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FASCICLE XXX

Radiation couples and electromagnetic moments

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RADIATION COUPLES

AND

ELECTROMAGNETIC MOMENTS

By Émile HENRIOT

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FOREWORD

The present work is the result of an effort that was undertaken to bring about an initial development of a question that has remained with no clear answer, even up to relatively recent times.

Chronologically, an experimental difficulty has been its point of departure. For a decade, I have been forced to measure the couples that a circularly-polarized wave can exert on matter directly. In those experiments, a circular wave that issued from a Fresnel parallelepiped crossed a half-wave or quarter-wave layer (*lame*) and left it inverse-circularly or rectilinearly polarized. Under those conditions, one would expect to exhibit a couple that the layer is subjected to that would be equal to $2\mathcal{P} / \omega$ in the former case and $\mathcal{P} / \omega$ in the latter one, where $\mathcal{P}$ is the power that is carried by the wave, and ω is its pulsation. The extreme smallness of the predicted couple will render the experiment difficult and its results indecisive, so a lack of time and adequate means have obliged me to defer a realization of that experiment. In the meantime, a doubt was born in my mind, which quickly dissipated, moreover. The principle of indeterminacy that was just stated makes the pursuit of some research chimerical – for example, the research that consists of exhibiting the spin of an electron by a method that is analogous to the one that Stern and Gerlach used for atoms. Should one not fear that some advanced consequence of the principle of indeterminacy might show the vanity of any experimental effort that was directed along those lines? Nonetheless, I can interpret an indirect experiment whose basic principle I shall describe at the end of this work as something that proves the observable character of light couples.

On the other hand, I have the very strong sentiment that it is necessary to subject the theoretical arguments that are invoked in order to predict such couples to an examination and a complete, critical revision. Indeed, those arguments lack coherence, and their deduction from the classical equations of electromagnetism is so indirect and so unnatural that it is impossible to escape the impression that the method that is utilized in the course of those arguments must be made more precise, if not corrected.

The better part of this monograph is devoted to an effort along those lines that is purely deductive and uses only the classical equations of Maxwell with no necessity for any supplementary hypothesis of a theoretical nature, and that will give its conclusions a certain degree of certainty. As for the interest that such an effort might present, one must not forget that, thanks to the correspondence principle, the results of classical electromagnetism retain all of their importance in the context of quantum theories. Despite it all, should one hope to find something in a realm that is as well-examined as classical electromagnetism that has not been seen clearly already and would also be of some interest? The conversations that circumstances have led me to carry on with Lorentz and the most knowledgeable theoreticians show clearly that they felt that the answer to the problem that had been posed was unclear and required more work.

On 19 December 1790, a forgotten physicist named Vassali, who was a contemporary of Volta, read a paper to the Turin Academy of Sciences on some experiments with electricity that began with these words: "After much research and many experiments that were preceded and followed by the deepest examination, electrical science seems to have reached its highest point of perfection. That would seem almost ridiculous to the common minds that have to deal with it, moreover, except that nature is hidden in its progress." That citation reminds us that we must never jump to conclusions in the answers that one gives to the questions that are posed by science.

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CHAPTER I

Statement of the problem

1. Moments of the first and second kind. – When the electromagnetic theories of light are substituted for Fresnel's elastic theories, most of the facts that are predicted by them will find a pre-existing framework and a well-adapted terminology in them, as well as an entirely natural explanation. Meanwhile, one must point out an important exception: The theory of elasticity easily predicts that when a plane, elliptic wave with semi-axes a and b propagates in the z direction inside of an isotropic solid, it will transport an impulse moment flux C_z along the z-axis and an energy flux Σ_z , and the ratio of the one to the other will be given by:

$$\frac{C_z}{\Sigma_z} = \frac{1}{\omega}\frac{2ab}{a^2+b^2}. \tag{1}$$

Is it possible to transpose that theorem into the context of electromagnetism? One can define an impulse density **g** in electromagnetism. Most authors up to now have referred to the vector product:

$$\boldsymbol{\lambda} = [\mathbf{r} \cdot \mathbf{g}], \tag{2}$$

in which $\mathbf{r} = \overrightarrow{OP}$, where P is the point where one calculated the density, by the name of the *density of the impulse moment with respect to a point O*. Now, in the case of a plane electromagnetic wave that is normal to z, **g** will be in the z direction, and the component λ_z will be zero. The definition (2) of impulse moment would lead to a zero impulse moment with respect to the z-axis, and theorem (1) could not be transposed into electromagnetism. However, there is a whole set of reasons that caution one against using formula (2) and its aptness for solving such problems. First of all, the introduction of the finite segment r has the consequence that the formulas that one deduces will not be written in the language of space-time. Now, the usual length scale that contemporary physicists use in the equations of special relativity denies any deep physical significance to such formulas. The less-physical character of the definition (2) will become even more noticeable when one takes the viewpoint of general relativity.

Later on, we shall define a moment of the second kind – or *momentor* – that we shall not call the "impulse moment" because it can be non-zero at a point where the electromagnetic impulse is zero and whose properties were transposed into general relativity in the course of my prior research ([1]). In addition, the definition (2) gives only one measure for the moment density, and it must be completed, as we shall do in Chapter VI, by an expression for the corresponding moment flux that permits one to translate the

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([1]) E. HENRIOT, "Les moments d'impulsion en théorie électromagétique," Bull. Cl. Sci. Acad. Roy. de Belgique **20** (1934), pp. 505 and 874. "L'aspect antisymmétrique de l'électromagnétisme," *ibid.*, **21** (1935), pp. 29 and 127. "Les moments électromagnétiques," *ibid.*, **21** (1935), pp. 363. – Y. DUPONT, "Les couples de forces et les moments d'impulsion électromagnétique dans la gravifique de Th. de Donder," *ibid.*, **20** (1934), pp. 773 and 1008. – TH. DE DONDER, "La gravifique tourbillionaire," *ibid.*, **20** (1934), pp. 986.

theorem of the conservation of moments. Other reasons will be given in the course of this study that will show that it is impossible to use formula (2) for the problems that we shall address.

