves to be of especial interest. On this, confer my paper: "Ueber die Beziehung des Nullsystems und linearen Stahlencomplexes zum Polarsystem des Rotationsparaboloids," in the Zeit. f. Math. u. Phys., 31 Jahrg., pp. 362.

§ 4. The reciprocal figures of graphical statics.

Two plane figures, one of which is a *planar rod framework* (*), while the other one represents its associated *force framework*, have precisely the same mutual relationship as far as their form is concerned as the ones that were discussed in the previous paragraphs and were illustrated by Fig. 1.

We can thus state the following theorems:

A given rod framework and its associated force framework can be regarded as the orthogonal projection and Neumann projection of one and the same polyhedral structure onto the same plane of projection,

or also:

A planar rod framework and its associated force framework can be regarded as the projections of two reciprocal polyhedral structures in the polar system of a sphere or a second-order surface of rotation onto the same plane of rotation, which is perpendicular to the axis of rotation, and indeed the rod framework can be regarded as the orthogonal parallel projection, while the force framework can be regarded as the central projection from the center of the polar system.

One is now confronted with the deeper problem of basing these theorems, which were found in an empirical way, *in statics directly.*

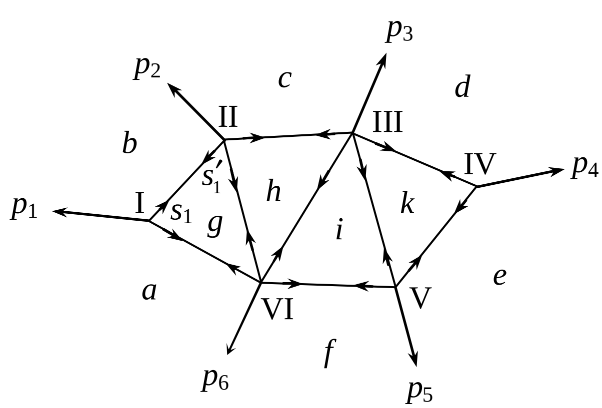

Figure 2.a

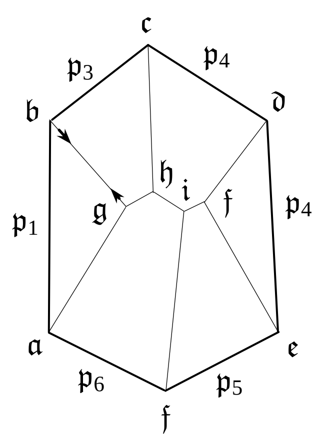

Figure 2.b

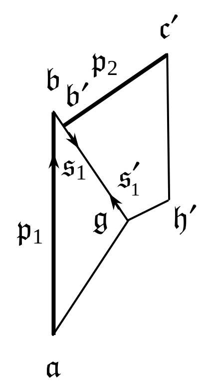

Figure 2.c

§ 5. Theorem on the force pyramid.

If one has a statically-determinate, planar, rod framework with k nodes (cf., Fig. 2.a) in the plane then the external forces that act upon the nodes and the stresses that act in the

(*) Here and in what follows, one might understand the term *rod framework* to mean the actual rod framework, *along with all of the lines of action of the external forces that are applied to its nodes.*

rods must fulfill the condition that the forces that come together at each individual node must be equilibrium. In relation to the stresses, one must then observe that each of them acts upon the two modes at the ends of the rod in question in the opposite sense (*). The rod framework then represents *a combination of k pencils of forces that are each in equilibrium*.

If one now considers the planar rod framework to be the orthogonal parallel projection of a corresponding spatial rod framework (**) then each of the aforementioned pencils of forces will represent the projection of a spatial sheaf of forces. Since the pencil of projections should be in equilibrium, the corresponding spatial sheaf of forces must possess a resultant that is perpendicular to the plane of projection. Accordingly, the spatial rod framework represents *a combination of k sheaves of forces, each of which possesses a resultant that is perpendicular to the plane of projection,* or also – if the resultant is direction oppositely to the sheaf – a combination of k sheaves of forces, *each of which is in equilibrium, and each of which contains force that is perpendicular to the plane of projection.*

We must then (corresponding to the former picture) seek the condition that any sheaf of forces must satisfy in order for its resultant V to be perpendicular to the plane of projection $\mathfrak{P}$.

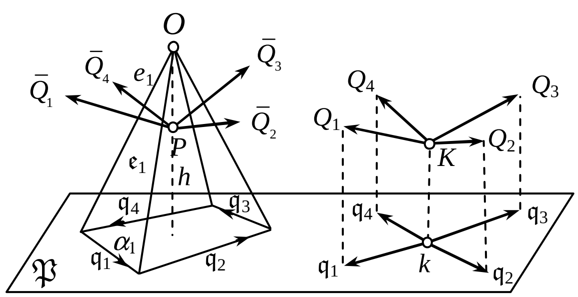

Figure 3.

Let the node considered be K (cf., Fig. 3), and let the forces that act upon it be Q_1, Q_2, …, Q_n .

We imagine a sphere with radius r that is described around an arbitrary point O on space and consider its polar system. Let h be the perpendicular that is dropped from O to the plane of projection $\mathfrak{P}$; the pole P of the plane of projection $\mathfrak{P}$, which we would like to refer to casually as the *pole*, lies along it.

We now displace the sheaf Q_1 Q_2 … parallel to itself to the pole P, at which position, the lines of action might be denoted by $\bar{Q}_1$, $\bar{Q}_2$, …, and carry out the examination in question in that position.

