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Consequences of the Dirac Theory of Positrons

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Abstract. From Dirac's theory of positrons, it follows that any electromagnetic field is inclined to pair creation, which is a discrepancy with Maxwell's vacuum equations. This discrepancy is computed for the special case in which no actual electrons and positrons are present and the field varies inappreciably at the level of the Compton wavelength. There is Lagrange function for the field:

$$\mathcal{L} = \tfrac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2)$$

$$+\frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\cdot\right)+c.c.}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\cdot\right)-c.c.}+|\mathfrak{E}_k|^2+\frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}\cdot$$

$$\left(\begin{array}{l}\mathfrak{E},\mathfrak{B}\quad\text{define the force on the electron.}\\ |\mathfrak{E}_k| = \dfrac{m^2c^3}{e\hbar} = \dfrac{1}{\text{"137"}}\dfrac{c}{(e^2/mc^2)^2} = \text{"Critical field strength."}\end{array}\right)$$

The terms in its development for weak fields (compared to $|\mathfrak{E}_k|$) describe processes of the scattering of light by light that are already known from perturbation theory. For large fields, the field equations that we derive here are very different from Maxwell's equations. They are equivalent to the ones that were proposed by Born.

The fact that matter can transform into radiation and radiation can transform into matter leads to some fundamental new trends in quantum electrodynamics. One of the most important consequences of this transformation is the fact that the Maxwell equations must be replaced with more complicated equations, even for events in empty space. In particular, it is not generally possible to separate events in empty space from events in matter since fields can create matter when their energy reaches a certain value; on the other hand, when the energy for the creation of matter is not attained, a sort of "vacuum polarization," and with it, a discrepancy from Maxwell's equation, will result from its virtual possibility. This vacuum polarization, which shall be studied in what follows, will be given the usual expression as a difference between the vectors $\mathfrak{B}$, $\mathfrak{E}$, on the one hand, and $\mathfrak{D}$, $\mathfrak{H}$, on the other, where one can set:

$$\left.\begin{aligned}\mathfrak{D} &= \mathfrak{E} + 4\pi\mathfrak{P},\\ \mathfrak{H} &= \mathfrak{B} - 4\pi\mathfrak{M}.\end{aligned}\right\} \tag{1}$$

The polarizations $\mathfrak{P}$ and $\mathfrak{M}$ can now be any sort of complicated functions the field strengths and their derivatives at a given point, and the field strengths in the neighborhood of that point. When the field strengths are very small (this means, as we

shall show later, that they are very small compared to the $e^2/\hbar c$ scale of the field strength at the "surface of the electron.") one can consider $\mathfrak{P}$ and $\mathfrak{M}$ to be approximated by linear functions of $\mathfrak{E}$ and $\mathfrak{B}$. Uehling [1]) and Serber [2]) have determined the discrepancy from Maxwell's equations in this approximation. One obtains another interesting special case when one considers the field strengths to be not necessarily small, but very slowly varying (i.e., almost constant over a distance of order $\hbar/mc$). One then defines $\mathfrak{P}$ and $\mathfrak{M}$ as functions of $\mathfrak{E}$ and $\mathfrak{B}$ at the same point; in this approximation, the derivatives of $\mathfrak{E}$ and $\mathfrak{B}$ no longer appear. As computation will show, the development of $\mathfrak{P}$ and $\mathfrak{M}$ can then be written in odd powers of $\mathfrak{E}$ and $\mathfrak{B}$; the third order terms are clearly due to the scattering of light by light and are already known [3]) The objective of the foregoing work is the complete determination of the functions $\mathfrak{P}(\mathfrak{E}, \mathfrak{B})$ and $\mathfrak{M}(\mathfrak{E}, \mathfrak{B})$ for the special case of very slowly varying field strengths. To that end, it suffices to compute the energy density of the field $U(\mathfrak{E}, \mathfrak{B})$ as a function of $\mathfrak{E}$ and $\mathfrak{B}$, since the wave equations can be derived from the energy density by the Hamiltonian method. One starts, perhaps, with the Lagrange function $\mathcal{L}(\mathfrak{E}, \mathfrak{B})$ and sets:

$$\mathfrak{D}_i = \frac{\partial \mathcal{L}}{\partial \mathfrak{E}_i}, \qquad \mathfrak{H}_i = -\frac{\partial \mathcal{L}}{\partial \mathfrak{B}_i}, \tag{2}$$

$$U(\mathfrak{E}, \mathfrak{B}) = \frac{1}{4\pi}[\sum_i \mathfrak{D}_i \mathfrak{E}_i - \mathcal{L}] = \frac{1}{4\pi}\left(\sum_i \mathfrak{E}_i \frac{\partial \mathcal{L}}{\partial \mathfrak{E}_i} - \mathcal{L}\right) \tag{3}$$

and then determines the Lagrange function from (3) and $\mathfrak{D}$ and $\mathfrak{H}$ from (2). Since the Lagrange function must be relativistically invariant, it can be at most a function of the two invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$ [4]). The computation of $U(\mathfrak{E}, \mathfrak{B})$ leads back to the question of the energy density of the matter field that is bound to the constant fields $\mathfrak{E}$ and $\mathfrak{B}$. Before we come to grips with this problem, we must first briefly recapitulate the mathematical formalism of positron theory (loc. cit. I) in order to improve on any miscalculations that were contained in the previous formulas.

1. The mathematical formalism of the theory of positrons.

The theory starts with the Dirac "density matrix," which is given by:

$$(x't'k' \mid R \mid x''t''k'') = \sum_{\substack{(n) \\ \text{def.state}}} \psi_n^*(x''t''k'')\psi_n(x't'k') \tag{4}$$

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[1]) E. A. Uehling, Phys. Rev. **48**, 55, 1935.
[2]) R. Serber. *ibid.*, **48**, 49, 1935.
[3]) H. Euler and B. Kockel, Naturwissensch. **23**, 246, 1935.
[4]) W. Heisenberg, ZS. f. Phys. **90**, 209, 1934; notated in the sequel by loc. cit. I.