The history of science shows that often a term that is poorly-chosen or too vague has been the source of a long series of errors in conception or methodology. It seems that we find ourselves in a typical case of that kind here, and that is what I would like to try to bring out in the sequel. We place ourselves in the case of the dynamics of solid bodies and suppose that we are using a solid body that moves in all of its directions around a fixed point O and plays the role of a ballistic body that is intended to measure the moments of impulse in the same way that the ballistic pendulum measures impulses. There is good reason to distinguish between the moment of the resultant applied force with respect to the point O and the moment of the resultant applied couple, which is defined independently of the point O. If one shoots unrifled projectiles that are animated with a translational velocity of v then they will communicate impulse moments that are defined and calculable only when one is given the point O and the point where they collide with the ballistic body, and which I shall call *impulse moments* – or *moments – of the first kind*. On the contrary, some rifled projectiles or tops will reach the ballistic body with a translational velocity that is zero, but an angular velocity ω that communicates *moments of the second kind*, which are independent of the position of the point O and the point of impact. Furthermore, in the case of the dynamics of solid bodies, the problems of the second kind can be reduced to ones of the first kind. Indeed, if one mentally decomposes the top into volume elements and considers the impulse or quantity of motion of each of them then upon adding their moments with respect to the point O geometrically, one will in fact find a resultant that is independent of the point O and the position of the rotor. The question that one poses is that of knowing whether a similar reduction of moments of the second kind to ones of the first kind is always possible in every case – notably, the case of electromagnetism. We see that in certain cases, the impulse moment can be zero, even though the moment of the second kind – or *momentor* – can be non-zero.

Another transposition from the dynamics of material bodies to electromagnetism that is very hasty seems to be the following one in many cases: The tensor of tensions that are applied to the faces of the elementary parallelepiped in the theory of elasticity is coupled to the density of force f by relations of the form:

$$f_x = -\frac{\partial p_{xx}}{\partial x} - \frac{\partial p_{xy}}{\partial y} - \frac{\partial p_{xz}}{\partial z} .$$

The nine quantities p, which are surface efforts per unit area, can be envisioned as impulse fluxes that traverse the faces of the elementary parallelepiped per unit time and area, and they express the theorem of the conservation of impulse. When f is finite, the quantities p will form a symmetric matrix; that is, $p_{xy} = p_{yx}$. The tensor p has only six independent components, and it is symmetric. It seems that such a symmetric tensor will suffice to solve all of the problems in elasticity on the scale of quantities that one usually envisions.

In electromagnetism, one defines a tensor that is called the impulse-energy tensor and whose components expression the fluxes and spatial densities of impulsion and energy. It corresponds to the tensor p of the elasticity and like that tensor, it is symmetric, especially when the medium in which the phenomena are located is the vacuum. That very close analogy has led many theoreticians to assume that the symmetric energy-impulse tensor will suffice to solve all problems in the case of electromagnetism. In what follows, we shall see that there is an entire class of problems that are insoluble by the use of only the symmetric tensor that yields the impulses, and their solutions can be obtained only by the use of antisymmetric tensors that will be defined later on. Those are the problems of the "second kind."

2. Exchange of moments of the second kind between an electromagnetic field and matter. – I would like to show how one can write the accounting for these exchanges in a simple example. Let **h** and $\mathfrak{H}$ be the electric and magnetic fields that act upon a substance that we assume to be electrically neutral and isolated, to simplify; i.e., the true density of electricity ρ is zero, as well as the conduction currents. Fields provoke the appearance of electric and magnetic polarizations – say **p** and $\mathfrak{P}$, resp. These polarizations do not have the same direction as the fields that induce them, so one will have:

$$\gamma = [\mathbf{p} \cdot \mathbf{h}] + [\mathfrak{P} \cdot \mathfrak{H}] \tag{3}$$

per unit volume for a couple ([1]), in which, the products are vectorial. If we envision things from the viewpoint of the Maxwell-Hertz theory, without appealing to Lorentz's electronic mechanism, then such a couple will be defined without one being obliged to fix the point O with respect to which one defined the force moments.

On the contrary, suppose that a solid body moves around a point O, and one gives an electric charge q to a point P. If the solid body is placed in the field **h** then it will be subjected to a couple whose moment will be equal to that of the force $q\mathbf{h}$ with respect to the point O. The moments that correspond to the latter case will be of the first kind, while the ones that are expressed by (3) will be of the second kind, with the terminology that we have adopted. Since the distinction is essential from the viewpoint that we have taken, so the couple of the first kind will be referred to by the term *force couple*, and we shall reserve the term *torque* for that of the second kind. Similarly, the moments that pertain to problems of the first kind will be referred to as *impulse moments*, and those of the second kind will receive the name of *momentors*. It is impossible to call both of them impulse moments, because the moments can be non-zero when the impulse is zero.

For reasons that we shall see later on, and which pertain to the axial nature of torque, its three components along x, y, z will be denoted by γ_{yz}, γ_{zx}, γ_{xy}, resp. For example:

$$\gamma_{yz} = p_y\, h_z - p_z\, h_y + \mathfrak{P}_y\, \mathfrak{H}_z - \mathfrak{P}_z\, \mathfrak{H}_y\,.$$

We now fix our attention on the elementary parallelepiped $dx\,dy\,dz$. The matter that it contains is subject to a torque; i.e., the momentor of the electromagnetic field will diminish, in particular. Let μ_{yz} be the density of the electromagnetic momentor along the

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([1]) That couple might not be the only one that is exerted on the matter by the field.

x-axis, let M_{yxz} denote the momentor flux along the x-axis that crosses the face that is normal to x per unit area and time, let M_{yyz} be the momentor flux along the z-axis that crosses a face that is normal to y, etc. M_{yxz} is equivalent to a flux of tops along the x-axis that cross the face that is normal to x. The extreme indices denote the direction of the momentor axis, while the middle one denotes the normal to the face envisioned. On first glance, it can seem that such a notation is unworkably complicated, but we will see later on that it is the only one possible.

