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Steps towards the material law for an elastic medium with moment-stresses

F. HEHL and E KRÖNER

Institut für Physik der Bergsakademie Clausthal – Technische Hochschule

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Abstract: If not only a force, but also a couple acts upon an area element in a continuum then we must introduce the so-called couple-stresses, in addition to the force-stresses. In this article, we shall emphasize the importance of couple-stresses in dislocated solids. – § 2 gives a short review of the present state of the theory of couple-stresses. In classical elasticity, couple-stresses are to be interpreted as a non-local effect that is intimately connected with the range of the atomic forces. The couple-stresses are of a higher order in that range than force-stresses and can therefore be usually neglected.

However, in the field theory of dislocations, couple-stresses are generally of the same order of magnitude as force-stresses. Hence, they can cause considerable effects. In § 3, we shall determine the macroscopically-observable couple-stresses of homogeneously-distributed screw and edge dislocations by averaging over their microscopically-fluctuating stress fields. In § 4, we use the PEIERLS model to show that the core of a dislocation produces an asymmetric state of stress, and for that reason, couple-stresses, as well, which are negligibly small under certain circumstances. In § 5, by introducing a polycrystalline model, we derive the constitutive relations for couple-stresses and dislocation density in an isotropic form. The results are discussed in § 6.

§1. Introduction. – At a sufficiently high temperature, a *plastically*-bent beam that is composed of crystalline material will possess surplus edge dislocations of one sign that are distributed in a macroscopically-homogeneous manner (Fig. 1).

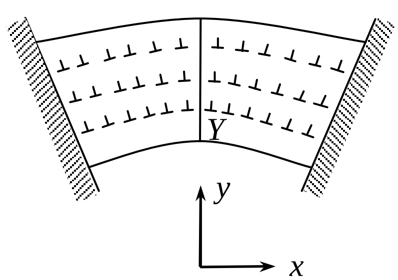

Figure 1. Macroscopically-homogeneous distribution of edge dislocations in a plastically-bent beam (schematic).

We can think of a dislocation state of this kind as being produced as follows: Take rectangular plates of density Y whose edges are parallel to the coordinate axes (Fig. 2a). If we let moments M_3 of well-defined magnitude act upon planes that are perpendicular to the x-axis (Fig. 2b) then the plates will take on an *elastic* bending curvature K_{13} according to:

$$M_3 = \frac{EY^3Z}{12} K_{13}\,. \tag{1.1}$$

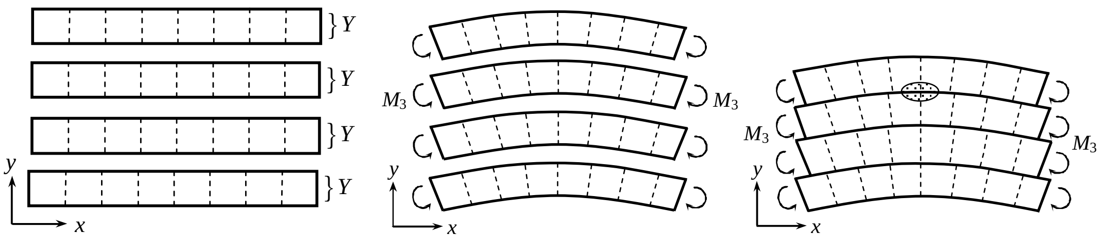

Figure 2.a
Rectangular plates of density Y.

Figure 2b.
Elastic bending of the plates.

Figure 2c.
Welding the plates together.

In this, E is the elastic modulus, and Z is the thickness of the plate in the z-direction. If we weld the plates together (Fig. 2c) then their bending states (1.1) will remain the same, since no forces acts upon the outer surfaces that are perpendicular to the y-axis.

Now, if the plates have a *crystalline* structure (for example, primitive cubic with BRAVAIS vectors that are parallel to the coordinate axes) then a jump in the lattice constants will appear in the y-direction at the weld locations, and it will produce an interaction between the plates. Since cohesive forces are short-range (at least, in metals), only forces that point in the x-direction will act approximately from the atoms of the lowest net-plane of one plate to the upper atomic plane of the neighboring plate, and conversely (Fig. 3a), which is an effect that can produce rotational moments M_3^p that act upon the x-surfaces, in addition. On energetic grounds, the state that is depicted in Fig. 3a is then known to be unstable, and it will lead to the formation of individual edge dislocations (Fig. 3b), but that will not change anything qualitatively in the state of affairs that was just described.

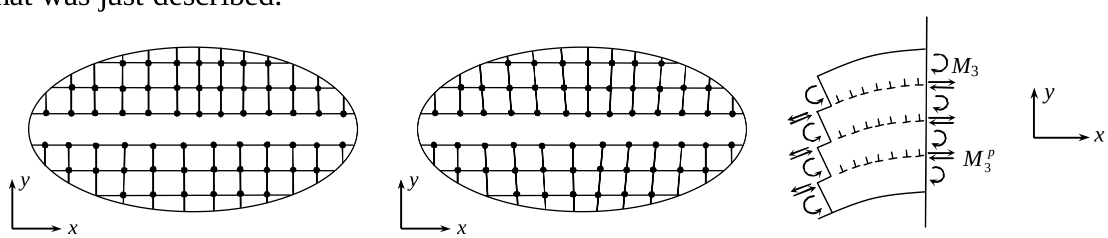

Figure 3a.
Unstable arrangement for a unicrystalline structure of the plates

Figure 3b.
Stable arrangement after formation of edge dislocations.

Figure 3c.
Measuring the moment-stresses in a test section

Therefore, our Gedanken experiment has produced a distribution of dislocations as in Fig. 1. We see two things, above all: The *geometric* state in Fig. 1 can be described macroscopically by a curvature K_{ij} [namely, the curvature of the plate in Fig. 2b (2c, resp.)], which can be expressed in terms of the dislocation density α_{ij} as follows, according to NYE ([1]) ([*]):

([1]) J. F. NYE, Acta Met. **1** (1953), 153.

$$K_{ij} = \tfrac{1}{2}\,\delta_{ij}\,\alpha_{kk} - \alpha_{ji}\,. \tag{1.2}$$

Only the component K_{13} of this is non-zero in Fig. 1.

The *static* state will be understood macroscopically in a corresponding way by means of moment-stresses τ_{ij} ([2,3]) that will, as a test section will show (Fig. 3c), result in two types of components: One of them comes from the moment M_3 that is necessary to bend the plate, and another one that comes from the additional moment M_3^p that appears at the location of the weld. M_3 can be ascertained by a calculation from the theory of elasticity, while M_3^p can be understood, at least qualitatively and up to its sign, by means of the half-lattice-theoretic dislocation model of PEIERLS ([4]).

If ΔF_i is an arbitrarily-oriented macroscopic surface element upon which a rotational moment Δt_j acts then the *moment-stresses* will be defined by:

$$\Delta t_j = \tau_{ij}\,\Delta F_i\,. \tag{1.3}$$

Hence, only the component:

$$\tau_{13} = (M_3 + M_3^p)\,/\,YZ \tag{1.4}$$

will be non-zero in Fig. 1. Since analogous statements are true for screw displacements, one can state the following: If dislocations are distributed homogeneously in a crystalline solid body then the material will respond with moment-stresses, when viewed macroscopically. This paper shall deal with the study of the corresponding *material law.*

In § **2**, we shall be concerned with a general overview of the realm of moment-stresses. In § **3**, the moment-stresses that are produced by macroscopically-homogeneous distributions of dislocations (Fig. 4, 5) shall be calculated by means of elasticity theory, and certain relationships to the two-dimensional COSSERAT continuum shall be revealed. The contribution from the center of the dislocation that is neglected in that analysis will be discussed in § **4** with the help of the PEIERLS model. In § **5**, we shall introduce a simple polycrystalline model and in that way obtain the material law that couples the dislocation density to the moment-stresses in isotropic form. In § **6**, we will discuss the results.

§ 2. Generalities in moment-stresses. – In last century, VOIGT ([5]) had already placed the concept of force-stresses σ_{ij} (which are usually referred to casually as "stresses") alongside the analogous concept of moment-stresses τ_{ij} [cf., (1.3)]. One will find the equilibrium conditions for forces and moments in a well-known way in the form of the fundamental equations of statics (f_j = volume force, c_j = volume moment):

([*]) We shall always calculate in Cartesian coordinates; doubled indices will always be summed over. δ_{ij} means the KRONECKER symbol, while ε_{ijk} means the totally-antisymmetric unit tensor of rank three.

([2]) E. KRÖNER, Arch. Rational Mech. Anal. **4** (1960), 273.

([3]) E. KRÖNER, Int. J. Eng. Sci. **1** (1963), 261.

([4]) R. E. PEIERLS, Proc. Phys. Soc. London **52** (1940), 34.