in intuitive wave theory, and by:

$$(x't'k' \mid R \mid x''t''k'') = \psi^*(x''t''k'')\psi(x't'k') \tag{5}$$

in the quantum theory of wave fields [1]). In addition to this matrix, an important role is also played by the matrix R_S, which is defined by:

$$(x't'k' \mid R_S \mid x''t''k'') = \frac{1}{2}\left(\sum_{\substack{(n) \\ \text{def.state}}} - \sum_{\substack{(n) \\ \text{indef.state}}} \right) \psi_n^*(x''t''k'')\psi_n(x't'k'), \tag{6}$$

and:

$$(x't'k' \mid R_S \mid x''t''k'') = \tfrac{1}{2}[\psi^*(x''t''k'')\psi(x't'k') - \psi(x't'k')\psi^*(x''t''k'')], \tag{7}$$

respectively. The matrix R_S is singular on the lightcone as a function of the differences $x'_\lambda - x''_\lambda = x_\lambda$, $t' - t'' = t$. If one sets:

$$ct = -\alpha_0 = -x_0; \qquad x_i = x^i; \qquad \xi_\lambda = \frac{x'_\lambda + x''_\lambda}{2}, \tag{8}$$

and further sets $A_0 = -A^0$; $A_i = A^i$, for the potentials, and $\alpha_0 = -\alpha^0 = 1$; $\alpha_i = \alpha^i$ for the Dirac matrices, then one has:

$$(x'k' \mid R_S \mid x''k'') = u\frac{\alpha^\rho x_\rho}{(x^\lambda x_\lambda)^2} - \frac{v}{x^\lambda x_\lambda} + w \log |x^\lambda x_\lambda|, \tag{9}$$

in which [2]):

$$u = -\frac{i}{2\pi^2} e^{+\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda dx_\lambda}. \tag{10}$$

(From now on, two equal Latin indices will always be summed from 1 to 3, and two equal Greek indices, from 0 to 3.) The integral is to be taken along the straight line from P' to P''.

One obtains the standard density matrix r for the phenomena of matter from the equation:

$$r = R_S - S, \tag{11}$$

in which S is given by:

$$S = e^{+\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda dx_\lambda} \cdot S_0 + \frac{\bar{a}}{x_\lambda x^\lambda} + \bar{b} \log \left| \frac{x_\lambda x^\lambda}{C} \right|. \tag{12}$$

Here, S_0 means the matrix R_S in field-free and matter-free space. $\bar{a}, \bar{b}$ and C are defined by the following equations [1])

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[1]) In loc. cit. I, the singly primed and doubly primed quantities in the right-hand side are incorrectly switched.

[2]) In loc. cit. I, the negative sign was incorrectly used in the exponent.

$$\left.\begin{aligned}
\bar{a} &= u\left\{\frac{ei}{24\hbar c}x_\rho x^\sigma \alpha^\lambda\left(\frac{\partial F_{\lambda\sigma}}{\partial \xi_\rho}-\delta^\rho_\lambda\frac{\partial F_{\tau\sigma}}{\partial \xi_\tau}\right)-\frac{e^2}{48\hbar^2c^2}x_\rho x_\sigma x^\tau \alpha^\rho F^{\mu\sigma}F_{\mu\tau}\right\},\\
\bar{b} &= u\left\{\frac{ei}{24\hbar c}\alpha^\lambda\frac{\partial F_{\tau\lambda}}{\partial \xi_\tau}+\frac{e^2}{24\hbar^2c^2}x_\lambda\alpha^\mu(F_{\tau\mu}F^{\tau\lambda}-\frac{1}{4}\delta^\lambda_\mu F_{\tau\lambda}F^{\tau\sigma}\right\},\\
C &= 4\left(\frac{\hbar}{\gamma mc}\right)^2,
\end{aligned}\right\} \qquad (13)$$

in which γ means the Euler constant $\gamma = 1.781\ldots$

One obtains the current density four-vector, as well as the energy-momentum tensor, from r by way of:

$$\left.\begin{aligned}
s_\lambda(\xi) &= -e\sum_{k'k''}\alpha^\lambda_{k'k'}(\xi k' \mid r \mid \xi k''),\\
U^\mu_\nu(\xi) &= \lim_{x\to 0}\left\{ic\hbar\frac{\partial}{\partial x_\mu}-\frac{e}{2}\left[A^\mu\left(\xi+\frac{x}{2}\right)+A^\mu\left(\xi-\frac{x}{2}\right)\right]\right\}\\
&\qquad \sum_{k'k''}\alpha^\nu_{k'k'}(\xi+\frac{x}{2},k' \mid r \mid \xi-\frac{x}{2},k'').
\end{aligned}\right\} \qquad (14)$$

In the quantum theory of wave mechanics, it is convenient to develop the wave function in an orthogonal system:

$$\psi(x,k)=\sum_n a_n u_n(x,k). \qquad (15)$$

One can represent the operators a_n in the form:

$$a_n^* = N_n\Delta_n V_n; \qquad a_n = V_n\Delta_n N_n, \qquad (16)$$

in which Δ_n transforms the number N_n into $1 - N_n$, and $V_n = \prod_{i\le n}(1-2N_i)$.

Furthermore, let:

$$a_n' = a_n^* = -V_n\Delta_n N_n'; \qquad a_n'^* = -N_n'\Delta_n V_n; \qquad N_n' = 1 - N_n.$$

With these variables, the Hamilton function of the total system is given by:

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[1]) Equation (38) in loc. cit. I contains an error that leads to another value for C; in that work, the notation γ was also used for the logarithm of the Euler constant, which contradicts the general usage.

$$
\begin{aligned}
H = \lim_{x\to 0}\int d\xi \quad & \left\{ -\left(ci\hbar\frac{\partial}{\partial x_l} - \frac{e}{2}\left[A^l\left(\xi+\frac{x}{2}\right) + A^l\left(\xi-\frac{x}{2}\right)\right]\right)\right. \\
& \sum_{k'k''}\alpha^l_{k''k'}\sum_{m,n}\frac{1}{2}(a_n^* a_m - a_m a_n^*)u_n^*\left(\xi-\frac{x}{2},k''\right)u_m\left(\xi+\frac{x}{2},k'\right) \\
& +\sum_{k'k''}\beta_{k''k'}mc^2\sum_{m,n}\frac{1}{2}(a_n^* a_m - a_m a_n^*)u_n^*\left(\xi-\frac{x}{2},k''\right)u_m\left(\xi+\frac{x}{2},k'\right) \qquad (17) \\
& -\sum_{k'}\left(ci\hbar\frac{\partial}{\partial x_0} - \frac{e}{2}\left[A^0\left(\xi+\frac{x}{2}\right) + A^0\left(\xi-\frac{x}{2}\right)\right]\right)\left(\xi+\frac{x}{2},k' \mid S \mid \xi-\frac{x}{2},k'\right) \\
& \left. +\frac{1}{8\pi}(\mathfrak{E}^2 - \mathfrak{B}^2)\right\},
\end{aligned}
$$