We write down that the torque that is imparted to the matter is equal to the loss that is suffered by the electromagnetic momentor. We obtain, from an argument whose form is classical:

$$\gamma_{yz} = -\frac{\partial}{\partial x} M_{yxz} - \frac{\partial}{\partial y} M_{yyz} - \frac{\partial}{\partial z} M_{yzz} - \frac{\partial}{\partial t} \mu_{yz} ,$$

along with two analogous equations. The form of the equation suggests a more symmetric notation. If we use Minkowski's variable $u = ict$, instead of t and set:

$$M_{yuz} = ic\, \mu_{yz}$$

then we will get expressions of the form:

$$\gamma_{ij} = -\frac{\partial}{\partial x} M_{ixj} - \frac{\partial}{\partial y} M_{iyj} - \frac{\partial}{\partial z} M_{izj} - \frac{\partial}{\partial u} \mu_{iuj}$$

or

(4) $$\gamma = -\Delta\text{iv}\, M$$

for the three components, in which the divergence is taken with respect to the middle index. Equation (4) corresponds to the one that expresses the density of force that is exerted on the matter by the field, and that one will write:

$$f = -\Delta\text{iv}\, T,$$

in which T is Maxwell's impulse-energy tensor. In writing with equation (4), we have implicitly supposed that the problems of the first and kind are completely separable; i.e., that the momentor that disappears per unit volume and time will give the torque that is exerted per unit volume on the matter by the field. In Chapter VI, we shall see that one can have exchanges between the moments of the second and first kind by the intermediary of matter.

The torque that the matter experiences can include other terms than the ones that are expressed by equation (3). Here, I would only like to show how one can write the conservation of moments in a definite example.

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CHAPTER II

THE ANTISYMMETRIC ASPECT OF ELECTROMAGNETISM

3. Review of some notions regarding tensors ([1]). – We confine ourselves to the case of special relativity. The passage from one system of axes S to another system of axes S' that moves with respect to the latter with a velocity v in the x direction will imply a transformation of the coordinates $x, y, z, u = ict$ that is provided by the Lorentz formulas:

$$x' = \quad x \cos\varphi + u \sin\varphi \qquad (y' = y),$$
$$u' = -x \sin\varphi + u \cos\varphi \qquad (z' = z),$$

in which:

$$\cos\varphi = \frac{1}{\sqrt{1-\beta^2}}, \qquad \sin\varphi = \frac{i\beta}{\sqrt{1-\beta^2}} \qquad \left(\beta = \frac{v}{c}\right).$$

4-vectors are physical quantities with four components p_x, p_y, p_z, p_u whose transformation when one changes axes is the same as that of the coordinates; i.e.:

$$p_{x'} = \quad p_x \cos\varphi + p_u \sin\varphi \qquad (p_{y'} = p_y),$$
$$p_{u'} = -p_x \sin\varphi + p_u \cos\varphi \qquad (p_{z'} = p_z).$$

6-vectors are physical quantities with six components that correspond in space-time to axial vectors in three-dimensional space. They are tensors with two antisymmetric indices whose components A_{ij} transform in the same way as $p_i\, q_j - q_i\, p_j$ in the course of a change of components, where p and q are two arbitrary 4-vectors. For example:

$$A_{x'u'} = A_{xu},$$
$$A_{y'u'} = A_{yu} \cos\varphi + A_{xy} \sin\varphi,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The transformation can be performed by using the table with double entries below:

	$A_{y'z'}$	$A_{z'x'}$	$A_{x'y'}$	$A_{x'u'}$	$A_{y'u'}$	$A_{z'u'}$
A_{yz} …	1	0	0	0	0	0
A_{zx} …	1	$\cos\varphi$	0	0	0	$-\sin\varphi$
A_{xy} …	0	0	$\cos\varphi$	0	$\sin\varphi$	0
A_{xu} …	0	0	0	1	0	0
A_{yu} …	0	0	$-\sin\varphi$	0	$\cos\varphi$	0
A_{zu} …	0	$\sin\varphi$	0	0	0	$\cos\varphi$

([1]) The reader will find all of the notions that are necessary for understanding the following section in von Laue's *Théorie de la Relativité* (translation by G. Létang, Gauthier-Villars, 1924).

This table will not change if one permutes the first three columns with the last three as a whole, and the first three rows with the last three as a whole.

As a result, when one has a 6-vector A_{yz} , A_{zx} , A_{xy} , A_{xu} , A_{yu} , A_{zu} , one can form a second one that one calls its *associate* A^*, and whose components will be:

$$A^*_{yz} = A_{xu}\,, \qquad A^*_{xu} = A_{yz}\,,$$
$$A^*_{zx} = A_{yu}\,, \qquad A^*_{yu} = A_{zx}\,,$$
$$A^*_{xy} = A_{zu}\,, \qquad A^*_{zu} = A_{xy}\,.$$

If one lets **b** and $\mathfrak{B}$ be electric and magnetic induction, resp., while **h** and $\mathfrak{H}$ are the corresponding fields, then the set of all their components will define two 6-vectors $\mathfrak{M}$ and $\mathfrak{N}$:

$$\mathfrak{M}_{yz} = \mathfrak{B}_x\,, \qquad \mathfrak{M}_{zx} = \mathfrak{B}_y\,, \qquad \mathfrak{M}_{xy} = \mathfrak{B}_z\,,$$
$$\mathfrak{M}_{xu} = -\,ih_x\,, \qquad \mathfrak{M}_{yu} = -\,ih_y\,, \qquad \mathfrak{M}_{zu} = -\,ih_z\,,$$

$$\mathfrak{N}_{yz} = \mathfrak{H}_x\,, \qquad \mathfrak{N}_{zx} = \mathfrak{H}_y\,, \qquad \mathfrak{N}_{xy} = \mathfrak{H}_z\,,$$
$$\mathfrak{N}_{xu} = -\,ib_x\,, \qquad \mathfrak{N}_{yu} = -\,ib_y\,, \qquad \mathfrak{N}_{zu} = -\,ib_z\,,$$

and their associates will be $\mathfrak{M}^*$ and $\mathfrak{N}^*$:

$$\mathfrak{M}^*_{yz} = -\,ih_x\,, \qquad \mathfrak{M}^*_{zx} = -\,ih_y\,, \qquad \mathfrak{M}^*_{xy} = -\,ih_z\,,$$
$$\mathfrak{M}^*_{xu} = \mathfrak{B}_x\,, \qquad \mathfrak{M}^*_{yu} = \mathfrak{B}_y\,, \qquad \mathfrak{M}^*_{zu} = \mathfrak{B}_z\,,$$

$$\mathfrak{N}^*_{yz} = -\,ib_x\,, \qquad \mathfrak{N}^*_{zx} = -\,ib_y\,, \qquad \mathfrak{N}^*_{xy} = -\,ib_z\,.$$
$$\mathfrak{N}^*_{xu} = \mathfrak{H}_x\,, \qquad \mathfrak{N}^*_{yu} = \mathfrak{H}_y\,, \qquad \mathfrak{N}^*_{zu} = \mathfrak{H}_z\,.$$

The tensorial character of those quantities is established by the fact that if one performs the operation Δiv on them then one will obtain:

$$(5) \qquad \Delta\text{iv}\,\mathfrak{M}^* = 0, \qquad c\,\Delta\text{iv}\,\mathfrak{N} = 4\pi\,\mathbf{C}$$

from Maxwell's equations. **C** is the current quadri-vector, whose components C_x, C_y, C_z are the true electric current densities, and $C_u = ic\rho$, where ρ is the true density of electricity. It is well-known that one can define a *symmetric tensor* from two 6-vectors by using the multiplication rule below. When that rule is applied to $\mathfrak{M}$, $\mathfrak{N}$, it will yield a symmetric tensor T that will be the impulse-energy tensor in the special case of the vacuum:

$$4T_{jk} = \left\{ \begin{array}{l} \mathfrak{M}_{jx}\mathfrak{N}_{kx} + \mathfrak{M}_{jy}\mathfrak{N}_{ky} + \mathfrak{M}_{jz}\mathfrak{N}_{kz} + \mathfrak{M}_{ju}\mathfrak{N}_{ku} \\ +\mathfrak{N}_{jx}\mathfrak{M}_{kx} + \mathfrak{N}_{jy}\mathfrak{M}_{ky} + \mathfrak{N}_{jz}\mathfrak{M}_{kz} + \mathfrak{N}_{ju}\mathfrak{M}_{ku} \end{array} \right\}$$

$$-\left\{\begin{array}{l}\mathfrak{M}^*_{jx}\mathfrak{N}^*_{kx}+\mathfrak{M}^*_{jy}\mathfrak{N}^*_{ky}+\mathfrak{M}^*_{jz}\mathfrak{N}^*_{kz}+\mathfrak{M}^*_{ju}\mathfrak{N}^*_{ku}\\+\mathfrak{N}^*_{jx}\mathfrak{M}^*_{kx}+\mathfrak{N}^*_{jy}\mathfrak{M}^*_{ky}+\mathfrak{N}^*_{jz}\mathfrak{M}^*_{kz}+\mathfrak{N}^*_{ju}\mathfrak{M}^*_{ku}\end{array}\right\}.$$

The 6-vectors $\mathfrak{M}$ and $\mathfrak{N}$ are not arbitrary, and their first three components can be regarded as a physical vector in *xyz*-space, as well as their last three components. That character is intrinsic and will be conserved in the course of a Lorentz change of axis. Starting from these 6-vectors, one can define a new antisymmetric tensor with two indices Γ_{jk} – viz., a 6-vector – by means of the multiplication rule:

$$4\,\Gamma_{jk}=\left\{\begin{array}{l}\mathfrak{M}_{jx}\mathfrak{N}_{kx}+\mathfrak{M}_{jy}\mathfrak{N}_{ky}+\mathfrak{M}_{jz}\mathfrak{N}_{kz}+\mathfrak{M}_{ju}\mathfrak{N}_{ku}\\-\mathfrak{N}_{jx}\mathfrak{M}_{kx}-\mathfrak{N}_{jy}\mathfrak{M}_{ky}-\mathfrak{N}_{jz}\mathfrak{M}_{kz}-\mathfrak{N}_{ju}\mathfrak{M}_{ku}\end{array}\right\}$$
$$+\ \{\text{associated quantities}\}.$$

The second set of brackets is identical with the first one and will simply double it when one takes the sum. If one substitutes the values of $\mathfrak{M}$ and $\mathfrak{N}$ then one will obtain:

$$\text{(6)}\qquad\left\{\begin{array}{l}2\Gamma_{yz}=[\mathbf{b}\cdot\mathbf{h}]_x+[\mathfrak{B}\cdot\mathfrak{H}]_x,\\2\Gamma_{xu}=i\{[\mathbf{b}\cdot\mathfrak{B}]_x-[\mathbf{h}\cdot\mathfrak{H}]_x\},\end{array}\right.$$

and four other components that are deduced by permutation.

4. Electromagnetic torque and momentor. – The physical nature of the components *yz*, *zx*, *xy* of Γ is displayed when one considers the relations between the inductions, the fields, and the polarization vectors:

$$\mathbf{b}=\mathbf{h}+4\pi\,\mathbf{p},\quad \mathfrak{B}=\mathfrak{H}+4\pi\,\mathfrak{P}.$$

If one substitutes this into the expressions for T then the real components will take the form:

$$2\,\Gamma_{yz}=4\pi\,\{[\mathbf{p}\;\mathbf{h}]_x+[\mathfrak{P}\;\mathfrak{H}]_x\}.$$

If one refers to equation (3) (Chap. I, 1) then those quantities will have the nature of a couple density of the second kind – or torque. One can have electromagnetic torques of another nature, but as we shall see, they will have a special importance. We let γ denote the antisymmetric tensor (or 6-vector) that is defined by:

$$\text{(7)}\qquad\left\{\begin{array}{l}4\pi\,\gamma_{yz}=2\Gamma_{yz},\\4\pi\,\gamma_{xu}=2\Gamma_{xu},\quad\cdots\end{array}\right.$$

and we shall refer to it by the name of *torque density*. The three real components have the significance that was given above, while the imaginary components will be interpreted in what follows. The existence is necessary; indeed, in the course of a Lorentz change of axes, they will yield a real contribution to the real components.

