"Ueber discontinuirliche Flüssigkeitsbewegungen," Monatsberichte d. königl. Akad. d. Wiss. zu Berlin (1868), 215-228; *Wissenschaftliche Abhandlungen*, v. 1., Barth, Leipzig, 1882, pp. 146-157.

On discontinuous fluid motions

By

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It is known that the equations of hydrodynamics yield precisely the same partial differential equation for the interior of an incompressible fluid that is not subject to viscosity and whose particles possess no rotational motions that exists for stationary currents of electricity or heat in conductors of uniform conductance. One can thus expect that the form of the currents of a drop-forming fluid, electricity, and heat should be the same for the same form of the space that is flowed through and the same boundary conditions, up to small differences that would depend upon supplementary conditions. However, in reality, easily-recognizable and very compelling differences exist between the distribution of current in a drop-forming fluid and that of the aforementioned imponderables in many cases.

Such differences, in fact, prove to be quite striking when the current enters a larger space through an opening with sharp edges. In such cases, the streamlines of electricity will radiate from the opening the same way in all directions, while a streaming fluid (e.g., water or air) will initially move forwards from the opening in a compact jet and then resolve into vortices at a lesser or greater distance from it. By contrast, the part of the fluid in the larger container that lies close to the opening, but outside of the jet, can remain almost completely at rest. Everyone knows about that kind of motion, as an air current that is impregnated with smoke will show quite intuitively. In fact, the compressibility of air does not come under consideration in these processes in an essential way, and air will thus exhibit the same form of motion as water with only minor deviations.

From the large deviations between reality and the results of theoretical analysis up to now, the hydrodynamical equations of the physicists must seem to be a very incomplete approximation to reality in practice. One would like to suspect that the root cause of that situation lies in the internal viscosity of the fluid, although there are all sorts of strange and jump-like irregularities that everyone must struggle with and which are imposed upon fluid motions by observations that cannot be explained by the viscosity, which acts continuously and uniformly, in any event.

The examination of the cases in which periodic motions are excited by a continuous air current – as, e.g., in organ pipes – allowed me to believe that such an effect can be provoked only by a discontinuous kind of air motion (or, at least, something of a closely-related nature), and that led me to search for a condition that must be considered in the integration of hydrodynamical equations, and has thus been overlooked (as far as I know). By contrast, by considering it in those cases where calculations can be performed, one will, in fact, deduce forms of motion that we have observed in reality. That is the case at hand.

In the equations of hydrodynamics, the velocities and pressure of the streaming particles are treated as continuous functions of the coordinates. On the other hand, there is nothing in the nature of a drop-forming fluid, when we consider it to be completely fluid (and thus, not subject to viscosity) for saying that two fluid layers that lie next to each other cannot slide past each other with finite velocities. At the very least, the properties of fluids that are considered in the equations of hydrodynamics – namely, the constancy of mass in every spatial element and the equality of the pressure in all directions – obviously define no impediment to saying that tangential velocities of a finite absolute difference cannot exist on both sides of a surface that extends through the interior. By contrast, the components of the velocities that are perpendicular to the surface and the pressure must naturally be the same on both sides of such a surface. In my paper on vortex motions ([*]), I have already pointed out that such a case must occur when two previously-separate and differently-moving masses of water come into contact at their outer surfaces. In that paper, I was led to the concept of a *separation surface* – or *vortex surface*, as I called it there – in such a way that I thought of vortex filaments as being distributed continuously along a surface whose mass could be vanishingly small without its rotational moment vanishing.

Now, a fluid that is initially at rest or moving continuously can produce a finite difference from the motion of the immediately-neighboring fluid particles only by means of moving forces that act discontinuously. Thus, among the external forces, only impacts will come under consideration.

However, a source that can generate discontinuities in the motion is also present in the interiors of fluids. Namely, the pressure can indeed assume any arbitrary positive value, and the density of the fluid will always vary continuously with it. However, a discontinuous variation in the density will come about as soon as the pressure passes the value zero and becomes negative; i.e., the fluids will rip apart.