or individually by developing in powers of the elementary charge:

$$
\left.
\begin{aligned}
H_0 &= \sum_{En>0} N_n E_n - \sum_{En>0} N'_n E_n + \sum_{even} M_{even} h\nu_{even}, \\
H_1 &= \int d\xi\, eA^l(\xi)\sum_{k''k'}\alpha^l_{k''k'}\left[\sum_{En>0} N_n u_n^*(\xi,k'')u_n(\xi,k')\right. \\
&\quad \left. -\sum_{En>0} N'N_n u_n^*(\xi,k'')u_n(\xi,k') + \tfrac{1}{2}\sum_{n\neq m}(a_n^* a_m - a_m a_n^*)u_n^*(\xi,k'')u_m(\xi,k')\right] \\
H_2 &= \int d\xi\left[i\frac{e^2}{2\hbar c}\left(\int A^\lambda dx_\lambda\right)^2\sum_{k'}\frac{\partial}{\partial x_0}\left(\xi+\frac{x}{2},k' \mid S_0 \mid \xi-\frac{x}{2},k''\right)\right. \\
&\quad +\frac{1}{12\pi^2}\frac{e^2}{\hbar c}\frac{x_\lambda x^\sigma}{x_\rho x^\rho}A^\lambda\left(\frac{\partial F_{0\sigma}}{\partial\xi_0} - \frac{\partial F_{\tau\sigma}}{\partial\xi_\tau}\right) + \frac{1}{24\pi^2}\frac{e^2}{\hbar c}\frac{x_\sigma x^\tau}{x_\rho x^\rho}F^{\mu\sigma}F_{\mu\tau} \\
&\quad \left. +\frac{1}{12\pi^2}\frac{e^2}{\hbar c}\log\left|\frac{x_\rho x^\rho}{C}\right|(F^{\tau 0}F_{\tau 0} - \frac{1}{4}F_{\tau\mu}F^{\tau\mu})\right], \\
H_3 &= -\frac{1}{6}\frac{e^2}{\hbar^2 c^2}\int d\xi(A^\lambda x_\lambda)^3\frac{\partial}{\partial x_0}\sum_{k'}\left(\xi+\frac{x}{2},k' \mid S_0 \mid \xi-\frac{x}{2},k'\right), \\
H_4 &= -\frac{1}{12\pi^2}\left(\frac{e^2}{\hbar c}\right)^2\frac{1}{\hbar c}\int d\xi\frac{(A^\lambda x_\lambda)^4}{(x_\rho x^\rho)^2}.
\end{aligned}
\right\} \qquad (18)
$$

2. Computation of the energy density in intuitive wave theory.

Since the Lagrange function, which is associated with the desired discrepancy from the Maxwell equations, must be a function of only the two invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$, it suffices to compute it by means of the energy density of the material field as a function

of two independent field quantities. For example, it suffices to determine the energy density of matter in a constant electrical field and a constant magnetic field that is parallel to it. With these constant fields, one then seeks the state of the matter field that corresponds to the nonexistence of matter, which is then obviously the state of lowest energy. If one next bases one's reasoning on intuitive wave theory – hence, on equations (4) and (6) – then the state of lowest energy in field-free space is given by the state in which all of the negative energy electron states are filled and no positive energy electron states are filled. When only one magnetic field is present then the stationary state of an electron can be further subdivided in just one way into negative and positive energies such that the state of lowest energy for the matter field can therefore be obtained in the magnetic field precisely as it is in field-free space.

Things are not the same for an electrical field. Here, the potential energy increases linearly in one coordinate, total energies from $-\infty$ to $+\infty$ are possible, and the eigenfunctions that belong to the various eigenvalues generally experience a simple spatial displacement. A unique separation into positive and negative eigenvalues is not possible here.

This difficulty depends physically upon the fact that pairs of positrons and electrons can originate from a constant electric field in its own right. One must thank Sauter [1]) for the exact treatment of this problem. In Fig. 1, the potential energy $V(x)$ is plotted as a function of the coordinate (the electric field is taken to be parallel to the x-axis), along with the lines $V(x) + mc^2$ and $V(x) - mc^2$.

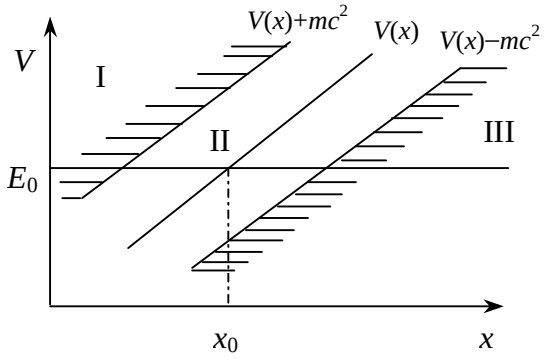

Fig. 1.

Sauter's computations show that the eigenfunction that belongs to, say, the eigenvalue E_0, is large only in the regions I and III, and decays exponentially in the region II. This has the consequence that a wave function that is perhaps large in the region I gradually goes over to region III, such that the transmission coefficient for region II, which plays the role of a Gamow mountain (?) here, has an order of magnitude that is given by Sauter as $e^{-\frac{m^2c^3}{\hbar e|\mathfrak{E}|}\pi}$. If one denotes the critical field strength by $|\mathfrak{E}_k| = \frac{m^2c^3}{\hbar e}$ then one can also write this expression as $e^{-\frac{|\mathfrak{E}_k|}{|\mathfrak{E}|}\pi}$. As long as $|\mathfrak{E}| \ll |\mathfrak{E}_k|$ one can neglect these extraordinarily rare

[1]) F. Sauter, ZS. f. Phys. **69**, 742, 1931.

pair production events at a certain level of approximation. It must therefore be possible to find solutions of the Dirac equation that take the place of eigenfunctions and are perhaps large only in region I and remain small in all of region III for a certain time with order $e^{-\frac{|\mathfrak{E}_k|\pi}{|\mathfrak{E}|2}}$; conversely, it must also be possible to find solutions that are large only in region III and almost vanish in I. When this is the case, one may characterize the state of lowest energy by saying that all electron states whose eigenfunctions are large only in region III exist, and none of the others exist. One lets E_0 denote the value of the energy of the aforementioned electron states in the computation of the energy density at the point x_0 [cf. equation (31)]. When one turns off the electric field adiabatically the total system state that was just characterized goes over to the state of a field-free space, in which only electron states of negative energy are present.