When the Δiv of a 6-vector is zero, the associated 6-vector can be considered to be the derivative of a potential 4-vector. In order to not prolong this discussion unnecessarily, we shall suppose that the true current **C** and the charge density ρ are zero, except in the cases where we shall specify to the contrary. The Maxwell equations (5) will then become:

$$\Delta\text{iv}\ \mathfrak{M}^* = 0, \qquad \Delta\text{iv}\ \mathfrak{N}^* = 0.$$

$\mathfrak{M}$ can be considered to have been derived from a vector potential **F** that is defined by:

$$\text{(8)} \qquad \left\{ \begin{array}{ll} \mathfrak{M}_{yz} = \mathfrak{B}_x = \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}, & \mathfrak{M}_{xu} = -ih_x = \dfrac{\partial F_u}{\partial x} - \dfrac{\partial F_x}{\partial u}, \\ \mathfrak{M}_{zx} = \mathfrak{B}_y = \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}, & \mathfrak{M}_{yu} = -ih_y = \dfrac{\partial F_u}{\partial y} - \dfrac{\partial F_y}{\partial u}, \\ \mathfrak{M}_{xy} = \mathfrak{B}_z = \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}, & \mathfrak{M}_{zu} = -ih_z = \dfrac{\partial F_u}{\partial z} - \dfrac{\partial F_z}{\partial u}. \end{array} \right.$$

Similarly, $\mathfrak{N}^*$ can be considered to be the derivative of a potential $\mathbf{F}^*$:

$$\text{(8}'\text{)} \qquad \left\{ \begin{array}{ll} \mathfrak{N}^*_{yz} = -ib_x = \dfrac{\partial F^*_z}{\partial y} - \dfrac{\partial F^*_y}{\partial z}, & \mathfrak{N}^*_{xu} = \mathfrak{H}_x = \dfrac{\partial F^*_u}{\partial x} - \dfrac{\partial F^*_x}{\partial u}, \\ \mathfrak{N}^*_{zx} = -ib_y = \dfrac{\partial F^*_x}{\partial z} - \dfrac{\partial F^*_z}{\partial x}, & \mathfrak{N}^*_{yu} = \mathfrak{H}_y = \dfrac{\partial F_u}{\partial y} - \dfrac{\partial F_y}{\partial u}, \\ \mathfrak{N}^*_{xy} = -ib_z = \dfrac{\partial F^*_y}{\partial x} - \dfrac{\partial F^*_x}{\partial y}, & \mathfrak{N}^*_{zu} = \mathfrak{H}_z = \dfrac{\partial F^*_u}{\partial z} - \dfrac{\partial F^*_z}{\partial u}. \end{array} \right.$$

The potentials **F**, $\mathbf{F}^*$ are not defined completely by these conditions, and one imposes the complementary Lorentz condition upon them:

$$\Delta\text{iv}\ \mathbf{F} = 0, \qquad \Delta\text{iv}\ \mathbf{F}^* = 0.$$

Recall the components of γ that are provided by (6) and (7); for example:

$$4\pi\,\gamma_{yz} = b_y\,h_z - b_z\,h_y + \mathfrak{B}_y\,\mathfrak{H}_z - \mathfrak{B}_z\,\mathfrak{H}_y\,.$$

If one substitutes the quantities **h** and $\mathfrak{B}$ in this, when they are expressed as functions of **F** as in (8), then after a simple transformation, and taking into account Maxwell's equations and the condition Δiv **F** = 0, one will have:

(9)
$$4\pi\,\gamma_{yz} = \frac{\partial}{\partial x}(-F_x\,\mathfrak{H}_z - F_z\,\mathfrak{H}_y + F_x\,\mathfrak{H}_x) + \frac{\partial}{\partial y}(F_x\,\mathfrak{H}_z + i\,F_u\,b_z + F_y\,\mathfrak{H}_x)$$
$$+\frac{\partial}{\partial z}(F_x\,\mathfrak{H}_z + iF_u\,b_y + F_z\,\mathfrak{H}_z) + \frac{\partial}{\partial u}(-iF_z\,b_y + i\,F_y\,b_z + F_u\,\mathfrak{H}_x)$$
$$-F_x\,\Delta\text{iv}\,\mathfrak{H} + F_u\left(-\frac{\partial\mathfrak{H}_x}{\partial u} + \frac{\partial ib_z}{\partial y} - \frac{\partial ib_y}{\partial z}\right).$$

The last two terms must be considered separately. We call them *complementary terms*. In most cases, they will be zero, but it can happen that they are different from 0. We denote them by the notation $-4\pi\,\theta_{yz}, \ldots, -4\pi\,\theta_{xu}$, so:

(10)
$$-4\pi\,\theta_{yz} = F_x\,\Delta\text{iv}_u\,\mathfrak{N}^* - F_u\,\Delta\text{iv}_x\,\mathfrak{N}^* = F_u\Box F_x^* - F_x\Box F_u^*,$$

in which $\Box$ is the d'Alembertian operator. The complementary terms constitute a complete 6-vector like γ. We see that these terms have the significance of a torque density that we call the *complementary torque*, and which we write:

$$4\pi\,(\gamma + \theta)_{yz} = \frac{\partial}{\partial x}(-F_z\,\mathfrak{H}_z - F_y\,\mathfrak{H}_y + F_x\,\mathfrak{H}_x) + \ldots,$$

which is an equation that, when it is compared to equation (4) in paragraph 2, will give a first expression for the electromagnetic momentor:

(11)
$$\begin{cases} 4\pi\,M_{yxz} = F_y\mathfrak{H}_y + F_z\mathfrak{H}_z - F_x\mathfrak{H}_x, \\ 4\pi\,M_{yyz} = -F_x\mathfrak{H}_y + iF_u b_z - F_y\mathfrak{H}_x, \\ 4\pi\,M_{yzz} = -F_x\mathfrak{H}_z - iF_u b_y - F_z\mathfrak{H}_x, \\ 4\pi\,M_{yuz} = iF_z\,b_y - iF_y b_z - F_u\mathfrak{H}_x, \end{cases}$$

and twenty other analogous components.