([5]) W. VOIGT, Abh. königl. Ges. Wiss. Göttingen (math. Kl.) **34** (1887), 3.

$$\frac{\partial \sigma_{ij}}{\partial x_i} \equiv \sigma_{ij,\,i} = -\,f_j\,, \tag{2.1}$$

$$\tau_{ij,\,i} + \varepsilon_{jkl}\;\sigma_{[kl]} = -\,c_j\,, \tag{2.2}$$

which were likewise given completely by VOIGT. One can also confer the *Handbuch* articles on that topic by HELLINGER ([6]) and HEUN ([7]), as well as TRUESDELL and TOUPIN ([8]). The square bracket in (2.2) means antisymmetrization:

$$\sigma_{[kl]} \equiv \tfrac{1}{2}\,(\sigma_{kl} - \sigma_{lk}), \tag{2.3}$$

while parentheses will be employed for symmetrization:

$$\sigma_{(kl)} \equiv \tfrac{1}{2}\,(\sigma_{kl} + \sigma_{lk}). \tag{2.4}$$

Generally, speaking, a material will react to stresses with changes to its geometrical structure. *Changes in distance* will be described by the strain tensor ε_{ij} , as one will see especially clearly from the definition:

$$ds^2 - ds_0^2 = 2\;\varepsilon_{ij}\;dx_i\;dx_j\,, \tag{2.5}$$

in which the ds_0 means the distance between two neighboring points of the continuum *before* the deformation, and *ds* means the distance between the same two points *afterwards.* It is known that the continuum will respond to that with symmetric force-stresses according to:

$$\sigma_{(ij)} = c_{ijkl}\;\varepsilon_{kl}\,. \tag{2.6}$$

c_{ijkl} is the (HOOKEAN) tensor of the elastic moduli in the linearized theory that is considered here.

The moment-stresses are now characterized according to the following argument: In the ordinary theory of elasticity, the strain can be derived from a displacement field u_j :

$$\varepsilon_{kl} = u_{(j,\,i)}\;. \tag{2.7}$$

Position-varying displacements then lead to a strain in the continuum, and therefore to force-stresses. In a similar way, position-varying *rotations* ω_j will give rise to a *curvature* k_{ij} :

$$k_{ij} = \omega_{j,\,i}\;. \tag{2.8}$$

The curvature, in its own right, will lead to a material law for the moment-stresses τ_{ij} :

([6]) E. HELLINGER, Enc. math. Wiss. IV, **4**, Art. 30, 1914, Teubner, Leipzig, 1907/14.
([7]) K. HEUN, Enc. math. Wiss. IV, **2**, Art. 11, 1914, Teubner, Leipzig, 1904/35.
([8]) C. TRUESDELL and R. TOUPIN, *Handbuch der Physik* (ed., FLÜGGE) III/1, 226, Springer-Verlag, Berlin 1960.

$$\tau_{ij} = a_{ijkl}\, k_{kl}\,, \tag{2.9}$$

which is linearized here. a_{ijkl} is a fourth-rank tensor of material constants that can be expressed in terms of three independent moduli a_1, a_2, a_3 according to:

$$a_{ijkl} = a_1\, \delta_{ij}\, \delta_{kl} + a_2\, \delta_{ik}\, \delta_{jl} + a_3\, \delta_{il}\, \delta_{jk} \tag{2.10}$$

in the isotropic case.

The various attempts at a theory of moment-stresses differ mainly by the way that they interpret the rotations ω_j that enter into (2.8)

§ 2.1 *Non-local moment-stresses in the classical theory of elasticity.* – The next idea to develop is probably the one that one adopts from the geometry of the usual theory of elasticity. The rotations ω_j are then derivable from the displacement field u_j :

$$\omega_j = \tfrac{1}{2}\, \varepsilon_{jkl}\, u_{[l,k]} = \tfrac{1}{2}\, \varepsilon_{jkl}\, u_{l,k}\,, \tag{2.11}$$

such that knowing the fundamental displacement field will also imply knowing the rotations. With (2.7), (2.8), and (2.11), we will then get the curvature as:

$$\kappa_{ij} = \omega_{j,\,i} = \tfrac{1}{2}\, \varepsilon_{jkl}\, u_{l,ki} = \tfrac{1}{2}\, \varepsilon_{jkl}\, \varepsilon_{il,\,k}\,, \tag{2.12}$$

from which, one sees that also κ_{ij} ultimately results from changes in distance, so it cannot be regarded as a fundamentally-new deformation quantity. The moment-stresses then come about as reactions to κ_{ij}, according to (2.9).

A theory of that kind was developed almost simultaneously by AERO and KUVSHINSKI ([9]) for infinitesimal deformations and by GRIOLI ([10]) for finite ones. Further work can be found in MINDLIN and TIERSTEN ([11]) (infinitesimal deformations, examples), TOUPIN ([12]) (finite deformations, propagation of waves), and KOITER ([13]) (boundary conditions, minimal principles). KRÖNER ([3]) has shown that the moment-stresses ([*]) that are treated here carry with them a certain *non-locality* as a correction to the otherwise-strictly-local classical theory of elasticity, and that will result in a very small, but still finite, *range* to the atomic forces of interaction.

VOIGT ([5]) has already proved that these moment-stresses can be neglected completely in comparison to the force-stresses. We quote him: "…the coefficients that are found in the expressions for the rotational moments are then to be regarded as infinitely-small in comparison to the ones that appear in the components X_x , …"

([9]) E. L. AERO and E. V. KUVSHINSKI, Soviet Phys. – Solid State **2** (1961), 1272.
([10]) G. GRIOLI, Ann. Mat. Pura Appl., ser. IV **50** (1960), 389.
([11]) R. D. MINDLIN and H. F. TIERSTEN, Arch. Rational Mech. Anal. **11** (1962), 415.
([12]) R. A. TOUPIN, Arch. Rational Mech. Anal. **11** (1962), 385.
([13]) W. T. KOITER, Proc. Kon. Ned. Akad. Wetenschap B **67** (1964), 17.
([*]) Generally, along with the rotational moments that are discussed here, other moments of the same rank will appear in ([3]) that give rise to an strained deformation.

VOIGT's components X_x , … correspond to our σ_{ij} in that statement. Furthermore, McCLINTOCK, et al ([14]) showed that:

$$| \partial \varepsilon_{ij} / \partial x_k | \geq | \varepsilon_{mn} | / l \qquad (\varepsilon_{mn} \neq 0) \tag{2.13}$$

is a *necessary condition* for the appearance of *moment-stresses* to become noticeable. l is interpreted as a linear dimension for a surface element in this, and for metals, it can have the order of magnitude of a few atomic distances:

$$l \approx 10 \text{ Å}. \tag{2.14}$$

Deformation gradients in the order of magnitude that is required in (2.13) and (2.14) appear in crystals at most at *singular locations* ([14]), and thus possibly along dislocation lines in the interior [McCLINTOCK ([15])] and at notches and similar things on the outer surface. However, since the strain field ε_{ij} of a singularity must drop off at least as fast as $1 / r$, it will follow from (2.13) that:

$$r \leq l; \tag{2.15}$$

i.e., moment-stresses are first noticeably present in domains in which the application of not only the linear theory of elasticity, but also the nonlinear theory, is normally no longer meaningful.

However, in connection with dynamical problems (which will not be treated in this paper), the condition (2.13) might possibly be applicable to *very short waves* in the domain of validity of elasticity theory. For this, we shall refer to a notice of KRUMHANSL ([16]).

In summary, one can then say that in the context of classical elastostatics, moment-stresses are, in essence, smaller by an order of magnitude than the force-stresses, and for that reason, they can be *neglected.* That is generally not the case in problems for which lengths appear that are comparable to l, such as perhaps the distance between two neighboring dislocations, and similar things.

§ 2.2. *Moment-stresses in an incompatible Cosserat continuum.* – Now one can, however, interpret the rotations in (2.8) in yet another way. Around the turn of the century, the Brothers COSSERAT ([17]) had already developed the theory of a continuum whose building blocks were not only displaced, but would also be rotated independently of each other. The deformation of such a COSSERAT *continuum*, as it is described in a modern way by ERICKSEN and TRUESDELL ([18]) and GÜNTHER ([19]), will then be represented by a displacement field u_j with its three functional degrees of freedom and a rotational field ϑ_j with just as many degrees of freedom. The COSSERAT curvature:

([14]) F. A. McCLINTOCK, P. A. ANDRE, K. R. SCHWERDT, and R. E. STOECKLY, Nature, London **182** (1958), 652.
([15]) F. A. McCLINTOCK, Acta Met. **8** (1960), 127.
([16]) J. A. KRUMHANSL, Solid State Comm. **1** (1963), 198.
([17]) E. COSSERAT and F. COSSERAT, *Théorie des Corps Déformables*, Hermann et Fils, Paris, 1909.
([18]) J. L. ERICKSEN and C. TRUESDELL, Arch. Rational Mech. Anal. **1** (1958), 295.
([19]) W. GÜNTHER, Abh. Braunschweig. Wiss. Ges. **10** (1958), 195.

$$K_{ij} = \vartheta_{j,\,i} \tag{2.16}$$

is clearly independent of the strain ε_{ij} here, and is therefore a truly new deformation quantity. Naturally, moment-stresses τ_{ij} appear once more as a reaction to the K_{ij} , and they generally make a contribution to the deformation energy that has the same order of magnitude as the force-stresses $\sigma_{(ij)}$, in contrast to the aforementioned moment-stresses.

The explicit form of the material law for a hypothetical COSSERAT continuum was examined by KOSTER ([20]), OSHIMA ([21]), SCHAEFER ([22]), COWIN ([23]), and KUVSHINSKI and AERO ([24]), as well as DJURITCH ([25]). Those considerations are of particular physical interest [and this has been known since GÜNTHER ([19])] since the field of dislocations works with a geometric model that must be regarded as a generalized – namely, *incompatible* – COSSERAT continuum. According to NYE ([1]), the structural curvatures K_{ij} that appear in them can be converted into the dislocation densities α_{ij} using eq. (1.2). One can confer the survey ([26]) for this, in which one can also find historical remarks.

The material law (2.9) of the field theory of dislocations, which is expressed in a concrete physical situation, and which allows one to calculate the moment-stresses from the structural curvatures was treated by KRÖNER ([3]): There, he provisionally determined the moment-stresses that are produced by distributions of edge dislocations. The present article represents continuation of that work.