Now, the magnitude of the pressure in a moving fluid depends upon the velocity, and indeed the reduction of the pressure in incompressible fluids will be directly proportional to the *vis viva* of the moving water particle, all other things being equal. Thus, if the latter exceeds a certain magnitude then, in fact, the pressure must become negative and the fluid must tear apart. At such a place, the accelerating force (which is proportional to the differential quotient of the pressure) will obviously become discontinuous, and will thus fulfill the condition that is necessary in order for a discontinuous motion of the fluid to come about. The motion of the fluid past such a place can happen only in such a way that a separation surface is defined from there onward.

The velocity that the tearing of the fluid must induce is then one that the fluid must assume when it flows into the empty space under the pressure that the fluid would have in the rest state at the same position. In general, that is a relatively appreciable velocity. However, it should probably be pointed out that if the drop-forming fluid is to flow continuously like electricity then the velocity at any sharp edge around which the current

([*]) Journal für die reine und angewandte Mathematik, Band LV.

bends must become infinitely large ([1]). It follows from this that *any geometrically complete sharply-defined edge at which fluids flow past must tear itself from the most typical velocity of the remaining fluid and define a separation surface.* By contrast, at incompletely formed, rounded edges the same thing will first occur at certain large velocities. Pointed projections on the wall of a stream channel must have similar effects.

As far as gases are concerned, the same situation will be true for them that is true for fluids, except that the *vis viva* of the motion of a particle will not be directly proportional to the reduction of the pressure p, but with consideration given to the cooling of the air by its expansion, to the decrease of p^m, where $m = 1 - 1 / \gamma$, and γ means the ratio of the specific heat at constant pressure to the specific heat at constant volume. (For the air in the atmosphere, the exponent m has the value 0.291.) Since it is positive and real, p^m, like p, can decrease only to zero for high values of the velocity, and not become negative. Things would be different if the types of gas simply followed *Mariotte's* law and suffered no change in temperature. The quantity $\log p$ would then enter in place of p^m, which can become negatively infinite without p having to become negative. A tearing of the air mass would not be necessary under that condition.

One becomes convinced of the actual existence of such discontinuities when one lets a jet of smoke-impregnated air emerge from a round opening or a cylindrical tube with a moderate velocity in such a way that no sputtering arises. Under advantageous circumstances, one can obtain thin jets of the kind of a diameter in a longitude with several feet. Inside of the cylindrical outer surface, the air is then in motion with a constant velocity, while outside of it, the air does not move at all, or only slightly, in close proximity to the jet. One also sees this sharp separation very clearly when one directs a calmly-flowing cylindrical jet of air through the tip of a flame, from which, it then cuts out a precisely-delimited piece, while the rest of the flame remains completely undisturbed, and at most a very thin lamella that corresponds to the boundary layers that are influenced by friction will be carried along somewhat.

As far as the mathematical theory of these motions is concerned, I have already given the boundary conditions for an internal separation surface of the fluid. They consist of saying that the pressure must be the same on both sides of the surface, and likewise for the component of the velocity that is directed normal to the separation surface. Now, since the motion in the entire interior of an incompressible fluid whose particles have no rotational motion will be determined completely when the motion of its entire outer surface and its internal discontinuities are given, it is only a matter of establishing the rule for the motion of the separation surface and the changes in the discontinuity on it for an externally-fixed fluid boundary.

Now, such a separation surface can be treated mathematically exactly as if it were a *vortex surface* – that is, as if it were continuously filled with vortex filaments of infinitely-small mass, but finite rotational moment. In each surface element, there will be a direction along which the components of the tangential velocities can be taken to be equal. At the same time, it will give the direction of the vortex filament at the corresponding place. The moment of that filament is set to be proportional to the difference that the components of the tangential velocities that are perpendicular to it exhibit on both sides of the surface.

([1]) At the very small distance ρ from a sharp edge whose faces come together at the angle α, the velocity will become infinite like ρ^{-m}, where $m = (n - a) / (2n - a)$.