As part of our mathematical examination, we summarize the argument of Sauter (loc. cit.). When an external magnetic field $\mathfrak{B}$ and an external electric field $\mathfrak{E}$, both of which are in the *x*-direction, are present the Dirac equation becomes:

$$\left\{\frac{i\hbar}{c}\frac{\partial}{\partial t}-\frac{e}{c}|\mathfrak{E}|x+\alpha_1 i\hbar\frac{\partial}{\partial x}+\alpha_2 i\hbar\frac{\partial}{\partial y}+\alpha_3\left(i\hbar\frac{\partial}{\partial z}-\frac{e}{c}|\mathfrak{B}|y\right)-\beta mc\right\}\psi=0. \tag{19}$$

The motion in the *y*- and *z*-direction may be separated from that in the *x*-direction by the Ansatz:

$$\psi = e^{\frac{i}{\hbar}(p_z z-Et)}\cdot u_n(y)\cdot\chi. \tag{20}$$

We therefore introduce a new operator *K* by way of:

$$K = +\alpha_2 i\hbar\frac{\partial}{\partial y}+\alpha_3\left(-p_z-\frac{e}{c}|\mathfrak{B}|y\right)-\beta mc. \tag{21}$$

This gives:

$$K^2\psi = \left\{-\hbar^2\frac{\partial^2}{\partial y^2}-i\hbar\alpha_2\alpha_3\frac{e|\mathfrak{B}|}{c}+\left(-p_z+\frac{e}{c}|\mathfrak{B}|y\right)^2+m^2c^2\right\}\psi. \tag{22}$$

One can consider this equation to be the wave equation for the as-yet-undetermined function $u_n(y)$. If one sets:

$$y=\eta\sqrt{\frac{\hbar c}{e|\mathfrak{B}|}}-\frac{ep_z}{e|\mathfrak{B}|}, \tag{23a}$$

then, since (22) actually comes from to the Schrödinger equation for oscillators, one obtains:

$$u_n(y)=H_n(\eta)e^{-\frac{\eta^2}{2}}(2^n\cdot n!\cdot\sqrt{\pi})^{-1/2}\left(\frac{\hbar c}{e|\mathfrak{B}|}\right)^{-1/4}, \tag{23}$$

($H_n(\eta)$ is the n^{th} Hermite polynomial), from which it follows that:

$$K^2 u_n \chi = \left\{ m^2 c^2 + \frac{e\,|\,\mathfrak{B}\,|}{c}(2n+1+\sigma_x) \right\} u_n \chi\,. \tag{24}$$

Furthermore, the operator K anticommutes with α_1. One can now perform a canonical transformation on the wave equation (19), which can be written in the form:

$$\left\{ \frac{E - e\,|\,\mathfrak{E}\,|}{c} + \alpha_1 i\hbar \frac{\partial}{\partial x} + K \right\} \psi = 0, \tag{25}$$

in such a manner that σ_x becomes diagonal, whereas one obtains the following representations for K and α_1:

$$K = \sqrt{m^2 c^2 + \frac{e\,|\,\mathfrak{B}\,|\,\hbar}{c}(2n+1+\sigma_x)} \cdot \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}; \qquad \alpha_1 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \tag{26}$$

Thus, we denote both matrices with another index that is independent of the spin direction (hence, it is independent of the "ρ-coordinate"). One can the consider σ_x to be an ordinary number ($\sigma_x = \pm 1$), and, with the abbreviations:

$$\left.\begin{aligned} \xi &= \sqrt{\frac{1}{\hbar c e\,|\,\mathfrak{E}\,|}}(e\,|\,\mathfrak{E}\,|\,x - E), \\ k &= \sqrt{\frac{c}{\hbar c e\,|\,\mathfrak{E}\,|}\left(m^2 c^2 + \frac{e\,|\,\mathfrak{B}\,|\,\hbar}{c}(2n+1+\sigma_x) \right)} \end{aligned}\right\} \tag{27}$$

one obtains the equations:

$$\left.\begin{aligned} \left(\frac{d}{d\xi} - i\xi \right) f + kg = 0, \\ \left(\frac{d}{d\xi} + i\xi \right) g + kf = 0, \end{aligned}\right\} \tag{28}$$

in which f and g mean the two components (indicated by the "ρ"-index) of the function χ. Equations (28) are formally identical with equations (12) in Sauter (loc. cit.). They differ from those of Sauter in the meaning of the quantity k and in the fact that the system (28) must actually be written twice, once for $\sigma_x = +1$ and once for $\sigma_x = -1$. Sauter obtained two systems of solutions for equation (28):

$$\left.\begin{aligned} f_1 &= -\frac{1}{2\sqrt{\pi}}\,|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}} ds, \\ g_1 &= -\frac{1}{2\sqrt{\pi}}\,\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-1} ds, \\ f_2 &= -\frac{1}{2\sqrt{\pi}}\,\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-1}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds, \\ g_2 &= -\frac{1}{2\sqrt{\pi}}\,|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds. \end{aligned}\right\} \tag{29}$$

The integrals are taken along a path that comes in from $+\infty$, goes around both of the singular points + $i/2$ and – $i/2$ in the positive sense and returns to $+\infty$.

Since we will be neglecting pair creation in our computations, as we said above, we shall sometimes consider our eigenfunctions to be only pieces of the functions f and g that vanish in a half-space. We thus set, perhaps:

$$f_1^{\,1} = \begin{cases} f_1 \text{ for } \xi > 0, \\ 0 \quad \text{" } \xi \le 0, \end{cases} \qquad f_1^{\,2} = \begin{cases} 0 \quad \text{for } \xi > 0, \\ f_1 \quad \text{" } \xi \le 0, \end{cases} \quad \text{etc.} \tag{30}$$

The new functions $f_1^{\,1}$, etc., do not correspond precisely to stationary states, but represent wave packets that have a very slight probability of diffusing into the originally empty space. For the construction of the density matrix we consider the states $f_1^{\,1}, g_1^{\,1}, \ldots$ as present and the states $f_1^{\,2}, g_1^{\,2}, \ldots$ as absent. Since we have doubled the number of "states" by the process (30), when one computes all of the $f_1^{\,1}, g_1^{\,1}, f_2^{\,1}, g_2^{\,1}$ as present and the $f_1^{\,2}, g_1^{\,2}, f_2^{\,2}, g_2^{\,2}$ as absent, one will obtain precisely twice the density matrix.