To our knowledge, the three-dimensional COSSERAT continuum was employed only in *rheology*. In that domain, we refer to the papers of OSHIMA ([21]) ("grained" media), ERICKSEN ([27]) ("anisotropic" fluids), and to the survey of DAHLER and SCRIVEN ([28]) ("structured" continuum).

§ 3. Moment-stresses of macroscopically-homogeneous distributions of dislocations with no regard for the center of dislocation. – Fig. 4 shows a macroscopically-homogeneous distribution of screw dislocations. The dislocation lines are parallel to the z-axis and possess the coordinates $[(m+\frac{1}{2})\,X,\,(n+\frac{1}{2})\,Y]$, with $m, n = 0, \pm 1, \pm 2, \pm 3, \ldots$ The BURGERS vector, which points in the z-direction, has the magnitude b. The corresponding relationships for edge dislocations are represented in Fig. 5.

([20]) W. KOSTER, Dissertation Utrecht 1920.

([21]) N. OSHIMA, Proc. 3rd Japan Nat. Congr. Appl. Mech. (1953), 77; Memoirs of the Unifying Study of the Basic Problems in Engineering Sciences by Means of Geometry I, Gakujutstu Bunken Fukyu-Kai, Tokyo, 1955, pp. 563.

([22]) H. SCHAEFER, Miszellaneen Angew. Mech. (Tollmien-Festschrift) (1962), pp. 277.

([23]) S. C. COWIN, Diss. Abstr. **23** (1963), 3838.

([24]) R. V. KUVSHINSKI and E. L. AERO, Soviet Phys. – Solid State **5** (1963), 1892.

([25]) S. DJURITCH, Dissertation, Belgrade, 1964.

([26]) E. KRÖNER, in SOMMERFELD, *Vorlesung über Theoretische Physik*, 5th ed., v. 2, Chapter 9, Akad. Verlagses., Leipzig 1964.

([27]) J. L. ERICKSEN, Trans. Soc. Rheol. **4** (1960), 29; Kolloid-Zeit. **173** (1960), 117; Trans. Soc. Rheol. **6** (1962), 275.

([28]) J. S. DAHLER and L. E. SCRIVEN, Proc. Roy. Soc. London A**275** (1963), 504.

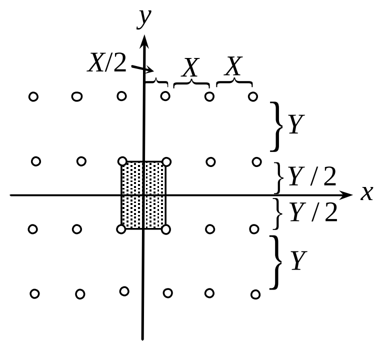

Figure 4.
Macroscopically-homogeneous distribution of screw dislocations

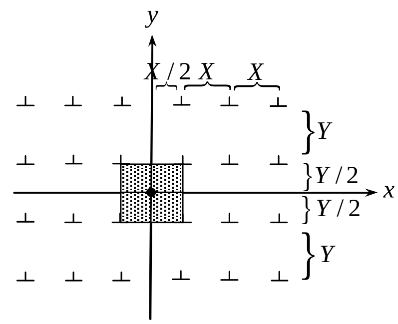

Figure 5.
Macroscopically-homogeneous distribution of edge dislocations (cf., Fig. 3c)

If we denote the stress field of a dislocation that runs along the *z*-axis by $\sigma_{ij}\,[x, y]$ then we will get the total stress field of the corresponding arrangement of dislocations from:

$$\sigma_{ij}^{\text{tot}} = \sum_{m,n=-\infty}^{+\infty} \sigma_{ij}[(x-(m+\tfrac{1}{2})X, y-(n+\tfrac{1}{2})Y]. \tag{3.1}$$

It is periodic in *X* and *Y*, which is why the macroscopic force-stresses $\bar{\sigma}_{ij}$, which are obtained by taking means over "large" surface elements ΔF_i, will vanish.

From now on, we will endow all macroscopic quantities that arise by taking means of microscopically-fluctuating ones with an overbar. On the one hand, the surface element over which one takes the mean must be pierced by many dislocations, but on the other hand, it must be small in comparison to the external dimensions of the probe. The error that one makes when one replaces ΔF_i with the differential dF_i will then become negligible.

One can largely account for the effect of *oscillating micro-stresses that vanish in the mean* σ_{ij}^{tot} by calculating the mean *moments* that the force-stresses σ_{ij}^{tot} exert upon a macroscopic cross-section. Those moments can be described by a third-rank tensor ([3]):

$$\bar{\sigma}_{ijk} = \frac{1}{\Delta V}\int_{\Delta V} dV\,\sigma_{ij}^{\text{tot}}\xi_k\,. \tag{3.2}$$

One infers from the Gedanken experiment in the introduction that the domain of integration for our homogeneous distributions of dislocations in Figs. 4 and 5 can be chosen as follows:

$$\left\{\begin{array}{c} -X/2 \le x_1 \le +X/2 \\ -Y/2 \le x_2 \le +Y/2 \\ -1/2 \le x_3 \le +1/2 \end{array}\right\}. \tag{3.3}$$

We can measure the position vector ξ_k from an arbitrary point (for example, the origin) since it can be chosen freely, due to the vanishing of the mean micro-stresses:

$$\bar{\sigma}_{ijk} = \frac{1}{\Delta V}\int_{\Delta V} dV\,\sigma_{ij}^{\mathrm{tot}}(\xi_k + c_k) = \frac{1}{\Delta V}\int_{\Delta V} dV\,\sigma_{ij}^{\mathrm{tot}}\xi_k\,. \tag{3.4}$$

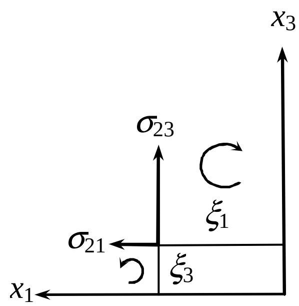

Figure 6. The derivation of eq. (3.8):
View on a plane x_2 = const.

In order to derive the third-rank tensor $\bar{\sigma}_{ijk}$ from a second-rank moment-stress tensor $\bar{\tau}_{ij}$, we consider Fig. 6. The force-stresses σ_{21} and σ_{32} with the lever arms ξ_3 (ξ_1, resp.) in a cross-section x_2 = const. are indicated in it. If we abstract from the special case that is represented in Fig. 6 then we can establish that the moments $\sigma_{ij}\ \xi_k$ and $\sigma_{ik}\ \xi_j$ that act upon the *i*-plane in the case of $j \neq k$ (when one considers them to be vectors) are both perpendicular to the *jk*-plane, and therefore possess opposite signs. Hence:

$$\bar{\sigma}_{ijk} - \bar{\sigma}_{ikj} \tag{3.5}$$

represents the moment-stress component that describes the moment in the *i*-plane that rotates around the direction that is perpendicular to the *jk*-plane; we would like to denote it by $\bar{\tau}_{ijk}$:

$$\bar{\tau}_{ijk} = \bar{\tau}_{i[jk]} = \bar{\sigma}_{ijk} - \bar{\sigma}_{ikj} = 2\bar{\sigma}_{i[jk]}\,. \tag{3.6}$$

If one goes from $\bar{\tau}_{ijk}$ to the tensor $\bar{\tau}_{ij}$ that is dual to it then one will get:

$$\bar{\tau}_{ij} = \tfrac{1}{2}\varepsilon_{jkl}\,\bar{\tau}_{ijk}\,. \tag{3.7}$$

Corresponding to (3.7) and (3.8), the moment-stress tensor $\bar{\tau}_{ij}$ then describes the rotational moment around the *j*-axis that acts upon the *i*-plane, in agreement with the definition (1.3).

Only $\bar{\sigma}_{i(jk)}$ enters into the expression (3.8) for $\bar{\tau}_{ij}$. However, we will see later in (3.16) and (3.35) that $\bar{\sigma}_{i(kl)}$ can be expressed in terms of the $\bar{\tau}_{ij}$, due to the vanishing of some $\bar{\sigma}_{ikl}$-components for the distribution of dislocations in Figs. 4 and 5. Therefore, $\bar{\tau}_{ij}$

contains the same physical information as $\bar{\sigma}_{ikl}$ in the case of dislocation theory. The deeper basis for that fact is that dislocations can be realized by curvatures according to (1.2). However, as gradients of rotations, curvatures will also induce *rotational* moments $\bar{\sigma}_{i[kl]}$, but not moments $\bar{\sigma}_{i(kl)}$, which are associated with stretching.

As we will see in § **4**, one must associate the center of any dislocation with an asymmetric stress tensor. We would like to ignore that fact in all of § **3** from now on; i.e., we will work with a symmetric tensor exclusively. If one then calculates the *trace* of the moment-stress tensor $\bar{\tau}_{ij}$ from (3.8) then since:

$$\bar{\sigma}_{[ik]l} = 0, \tag{3.9}$$

one will have:

$$\bar{\tau}_{ii} = \varepsilon_{ikl}\,\bar{\sigma}_{ikl} = \varepsilon_{ikl}\,\bar{\sigma}_{[ik]l} = 0. \tag{3.10}$$

§ 3.1. *Moment-stresses in Fig. 4 (screw dislocations).* – If one treats dislocations with the methods of linear elastostatics ([29]) then one will get the force-stress field:

$$\sigma_{13}\,[x, y] = \sigma_{31}\,[x, y] = +\frac{Gb}{2\pi}\frac{y}{x^2+y^2}, \tag{3.11}$$

$$\sigma_{23}\,[x, y] = \sigma_{32}\,[x, y] = -\frac{Gb}{2\pi}\frac{x}{x^2+y^2} \tag{3.12}$$

for a screw dislocation that runs along the z-axis. All other components vanish. G is the shear modulus, while b is length of the BURGERS vector.