The existence of such vortex filaments is a mathematical fiction for an ideal, inviscid fluid that integration will alleviate. In a real fluid that is subject to viscosity, that fiction will rapidly become a reality when the boundary particles are set into rotation by friction, and thus vortex filaments of finite, gradually-decreasing masses will arise there, while the discontinuity in the motion will become simultaneously balanced out by it.

The motion of a vortex surface and the vortex filaments that lie in it is to be determined by the rules that were established in my paper on vortex motions. Admittedly, the mathematical difficulty in that problem can be first overcome in some of the simpler cases. By contrast, in many other cases, one can at least infer the direction of the variations that will occur from the given way of looking at things.

In fact, it should be mentioned that according to the laws that were proved for the motions of vortices, the filaments – and with them, the vortex surface – do not arise in the interior of an inviscid fluid, and cannot vanish, but rather, each vortex filament must constantly maintain the same rotational moment. Furthermore, one has the fact that the vortex filament along a vortex surface itself swims forward with a velocity that is the mean of the velocities that exist on the two sides of the surface. It follows from this that *a separation surface can always lengthen only in the direction along which the stronger of the two currents that contact it is directed.*

I then sought to find examples of separation surfaces in stationary currents that exist unchanged and for which the integration can be performed in order to test whether the theory yields forms of current that correspond to experience better than when one leaves the discontinuity of the motion unconsidered. If a separation surface that divides calm and moving water from each other is to remain stationary then the pressure along it must be the same in the moving part as it is in the calm part, from which, it will follow that the tangential velocity of the water particles must be constant along the entire extent of the surface; the same must be true for the density of the fictitious vortex filament. The beginning and the end of such a surface can only lie on the wall of the vessel or at infinity. When the former is the case, it must be tangent to the wall of the vessel, assuming that it is curved continuously there, since the component of the velocity that is normal to the wall of the vessel must be equal to zero.

Moreover, as experiment and theory consistently reveal, the stationary forms of the separation surfaces are distinguished by a strikingly high degree of variability for the most insignificant currents, such that they will maintain bodies that are in a state of unstable equilibrium somewhat similarly. The astounding sensitivity of a cylindrical jet of air that is impregnated with smoke to sound has already been described by *Tyndall*; I have confirmed the same thing. This is obviously a property of the separation surface that has the greatest importance for the blowing of whistles.

The theory reveals that everywhere that an irregularity is defined on the outer surface of an otherwise stationary jet, it must lead to a progressive spiral unrolling of the part of the surface in question (slipping away elsewhere on the jet). This striving for spiral unrolling of that current is, moreover, easy to discern for the observed jets. According to the theory, a prismatic or cylindrical jet can be infinitely long. In fact, such a thing cannot be exhibited, since small currents can never be eliminated completely in a moving element that is as light as air.

It is easy to see that such an infinitely-long cylindrical jet that emanates from a tube of corresponding cross-section into an external fluid at rest and contains fluid moving with a uniform velocity that is everywhere parallel to its axis corresponds to the conditions of the stationary state.

Here, I would only like to sketch out the mathematical treatment of a case of the opposite kind, in which the current flows from an open space into a narrow channel, in order to also simultaneously give an example of a method by which some problems of the theory of potential functions can be solved that previously created complications.

I restrict myself to the case in which the motion is stationary and depends upon only two rectangular coordinates x, y, and in which no rotating particles are present in the inviscid fluid from the onset, so no such particles can then be formed either. If we denote the velocity component of the fluid particle that is found at the point (x, y) and is parallel to x by u and the one parallel to y by v then, as is known, one can find two functions of x and y in such a way that:

$$\left.\begin{aligned} u &= \frac{d\varphi}{dx} = \frac{d\psi}{dy}, \\ v &= \frac{d\varphi}{dy} = -\frac{d\psi}{dx}. \end{aligned}\right\} \tag{1}$$

The condition that the mass in any spatial element in the interior of the fluid must remain constant will also be fulfilled immediately as a result of these equations, namely:

$$\frac{du}{dx} + \frac{dv}{dy} = \frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0. \tag{1a}$$

The pressure in the interior will be h for constant density, and if the potential of the external force is denoted by V then it will be given by the equation:

$$V - \frac{p}{h} + C = \tfrac{1}{2}\left[\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2\right] = \tfrac{1}{2}\left[\left(\frac{d\psi}{dx}\right)^2 + \left(\frac{d\psi}{dy}\right)^2\right]. \tag{1b}$$

The curves:

$$\psi = \text{const.}$$

are the streamlines of the fluid, and the curves:

$$\varphi = \text{const.}$$

are orthogonal to them. The latter will be the curves of equal potentials when electricity flows in a conductor of constant conductance, or equal temperature, in the case of heat.