In order to compute the energy density of the vacuum, one must next compute the density matrix for a finite distance between the points $\mathfrak{r}'$ and $\mathfrak{r}''$ using the method that was described in sec. 1. For the following calculations, it is more convenient to set $\mathfrak{r}' = \mathfrak{r}''$ from the outset, but, for that reason, the summation over the stationary states is to be taken only over the finite energies, or, what is roughly the same thing, one must make this summation convergent by means of an additional factor of $e^{-\text{const.}\cdot[k^2-(mc^2)^2]}$. If one lets the constant in the exponent go to zero then some of the terms of the energy matrix will become singular, and they will be compensated for by the corresponding terms of the S matrix. The remaining regular terms give the desired result.

Before the density matrix can be written down, the eigenfunctions must be normalized. One can perhaps think of the space of eigenfunctions in the x- and z-directions as shrunk down to a very long length L [the eigenfunction $u_n(y)$ is already normalized]. One then obtains the normalization factor $1/\sqrt{L}$ in the z-direction; in the x-direction, one has the asymptotic expressions of the Sauter eigenfunctions [loc. cit. (22)]

with the normalization factor $2\frac{1}{\sqrt{L}}e^{-\frac{k^2\pi}{4}}$. The sum over all states is then to be taken over all momenta of the form:

$$p_z = \frac{h}{L}\cdot m + \text{const.}$$

and over all energies of the form $E = \frac{hc}{(L/2)}m + \text{const.}$ One can therefore introduce these two sums into the integral whose differential is $\frac{dp_z}{h}\cdot\frac{dE}{2hc}$ when one omits the $\frac{1}{\sqrt{L}}$ factors in the eigenfunction. If one ultimately regards the energy of a state, when we compute the density at the point x_0, as the difference $E - e|\mathfrak{E}|x_0$ then one obtains the following expression for the corresponding energy density for the matrix R_S (cf. sec. 1) [for the meaning of α, cf. (33)]:

$$U = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{\sigma=-1}^{1}\int_{-\infty}^{+\infty}\frac{dp_z}{h}\int\frac{dE}{hc}(E - e\,|\,\mathfrak{E}\,|\,x)u_n^2(y)e^{-\frac{k^2\pi}{2}} \begin{bmatrix} |f_1^1|^2 + |g_1^1|^2 - |f_1^2|^2 - |g_1^2|^2 \\ |f_2^1|^2 + |g_2^1|^2 - |f_2^2|^2 - |g_2^2|^2 \end{bmatrix} e^{-\alpha\left(\xi^2 - \frac{1}{a}\right)}, \tag{31}$$

which, due to (23a), goes to:

$$U = -\sum_{n=0}^{\infty}\sum_{\sigma=-1}^{1}\frac{e\,|\,\mathfrak{B}\,|}{c}\frac{\hbar e\,|\,\mathfrak{E}\,|}{h^2}\int_{-\infty}^{+\infty}d\xi\,|\,\xi\,|\,e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}} \frac{1}{2}\left[|f_1|^2 + |g_1|^2 - |f_2|^2 - |g_1|^2\right]e^{-\alpha\xi^2}. \tag{32}$$

We will discuss what results from this expression when $\alpha \gg 0$ later. We introduce the following abbreviations:

$$\frac{e\,|\,\mathfrak{E}\,|\,\hbar}{m^2c^3} = a; \qquad \frac{e\,|\,\mathfrak{B}\,|\,\hbar}{m^2c^3} = b. \tag{33}$$

a and *b* are dimensionless and mean the ratios of the field strengths to the critical field strengths $|\mathfrak{E}_k|$, i.e., to the "137th part of the field strength at the surface of the electron."

By substitution in equation (29) one ultimately obtains:

$$U = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\sigma=-1}^{1}a\cdot b\cdot mc^2\left(\frac{mc}{\hbar}\right)^3\cdot\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}d\xi\,|\,\xi\,|\,e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}-\alpha\xi^2} \int ds_1\int ds_2 e^{-\xi^2(s_1+s_2)}e^{-\frac{k^2}{4i}\log\frac{\left(s_1+\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}{\left(s_1-\frac{i}{2}\right)\left(s_2-\frac{i}{2}\right)}}\cdot h(s_1s_2), \tag{34}$$

in which $h(s_1 s_2)$ is given by:

$$h(s_1 s_2) = \frac{1}{2\pi\left(s_1 + \frac{i}{2}\right)^{1/2}\left(s_2 - \frac{i}{2}\right)^{1/2}}\left[\xi^2 + \frac{k^2}{4\left(s_1 - \frac{i}{2}\right)\left(s_2 + \frac{i}{2}\right)}\right].$$

The integration over ξ gives:

$$U = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\sigma=-1}^{1} a\cdot b\cdot mc^2\left(\frac{mc}{\hbar}\right)^3 \cdot \int ds_1 \int ds_2 e^{-\xi^2(s_1+s_2)} e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}-\frac{k^2}{4i}\log\frac{\left(s_1+\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}{\left(s_1-\frac{i}{2}\right)\left(s_2-\frac{i}{2}\right)}}$$

$$\frac{1}{8\pi^3}\left(s_1 + \frac{i}{2}\right)^{1/2}\left(s_2 - \frac{i}{2}\right)^{1/2}\left[\frac{1}{(s_1+s_2+\alpha)^2} + \frac{k^2}{4(s_1+s_2+\alpha)\left(s_1-\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}\right]. \qquad (35)$$

The first of the two s-integrals, perhaps the one over s_1, may be easily completed when one deforms the path of integration in such a way that only one loop remains around the pole $s_1 = -s_2 - \alpha$. If one replaces s_2 with $s = s_2 + \alpha/2$ in the result then one obtains:

$$U = \sum_{n=0}^{\infty}\sum_{\sigma=-1}^{1} a\cdot b\cdot mc^2\left(\frac{mc}{\hbar}\right)^3 \cdot f(k)\cdot e^{\frac{\alpha}{a}}, \qquad (36)$$

in which:

$$f(k) = -\int ds \frac{1}{32\pi^2}\left(s_1 - \frac{i}{2} + \frac{\alpha}{2}\right)^{-\frac{3}{2}}\left(s_2 - \frac{i}{2} - \frac{\alpha}{2}\right)^{-\frac{1}{2}}\left(s + \frac{i}{2} + \frac{\alpha}{2}\right)^{-1}\left(s + \frac{i}{2} - \frac{\alpha}{2}\right)^{-1}$$

$$\left[k^2(i-\alpha) + 2\left(s+\frac{i}{2}\right)^2 - \frac{\alpha^2}{a}\right] e^{-\frac{k^2\pi}{2}-\frac{k^2}{4i}\log\frac{s^2-\left(\frac{i-\alpha}{2}\right)^2}{s^2-\left(\frac{i+\alpha}{2}\right)^2}}. \qquad (37)$$

Here, the integration is along the path that is given in Fig. 2, which comes in from $+\infty$, goes between the four poles in the integrand ($s = \pm i/2 \pm \alpha/2$), and goes back to $+\infty$.