To that end, if we construct the polar figure of the sheaf then since $\bar{Q}_1$, $\bar{Q}_2$, … go through P, the polars $\mathfrak{q}_1$, $\mathfrak{q}_2$, … must lie in the plane $\mathfrak{P}$. $\mathfrak{q}_1$, $\mathfrak{q}_2$, … then likewise represent

(*) For example, if rod 1 in Fig. 2 .a suffers a *tension* then it will, conversely exert two forces s_1 and s_1' on the nodes I and II whose common line of action is the line 1 and which have equal intensities *in the opposite directions to each other*.

(**) It should be thought of as an open, *non-rigid*, polyhedral structure.

the Neumann projections of $\bar{Q}_1$, $\bar{Q}_2$, … onto the plane $\mathfrak{P}$. According to the properties of the polar system of the sphere, two conjugate lines $\bar{Q}_i$ and $\mathfrak{q}_i$ must intersect perpendicularly. The plane that is laid through O and $\mathfrak{q}_i$ is perpendicular to the $\bar{Q}_i$, just as the plane that is laid through O and $\bar{Q}_i$ is perpendicular to $\mathfrak{q}_i$. The product of the distances of both lines from O is equal to the square of the radius of the sphere. If one then denotes the distances from the lines $\bar{Q}_1$, $\bar{Q}_2$, … to O by e_1, e_2, …, resp., and the distances from the lines $\mathfrak{q}_1$, $\mathfrak{q}_2$, … to O by $\mathfrak{e}_1$, $\mathfrak{e}_2$, …, resp., then one will have:

(1) $$e_i\,\mathfrak{e}_i = r^2 .$$

If one fixes a particular sequence to the n forces $\bar{Q}_1$, $\bar{Q}_2$, … and then makes each of the polars $\mathfrak{q}_1$, $\mathfrak{q}_2$, … intersect the next one in that sequence then one will obtain a closed n-angle in the plane $\mathfrak{P}$ whose sides might also be denoted $\mathfrak{q}_1$, $\mathfrak{q}_2$, … according to their lengths. This n-angle defines the base surface of a pyramid whose vertex is O and whose faces are perpendicular to $\bar{Q}_1$, $\bar{Q}_2$, …, resp. In the faces, $\mathfrak{e}_1$, $\mathfrak{e}_2$, … represent the altitudes that belong to the base lines $\mathfrak{q}_1$, $\mathfrak{q}_2$, …, resp. If one then denotes the areas of the faces by Δ_1, Δ_2, … then one will get:

(2) $$2\,\Delta_i = \mathfrak{q}_i\,\mathfrak{e}_i .$$

Now, should the n forces $\bar{Q}_1$, $\bar{Q}_2$, … possess a resultant that is perpendicular to the plane $\mathfrak{P}$, so it is in the direction PO, then the sum of their rotational moments about an arbitrary point of the that line – e.g., around O – must be equal to zero. If we represent the rotational moments by their instantaneous axes then they must combine into a closed polygon. However, since the instantaneous axes are perpendicular to the planes that the forces $\bar{Q}_1$, $\bar{Q}_2$, … project onto from O, they will be parallel to the polars $\mathfrak{q}_1$, $\mathfrak{q}_2$, …, resp. The n-angle $\mathfrak{q}_1\,\mathfrak{q}_2$… can thus be regarded as the axis polygon. That is: If the moments of $\bar{Q}_1$, $\bar{Q}_2$, … relative to the point O are proportional to the segments $\mathfrak{q}_1\,\mathfrak{q}_2$… then the resultant of $\bar{Q}_1$, $\bar{Q}_2$, … is perpendicular to $\mathfrak{P}$.

(The direct converse of the conclusion is valid only for a sheaf of three forces. For more than three forces, the converse must read: If the resultant is perpendicular to $\mathfrak{P}$ and the $n - 2$ sides of the n-angles $\mathfrak{q}_1\,\mathfrak{q}_2$… represent the moments of the corresponding forces relative to O then this will also be true for the remaining two sides.)

If we accept that $\mathfrak{q}_1$, $\mathfrak{q}_2$, … are instantaneous values then if we understand Q_1, Q_2, …, resp., to be likewise the intensities of the forces then:

$$Q_i\,e_i = \mathfrak{q}_i ,$$

or [due to (1)]:

$$Q_i \frac{r^2}{\mathfrak{e}_i} = \mathfrak{q}_i ,$$

or

$$Q_i = \frac{\mathfrak{q}_i \, \mathfrak{e}_i}{r^2} ,$$

or [due to (2)]:

(3) $$Q_i = \frac{2}{r^2} \Delta_i .$$

If α_1, α_2, … are the angles that Q_1, Q_2, …, resp., make with the perpendiculars to the plane of projection then the base inclination angles of the corresponding side faces of the pyramid will also be equal to α_1, α_2, …, resp. If one then denotes the area of the base surface of the pyramid by Π then the resultant of the forces $\bar{Q}_1$, $\bar{Q}_2$, … will be:

$$V = Q_1 \cos \alpha_1 + Q_2 \cos \alpha_2 + \ldots$$

or [due to (3)]:

$$= \frac{2}{r^2} (\Delta_1 \cos \alpha_1 + \Delta_2 \cos \alpha_2 + \ldots),$$

or:

(4) $$V = \frac{2}{r^2} \Pi.$$

(3) and (4) yield the theorem:

The forces Q_1, Q_2, …, and their resultant V are proportional to the areas of the pyramid faces that are perpendicular to them.