Corresponding to (3.2), (3.11), and (3.12), at most twelve of the moment-stress components:

$$\bar{\sigma}_{13i} = \bar{\sigma}_{31i}, \qquad \bar{\sigma}_{23i} = \bar{\sigma}_{32i}, \qquad i = 1, 2, 3 \tag{3.13}$$

can then be non-zero. If one divides a body into two parts with an arbitrary surface F then one will ALBENGA's theorem ([29]) for the proper stresses:

$$\int_F \sigma_{ij}\, dF_i = 0. \tag{3.14}$$

With its help, one can easily show that the eight moment-stress components are:

$$\bar{\sigma}_{131} = \bar{\sigma}_{311} = \bar{\sigma}_{133} = \bar{\sigma}_{313} = \bar{\sigma}_{232} = \bar{\sigma}_{322} = \bar{\sigma}_{233} = \bar{\sigma}_{323} = 0, \tag{3.15}$$

such that, from (3.13), at most:

$$\bar{\sigma}_{132} = \bar{\sigma}_{312} \qquad \text{and} \qquad \bar{\sigma}_{231} = \bar{\sigma}_{321} \tag{3.16}$$

([29]) G. ALBENGA, Atti Accad. Sci. Torino, Classe Sci. Fis. Mat. Nat. **54** (1918/19), 864.

are non-zero.

If we go over to $\bar{\tau}_{ij}$, using (3.8), then that will yield the following non-vanishing components:

$$\bar{\tau}_{11} = \varepsilon_{1kl}\,\bar{\sigma}_{1kl} = -\bar{\sigma}_{132}\,, \tag{3.17}$$
$$\bar{\tau}_{22} = \varepsilon_{2kl}\,\bar{\sigma}_{2kl} = +\bar{\sigma}_{231}\,, \tag{3.18}$$
$$\bar{\tau}_{33} = \varepsilon_{3kl}\,\bar{\sigma}_{3kl} = +\bar{\sigma}_{312} - \bar{\sigma}_{321} = -\,\bar{\tau}_{11} - \bar{\tau}_{22}\,. \tag{3.19}$$

The dislocation density is known ([1]) to be defined by:

$$\Delta b_j = \bar{a}_{ij}\,\Delta F_i\,. \tag{3.20}$$

Δb_j is then the total BURGERS vector of the dislocations that cross an arbitrary oriented surface element ΔF_i . Therefore, only the component $\bar{\sigma}_{33}$ of the dislocation density is non-zero in Fig. 4. Since a dislocation exists in the mean on a surface of area XY, one will have:

$$\bar{\sigma}_{33} = b\,/\,XY. \tag{3.21}$$

If one considers (3.17)-(3.19), (3.2), (3.1), (3.11), (3.12), and (3.21) then, after a simple calculation that is based upon Fig. 4, one will get the moment-stresses:

$$\bar{\tau}_{ij} = \bar{\sigma}_{33}\frac{G}{12}\begin{bmatrix} F(Y,X) & 0 & 0 \\ 0 & F(X,Y) & 0 \\ 0 & 0 & -F(Y,X)-F(Y,X) \end{bmatrix}, \tag{3.21}$$

with

$$F\,(Y, X) \equiv -\frac{6}{\pi}\sum_{m,n=-\infty}^{+\infty}\int_{-X/2}^{+X/2}dx\int_{-Y/2}^{+Y/2}dy\,\frac{y[y-(n+\frac{1}{2})Y]}{[x-(m+\frac{1}{2})X]^2+[y-(m+\frac{1}{2})Y]^2}\,. \tag{3.23}$$

We shall return to the calculation of $F\,(Y, X)$.

§ 3.2. *Moment-stresses in Fig. 5 (edge dislocations).* – The moment-stresses of the dislocation arrangement in Fig. 5 have already be determined approximately for $Y / X \ll 1$ ([3]). An exact solution will be given here.

The crystal in Fig. 5 is not to be perturbed macroscopically (i.e., $\bar{\varepsilon}_{ij} = 0$). Therefore, we must let boundary forces p_3 of suitable magnitudes act upon the planes $z = +\infty$ and $z = -\infty$ in order to hinder a lateral contraction in the z-direction. The boundary conditions $p_j = \sigma_{ij}\, n_i$ imply that:

$$p_3\,(z = \pm\,\infty) = \pm\sigma_{33}^{\text{tot}}\,, \tag{3.24}$$

while it follows from the theory of planar *strain* states that one has:

$$\sigma_{33}^{\text{tot}} = \nu\,(\sigma_{11}^{\text{tot}} + \sigma_{22}^{\text{tot}}) \tag{3.25}$$

for σ_{33}^{tot}; ν is the POISSON number in this. It should be emphasized that the macroscopic quantities $\bar{p}_3$ that correspond to the microscopic boundary stresses p_3 will vanish, due to the fact that $\bar{\varepsilon}_{ij} = 0$:

$$\bar{p}_3(z = \pm\infty) = \pm\sigma_{33}^{\text{tot}} = \pm\frac{1}{XY}\int_{-X/2}^{+X/2} dx \int_{-Y/2}^{+Y/2} dy\, \sigma_{33}^{\text{tot}} = 0. \tag{3.26}$$

Linear elastostatics gives a stress field for an edge dislocation that runs along the z-axis ([26]):

$$\sigma_{11}[x, y] = +A\frac{y(3x^2 + y^2)}{(x^2 + y^2)^2}, \tag{3.27}$$

$$\sigma_{12}[x, y] = \sigma_{21}[x, y] = -A\frac{x(x^2 - y^2)}{(x^2 + y^2)^2}, \tag{3.28}$$

$$\sigma_{22}[x, y] = +A\frac{y(x^2 - y^2)}{(x^2 + y^2)^2}, \tag{3.29}$$

$$\sigma_{33}[x, y] = \nu(\sigma_{11} + \sigma_{22}) = 2\nu A\frac{y}{x^2 + y^2}, \tag{3.30}$$

with:

$$A \equiv \frac{b}{2\pi}\frac{G}{1-\nu}.$$

b is once more the length of the BURGERS vector. All other σ_{ij} will vanish.

One has:

$$\bar{\sigma}_{13i} = \bar{\sigma}_{31i} = \bar{\sigma}_{23i} = \bar{\sigma}_{32i} = 0 \tag{3.31}$$

for the moment-stresses, since the associated components of the stress tensor are equal to zero. If one applies ALBENGA's theorem (3.14), in turn, and observes (3.26) then one can prove the vanishing of the following 11 components:

$$\bar{\sigma}_{111} = \bar{\sigma}_{113} = \bar{\sigma}_{121} = \bar{\sigma}_{122} = \bar{\sigma}_{123} = \bar{\sigma}_{211} = \bar{\sigma}_{212} = \bar{\sigma}_{213} = \bar{\sigma}_{222} = \bar{\sigma}_{223} = \bar{\sigma}_{333} = 0. \tag{3.32}$$

If one switches the integration variables x and y in the component $\bar{\sigma}_{122}$ in the integrand then (3.32) will give:

$$\bar{\sigma}_{221} = 0. \tag{3.33}$$

With (3.2), (3.25), and (3.32), one will get:

$$\bar{\sigma}_{331} = \nu(\bar{\sigma}_{111} + \bar{\sigma}_{221}) = 0 \tag{3.34}$$

from this. Hence, from (3.31)-(3.34), only the components:

$$\bar{\sigma}_{112} \text{ and } \bar{\sigma}_{332} \tag{3.35}$$

will remain, and (3.8) will yield at most the following non-vanishing $\bar{\tau}_{ij}$:

$$\bar{\tau}_{13} = \varepsilon_{3kl}\,\bar{\sigma}_{1kl} = +\ \bar{\sigma}_{112}\,, \tag{3.36}$$

$$\bar{\tau}_{31} = \varepsilon_{1kl}\,\bar{\sigma}_{3kl} = -\ \bar{\sigma}_{332}\,. \tag{3.36}$$

Due to (3.25), one can give a very simple connection between the two components $\bar{\tau}_{13}$ and $\bar{\tau}_{31}$. It results from (3.37), along with (3.2), (3.25), (3.32), and (3.36):

$$\bar{\tau}_{31} = -\ \bar{\sigma}_{332} = -\ \nu\,(\bar{\sigma}_{111} + \bar{\sigma}_{221}) = -\ \nu\,\bar{\sigma}_{112} = -\ \nu\,\bar{\tau}_{13}\,. \tag{3.38}$$

The single non-vanishing component of the dislocation density in Fig. 5 is:

$$\bar{a}_{31} = b\,/\,XY. \tag{3.39}$$

One then obtains the moment-stresses in Fig. 5 from (3.38), (3.2), (3.1), (3.30), and (3.39):

$$\bar{\tau}_{ij} = \bar{a}_{31}\,\frac{G}{6\,(1-\nu)}\begin{bmatrix} 0 & 0 & -F(Y,X) \\ 0 & 0 & 0 \\ \nu\,F(Y,X) & 0 & 0 \end{bmatrix}. \tag{3.40}$$

F (Y, X) is, in turn, the function that was defined in (3.23). We then see that the moment-stresses in Figs. 4 and 5 can be calculated with the same function, which is a result that ultimately originates from the condition $\bar{\tau}_{ij} = 0$, and therefore from the symmetry of the stress tensor.