It follows from equations (1) as an integral equation that the quantity $\varphi + \psi i$ can be a function of $x + y i$ (where $i = \sqrt{-1}$). As a rule, the solutions that were found previously can be expressed as a sum of terms that are themselves functions of x and y. However, conversely, one can also consider $x + y i$ to be a function of $\varphi + \psi i$ and develop it. For the problems of currents between two fixed walls, ψ is constant along the boundaries, and if one represents φ and ψ as rectangular coordinates in a plane then one must look for the function $x + y\,i$ in a strip in that plane that is bounded by two parallel straight lines $\psi = c_0$ and $\psi = c_1$ that correspond to the equations of the wall on the edge such that it assumes given discontinuities in the interior.

One case of this kind is obtained when we set:

$$x + y\,i = A\,\{\varphi + \psi\,i + e^{\varphi + \psi\,i}\} \tag{2}$$

or

$$x = A\varphi + A\,e^{\varphi} \cos \psi,$$
$$y = A\psi + A\,e^{\varphi} \sin \psi.$$

y will be constant for the values $\psi = \pm\,\pi$, and:

$$x = A\,\varphi - A\,e^{\varphi}.$$

If φ runs from $-\,\infty$ to $+\,\infty$ then x will simultaneously run from $-\,\infty$ to $-\,A$ and then back to $-\,\infty$. The stream curves $\psi = \pm\,\pi$ will then correspond to the flow along two straight walls for which $y = \pm\,A\,\pi$, and x runs between $-\,\infty$ and $-\,A$.

If we consider ψ to be the expression for the stream curves then equation (2) will correspond to the current from a channel that is bounded by two parallel planes into infinite space. However, on the edge of the channel, where $x = -\,A$ and $y = \pm\,A\,\pi$, and one has:

$$\varphi = 0 \quad \text{and} \quad \psi = \pm\,\pi,$$

moreover, one will have:

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = 0,$$

so:

$$\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2 = \infty\,.$$

Electricity and heat can flow in that way; however, drop-forming fluids must tear apart.

Should stationary separation lines emanate from the edge of the channel, which will naturally be continuations of the streamlines $\psi = \pm\,\pi$ that run along the wall, and should one find rest outside of these separation lines that bound the flowing fluid, then the pressure on both sides of the separation lines would have to be the same. That is, according to (1b), one would need to have:

$$\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2 = \text{const.} \tag{3}$$

along those parts of the lines $\psi = \pm\,\pi$ that correspond to the free separation lines.

Now, in order to maintain the basis for the motion that was given in equation (2), we add another term $\sigma + \tau\,i$ to the expression for $x + y\,i$ above that is likewise a function of $\varphi + \psi\,i$.

We will then have:

$$\left.\begin{aligned} x &= A\varphi + Ae^{\varphi}\cos\psi + \sigma, \\ y &= A\psi + Ae^{\psi}\sin\psi + \tau, \end{aligned}\right\} \tag{3a}$$

and we must determine $\sigma + \tau\, i$ in such a way that we will have:

$$\left(A - Ae^{\varphi} + \frac{d\sigma}{d\varphi}\right)^2 + \left(\frac{d\tau}{d\varphi}\right)^2 = \text{const.}$$

along the free part of the separation surface, where $\psi = \pm\,\pi$.

This condition will be fulfilled when we set:

$$\frac{d\sigma}{d\varphi} = 0 \qquad \text{or} \qquad \sigma = \text{const.} \tag{3b}$$

and

$$\frac{d\tau}{d\varphi} = \pm A\sqrt{2e^{\varphi} - e^{2}} \tag{3c}$$

in it.