Since $\mathfrak{q}_1$, $\mathfrak{q}_2$, … have corresponding senses as sides of the axis polygon, the forces $\bar{Q}_1$, $\bar{Q}_2$, …, resp., must also be directed in the same senses as the surfaces Δ_1, Δ_2, …, resp.; that is: Either all of them point from the inside of the pyramid to outside or all of them point from outside to inside, while the resultant V is directed in the opposite sense to the base surface Π. For the case in which the base edge, and accordingly also the side faces of the pyramid, intersect themselves, the area of the base surface must be determined in the Möbius sense. If one then counts the base surface as having the positive sense under traversing its perimeter for any base edge with an external side and an inner side, then it must also be determined what one means by external side and inner side of the side face that has the base edges in question, and accordingly also which direction of the arrow that is perpendicular to the side face points from inside to outside or from outside to inside.

We shall now go on to the *planar* pencil of forces q_1 q_2 …, which is defined by the orthogonal parallel projection of the sheaf Q_1Q_2…, resp. The lines of action of q_1, q_2, … are (from § 3) perpendicular to $\mathfrak{q}_1$, $\mathfrak{q}_2$, …, resp. Their magnitudes are equal to $Q_1 \sin \alpha_1$, $Q_2 \sin \alpha_2$, …, resp. If one then observes that in the right triangle that is defined by $\mathfrak{e}_i$ and h one has $\sin \alpha_i = h / \mathfrak{e}_i$ then one will get:

$$q_i = Q_i \cdot \sin \alpha_i ,$$

or [due to (3)]:

$$= \frac{2}{r^2} \Delta_i \cdot \frac{h}{\mathfrak{e}_i} ,$$

or [due to (3)]:

$$= \frac{\mathfrak{q}_i \mathfrak{e}_i}{r^2} \cdot \frac{h}{\mathfrak{e}_i} ,$$

or:

(5) $$q_i = \frac{h}{r^2} \mathfrak{q}_i .$$

That is: The forces q_1, q_2, … are *proportional to the segments* $\mathfrak{q}_1$, $\mathfrak{q}_2$, …, *resp. that are perpendicular to them*. The *n*-angle $\mathfrak{q}_1$ $\mathfrak{q}_2$ … can thus be regarded as the force polygon for the plane pencil of forces.

If we add the force *V* with *the opposite arrow direction* to the sheaf of forces Q_1, *Q*, … then we will obtain a sheaf that is found to be in equilibrium, relative to which, we can formulate the total result of the foregoing argument as follows:

If the lines of action of the n + 1 *forces of a sheaf of forces are individually perpendicular to the n* + 1 *faces of an n-sided pyramid with arrow directions that either all point from inside to outside or all point from outside to inside, and if the magnitudes of the forces are proportional to the areas of the faces that they are perpendicular to then the sheaf of forces will be in equilibrium.*

If one projects the sheaf orthogonal to the base surface of the pyramid then the plane pencil of forces that the projection produces will likewise be in equilibrium, and the magnitudes of its n forces will be proportional to the n base edges of the pyramid that are perpendicular to them.

We shall refer to this theorem (for whose converse, the same thing will be true that was stated above for the polygon of instantaneous axes) as the *force pyramid theorem* (*).

Finally, let the following be recalled in relation to Fig. 3: The polygon $\mathfrak{q}_1$ $\mathfrak{q}_2$ … represents the Neumann projection of the spatial sheaf of forces *in its original position*, as well as in its parallel-displaced one. However, only the sheaf $\bar{Q}_1 \bar{Q}_2 \ldots$ that was parallel displaced to *P* will correspond to it as its direct polar figure. If we were to construct the polar figure to the sheaf of forces in its original position Q_1 Q_2 … then it would have a different form and position $\mathfrak{Q}_1$ $\mathfrak{Q}_2$ … The individual polars, however,

(*) The first part of our theorem can be easily generalized to an arbitrary polyhedron, as one deduces with no difficulty when one decomposes the polyhedron into pyramids from a point *O*, or even more simply in a hydrodynamic way when one thinks of the polyhedron as being immersed in a fluid of constant pressure. (Cf., Poisson, *Traité de Mécanique*, 2nd. ed., t. V, nos. 577 and 580. Furthermore: *Maxwell*, in the third paper that was cited in the beginning of this article, pp. 23.)

would lie in the same projecting planes as $\mathfrak{q}_1$, $\mathfrak{q}_2$, … Therefore, the polygon $\mathfrak{q}_1$ $\mathfrak{q}_2$ … will represent the *projection of the polar figure* $\mathfrak{Q}_1$ $\mathfrak{Q}_2$ … relative to *O* as its center of projection.

§ 6. The rod framework.

We now consider a *statically-determinate, planar, rod framework* (cf., Fig. 2.a) whose *k* nodes are denoted by I, II, …, and whose $2k - 3$ rods are denoted by 1, 2, … The external forces p_1, p_2, … might act upon the nodes; let the stresses in the rods be s_1, s_2, …

We fix the rod framework as the orthogonal parallel projection of a corresponding spatial rod framework at whose nodes the external forces P_1, P_2, … act and in whose rods the stress forces S_1, S_2, … act. Let p_1, p_2, … and s_1, s_2, … be the projection of P_1, P_2, … and S_1, S_2, …, resp.