§ 3.3. *Combining the moment-stresses in Figs. 4 and 5.* – The explicit calculation of F (Y, X), which is, above all, in no way trivial due to the double summation that appears in it, can be circumvented by a trick, as we will see below. (3.60) will then yield:

$$F\,(Y, X) = +\,Y^2. \tag{3.41}$$

If one substitutes this in (3.22) and (3.40) then one will get:

$$\bar{\tau}_{ij} = \bar{a}_{33}\,\frac{G}{12}\begin{bmatrix} Y^2 & 0 & 0 \\ 0 & X^2 & 0 \\ 0 & 0 & -Y^2 - X^2 \end{bmatrix} \tag{3.42}$$

for the moment-stresses in Fig. 4 and:

$$\bar{\tau}_{ij} = \bar{a}_{33}\frac{G}{6(1-\nu)}\begin{bmatrix} 0 & 0 & -Y^2 \\ 0 & 0 & 0 \\ \nu Y^2 & 0 & 0 \end{bmatrix} \qquad (3.43)$$

for the ones in Fig. 5. Cyclic permutation of this will then yield the moment-stresses of all homogeneous distributions of dislocations for which the BURGERS vector and the line directions of the individual dislocations run parallel to the coordinate axes, as in Figs. 4 and 5.

One reads off from (3.42) and (3.43) that the trace of the moment-stress tensor will always vanish, in agreement with (3.10). $\bar{\tau}_{ij}$ will then be a *deviator*, which we would like to abbreviate with a superscript of *D*:

$$\bar{\tau}_{ij} = \bar{\tau}_{ij}^{\,D} + \tfrac{1}{3}\delta_{ij}\,\bar{\tau}_{kk} = \bar{\tau}_{ij}^{\,D}. \qquad (3.44)$$

Now, for suitably-chosen constants b_1, b_2, b_3, (3.42) and (3.43) can be represented in the form:

$$\bar{\tau}_{ij} \overset{*}{=} b_1\,\delta_{ij}\,\bar{a}_{kk} + b_2\,\bar{a}_{ij} + b_3\,\bar{a}_{ji}, \qquad (3.45)$$

in which the asterisk over the equal sign means that the equation is true only in the x_i coordinate system. If one constructs the deviator on both sides of (3.45) and considers (3.44) then that will yield:

$$\bar{\tau}_{ij} \overset{*}{=} b_2\,\bar{a}_{ij}^{\,D} + b_3\,\bar{a}_{ji}^{\,D}. \qquad (3.46)$$

This equation allows one to think of all moment-stress states as being produced by deviatoric dislocation densities, and thus, by *edge* dislocations, as long as the center of the dislocation is neglected, as was emphasized above. As a result, (3.42) will be included in (3.43) implicitly!

In the field theory of dislocations, the diagonal components of the dislocation tensor $\bar{a}_{ij}$ appear as screw dislocations, while the remaining components appear as edge dislocations. However, that is not an invariant decomposition: If one transforms, e.g., a dislocation density:

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by a rotation of 45° around the *z*-axis then one will get a matrix:

$$\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

i.e., the dislocations that were screws originally appear to be edge dislocations in the new coordinate system. By contrast, the trace of $\bar{a}_{ij}$ is invariant, so the three mutually-perpendicular families of screw dislocations will describe the same density. One should compare this to the corresponding behavior of the strain tensor $\bar{\varepsilon}_{ij}$.

§ 3.4. *Relationships to the two-dimensional COSSERAT continuum.* – We bring the edge dislocations in Fig. 5 ever closer together with constant *Y* in the *x*-direction and thus reduce their BURGERS vector *b* in such a way that the magnitude of the macroscopic dislocation density:

$$\bar{a}_{31} = \frac{b}{X\,Y} = \frac{b}{X} \cdot \frac{1}{Y} = \frac{\bar{b}}{Y} \tag{3.47}$$

remains preserved.

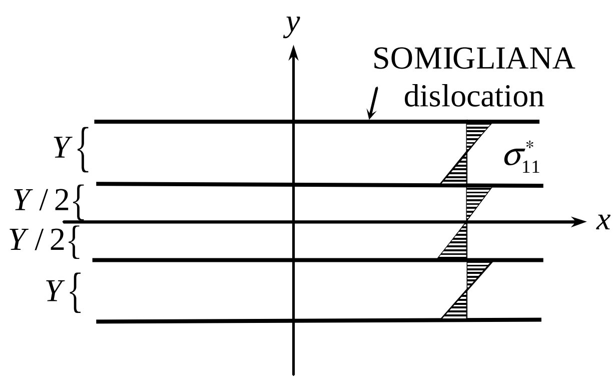

Figure 7. Passing to the limit of a SOMIGLIANA dislocation (cf., Fig. 2c)

If we perform the corresponding passage to the limit then we will get the infinitesimal BURGERS vector *db* for any dislocation:

$$db / dx = b / X = \bar{b}\,. \tag{3.58}$$

Now, a so-called *SOMIGLIANA dislocation* (Fig. 7) will arise in the surface $y = (n + \frac{1}{2})\,Y$ with a linearly-increasing jump displacement in the *x*-direction:

$$[u_1] = (b / X)\,x + \text{const.}; \tag{3.49}$$

the square bracket in this shall represent an abbreviation for "jump."

It is known sufficiently from the literature [MASING and POLANYI ([30]), MANN ([31]), NYE ([32]), and INDENBOM ([33])] that the unperturbed parts of the material that lie

([30]) G. MASING and M. POLANYI, Ergeb. exakt. Naturw. **2** (1923), 177.
([31]) E. H. MANN, Proc. Roy. Soc. London A **199** (1949), 376.
([32]) J. F. NYE, Proc. Soc. London A **200** (1949), 47.

between the (unstable) SOMIGLIANA dislocations in Fig. 7 will undergo an elastic deformation that corresponds to a pure *bending* of those "plates." For that reason, in the absence of lateral contraction in the z-direction, (3.25) will yield [cf., e.g., TIMOSHENKO and GOODIER ([34])]:

$$\sigma^*_{11} = \frac{b}{XY}\frac{2G}{(1-\nu)}y\,, \qquad \sigma^*_{33} = \nu\,\sigma^*_{11} \qquad (3.50), (3.51)$$

for the plate $-Y/2 < y < +Y/2$, and with HOOKE's law, that will give:

$$\varepsilon^*_{11} = \frac{b}{XY}y\,, \qquad \varepsilon^*_{33} = -\frac{\nu}{1-\nu}\varepsilon^*_{11} \qquad (3.52), (3.53)$$

The constants in (3.50) were arranged so that the displacement:

$$u^*_1 = \frac{b}{XY}xy \qquad (3.54)$$

that (3.52) implies would describe precisely the jump displacement that is required by (3.49). Our plate will then take on an elastic curvature b / XY whose magnitude agrees with the structural curvature $\bar{K}_{13}$ in Fig. 5.

This is closely related to the idea that the *distribution of dislocations in Figs. 4 and 5 are equivalent*, as long as the contribution of the dislocation center can be neglected in comparison to the moment-stresses, as before.

Proof: From (3.1), (3.2), and (3.36), the component $\bar{\tau}_{13}$ for the dislocations in Fig. 5 is defined as follows:

$$\bar{\tau}_{13} = \bar{\sigma}_{112} = \frac{1}{XY}\sum_{m,n=-\infty}^{+\infty}\int_{-X/2}^{+X/2}dx\int_{-Y/2}^{+Y/2}dy\,\sigma_{11}[x-(m+\tfrac{1}{2})X,\,y-(n+\tfrac{1}{2})Y]\,y\,. \qquad (3.55)$$

In order to take the quantity m over which we are summing out of the integrands, we displace the limits of integration of x:

$$\bar{\tau}_{13} = \frac{1}{XY}\sum_{m,n=-\infty}^{+\infty}\int_{-(m+1)X}^{-mX}dx\int_{-Y/2}^{+Y/2}dy\,\sigma_{11}[x,\,y-(n+\tfrac{1}{2})Y]\,y\,. \qquad (3.56)$$

With that, the summation over m can be evaluated in an elementary way:

([33]) V. L. INDENDOM, *Plasticity of Crystals* (ed. Klassen-Neklyudova) Consultants Bureau, New York, 1962, pp. 105.

([34]) S. TIMOSHENKO and J. N. GOODIER, *Theory of Elasticity*, McGraw-Hill, New York, 1951.

$$\overline{\tau}_{13} = \frac{1}{XY} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \int_{-Y/2}^{+Y/2} dy\, \sigma_{11}[x, y-(n+\tfrac{1}{2})Y]\, y\,. \tag{3.57}$$

On the other hand, one easily sees that (3.57) is also an expression for the moment-stresses in Fig. 7. Any infinitesimal dislocation with the abscissa x and ordinate $(n + \tfrac{1}{2})\, Y$ will generate a moment with respect to the origin of:

$$db \cdot (1/b) \cdot \sigma_{11}\, [-x, y-(n+\tfrac{1}{2})\, Y] \cdot y \tag{3.58}$$

on the surface $x = 0$. If we take the mean of this moment over y from $-Y/2$ to $+Y/2$ and sum (integrate, resp.) over all dislocations then that will yield eq. (3.57), because of (3.48).

$\overline{\tau}_{31}$ can be converted in just that way. Since no further moment-stresses appear in Fig. 7, as we see from (3.50) and (3.51), the assertion above has been proved.

The function $F(Y, X)$ that is defined in (3.23) can now be easily determined. The *moment-stress* $\overline{\tau}_{13}$ is clearly the negative *bending moment* M_3 per unit cross-sectional area:

$$\overline{\tau}_{13} = -\frac{M_3}{XY} = -\frac{1}{Y} \int_{-Y/2}^{+Y/2} dy\, \sigma_{11}^{*} \cdot y\,. \tag{3.59}$$

If one substitutes (3.50) and observes (3.47) then a comparison with (3.40) will give:

$$F\,(Y, X) = Y^2, \tag{3.60}$$

which shows us that $F\,(Y, X)$ does not depend upon X at all.