When the complementary torque θ is annulled, what will remain is only the principal torque γ, and we will have the relation:

$$\gamma_{ij} = -\,\Delta\text{iv}_{ij}\,M,$$

in which the divergence is taken with respect to the middle index. The tensorial character of M is fixed in the following manner:

Form the expressions:

$$\tau_{ijk} = F_i\,\mathfrak{N}_{jk} + F_j\,\mathfrak{N}_{ik} + F_k\,\mathfrak{N}_{ij},$$

while respecting the order of the indices. For example:

$$\tau_{yxz} = F_y\,\mathfrak{N}_{xz} + F_x\,\mathfrak{N}_{yz} + F_z\,\mathfrak{N}_{yx}.$$

Moreover, consider the 4-vector:

$$\mathbf{J} = [\mathbf{F} \cdot \mathfrak{N}];$$

i.e.:

$$J_x = F_x\, \mathfrak{N}_{xx} + F_y\, \mathfrak{N}_{xy} + F_z\, \mathfrak{N}_{xz} + F_u\, \mathfrak{N}_{xu}\,.$$

It is easy to see that:

$$4\pi\,(\gamma_{ij} + \theta_{ij}) = \Delta\text{iv}_{ij}\; t + \text{rot}_{ij}\; \mathbf{J}.$$

However, since **J** is itself a tensor with three degenerate indices, if one sets:

$$\begin{aligned} -J_x &= \sigma_{xyy} = -\sigma_{zzx} = \sigma_{xuu}\,, \\ -J_y &= \sigma_{yzz} = -\sigma_{xxy} = \sigma_{yuu}\,, \\ -J_z &= \sigma_{zxx} = -\sigma_{yyz} = \sigma_{zuu}\,, \\ -J_u &= -\sigma_{xxu} = -\sigma_{yyu} = -\sigma_{zzu}\,, \end{aligned}$$

while all of the other components of σ are zero, then:

(11′) $$4\pi\,(\gamma + \theta)_{ij} = \Delta\text{iv}_{ij}\,(\sigma + \tau), \qquad \text{and} \qquad 4\pi M = -(\sigma + \tau).$$

3. Case in which the complementary torque is annulled. – In the case where the isolated medium is the vacuum, if one refers to equation (9) then the complementary torque will be annulled, and in effect one will then have:

$$\text{div}\, \mathfrak{H} = \text{div}\, \mathfrak{B} = 0,$$

$$-\frac{\partial \mathfrak{H}_x}{\partial u} + \frac{\partial ib_z}{\partial y} - \frac{\partial ib_y}{\partial z} = 0.$$

The principal term in the torque γ is annulled along with its six real and imaginary components, because then **b** = **h** and $\mathfrak{B} = \mathfrak{H}$. That will be true in all systems of Lorentz axes, and will possess an intrinsic character.

The complementary torque and the real components of θ will be annulled in the case where the medium is refringent, absorbent, or pseudo-isotropic, and when one imagines a plane wave (see Chap. IV). That annihilation does not have an intrinsic or invariant character and will persist in the case when the medium is in motion in the system of axes envisioned (i.e., the case of the dragging of waves by matter).

In the case of an anisotropic medium, when one imagines a plane wave that is propagating along a principal direction of a crystal, θ will be likewise annulled, as far as its real components are concerned.

As an application of the preceding results, in what follows, we propose to treat some simple problems that are approachable when one specifies the antisymmetric tensor Γ, but insoluble when one specifies only the symmetric tensor T, and which are indeed problems of the second kind.

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CHAPTER III.

APPLICATION TO THE CASE IN WHICH THE MEDIUM IS A VACUUM

6. Case of a plane elliptic wave. – In any case where the medium is a vacuum, the force density f will be zero, and it is related to the symmetric energy-impulse tensor T by the relation $f = -\Delta\text{iv}\, T$. Even when f is zero, T cannot be zero, since its components represent the flux and densities of impulse and energy, resp., which cannot be zero, because the wave transports energy and quantity of motion. Similarly, in the case of the vacuum, γ will be zero, but M cannot be zero since the conservation of moments would not be assured then. Take the case of a plane elliptic wave with its semi-major axis a along x and its semi-minor axis b along y, and let it propagate in the z direction, with $\mathbf{b} = \mathbf{h}$, $\mathfrak{B} = \mathfrak{H}$:

$$h_x = a \cos \omega\left(t-\frac{z}{c}\right), \qquad \mathfrak{H}_x = -b \sin \omega\left(t-\frac{z}{c}\right),$$

$$h_y = b \cos \omega\left(t-\frac{z}{c}\right), \qquad \mathfrak{H}_y = a \cos \omega\left(t-\frac{z}{c}\right),$$

$$F_x = -\frac{ca}{\omega} \sin \omega\left(t-\frac{z}{c}\right), \qquad F_z = F_u = 0,$$

$$F_y = \frac{cb}{\omega} \cos \omega\left(t-\frac{z}{c}\right), \qquad F_z^* = F_u^* = 0,$$

$$F_x^* = -\frac{icb}{\omega} \cos \omega\left(t-\frac{z}{c}\right),$$

$$F_y^* = -\frac{ica}{\omega} \sin \omega\left(t-\frac{z}{c}\right).$$

From the formulas that are deduced from (11) (§ 4) by permutation:

$$M_{xzy} = \frac{1}{4\pi}(F_y\, \mathfrak{H}_y + F_x\, \mathfrak{H}_x - F_z\, \mathfrak{H}_z)$$

represents the momentor flux along the z (x, y, resp.) axis that crosses a surface that is normal to the z axis per unit time and area C_z :

$$M_{xzy} = C_z = \frac{ab}{4\pi\omega}c.$$

The momentor density μ_{xy} along the z axis is:

$$\mu_{xy} = \frac{1}{ic}M_{xzy} = \frac{1}{4\pi ic}(-\, ib_x\, F_y + ib_y\, F_x + F_u\, \mathfrak{H}_z) = \frac{ab}{4\pi\omega}.$$

Since the ratio of the flux to the density is c, one can say that the momentor propagates with a velocity of c. The fluxes and densities along the x and y axes are zero. The mean energy flux, which is given by the Poynting vector (which is deduced from the symmetric tensor), is:

$$\Sigma_z = \frac{a^2+b^2}{8\pi}c.$$

The ratio of momentor flux to energy flux is:

$$\frac{C_z}{\Sigma_z} = \frac{1}{\omega}\frac{2ab}{a^2+b^2},$$

as in the case of elasticity (§ 1).