The spatial rod framework represents a combination of *k* sheaves of forces, each of which possesses a resultant that is perpendicular to the plane of projection $\mathfrak{P}$. We perform precisely the same operation on these *k* sheaves of forces that we did in the previous paragraph with the sheaf Q_1, Q_2, … We then construct the Neumann projections for all of the sheaves relative to the center of projection *O* and the plane of projection $\mathfrak{P}$. We then obtain a figure of projection (see Fig. 2.b) that likewise exhibits the central projection of the polar figure of the spatial rod framework relative to the sphere *O* and which possesses the following properties:

Any line p_1, p_2, …, s_1, s_2, … in the planar rod framework corresponds to a well-defined line $\mathfrak{q}_1$, $\mathfrak{q}_2$, … , $\mathfrak{s}_1$, $\mathfrak{s}_2$, …, resp., in the Neumann projection figure that is perpendicular to it. (In Figures 2.a and 2.b, the lines s_1, s_2, … and $\mathfrak{s}_1$, $\mathfrak{s}_2$, … are denoted by just the relevant numerals 1, 2, …) All lines that lie in the same planar surface in the spatial rod framework correspond to lines in the projection figure that meet at a point – viz., the associated *surface location*. (Such associated surfaces and surface locations are denoted in Figs. 2.a and 2.b with Latin and German symbols for the same letters *a*, *b*, *c*, … and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, …) All lines that meet at a node in the rod framework correspond to lines in the projection figure that define a closed polygon.

Now, according to § 5, each of these latter polygons can be regarded as the direct polar figure of the corresponding sheaf of forces in the spatial rod framework that has been parallel-displaced to *P*, and as a result, *as force polygons for the planar sheaves of forces*, which represent the orthogonal projection of that sheaf. The condition that was posed at the beginning of § 5 – that the forces that come together at each node of the planar rod framework must be in equilibrium – is fulfilled. Since the proportionality coefficient h / r^2 in equation (5) of § 5 is independent of the position of the node in question, all of the force polygons refer to the same unit of force. They can then collectively be regarded as the force framework that belongs to the planar rod framework and must be regarded as such as long as the rod framework is statically-determinate.

This point then demands a more detailed examination.

Each stress s_i belongs to two pencils of forces at once. It must then correspond to the same segment $\mathfrak{s}_i$ in the polygons of the force framework that belong to these pencils; that is: The two polygons must have the side $\mathfrak{s}_i$ in common. In order for this to be the case, the spatial rod framework, as whose projection the planar rod framework is regarded, must satisfy certain conditions that we have still not encountered up to now. We consider matter more closely for the stress in rod 1! (Cf., Fig. 2.a, b, and c.)

The stress in rod 1 belongs to the two pencils of forces at the nodes I and II; for I, let it be denoted by s_1 and for II, let it be denoted by s_1'. The node I corresponds to the triangle $\mathfrak{abg}$ in the force framework (see Fig. 2.b and c), the force s_1 corresponds to the segment $\mathfrak{bg} = \mathfrak{s}_1$. If one now goes over to node II then the segment $\mathfrak{s}_1'$ that belongs to the force s_1' in the polygon that corresponds to this node will fall, *eo ipso*, on the line $\mathfrak{bg}$. One endpoint will also coincide with $\mathfrak{g}$. (Then, since rod 3 in the spatial rod framework must lie in the same plane with 1 and 2, the Neumann projection of 3 must go through the point of intersection of the projections of 1 and 2.) By contrast, the other endpoint must not necessarily coincide with $\mathfrak{b}$ – at least, under the assumptions that we have made up to now for the spatial rod framework; moreover, perhaps the tetrangle $\mathfrak{gb}'\mathfrak{c}'\mathfrak{h}'$ (Fig. 2.c), where $\mathfrak{gb}' = \mathfrak{s}_1'$, would be able to appear as the polygon for the node II. Should both s_1' and s_1 yield the same value $\mathfrak{s}_1' = \mathfrak{s}_1$ then $\mathfrak{b}'$ would have to coincide with $\mathfrak{b}$, or: $\mathfrak{p}_2$ must go through the point of intersection of the lines $\mathfrak{p}_1$ and 1. This is, however, possible only when the *line of action of* P_2 *lies in the plane of* P_1 *and the rod* 1 in the spatial rod framework.

If one consider the remaining nodes in the same way then that will generally yield the condition that the individual rods and lines of action of the external forces in the spatial rod framework must enclose nothing but planar surfaces. We can formulate the condition for the external forces briefly as: *The lines of action of any two successive forces must lie in a plane such that the lines of action define a closed, spatial polygon, in general* (individual vertices of which can indeed lie at infinity, moreover).

According to the polar relationship, it follows from this immediately that the external forces in the force framework must also appear to form a closed polygon one after the other.

If external forces do not act at all of the nodes then the band rods (*Gurtungsstäbe*) with two external forces that are associated with the free nodes in the spatial rod framework, between which they lie, must lie in a plane. (If no external forces were applied to, e.g., nodes V and VI in Fig. 2.a then the rods 2, 6, 9 would have to lie in a plane with P_1 and P_4.)

In the example of Fig. 2.a, the nodes are found only on the bands. Therefore, a node can very well also fall inside of the rod framework. Now, no external force can act upon it, since that would contradict the condition above.

The spatial rod framework, as whose orthogonal projection a planar rod framework is regarded, is then to regarded in such a way that even more planar surfaces can be attached to the band rods whose external boundary lines will be defined by the lines of action of any two external forces, such that *the total figure will be presented as a simply-*

connected (possibly extended to infinity), polyhedral, surface piece whose boundary is defined by the lines of action of the external forces.

If we subject the spatial rod framework to the stated conditions then its Neumann projection will, moreover, satisfy *all of the requirements that are placed upon the force framework.*

Therefore, the empirically-presented theorems in § 4 are proved, as long as the term *polyhedral structure* that was used there is understood in the sense of the foregoing discussion.