The mean *energy density* of the plates in Fig. 7 is determined from:

$$\overline{f} = \frac{1}{2} \cdot \frac{1}{Y} \int_{-Y/2}^{+Y/2} dy\, \sigma_{11}^{*}\, \varepsilon_{11}^{*}, \tag{3.61}$$

which can be recalculated as:

$$\overline{f} = \frac{1}{2} \cdot \frac{1}{XY} \cdot \frac{1}{Y} \int_{-Y/2}^{+Y/2} dy\, \sigma_{11}^{*}\, y = \tfrac{1}{2} \overline{K}_{13}\, \overline{\tau}_{13} \tag{3.62}$$

by using (3.52), (3.59), and (1.2) with (3.39). The energy density that is associated with the structural curvature $\overline{K}_{ij}$ for edge dislocations is then obtained from the corresponding formula:

$$\overline{f} = \tfrac{1}{2} \overline{\tau}_{ij} \overline{K}_{ij} = \tfrac{1}{2} a_{ijkl}\, \overline{K}_{ij}\, \overline{K}_{kl}\ . \tag{3.63}$$

The results of this section are then hardly unexpected, since the dislocation arrangement in Fig. 5 macroscopically represents a curvature state $\overline{K}_{13}$ that is compatible ([26]), and can therefore be derived from a rotation field. In that respect, Fig. 5 is the verification of a COSSERAT continuum. However, a plate that is considered to be two-dimensional can likewise be regarded as a COSSERAT continuum ([17-19]), such that the three-dimensional COSSERAT state in Fig. can be thought of as being constructed from two-dimensional COSSERAT states.

At the same time, we see that the static state of the dislocated continuum in Fig. 5 will be better described by only the first-order moment-stresses that we have used up to now the more that the material pieces between the glide planes behave like bending plates, so the closer together that the dislocations in the x-direction are for a given Y:

$$X/Y \ll 1. \tag{3.64}$$

If that relationship is not true then we will also have to include moment-stresses of even *higher order* in our calculations ([3]). Since one can generally prove that all moment-stresses of even-number order vanish in Figs. 4 and 5, the first moment-stresses to become considerable would be in order three.

In retrospect, one now also recognizes the relationship between this section and our Gedanken experiment in the introduction. The calculations of the moment-stresses using (3.2) and (3.8) (which have been purely elasticity-theoretic, up to now) gave us only the bending moment M_3 that appeared in Fig. 2c, since we neglected the center of the dislocation, as we stated expressly. If we also draw our attention to the latter, as we will do in § **4**, then we will get the moment M_3^P that originates in the crystalline structure of the welded plates, in addition (cf., Fig. 3b and 3c).

With the help of the results of this section, one can easily see how big the moment-stresses of distributions of dislocations will be when they are *not arranged as regularly* as in Fig. 5. For example, if we displace any two rows of dislocations by X / 2 to the right then the moment-stresses will *not* change; we must then generally take the mean in the y-direction from – Y / 2 to 3Y / 2. We can verify the validity of this assertion by employing the argument that led from (3.55) to (3.57), which is clearly reasonable since the original abscissa of the periodically-arranged individual dislocations is naturally inessential when one goes to the SOMIGLIANA dislocations. In that way, we will see that static distributions of the $\bar{a}_{31}$–dislocations will yield the same first-order moment-stresses, as long as Y is the mean lattice separation distance. We must expect at most differences in the higher moments.

§ 4. Contribution of the dislocation center to the moment-stresses. – In the elasticity-theoretic model for dislocations, they diverge at the center of the stresses, which naturally does not correspond to reality. In order to include the center (at least, to some extent), one can employ the dislocation model of PEIERLS ([4]), which considers the lattice structure, and one can find it described in, e.g., the works of NABARRO ([35, 36]), COTTRELL ([37]), and SEEGER ([38]).

([35]) F. R. N. NABARRO, Proc. Phys. Soc. London **59** (1947), 256.
([36]) F. R. N. NABARRO, Adv. Phys. (Phil. Mag. Supp.) **1** (1952), 269.
([37]) A. H. COTTRELL, *Dislocations and Plastic Flow in Crystals*, Clarendon Press, Oxford, 1953.
([38]) A. SEEGER, *Handbuch der Physik* (ed., Flügge) VII/I, Springer-Verlag, Berlin, 1955, pp. 383.

Entirely within the spirit of this model, we treat the bent plates in Fig. 7 that lie between the SOMIGLIANA dislocations, which will make a jump in the lattice constants appear in the *y*-direction at each of the SOMIGLIANA dislocations, as we saw already in the introduction. We will then suddenly "switch on" the cohesion forces that act between the outer surfaces of the plates, which therefore act the same, and which should obey the nonlinear PEIERLS sine law ([4]). The continuum will then go to the state of least-possible energy, namely, as VAN DER MERWE ([39]) has calculated in detail, the one that is depicted in Fig. 5. As such, VAN DER MERWE has discussed this problem for two (primitive cubic) *half*-crystals with different lattice constants. However, we can certainly adapt his results to the case (3.64) of $X / Y \ll 1$, since the "bent plate" approximation is reasonable then, as we saw in § 3.4.

§ 4.1 *Energy.* – The energy per dislocation in the continuum of Fig. 5 is composed accordingly of the elastic bending energy (3.61):

$$XY f = \frac{X}{2}\int_{-Y/2}^{+Y/2} dy\, \sigma^*_{11}\, \varepsilon^*_{11} = \frac{Gb^2}{12(1-\nu)}\frac{Y}{X} \tag{4.1}$$

and the boundary-surface energy E_a of the glide plane:

$$E_a = \frac{GbX}{4\pi^2}\left\{1+\frac{\pi b}{(1-r)X}-\sqrt{1+\left(\frac{\pi b}{(1-r)X}\right)^2}\right\}, \tag{4.2}$$

which we deduce from VAN DER MERWE's paper ([39]) [§ 5, eq. (27)]. When (4.2) is developed in powers of b / X that will give:

$$E_a = \frac{Gb^2}{4\pi(1-r)}\left\{1-\frac{\pi}{2(1-r)}\left(\frac{b}{X}\right)+\cdots\right\} \qquad b / X < 2\,(1-\nu) / \pi. \tag{4.3}$$

A comparison shows clearly that for the case $X/Y \ll 1$ that is of interest to us, the boundary-surface energy (4.3) can be neglected in comparison to the elastic energy (4.1), so the contribution of the *center* will play *no role.* That is understandable, since the stress field of any dislocation in Fig. 5 will possess a relatively-large range that has order of magnitude *Y.* Corresponding arguments will be true for screw dislocations.

§ 4.2 *Moment-stress.* – However, what gives us the right to say that the energy (4.2) [(4.3), resp.] can be attributed to moment-stresses? In order to answer this question, we cut out a plate from the crystal in Fig. 5 that includes the inelastic domain of the PEIERLS model (Fig. 8). It possesses a thickness of *b* and consists of only two neighboring net-planes. We infer the shear-stresses that act upon its upper plane S_A from the paper of VAN DER MERWE ([39]) [§ 4, eq. (23), with $Z = 0$]:

([39]) J. H. VAN DER MERWE, Proc. Phys. Soc. London **63** (1950), 616.

$$\sigma_{21}\,(x, S_A) = -\frac{G}{4(1-\nu)}\frac{b}{X}\frac{\sin(2\pi x/X)}{\sin^2(\pi x/X)+\pi^2\lambda^2}\,, \tag{4.4}$$

with

$$\lambda^2 \equiv \frac{1}{4\pi^2}\left\{\frac{1}{\sqrt{1+[\pi b/(1-\nu)X]^2}}+\sqrt{1+[\,]^2}-[\,]-2\right\}$$

or

$$\lambda^2 = \left(\frac{1}{2(1-\nu)}\frac{b}{X}\right)^2 + O\,([\,]^3) + \ldots \qquad \text{for } [\,] < 1. \tag{4.5}$$

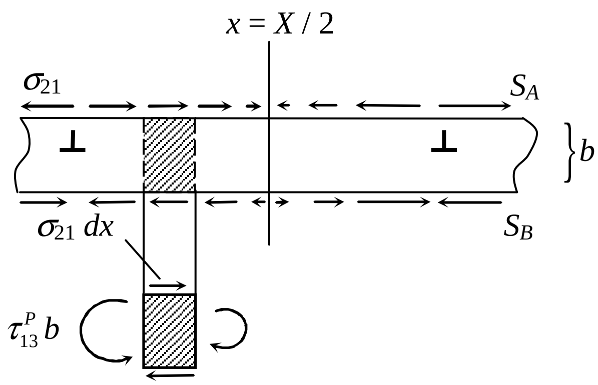

Figure 8. Antisymmetric stress state in the PEIERLS model of an edge dislocation

When regarded continuum-theoretically, an *antisymmetric force-stress state:*

$$\sigma^P_{[21]}\,(\text{plate}) = \tfrac{1}{2}(\sigma_{21} - \sigma_{12}) = \tfrac{1}{2}\,\sigma_{21}(x, S_A) \tag{4.6}$$