The impulse moment that is defined by means of the formula (2) has a z-component that equals zero. That shows very clearly that the impulse moment that is defined by (2) by starting with the symmetric tensor T and the momentor that is defined by starting with the asymmetric tensor Γ are two very essentially different things that are not reducible to each other. Any terminology that confuses the two in the same application will be a source of error: The preoccupation that I have in denoting them by different terms is not merely rooted in a tendency to create useless neologisms.

7. Case of a harmonic or Keplerian point-like radiator. – A point-like radiator is composed of an electric charge e that describes closed orbits around a center O under the action of a central force that will be quasi-elastic in the case of a harmonic radiator and inversely proportional to the square of the distance in the case of a Keplerian radiator. Let ξ, η, ζ be the coordinates of the moving charge, which is assumed to move in the xy-plane, while $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ and $\ddot{\xi}$, $\ddot{\eta}$, $\ddot{\zeta}$ are the components of its velocity and acceleration, resp. We shall also make the hypothesis that the motion is quasi-stationary, moreover. The velocity and acceleration are assumed to be small enough that the radiator will emit quantities of energy during time intervals of order its period that are small in comparison to the amount of energy that it contains. During such intervals, the trajectory will deform very little and the means of the various quantities that define the trajectory will be well-defined at each instant. Let a point P have coordinates x, y, z, and set $\mathbf{r} = \overrightarrow{OP}$, where ξ and η are very small in comparison to r, and $\dot{\xi}$ and $\dot{\eta}$ are very small in comparison to c. The Lorentz potentials at the point P are given by:

$$F_x = e\left[\frac{\dot{\xi}}{r\left(1-\frac{v_r}{c}\right)}\right], \quad F_y = e\left[\frac{\dot{\eta}}{r\left(1-\frac{v_r}{c}\right)}\right], \quad F_z = 0, \quad F_u = ice\left[\frac{1}{r\left(1-\frac{v_r}{c}\right)}\right],$$

in which v_r / c is negligible in comparison to 1.

$$h_x = \frac{ce}{r}(\gamma_r - \gamma)_x = \frac{ce}{r^2}(\ddot{\xi}x + \ddot{\eta}y + \ddot{\xi}z - \ddot{\xi}r),$$

$$\mathfrak{H}_x = [\boldsymbol{\gamma} \cdot \mathbf{1}_r]_x = \frac{e}{r^2}(\ddot{\eta}z - \ddot{\xi}y),$$

while the other components are deduced by permutation. We calculate the momentor flux Φ_z along the z-axis that crosses a sphere of radius r.

In order to do that, we imagine an element dS on that sphere. Upon using formulas (11) and the ones that are deduced from them by cyclic permutation, we will find that:

$$\begin{aligned} 4\pi\Phi_z\, dS = \frac{dS}{r}\{ & (-F_z\,\mathfrak{H}_x - F_x\,\mathfrak{H}_z + i\,F_u\,h_y)\,x \\ & + (-F_z\,\mathfrak{H}_y - F_y\,\mathfrak{H}_z + i\,F_u\,h_x)\,y \\ & + (\;F_y\,\mathfrak{H}_y + F_z\,\mathfrak{H}_z - \;F_x\,\mathfrak{H}_x)\,z\}. \end{aligned}$$

If one substitutes the values of **h**, $\mathfrak{H}$, and **F** and integrates over the entire sphere at x, y, z then some of the terms will contribute zero, since they are odd in x, y, z, while other terms will give a mean value of zero in time. If one abstracts from that, then what will remain is:

$$4\pi\Phi_z = \frac{e}{r^2}\{x^2(\ddot{\xi}\ddot{\eta} - c\ddot{\xi}) + y^2(-\dot{\eta}\ddot{\xi} - \ddot{\eta}c) + z^2(\dot{\xi}\ddot{\eta} - \dot{\eta}\ddot{\xi})\}.$$

The terms that contain c as a factor have a mean value of zero in time; for example:

$$\frac{1}{t}\int_0^t \ddot{\xi}\,dt = \frac{\dot{\xi}}{t},$$

which will tend to zero when time increases indefinitely, since $\dot{\xi}$ is finite. If one then takes into account the fact that:

$$\int x^2 dS = \int y^2 dS = \int z^2 dS = \frac{4}{3}\pi r^3$$

then the mean momentor that is radiated per unit time is will be:

(12) $$\bar{\Phi}_z = \frac{2e^2}{3}\overline{(\dot{\xi}\ddot{\eta} - \dot{\eta}\ddot{\xi})}.$$

The mean radiated energy flux per unit time that is obtained by means of the Poynting vector will be:

(13) $$\bar{\Sigma} = \frac{2e^2}{3}\overline{(\ddot{\xi}^2 + \ddot{\eta}^2)}.$$

The case of the harmonic radiator is the one in which one has:

$$\xi = a\cos\omega t, \qquad \eta = a\sin\omega t,$$

$$\bar{\Phi}_z = \frac{2e^2}{3} ab\omega^2,$$

$$\bar{\Sigma} = \frac{2e^2}{3}\frac{a^2+b^2}{2}\omega^2,$$

and

(14)
$$\frac{\bar{\Phi}_z}{\bar{\Sigma}} = \frac{1}{\omega}\frac{2ab}{a^2+b^2}.$$