§ 7. Special case of the funicular polygon.

One can consider the lines of the planar rod framework to be the lines of action of a distributed system of forces in equilibrium that acts in the plane. When any two equal and opposite stresses have been removed, the external forces will then represent a system in equilibrium, in their own right.

Conversely, the equilibrium of a planar system of forces can always be easily verified with the help of an arbitrary rod framework that is woven into the forces and its constructed force framework. As a result, the relevant construction can be simplified essentially by specializing the rod framework.

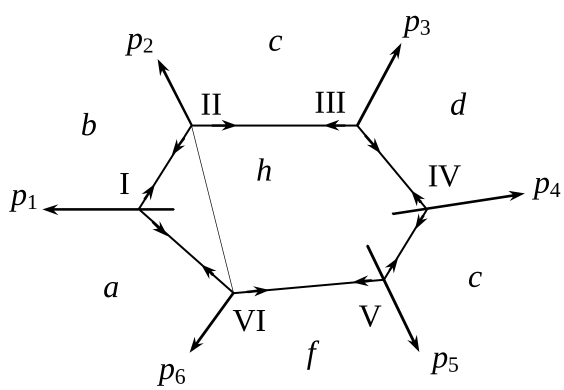

Figure 4.a

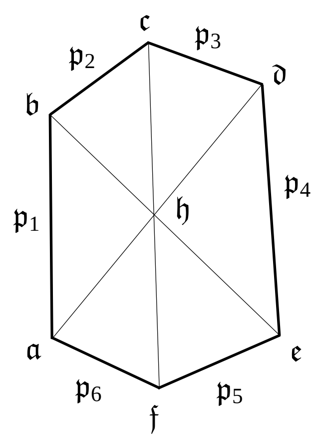

Figure 4.b

In Fig. 3.a, the planar rod framework is so arranged that the rods define nothing but triangles, so the rod framework is a *rigid* one. However, this is not necessarily required. If we imagine, e.g., that the two triangles *h* and *i* lie in the same plane in the *spatial* rod framework then the Neumann projections of the four sides 3, 4, 7, 6 of the planar tetrangle $h + i$, and therefore the four lines 3, 4, 7, 6 of the force framework (Fig. 2.b) must intersect in the same point (viz., the surface position of the tetrangle); this has the consequence that the line 5, which represents the stress in rod 5, would be zero. This rod can thus be omitted as completely irrelevant. The remaining inner rods can be likewise eliminated: Namely, if all of the five of the rods in the spatial rod framework (Fig. 2.a) that define a triangle fall in a plane then the six lines 1, 2, 3, 6, 8, 9 in Fig. 2.b that correspond to the six bands must intersect at a point, namely, the surface position. The stresses in the inner rods 3, 5, 7 will be zero, so those rods can be omitted. This case is illustrated by Figs. 4.a. and 4.b. (The system of external forces, as well as the triangle II

III VI, agree with Fig. 2.a, while the three remaining triangles are altered in such a way that one lets the nodes I, IV, V move along the lines of action of their forces until the triangle in the spatial rod framework coincides with the triangle in the plane of the triangle II III VI. The position of the surface location 𝔥 in Fig. 4.b is, accordingly, identical with the point 𝔥 in Fig. 2.b.)

One recognizes that the planar rod framework now represents nothing but a *funicular polygon* that is woven into the system of forces in such a way that its sides are perpendicular to the rays that are drawn from the *pole* 𝔥 to the vertices of the force polygon. *A funicular polygon that is woven into a planar force system can therefore always be regarded as the orthogonal projection of a planar section of the spatial force system whose projection figures as the planar force system.*

This immediately yields the theorem that when a planar force system is woven into two funicular polygons, the points of intersection of any two corresponding funicular polygon sides will lie along a straight line that is to be regarded as the projection of the line of intersection of the two planes of the funicular polygons, and will accordingly be perpendicular to the connecting line of two associated surface locations in the force framework.

With that, our considerations have arrived at the point where they encounter the analogous investigations of *Cremona*.

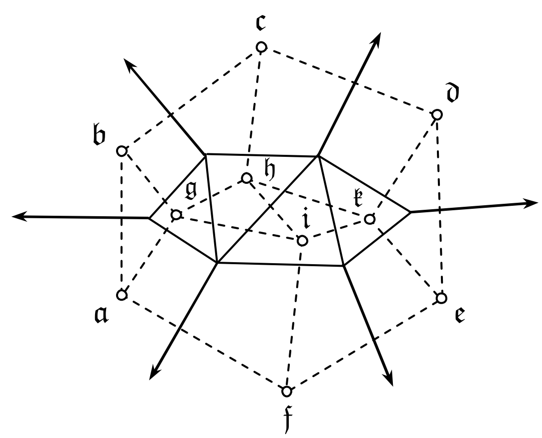

Figure 5.

One can scarcely draw the conclusion from the theory that was just presented that it is recommended in practical constructions for one to draw the lines of the force framework as being no longer *parallel*, but perpendicular, to the corresponding lines of the rod framework. Indeed, for the perpendicular position, the detailed relationships between the rod framework and the force framework emerge quite clearly, as a rule. For example, the sketch of the general configuration of the force framework is then also produced most simply from the perpendicular position when one sketches a relevant frame figure into the rod framework directly, such that surface location, where possible, falls inside of all the faces of the rod framework in question, while its connecting lines intersect the relevant separation lines of the faces perpendicularly, as is illustrated by Fig. 5 (or also Fig. 1) in more detail. From the purely theoretical standpoint, the perpendicular position also involves all of the same calculations as the parallel position, since the former corresponds to the theory of the composition of rotational moments, while the latter corresponds to the theory of the composition of isolated forces. Nevertheless, for practical constructions, one would do well to keep to the tried and tested schema that *Cremona* introduced in his splendid paper.