(no σ_{12} component will appear, due to the vanishing of the vertical force transfer) will prevail in the plate in Fig. 8, which can be made no thinner than b physically. From (2.2), (4.6) implies moment-stresses; since τ_{23} will be zero, on the grounds of symmetry, one will have:

$$\frac{\partial \tau^P_{13}}{\partial x_1} = 2\,\sigma^P_{[21]} = \sigma_{21}(x, S_A) \tag{4.7}$$

for

$$(n+\tfrac{1}{2})\,Y - \frac{b}{2} \le y \le (n+\tfrac{1}{2})\,Y + \frac{b}{2}\,,$$

whereas the right-hand side of the equation will vanish everywhere else. Integration over the aforementioned domain will yield:

$$\tau^P_{13}(x, y) = \begin{cases} \displaystyle\int_{+X/2}^{x} \sigma_{21}(\xi, S_A)\,d\xi + \tau^P_{13}(X/2, y), \\ 0 \end{cases}.$$

If one takes the mean over the volume element (3.3) then one will get from (4.8) the macroscopic (PEIERLS) moment-stress:

$$\bar{\tau}_{13}^{P} = \frac{1}{XY}\int_{-X/2}^{+X/2} dx \int_{-Y/2}^{+Y/2} dy\, \tau_{13}^{P}(x, y) = \frac{b}{Y}\left\{\frac{1}{X}\int_{-X/2}^{+X/2} dx \int_{-X/2}^{x} d\xi\, \sigma_{21}(\xi, S_A) + \tau_{13}^{P}\left(\frac{X}{2}, \frac{Y}{2}\right)\right\}, \tag{4.9}$$

or, with (4.4):

$$\bar{\tau}_{13}^{P} = \frac{G}{2\pi(1-\nu)}\frac{b^2}{Y}\left[\ln 2 - \ln\left\langle 1 + \frac{\pi\lambda}{\sqrt{1+\pi^2\lambda^2}}\right\rangle\right] + \frac{b}{Y}\tau_{12}^{P}\left(\frac{X}{2}, \frac{Y}{2}\right). \tag{4.10}$$

If one develops (4.10) into a power series in b / X then one will see that in the first approximation, the second term in the square bracket can be neglected in comparison to the first one. In the event that one introduces the structural curvature K_{13} according to (1.2) and (3.39), one will get:

$$\bar{\tau}_{13}^{P} = \left[\frac{\ln 2}{2\pi}\cdot\frac{G(bX)}{1-\nu} + X\,\tau_{13}^{P}\left(\frac{X}{2}, \frac{Y}{2}\right)\right]K_{13}, \tag{4.11}$$

such that $\bar{\tau}_{13}^{P}$ will then possess the same sign as $\bar{\tau}_{13}$ in (3.43).

The integration constant $\bar{\tau}_{13}^{P}(X/2,\ Y/2)$ cannot be determined in the context of the PEIERLS model, since the moment-stress of an *individual* dislocation will diverge at an infinitely-large distance from the center, as will the corresponding energy. However, if one assumes that both terms in (4.11) have the same order of magnitude then $\bar{\tau}_{13}^{P}$ in (4.11) can be neglected in comparison to $\bar{\tau}_{13}$ in (3.43), due to (3.64), which agrees with the argument in § 4.1.

In the PEIERLS model, the center of the dislocation, and therefore also the moment-stresses, are "smeared" somewhat in the glide plane. Naturally, that represents an idealization, since in reality those moment-stresses (which generally decay rapidly to zero with increasing distance from the center) must be thought of as being distributed over all space. Therefore, we might employ (4.11) only to estimate the order of magnitude of $\bar{\tau}_{13}^{P}$ and to determine its sign.

In summation, we can then state the following: With the help of the PEIERLS dislocation model, we have shown (in a semi-quantitative way) that antisymmetric force-stresses $\sigma_{[21]}^{P}$ will be generated in crystals by the center of a dislocation according to (4.6), and therefore also macroscopically-observable moment-stresses $\bar{\tau}_{13}^{P}$, corresponding to (4.11). However, that effect can be neglected in the case of $X/Y \ll 1$.

Since it is known that an elasticity-theoretic model for an edge dislocation can be obtained from the PEIERLS model by passing to the limit of vanishing dislocation width, it must exhibit a singular diverging $\sigma_{[21]}$ at its dislocation center. In sees from (3.28) that, in fact, no shear-stresses are associated with the center – i.e., σ_{12} vanishes, but not σ_{21} (Fig. 9):

$$\sigma_{[21]}\,[\pm 0,\,0] = \tfrac{1}{2}(\lim_{x\to\pm 0}\sigma_{21}[x,0]-\lim_{y\to 0}\sigma_{12}[0,y]) = \tfrac{1}{2}\lim_{x\to\pm 0}\sigma_{21}[x,0] = -\,\frac{A}{2}\lim_{x\to\pm 0}\frac{1}{x}. \tag{4.12}$$

That implies a diverging moment-stress τ_{13} at the center with a delta-function character, in analogy with (4.7). Moreover, one likewise gets (4.12) when one employs the stress field of an edge dislocation that is computed with the help of the nonlinear theory of elasticity of PFLEIDERER, *et al.* ([40]) in the second approximation.

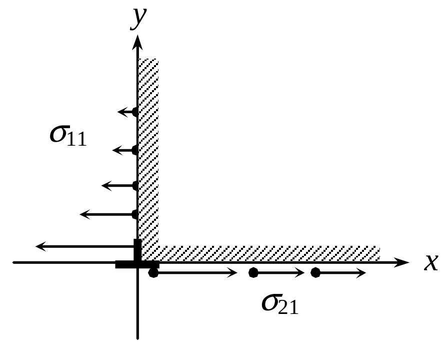

Figure 9. Proving the lack of associated edge stresses at the center of a dislocation.

The stress distribution that is depicted in Fig. 9 has a certain similarity to the one in an example that was given by REISSNER ([41]). In that example, REISSNER showed that the symmetry of the stress tensor was probably sufficient, but not necessary, for the equilibrium of moments in the classical theory of elasticity, in the event that the stress gradients were allowed to become infinite, as in Fig. 9.

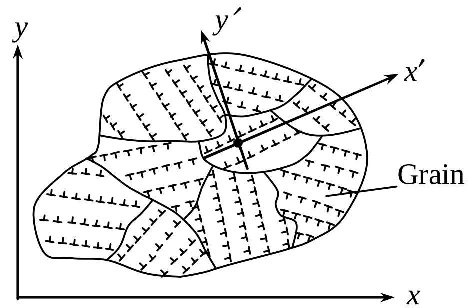

Figure 10. Two-dimensional schema for our polycrystal model.

§ 5. Material law for moment-stresses in isotropic form. – The volume element ΔV of a macroscopically-isotropic polycrystal consists of very many grains whose orientations and forms are distributed statically at random. Let dislocations be present in each grain. Their effect on the moment-stresses can, as we saw in (3.46), be described by edge dislocations alone, as long as we can neglect the contribution of the dislocation center, in the sense of § **4**. For the time being, we would like to assume that only one glide system is present in each grain (Fig. 10). One can then lay out a Cartesian coordinate system in each of them with axes x', y', z', in such a way that the active

([40]) H. PFLEIDERER, A. SEEGER, and E. KRÖNER, Z. Naturforschg. **15a** (1960), pp. 758.
([41]) E. REISSNER, J. Math. Phys. **23** (1944), 192.

dislocations that are present seem to be $\alpha_{z'x'}$ dislocations (viz., x'-axis in the direction of the BURGERS vector, z'-axis parallel to the line direction of the edge dislocation).

We now choose an arbitrary *xyz*-coordinate system, relative to which the orientation of the $x'y'z'$-system is established perhaps by the three EULER angles Φ, Ψ, and Θ, which we shall collectively denote by Ω. One will then have:

$$d\Omega = \sin\Theta\, d\Theta\, d\Phi\, d\Psi \quad \int_{\Phi=0}^{2\pi}\int_{\Psi=0}^{2\pi}\int_{\Theta=0}^{\pi} d\Omega = 8\pi^2.$$

We denote the corresponding direction cosines by:

$$A_i^{j'} \equiv \cos(x_i, x_{j'}),$$

and we abbreviate their product as follows:

$$A_{i\,j\cdots}^{m'n'\cdots} \equiv A_i^{m'} A_j^{n'} \ldots$$

In the volume element Δ*V*, a set of grains lie between the orientations Ω and Ω + *d*Ω and have a dislocation density:

$$\alpha^D_{z'x'}(\Omega)\, d\Omega, \tag{5.1}$$

which we write as a deviator, due to (3.46). If one would like to obtain the mean dislocation density $\bar{\alpha}^D_{ij}$ over Δ*V* relative to the *xyz*-coordinate system then we would have to calculate the α^D_{ij}–component of (5.1) corresponding to the transformation law for tensors and subsequently take the mean over all angles:

$$\bar{\alpha}^D_{ij} = \frac{1}{8\pi}\int_\Omega A_{i\,j}^{z'x'} \alpha^D_{z'x'}\, d\Omega\ . \tag{5.2}$$

On the other hand, from (3.43) the moment-stresses that are produced by (5.1) amount to:

$$\tau_{z'x'}(\Omega)\, d\Omega = \frac{G\nu}{6(1-\nu)} D^2(\Omega)\, \alpha^D_{z'x'}(\Omega)\, d\Omega\,, \tag{5.3}$$

$$\tau_{x'z'}(\Omega)\, d\Omega = -\frac{G\nu}{6(1-\nu)} D^2(\Omega)\, \alpha^D_{z'x'}(\Omega)\, d\Omega\,, \tag{5.4}$$

since the distance *D* between glide planes can depend upon only the orientation, due to the homogeneity of the volume element. One calculates the mean moment-stress over Δ*V* in the *xyz*-system in analogy to (5.2):

$$\bar{\tau}_{ij} = \frac{1}{8\pi}\int_{\Omega} A_{i\,j}^{z'x'}\tau_{z'x'}\,d\Omega + \frac{1}{8\pi}\int_{\Omega} A_{i\,j}^{x'z'}\tau_{x'z'}\,d\Omega\,. \tag{5.5}$$

In retrospect, we now know that one can admit arbitrarily many glide systems in all grains of our volume element as long as we associate each of them with a $x_{i'}$-coordinate system.