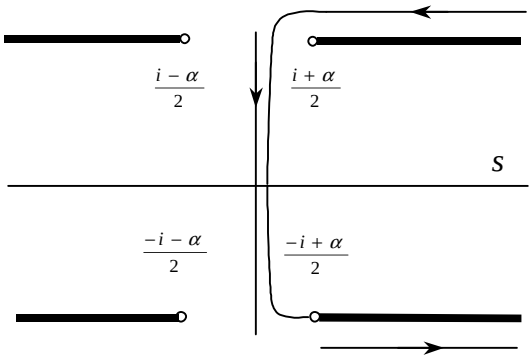

Fig. 2.

Therefore, one can also simply integrate along the imaginary axis from $+\infty$ to $-\infty$. The main contribution is given by the part of the integration path that lies between the poles. There, one can develop the logarithm in the exponent in powers of α and obtain:

$$\log\frac{s^2-\left(\frac{i-\alpha}{2}\right)^2}{s^2-\left(\frac{i+\alpha}{2}\right)^2}=-2\pi i+\frac{i\alpha}{s^2+\frac{1}{4}}+\frac{i\alpha^3}{4(s^2+\frac{1}{4})^2}-\frac{i\alpha^3}{12(s^2+\frac{1}{4})^3}+\cdots \tag{38}$$

For the computations that follow, we shall assume that the electric field is small compared to the critical field $|\mathfrak{E}_k|$, i.e., $\alpha \ll 1$, and therefore:

$$k^2=\frac{1}{a}[1+b(2n+1+\sigma_x)] \ll 1.$$

One must therefore leave the expression $-\dfrac{k^2\alpha}{4s^2+1}$ in the exponent; however, one can regard the higher terms in the exponent as small and omit them, since ultimately one will consider only the lim $\alpha \to 0$. In this way, one obtains an expression for $f(k)$ that has the form:

$$f(k)=\frac{1}{2\pi^2}\int_{+i/2}^{-i/2} ds\, e^{-\frac{k^2\alpha}{4s^2+1}}(1+4s^2)^{-2}[A+Bk^2+Ck^4+\ldots] \tag{39}$$

In order to perform this integral, it is useful to invert the summations over n and σ; this results from the schema:

$$\sum_{\sigma=-1}^{+1}\sum_{n=0}^{\infty} g\left(n+\frac{1+\sigma_x}{2}\right)=\left(\sum_0^\infty+\sum_1^\infty\right)g(n)$$

$$=\lim_{n'\to\infty}\left\{g(0)+2\sum_1^{n'}g(n)+2\int_{n'+\frac{1}{2}}^{\infty}g(n)dn+\tfrac{1}{2}g'(n'+\tfrac{1}{2})+\cdots\right\}. \tag{40}$$

From this, it appears that the higher terms of the Eulerian summation formula do not contribute to the end result. Ultimately, when one sets $\alpha / a = \varepsilon$ and lets $\varepsilon \to 0$, one obtains (γ means the Euler constant $\gamma = 1.781$):

$$\begin{aligned}
&\frac{U}{4\pi mc^2}\left(\frac{h}{mc}\right)^3 \\
&\quad = \lim_{n'\to\infty}\left\{-\frac{1}{\varepsilon^2}-\frac{1}{2\varepsilon}-\left(\frac{1}{8}+\frac{a^2}{12}+\frac{b^2}{12}\right)\log\frac{\gamma\varepsilon}{4}-\frac{1}{16}-\frac{b^2}{6}-\frac{a^2}{12}\right. \\
&\qquad -\frac{1}{16}+\frac{b^2}{8}+\frac{[1+(2n'+1)b]^2}{16} \\
&\qquad -\frac{1}{8}[1+(2n'+1)b]^2\log[1+(2n'+1)b] \\
&\qquad +\frac{b^2}{24}\log[1+(2n'+1)b]+\frac{b}{2}\sum_1^{n'}(1+2nb)\log(1+2nb) \\
&\qquad +\frac{a^2}{12}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)}-\log[1+(2n'+1)b]\right\} \\
&\qquad +\frac{a^4}{80}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)^3}\right\} \\
&\qquad \left.+\sum_{m=3}^{\infty}c_m a^{2m}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)^{2m-1}}\right\}\right. \qquad (41).
\end{aligned}$$

(The coefficients c_m are yet to be determined.)

Now we have to subtract all of the terms from this result that correspond to the singular matrix *S*. One easily obtains the part of this truncated singular energy density that is independent of the field when one carries out the previous computations with even waves:

$$\begin{aligned}
U_S' &= -2\int c\sqrt{m^2c^2+p^2}\; e^{-\varepsilon\left(\frac{p}{mc}\right)^2}\frac{dp_x dp_y dp_z}{h^3} \\
&= 4\pi mc^2\left(\frac{mc}{h}\right)^3\left(-\frac{1}{\varepsilon^2}-\frac{1}{2\varepsilon}-\frac{1}{8}\log\frac{\gamma\varepsilon}{4}-\frac{1}{16}\right).
\end{aligned}$$

On the other hand, it is more difficult to ascertain the field-independent part of *S*. Since equation (13) implies that only the squares of the field strengths enter into $\bar{a}$ and $\bar{b}$, the same is true for U_S. Furthermore, since the constant *C* in equation (13) is chosen in such a way that for constant fields a field that is proportional to the vacuum polarization does not enter, it then follows that the terms in the squares of the fields should all be omitted, and only the higher terms remain. When one anticipates the development in *b* for $b \ll 1$, this means that the first terms of the right-hand side of equation (41) are to be completely ignored. We have also proved this result in the case $a = 0$, $b \neq 0$, when we assumed that the points $\mathfrak{r}'$ and $\mathfrak{r}''$ in the density matrix were different along the *x*-axis [$\mathfrak{r}' - \mathfrak{r}'' = (x, 0, 0)$]

and then set $\alpha = \varepsilon = 0$. By deducting the terms that originate in the matrix *S*, all that remains is the part of *U* that we just spoke of. This computation proves to be very complicated for the electric field. For the total energy density, which combines the ordinary Maxwellian energy density $\frac{1}{8\pi}(\mathfrak{E}^2 + \mathfrak{B}^2)$ and the Dirac energy density $U - U_S$, we can likewise go over to the Lagrange function by means of relation (3):

$$a\frac{\partial \mathcal{L}}{\partial a} - \mathcal{L} = 4p\left(U - U_S + \frac{1}{8\pi}(\mathfrak{E}^2 + \mathfrak{B}^2)\right).$$