§ 8. The forces on the spatial rod framework. Rankine's theorem.

It still remains for us to subject the forces that act upon the *spatial rod framework* to a brief consideration.

It follows immediately from the discussion in § 5 (viz., *the force pyramid theorem*) that the external forces P_1, P_2, … that act upon the nodes of the *spatial* rod framework and the stresses S_1, S_2, … that act inside of its rods can be represented by the *areas of the triangles (pyramid side faces) that project to the corresponding lines of the force framework (Fig. 2.b) from the point O.*

The forces are next found to be in equilibrium in the spatial rod framework. However, equilibrium can be exhibited in such a way that one adds a force V_i at each node whose line of action is perpendicular to the plane of projection and whose magnitude is equal to the area of the polygon in the force framework (viz., the pyramid base face) that corresponds to the node.

If one combines the two external forces P_i and V_i – which act upon each node, moreover –into a resultant R_i then the totality of all forces R_i will define a *general* spatial system in equilibrium (as long as no two successive forces intersect, in general). However, since the projections of the forces R_1, R_2, … are identical with the projection p_1, p_2, … of the forces P_1, P_2, …, the planar equilibrium system that acts upon the planar rod framework will be represented as the projection of a *general* equilibrium system that acts upon the spatial rod framework.

We are then confronted with the direct problem of calculating an originally given *spatial* rod framework whose external forces R_1, R_2, … define a complete, general, spatial, equilibrium system, while in the rest of the rod framework the conditions of § 6 might prevail, so that can happen, conversely, in the following way:

One first decomposes every external force R_i into two components, one of which V_i is perpendicular to an arbitrarily-chosen plane of projection $\mathfrak{P}$, while the other one P_i intersects the component P_k of the next force R_k in the sequence. To that end, one lays a plane through each external force R_i that is perpendicular to the plane $\mathfrak{P}$, labels the edges of intersection of any two successive planes, and draws a spatial polygon whose edges lie along the edges of intersection, and whose sides go through the point of application of the external forces. One then decomposes each external force R_i into two components, – V_i which is perpendicular to the plane of projection, and P_i, which lies along the polygonal side that goes through its point of application. Therefore, the same relationships are now exhibited that were assumed at the beginning of this paragraph. The calculations can thus proceed in the manner described above, moreover, when one exhibits the Neumann projection of the spatial rod framework from an arbitrary center of projection O onto the plane $\mathfrak{P}$, and then obtains the stresses S_1, S_2, … in the individual rods as the areas of the relevant projected triangles.

Our *force pyramid theorem* can also be applied to spatial rod frameworks whose arrangement does not satisfy the conditions of § 6 – e.g., rod frameworks whose rods define the edges of a *closed* polyhedron. For example, it easily yields the proof of *Rankine's theorem*, as well as the determination of its converse (which has not be noticed, up to now, to my knowledge):

Namely, if one is given a spatial rod framework whose rods define the edges of a closed polyhedron of arbitrary shape, and at all of whose nodes external forces act, then one would have to apply the force pyramid theorem to the sheaf of forces at the individual nodes by constructing a pyramid at each node whose vertex always lies at the same point O, whose side faces are perpendicular to the associated rods, and whose base face is perpendicular to the external force that acts at the node, and indeed in such a way that two such pyramids that correspond to two nodes that are connected by a rod will always have side faces that are perpendicular to that rod in common, such that the associated two base faces will then lie together with a common base edge. All of the base faces will then be connected by their edges, and as a result, they will collectively define the outer surface of a polyhedron, *just as Rankine's theorem states.*

However, one easily recognizes that this can be done only when the external forces satisfy certain conditions. Three faces would then go through the edge that is common to two base faces, namely, the two base faces and the side face that is common to the two pyramids. As a result, the three lines that are perpendicular to these faces – namely, the two external forces and the rod that connects their nodes – *must lie in a plane* that is perpendicular to the common base edge. This then gives:

A Rankine force polyhedron is constructible only when any two external forces whose nodes are connected by a rod lie in a plane.

In the special case in which the rods of the framework define nothing but triangles, this condition is specialized by the fact that the external forces must all go through a point.

The difficulty that is rooted in this restriction would be bypassed by us for *open* spatial rod frameworks by the decomposition of the external forces R_i into the components V_i and P_i . This sheds light upon the fact that our Fig. 2.b, with its projecting pyramids, as were considered in the beginning of this paragraph, *represents nothing but a Rankine polyhedral structure*, except that the force polyhedron is not closed, but possesses an open boundary in the polygon $\mathfrak{abcdef}$, and its faces all lie in the same plane.

With that, the theory of reciprocal figures in graphical statics is related to Rankine's theorem in the desired way.

§ 9. Practical example. (Cf., Fig. 6).

The manner by which the spatial rod framework, whose projection is regarded as a planar rod framework, can be constructed in reality, and by which one can make use of the special properties of the associated force framework, might ultimately be illustrated by a simple practical example. It involves an *English roof truss* (cf., Fig. 6, below), to whose upper nodes I, II, …, IX a system of parallel forces are applied in a perpendicular direction. The forces p_2, p_3, …, p_8, which represent a uniform load, are equally large. Each of the two support reactions p_1 and p_9 is equal to one-half the sum of p_2, p_3, …, p_8 .