Due to (5.2)-(5.4), (5.5) implies that:

$$\bar{\tau}_{ij} = \frac{G\nu}{6(1-\nu)}[\nu\bar{D}^2(i,j)\,\bar{\alpha}_{ij} - \bar{D}^2(j,i)\,\bar{\alpha}_{ji}], \tag{5.6}$$

in which:

$$\bar{D}^2(i,j) \equiv \frac{\int_{\Omega} D^2(\Omega)\,A_{i\,j}^{z'x'}\,\alpha_{z'x'}^D\,d\Omega}{\int_{\Omega} A_{i\,j}^{z'x'}\,\alpha_{z'x'}^D\,d\Omega}$$

means the mean-square value of an effective glide plane distance. In the future, we would like to assume that D is independent of the orientation, because only then will the material tensor be macroscopically isotropic:

$$\bar{\tau}_{ij} = \frac{G\,D^2}{6(1-\nu)}[\nu\,\bar{\alpha}_{ij}^D - \bar{\alpha}_{ji}^D]\,. \tag{5.7}$$

If we express the dislocation density in (5.7) in terms of the structural curvature $\bar{K}_{ij}$ as in (1.2) then we will get the main result of our article from the validity of (3.64) in the form of the *material law* for moment-stresses in *isotropic* form:

$$\bar{\tau}_{ij} = \frac{G\,D^2}{6(1-\nu)}[\,\bar{K}_{ij}^D - \nu\,\bar{K}_{ji}^D\,]\,. \tag{5.8}$$

A comparison of coefficients will yield the moduli that were introduced in (2.10):

$$a_1 = -\frac{G\,D^2}{18}, \quad a_2 = \frac{G\,D^2}{6(1-\nu)}, \quad a_3 = -\frac{\nu\,G\,D^2}{6\,(1-\nu)}\,. \tag{5.9}$$

When split covariantly, (5.8) can be written:

$$\left\{\begin{array}{l} \bar{\tau}_{ij}^D = a_2\bar{K}_{ij}^D + a_3\bar{K}_{ji}^D, \\ \bar{\tau}_{ii} = a_0\bar{K}_{ii} = (3a_1 + a_2 + a_3)\bar{K}_{ii}, \end{array}\right\} \tag{5.10}$$

such that the three moduli a_0, a_2, a_3 can also be employed to characterize the material in terms of its moment-stresses. As we saw already in (3.10) when we neglected the dislocation center (cf., § **4**), the modulus a_0 of the "torsional curvature," while the modulus a_2 of the "longitudinal bending curvature" (a_3 of "transversal bending curvature," resp.) has the value that was given in (5.9).

From (3.63) and (5.8), that implies an *energy density:*

$$\bar{f} = \tfrac{1}{2}\bar{K}_{ij}\,\bar{\tau}_{ij} = \frac{G\,D^2}{12(1-\nu)}[\bar{K}_{ij}^D\,\bar{K}_{ij}^D - \nu\,\bar{K}_{ij}^D\bar{K}_{ji}^D]. \tag{5.11}$$

Since it must naturally be positive-definite, one can derive the inequality:

$$a_2 > 0, \qquad a_2 \geq |\,a_3\,|, \qquad a_2 \geq -\tfrac{3}{2}a_1 \tag{5.12}$$

for the moduli, which is also actually fulfilled for our moduli, as a glimpse at (5.9) will show.

§ 6. Discussion ([*,42,43]). – The field of dislocations in its differential form teaches us that one can interpret a dislocated solid body in its natural state as a *non-Euclidian* material space [cf., e.g., ([26])]. From KONDO ([44]) and BILBY, *et al.*([45]), the *torsion* $\Gamma_{[ml]k}$ of that space corresponds to the dislocation density:

$$\bar{a}_{nk} = \varepsilon_{nml}\,\bar{\Gamma}_{[ml]k}\,, \tag{6.1}$$

while the *metric* is described by the strain tensor $\bar{\varepsilon}_{ij}$. That space is defined uniquely by being given its affine *connection:*

$$\bar{\Gamma}_{mlk} = -(\bar{\varepsilon}_{lk,m} + \bar{\varepsilon}_{mk,l} - \bar{\varepsilon}_{lm,k}) - \varepsilon_{lkn}\bar{K}_{mn}\,. \tag{6.2}$$

The expression in parentheses represents twice the CHRISTOFFEL symbol of the second kind that belongs to $\bar{\varepsilon}_{ij}$. As usual, $\bar{K}_{mn}$ is the structural curvature which is, from (6.2), equivalent to the torsion, and therefore, from (6.1), to the dislocation density ([1]), as well.

One can now show [cf., e.g., AMARI ([46])] that the relative elastic rotation $d\omega_{lk}$ of two neighboring volume elements can be calculated as follows:

$$d\omega_{lk} = -\,\Gamma_{m[lk]}\,dx_m\,. \tag{6.3}$$

([*]) In regard to moment-stresses, one can refer to the recent papers of ERINGEN ([42]) and MINDLIN ([43]), which unfortunately could not be considered further here.

([42]) A. C. ERINGEN, Int. J. Eng. Sci. **2** (1964), 189.

([43]) R. D. MINDLIN, Arch. Rational Mech. **16** (1964), 51.

([44]) K. KONDO, Proc. 2[nd] Japan Nat. Cong. Appl. Mech. (1953), pp. 41.

([45]) B. A. BILBY, R. BULLOUGH, and E. SMITH, Proc. Roy. Soc. London A **231** (1955), 263.

([46]) S. AMARI, RAAG Memoirs 3, Gakujutsu Bunken Fukyu-Kai, Tokyo, 1962, pp. 99.

$-\bar{\Gamma}_{m[lk]}$ will then play the role of a curvature tensor; from (6.2), one will have:

$$-\Gamma_{m[lk]} = \bar{\varepsilon}_{mk,l} - \bar{\varepsilon}_{ml,k} + \varepsilon_{lkm}\,\bar{K}_{mn} \tag{6.4}$$

for it. With

$$\bar{\kappa}_{mn} = \varepsilon_{nrs}\,\bar{\varepsilon}_{ms,r} = \varepsilon_{nrs}\,\bar{\varepsilon}_{m[s,r]}, \tag{6.5}$$

one can write:

$$-\bar{\Gamma}_{m[lk]} = \varepsilon_{lkn}\,(\bar{\kappa}_{mn} + \bar{K}_{mn}) \tag{6.6}$$

for it. In $\bar{\kappa}_{mn}$, we again recognize the tensor that was defined in (2.12), which measures the curvatures that are produced by strains. Generally, in the field theory of dislocations, as opposed to the classical and COSSERAT theories of elasticity, the strain is no longer derivable from a single-valued displacement field. (6.6) says that the *total* curvature $-\bar{\Gamma}_{m[lk]}$ is composed additively of the *strain* curvature $\bar{\kappa}_{mn}$ and the COSSERAT-NYE *structural* curvature $\bar{K}_{mn}$.

However, one must observe that $\bar{\kappa}_{mn}$ and $\bar{K}_{mn}$ are, in essence, completely different quantities. Whereas $\bar{\kappa}_{mn}$ is causally linked with strains whose reactions are the force-stresses, $\bar{K}_{mn}$ has nine proper functional degrees of freedom and can obviously also appear for vanishing macroscopic strains, as in e.g., Fig. 5.

In the special case of the classical theory of elasticity, the structural curvatures $\bar{K}_{mn}$ are equal to zero, and what will remain are only the curvatures $\bar{\kappa}_{mn}$. From § 2.1, the moment-stresses that are associated with those curvatures can be neglected. However, dislocations will produce structural curvatures $\bar{K}_{mn}$ that, from (5.8), will demand moment-stresses as static reactions in their own right that should not be neglected. We see from this that one *cannot* regard the moment-stresses as reactions to the total curvature $-\bar{\Gamma}_{m[lk]}$. Moreover, the moment-stresses correspond to the geometric aspect of the structural curvatures.

As a result, one should understand [in the sense of LAGRANGE's liberation principle ([47])] the force-stresses $\bar{\sigma}_{(ij)}$ ([*,48]) to be reactions to the strains $\bar{\varepsilon}_{ij}$ and the moment-stresses $\bar{\tau}_{ij}$ to be reactions to the *structural curvatures* $\bar{K}_{ij}$, which are independent of them. Hence, one must also not associate our considerations with the total curvature in the context of the original compatible COSSERAT theory of moment-stresses, as is done occasionally. By contrast, the "rotational stresses" that DJURITCH ([25]) likewise introduced in a COSSERAT continuum, but whose physical interpretation he left open,

([47]) G. HAMEL, *Theoretische Mechanik*, Springer-Verlag, Berlin, 1949.

([*]) The material law of the antisymmetric part of the force-stress tensor $\sigma_{[ij]}$ was skipped here; in the meantime, it was dealt with in another publication ([48]). In the later, it was shown that $\sigma_{[ij]}$ vanishes in the field theory of dislocations, since the geometric quantities that are associated with the $\sigma_{[ij]}$ have a plastic nature. Therefore, one cannot apply LAGRANGE's liberation principle to the $\sigma_{[ij]}$, either.

([48]) E. KRÖNER, Proc. 11th Cong. Appl. Mech., Munich, 1964, in press.

seem to correspond to our moment-stresses that are produced by dislocations, while "his" moment-stresses arise from classical kinematics, and can therefore be neglected.

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