Since ψ is constant along the wall, we can integrate the last equation over φ, and convert the integral into a function of $\varphi + \psi\, i$ when we replace φ with $\varphi + i\,(\psi + \pi)$ everywhere. For a suitable determination of the integration constant, we will get:

$$\sigma + \tau\, i = Ai\left\{\sqrt{-2e^{\varphi+\psi i} - e^{2\varphi+2\psi i}} + 2\arcsin\left[\frac{i}{\sqrt{2}}e^{(\varphi+\psi i)/2}\right]\right\}. \tag{3d}$$

The branch points of this expression lie where $e^{\varphi + \psi i} = -\,2$, that is, where $\psi = \pm\,(2\mathfrak{a} + 1)\,\pi$ and $\varphi = \log 2$. Thus, none of them lie inside the interval from $\psi = +\,\pi$ to $\psi = -\,\pi$. The function $\sigma + \tau\, i$ is continuous here.

Along the wall, one will have:

$$\sigma + \tau\, i = \pm\, Ai\left\{\sqrt{2e^{\varphi} - e^{2\varphi}} - 2\arcsin\left[\frac{i}{\sqrt{2}}e^{\varphi/2}\right]\right\}.$$

If $\varphi < \log 2$ then this value will be pure imaginary, so $\sigma = 0$, while $d\tau / d\varphi$ will take on the value that is prescribed in (3c) above. That part of the lines $\psi = \pm\,\pi$ will then correspond to the free part of the jet.

If $\varphi > \log 2$ then the expression will be real, up to the summands $\pm\, Ai\pi$ that are added to the value of $\tau\, i$ ($y\, i$, resp.).

Equations (3a) and (3d) will then correspond to the flow from an unbounded reservoir into a channel that is bounded by two planes whose width is $4A\pi$, and whose walls reach from $x = -\,\infty$ to $x = -\,A\,(2 - \log 2)$. The free separation line of the streaming fluid will curve slightly from edge of the opening towards a side that is opposite to the positive x, where it attains its largest x-value for $\varphi = 0$, $x = -A$ and $y = \pm A\,(\frac{3}{2}\pi + 1)$, in order to then turn into the interior of the channel, and finally asymptotically approach the two lines $y = \pm A\pi$, such that ultimately the width of the flowing jet is only one-half that of the channel.

The velocity along the separation surface and at the straight end of the flowing jet is $1 / A$. Along the fixed wall, and in the interior of the fluid, it is everywhere smaller than $1 / A$, such that this form of motion can take place for any magnitude of flow velocity.

I emphasize that this example, in fact, shows that the form of the fluid current in a tube can be determined from the form of the initial part along very long extents.

Appendix concerning electrical distributions. If one considers the quantity ψ in equation (2) to be an electric potential then one will obtain the distribution of electricity in the vicinity of the edge of two planar and closely-spaced plates, assuming that their separation can be considered to be vanishingly small in comparison to the radius of curvature of their edge curves. That gives a very simple solution to the problem that *Clausius* ([1]) treated. Moreover, one obtains the same distribution of electricity that he found, at least, to the extent that it is independent of the curvature of the edge.

I would like to add that the same method suffices to also find the distribution of electricity on two parallel, infinitely-long, planar strips whose four edges define the corners of a rectangle in the cross-section. Its potential function ψ will be given by an equation of the form:

$$x + y\,i = A\,(\varphi + \psi\,i) + B\,\frac{1}{H(\varphi + \psi i)}, \tag{4}$$

where $H(u)$ denotes the function that *Jacobi* developed in *Fundamenta nova* (pp. 172) as the numerator of sin *am u*. With the notation that one finds there, the strip that it occupies corresponds to the values $\varphi = \pm 2\,K$, such that $x = \pm 2\,AK$ yields the half-distance of the strip, while the ratio of the constants A and B depends upon the width of the strip.

The form of equations 2 and 4 is revealed in such a way that φ and ψ can be expressed as functions of x and y only as especially complicated series developments.

([1]) Poggendorf's Annalen, Bd. LXXXVI.