One obtains:

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) \\ & + 16\pi^2 mc^2 \left(\frac{mc}{h}\right)^3 \lim_{n'\to\infty}\left\{\frac{1}{16} - \frac{b^2}{8} - \frac{[1+(2n'+1)b]^2}{16}(1 - 2\log[1+(2n'+1)b])\right. \\ & - \frac{b^2}{24}\log[1+(2n'+1)b] - \frac{b}{2}\sum_1^{n'}(1+2nb)\log(1+2nb) \\ & + \frac{ba^2}{12}\left(1 + 2\sum_1^{n'}\frac{1}{1+2nb} - \frac{1}{b}\log[1+(2n'+1)b]\right) \\ & + \frac{ba^4}{90}\left(1 + 2\sum_1^{n'}\frac{1}{(1+2nb)^3}\right) \\ & \left. + \sum_{m=3}^{\infty}\frac{c_m}{2m-1}a^{2m}b\left(1 + 2\sum_1^{n'}\frac{1}{(1+2nb)^{2m-1}}\right)\right\}. \end{aligned} \qquad (42)$$

For small magnetic fields, one can find the power series development of *b* by repeated application of the Eulerian summation formula in equation (42). Furthermore, since $\mathcal{L}$ can depend only on the two invariants $a^2 - b^2$ and a^2b^2 – from which, it follows for $\mathcal{L}(a, b)$ that $\mathcal{L}(a, 0) = \mathcal{L}(0, ia)$ – one can then indirectly ascertain the missing coefficients c_m, whose direct computation is very long-winded, from this relation. Since the direct computation of c_m gives the same result, we have checked the computation of c_2 and c_3; we have not found the general proof of this. In this way, one next obtains, for small fields ($a \ll 1$, $b \ll 1$):

$$\begin{aligned}\mathcal{L} \approx {} & \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) \\ & + 16\pi^2 mc^2 \left(\frac{mc}{h}\right)^3 \left[\frac{(a^2-b^2)^2 + 7(ab)^2}{180} + \frac{13(ab)^2(a^2-b^2) + 2(a^2-b^2)^3}{630} + \cdots\right]. \end{aligned} \qquad (43)$$

In the opposite case ($a \gg 1$, $b \gg 1$), one obtains:

$$\mathcal{L} \approx \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 16\pi^2 mc^2 \left(\frac{mc}{h}\right)^3 \left\{ b^2 \left[\frac{1}{12}\log b - 0.191 \right] + \frac{b}{4}[\log b - 0.145] + \frac{\log b}{8} + 0.202 - \frac{a^2}{12}[\log b + 0.116] + b\left[\frac{a^2}{12} + \frac{a^4}{90} + \cdots \right] \right\}. \tag{44}$$

In order to get a better overview of the general expression for $\mathcal{L}$, we have sought to obtain an integral representation for $\mathcal{L}$. This is possible when one starts with the usual integral representation for the zeta function. One obtains:

$$\mathcal{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 mc^2 \left(\frac{mc}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -a\eta \cot a\eta \cot b\eta + 1 + \frac{\eta^2}{8}(b^2 - a^2) \right\}$$

$$= \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 mc^2 \left(\frac{mc}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -iab\eta^2 \frac{\cos(b+ia)\eta + \cos(b-ia)\eta}{\cos(b+ia)\eta - \cos(b-ia)\eta} + 1 + \frac{\eta^2}{8}(b^2 - a^2) \right\}. \tag{45}$$

In the latter form, it is particularly simple to see that $\mathcal{L}$ depends only upon the two invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$. The cosine terms allow a development in the squares of the arguments $(b + ia)^2 = b^2 - a^2 + 2i(ab)$, $(b - ia)^2 = b^2 - a^2 - 2i(ab)$, resp. Since the total result is real, it can therefore be represented as a power series in $b^2 - a^2$ and $(ab)^2$, which can generally be replaced with $\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{|\mathfrak{E}_k|^2}$, $\frac{(\mathfrak{E}\mathfrak{B})^2}{|\mathfrak{E}_k|^4}$, resp. $\left(|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} \right)$. For arbitrarily directed fields, the Lagrange function then looks like:

$$\mathcal{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|} \sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})} \right) + c.c.}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|} \sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})} \right) - c.c} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{8}(\mathfrak{B}^2 - \mathfrak{E}^2) \right\}. \tag{45a}$$

The first term of the development in equation (43) agrees with the results of Euler and Kockel (loc. cit.)

All of this begs the question of the convergence of this power series under a closer examination. If one sets $a = 0$ then the integral (45) converges for all values of b. However, when $a \neq 0$ the integral becomes undefined when $\eta = \pi / a, 2\pi / a, \ldots$, which makes $\cot a\eta$ infinite. Correspondingly, the power series expansion in a that we started with can also be only semi-convergent. One can give the integral (45) a unique sense when one traces out an integration path that avoids the singular places π / a, $2\pi / a$. However, the integral (45) then includes additional imaginary terms that cannot be given any immediate physical interpretation. One immediately comprehends their sense when one estimates their magnitudes. At the pole $\eta = \pi / a$ of (45) the integral has (for $b = 0$) the value $-\frac{2i}{\pi} \cdot 4a^2 mc^2 \left(\frac{mc}{h}\right)^3 e^{-\frac{\pi}{a}}$. This is the order of magnitude of the terms that describe pair creation in an electric field. The integral (45) must therefore be seen as similar to an integration over a resonance denominator in perturbation theory. One can assume that a damping term, which corresponds to the frequency in the resonance process, suffices for the convergence of the integral and that the result that one obtains by going around the singular places is correct up to terms whose magnitude corresponds to that of the frequency in the resonance processes.