We carry out the construction in the ground plan and the elevation, where the ground plane will serve as the plane of projection that was denoted by $\mathfrak{P}$, from now on. Let the axis of projection be A.

The given rod framework N will be considered to be the ground plan of a spatial rod framework whose external forces lie in planes that are perpendicular to the axis of projection A and which are thought of as decomposed into two components: $-V_i$, which is perpendicular to the ground plane and P_i , which is parallel to the ground plane. The components P_i project to their true sizes in the ground plane N as p_i, while they project to points in the elevation N'. The components V_i no longer come under consideration for the calculation of the *planar* rod framework, and are thus no longer indicated in Fig. 6.

We next think of the elevation N' as being known. The force framework $\mathfrak{N}$ that belongs to N is then obtained as the Neumann projection of the spatial rod framework (N, N') onto the ground plane. We choose the center of projection O for the Neumann projection onto the elevation plane. In order to find the Neumann projection of any line then – e.g., the rod (1, 1′) – we lay a plane through O that is perpendicular to (1, 1′). Its traces must be perpendicular to the projections 1 and 1′, respectively, and the elevation trace must go through O. If one then draws $O\mathfrak{a}_1$ perpendicular to 1′ up to the axis of projection A and draws a perpendicular to 1 through $\mathfrak{a}_1$ then that will represent the Neumann projection of (1, 1′); it is likewise denoted by 1. One does the same thing with all of the lines of rod framework. The Neumann projections $\mathfrak{p}_1$, $\mathfrak{p}_2$, … of the external forces P_1, P_2, … will then fall along the axis of projection if their reciprocal projecting planes coincide with the elevation plane. In order to get the projections of the rods 5, 9, 13, 17, 21 that are perpendicular to the axis of projection, one can employ a side view plane, which is, however, not indicated specially in Fig. 6.

The three figures N, N', $\mathfrak{N}$ now represent *three inequivalent trilinear figures*, in the sense of § 2. The third of them is always determined completely by two of them. As we just saw, $\mathfrak{N}$ can be ascertained from N and N', or also conversely, N', from N and $\mathfrak{N}$. However, if one recalls the properties of the spatial rod framework (see § 6) then the latter construction will allow one to know only a part of $\mathfrak{N}$, and indeed *just the external force polygon*. Thus, if only N is given originally then one proceed as follows: One first draws only the external force polygon of the figure $\mathfrak{N}$, then constructs the line N' that corresponds to the two existing lines of N and $\mathfrak{N}$, finishes N' with their help, and finally determines the still-missing lines of $\mathfrak{N}$ from the ones that correspond to N and N'.

In our special example, one would accordingly begin by ordering the given external forces of the force polygon $\mathfrak{p}_1\mathfrak{p}_2\ldots\ \mathfrak{p}_9$ along the axis of projection. If one then draws rays from O to its vertices and (once the point I′ is established arbitrarily along the altitude of projection through I) draws I′ II′ perpendicular to the next ray $O\mathfrak{a}_1$ up to the altitude of projection through II, then draws II′ III′ perpendicular to the next ray $O\mathfrak{a}_4$ up to the altitude of projection III, etc., then one will obtain the individual nodes that are connected by rods in the same way as the ground plan. One thus remarks that, from the discussion in § 6, the bands 2, 6, 10, 16, 20, 24 of the spatial rod framework, at whose intermediate nodes no external forces act, must lie in the same plane with P_1 and P_9, which has the consequence that these rods project onto a straight line in the elevation plane. If N' is obtained in that way then the construction of $\mathfrak{N}$ will ultimately result in the way that was discussed above (*).

The following remarks about that might find a place here:

Since the bands of N' are perpendicular to the rays $O\mathfrak{a}_1$, $O\mathfrak{a}_2$, … that are drawn from O to the vertices of the force polygon $\mathfrak{p}_1$ $\mathfrak{p}_2$… $\mathfrak{p}_9$, resp., and since the nodes of N' lie along the lines of action of p_1, p_2, …, the belt (*Gurtung*) can be regarded as a funicular polygon (for the point O as its pole) that is woven into the external forces p_1, p_2, … This yields that: If the fixed external forces P_1, P_2, … are perpendicular to the elevation plane then *the belt of the rod framework in the elevation N' will always be identical to a funicular polygon that is woven into the forces p_1, p_2, …, which is a form that the given rod framework might also take.* In our special case of a uniform load, the rod framework accordingly projects onto the elevation as *the carrier of a parabola.*

If the external forces p_1, p_2, … are not mutually parallel – as in our example – but define a *general*, planar, equilibrium system then one would proceed to ascertain the elevation figure N' as follows: One would once more begin by drawing the external force polygon $\mathfrak{p}_1$ $\mathfrak{p}_2$ … in the ground plane, make its individual sides intersect the axis of projection, and draw rays from O to the points of intersection. If one then determines the point of intersection of the lines of action of any two successive forces p_i, p_k, draws the projection altitude through that point of intersection that is perpendicular to A, and draws perpendiculars to the corresponding rays of the pencil O from one altitude of projection to another (with an arbitrary choice of starting point) then one will get the lines of action of the external forces in the elevation, onto which, one finally projection the nodes of the rod framework from the ground plan upwards.

This likewise yields the general remark that when the planar rod framework is given (including the external forces that act upon its nodes), the form of the associated spatial rod framework will depend upon just the position of the center of projection O. *The form of the spatial rod framework is determined completely from the form of the planar rod framework and the position of the center of projection O.*

Finally, let the following remark be made in regard to the *special properties of the force framework* in the example that was treated above:

The vertices $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{v}$ (cf., Fig. 6) in the force framework $\mathfrak{N}$ lie along a straight line, and indeed, along the extension of line 3. Moreover, the line 4 goes through the vertex $\mathfrak{w}$. This is required by the facts that in the spatial rod framework the faces (t, t'), (u, u'), (v, v') that correspond to the stated vertices are parallel to the rod (3, 3′), and that the face (w, w') is parallel to the rod (4, 4′). It is logical (cf., § 2, Theorem 4) that the associated surface location in the Neumann projection must lie on the relevant zone line. The first of the aforementioned parallelisms is immediately obvious. The other three are proved

(*) The compressions are indicated by double lines in figure $\mathfrak{N}$, and likewise for the tension rods in figures N and N'.

by the fact that the sides 13 and 13′ of the triangles *u* and *u*′, resp., will be cut by the lines that are drawn through opposite vertices IV and IV′, resp., parallel to 3 and 3′, resp., in the same ratio 1 : 1. Likewise, the parallels to the sides 9 and 9′ that are drawn in the triangles *v* and *v*′ from III and III′, resp., to 3 and 3′, resp., will both intersect in the ratio 1 : 2. Finally, the sides 9 and 9′ of the triangles *w* and *w*′, resp., will both be cut by the lines that are drawn from the opposite vertices parallel to 4 and 4′, resp., in the ratio – 1 : 4. In the elevation, in order to prove this, one remarks that the ordinates of the points IV′, III′, II′, I′ of the parabola on the horizontal that is drawn through V′ (viz., the vertex tangent) as the abscissa axis relate like 1 : 4 : 9 : 16.

Berlin, April 1886.

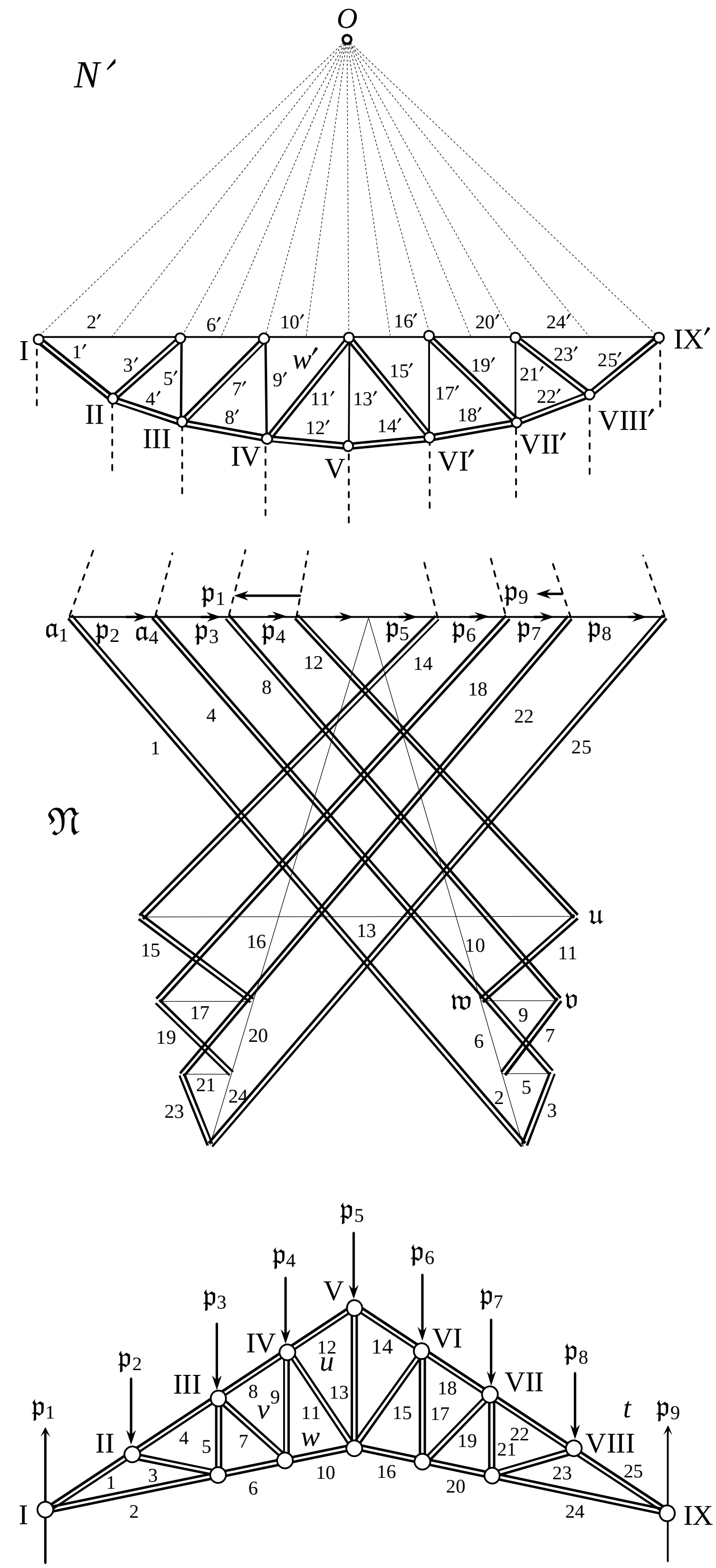
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Figure 6.