From (43) and (44), the discrepancy from Maxwell's theory remains very small as long as $\mathfrak{E}$ and $\mathfrak{B}$ are small compared to the electric field that one finds at a distance of $\sqrt{137} \cdot e^2/mc^2$ from the center of mass of an electron. However, when, say, the magnetic field exceeds this value then the additional terms in the Maxwell equations remain small (when compared to the original terms), with a relative order of $\frac{1}{3\pi}\frac{e^2}{\hbar c}$, as long as $\log b$ has the order of magnitude 1. The discrepancy from, e.g., the usual Coulomb force between two protons, which originates in the terms (43) and (44), therefore also remains very small. Everything must be considered in this estimate, since even for a Coulomb field the additional terms that the derivation of the field strengths imply can be more important than the ones that were considered in equations (43) and (44).

3. Meaning of the results in the quantum theory of wave fields.

The results that were derived in the last section cannot be interpreted in the quantum theory of wave fields without further assumptions. Rather, one can easily show that in the quantum theory of wave fields the state of matter in a homogeneous field cannot be described by the equations just obtained. Namely, if one starts with the state of matter that was dealt with in the last section as an "unperturbed state" then there are matrix elements in perturbation theory that refer to a simultaneous creation of a light quantum and a pair. Now, when the energy is not also sufficient for the production of these particles these matrix elements give rise to a perturbing energy of second order. This therefore leads to the virtual possibility of the creation and annihilation of light quanta, and the calculation gives a divergent result for them. One makes the appearance of these perturbing terms intuitively plausible when one makes the assumption that, e.g., the loops in a magnetic field are actually not stationary states, but that electrons can radiate in these states. In the classical theory of wave fields one needs – and this is crucial in order for

the calculations of the last section to have any physical content – to overlook these rays, since it is to be understood that the charge density and current density that are obtained from the ultimate solution must vanish, so therefore no radiation can appear. On the other hand, in the quantum theory of waves a residue of this radiation remains in the form of a divergent perturbing energy of second order.

In general, one can also realize that a perturbing energy of precisely the same type as in a *field-free* vacuum enters in (the "self-energy of the vacuum"). Such self-energies then always appear when one calculates the contributions of second or higher order that originate in the virtual transition to another state and a *return to the initial state.* Until now, one has always neglected this self-energy. For example, one calculates the cross-section for Compton scattering, which leads to a perturbation calculation up to second order. If one also considers the terms of fourth order then one obtains contributions of the aforementioned sort that give no convergent result. The calculation of the scattering of light by light (loc. cit.) is performed by a perturbation calculation up to fourth order (hence, up to first order, which makes a contribution to the process in question); the second order contributions will already diverge here. The previous results of these calculations – e.g., the Klein-Nishina formula – seem to show that these omissions of the divergent terms of higher order lead to the correct result. When this is the case, then, from the foregoing, the results in sec. 2 can also be interpreted in terms of the quantum theory of waves. This is also physically plausible, since the appearance of the so-called radiation terms remains correspondingly hard to understand. Every single term in the development of the energy density in powers of $\mathfrak{E}$ and $\mathfrak{B}$ can then be intuitively associated with a scattering process whose cross-section is determined from that term. The terms of fourth order, for example, give the usual scattering of light by light, the terms of sixth order determine the cross-section for the process in which three light quanta are scattered by each other, and so on. Completely independent of the question of whether omitting the terms of higher order is physically permissible, is the question of whether every term of the development in the results of the last section corresponds to results of a direct calculation of the scattering processes in question in terms of the quantum theory of waves, when the perturbation computation is carried out only up to terms of first order that give a contribution to the process in question; contributions from terms in both methods of calculation are then neglected if they correspond to the creation and annihilation of light quanta and pairs. [The correspondence of the terms of fourth order with the ones that are obtained by direct computation of the scattering of light by light thus obtained [1]) thus gives a test of the correctness of the computation.] On this basis, it seems not impossible that the results for $|\mathfrak{B}| \gg |\mathfrak{E}_k|$ can be applied in practice. This is, however, certainly *not* possible for $|\mathfrak{E}| \gg |\mathfrak{E}_k|$, since the appearance of pair terms in the previous calculations has no basis for large electric fields.

[1]) H. Euler and B. Kockel, loc. cit.

4. The physical consequences of the results.

The results that were presented in the second section have considerable formal similarity with the Ansätze that Born [1]) has based his alteration of the Maxwell equations upon. Also, for Born, a complicated function of the two invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$ appears in place of the classical Lagrange function $\mathfrak{E}^2 - \mathfrak{B}^2$ that usually [cf. [1])] makes (43) agree with the actual numerical values of e^2/hc up to the order of magnitude of the first terms. It is therefore important to also emphasize the differences between the results. For Born, the altered Maxwell equations were made the starting point of the theory, whereas in the Dirac theory this alteration seems to be an indirect consequence of the virtual possibility of pair production. This is connected with the fact that in the Dirac theory the alteration of the Maxwell equations that was calculated here is in now the only one – one comes to terms that include higher derivative of the field strengths [2]) – and therefore the question of the self-energy of the electron cannot be decided by the consideration of only this discrepancy. Hence, the result of Born's theory, that alterations of the Maxwell equations of the order of magnitude that was discussed here suffice under the circumstances to remove the difficulty of the infinite self-energy, remains an important suggestion for the further development of the theory.

In connection with this, one must also pose the question of whether results that were derived from the Dirac theory concerning the scattering of light by light, etc., can be considered as ultimately valid, or if one is to expect that some later theory will lead to other outcomes. Undoubtedly, the theory of positrons and the present form of quantum electrodynamics are to be regarded as provisional. In particular, it seems that the prescription for constructing the *S* matrix (the inhomogeneity in the Dirac equation) seems arbitrary in this theory. For the particular matrix that was introduced in loc. cit. I, one can hardly recommend much beyond its relative mathematical simplicity (together with any claims to the formulation of the conservation laws). On this basis, deviations of the later theory from the earlier one seem possible at this point. Since such deviations can exert a definite influence on the altered Maxwell equations, one can ignore these discrepancies in the earlier theory only in their order of magnitude and qualitative form. It is therefore also hardly possible to make a definite statement of the ultimate form of the Maxwell equations in future quantum field theory since it is completely indispensable to have an inkling of the totality of processes in which particles of very high energy participate (e.g., the appearance of "jets").

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[1]) M. Born, Proc. Roy. Soc. London (A) **143**, 410, 1933; M. Born and L. Infeld, *ibid.*, **144**, 425, 1934; **147**, 522; **150**, 141, 1935.

[2]) E.A. Uehling, loc. cit., and R. Serber, loc. cit.