That problem has a certain historical importance. It was upon using formula (14), which he obtained by a different method, that Sommerfeld could state the selection principle that related to azimuthal spin. The exactitude of formula (14) is obvious *a priori* if one assumes that the radiated moment has its counterpart in a diminishing of the moment of the electron along its trajectory, and that the radiated energy has its counterpart in a diminishing of the energy of the radiator. Sommerfeld obtained (14) by using the impulse moment that is defined by the formula $\boldsymbol{\lambda} = [\mathbf{r} \cdot \boldsymbol{g}]$; i.e., upon treating that problem as a problem of the first kind, and on the other hand, making use of the Hertz potentials. His calculations presented the unsettling character that they introduced mixed quantities into the nonlinear equations that were neither pure real nor pure imaginary, and they were extremely lengthy. If, instead of proceeding as he did, one introduces the Lorentz potentials **F**, instead of the Hertz potentials, into formula $\boldsymbol{\lambda} = [\mathbf{r} \cdot \boldsymbol{g}]$ that defines the *impulse moment*, one will get a radiated impulse moment that equals zero.

Now, imagine the case of the Keplerian radiator. Let **L** denote the impulse moment of the charge that is moving with respect to the point *O*, and let ρ be its distance from the attracting center:

$$L_z = m\,(\xi\dot{\eta} - \dot{\xi}\eta),$$

$$m\ddot{\xi} = -\frac{e^2}{\rho^3}\xi, \qquad m\ddot{\eta} = -\frac{e^2}{\rho^3}\eta.$$

Formula (12) gives:

$$\bar{\Phi}_z = \frac{2e^4}{3m^2}\cdot L\cdot\left(\overline{\frac{1}{\rho^3}}\right).$$

The mean of $1/\rho^3$ will be taken over a closed trajectory.

From (13):

$$\bar{\Sigma} = \frac{2e^6}{3m^2}\left(\overline{\frac{1}{\rho^4}}\right)$$

and

(15) $$\frac{\bar{\Phi}_z}{\bar{\Sigma}} = \frac{L}{e^2}\frac{\left(\overline{1/\rho^3}\right)}{\left(\overline{1/\rho^4}\right)}.$$

All of the calculations reduce to an evaluation of the mean.

Take ρ and φ to be Lagrange coordinates, where φ is the angle between the radius vector and the major axis, and let p_ρ and p_φ are the canonically-conjugate moments:

$$p_\rho = m\dot{\rho}, \qquad p_\varphi = m\rho^2\dot{\varphi} = L = \text{const.}$$

Let $\mathcal{E}$ be the Hamilton function; i.e., the energy function that is expressed in terms of the coordinates and conjugate moments:

$$\mathcal{E} = -\frac{e^2}{\rho} + \frac{p_\rho^2}{2m} + \frac{p_\varphi^2}{2m\rho^2},$$

$$p_\rho = -\frac{\partial\mathcal{E}}{\partial\rho} = -\frac{e^2}{\rho^2} + \frac{L^2}{m\rho^3}.$$

The mean value of $\dot{p}_\rho = m\ddot{\rho}$ is zero, so:

(16) $$e^2\left(\overline{\frac{1}{\rho^2}}\right) = \frac{L^2}{m}\left(\overline{\frac{1}{\rho^3}}\right),$$

but on the other hand:

(17) $$\frac{\dot{p}_\rho}{\rho} = -\frac{e^2}{\rho^3} + \frac{L^2}{m\rho^4}.$$

By means of a simple calculation, one can assure that:

$$\left(\overline{\frac{\dot{p}_\rho}{\rho}}\right) = m\left(\overline{\frac{\dot{\rho}^2}{\rho^2}}\right),$$

so, upon using (17):

(18) $$m\left(\overline{\frac{\dot{\rho}^2}{\rho^2}}\right) = -e^2\left(\overline{\frac{1}{\rho^3}}\right) + \frac{L^2}{m}\left(\overline{\frac{1}{\rho^4}}\right).$$

On the other hand, if one divides $\mathcal{E}$ by ρ^2 and takes means then one will have:

(19) $$m\left(\overline{\frac{\dot{\rho}^2}{\rho^2}}\right) = 2\mathcal{E}\left(\overline{\frac{1}{\rho^2}}\right) + 2e^2\left(\overline{\frac{1}{\rho^3}}\right) - \frac{L^2}{m}\left(\overline{\frac{1}{\rho^4}}\right).$$

If one equates (18) and (19) and combines that with (16) then one will immediately get:

(20) $$\frac{\bar{\Phi}_z}{\bar{\Sigma}} = \frac{L}{e^2}\frac{\left(\overline{1/\rho^3}\right)}{\left(\overline{1/\rho^4}\right)} = \frac{2L^2}{2L^2\mathcal{E} + 3me^4}.$$

The mean momentor is assigned to L, and the energy, to $\mathcal{E}$, so one can write:

$$\frac{\bar{\Phi}_z}{\bar{\Sigma}} = \frac{-dL}{-d\mathcal{E}}.$$

When that relation is substituted in (20), that will give a differential equation in $\mathcal{E}$ and L that can be integrated by a quadrature, and will give:

(21) $$\frac{\mathcal{E}}{L} + \frac{me^4}{2}\frac{1}{L^3} = \text{const.}$$

However, if a is the semi-major axis of the ellipse, and ε is its eccentricity then one will have:

$$L = [e^2\, ma\,(1 - \varepsilon^2)]^{1/2}, \qquad \mathcal{E} = -\frac{e^2}{2a}.$$

If one substitutes these values in (21) then one will get:

(22) $$\frac{\varepsilon^4}{[a(1-\varepsilon^2)]^2} = \text{const.}$$

In the course of time, $\mathcal{E}$ will diminish, while a will decrease, and ε will change in such a manner as to insure the constancy of the expression (22).

We can imagine two extreme cases:

1. The orbit is initially circular: $\varepsilon_0 = 0$, so the constant is zero, and the orbit remains circular.

2. The orbit is initially rectilinear: $\varepsilon_0 = 1$, so the constant is infinite, and the orbit will remain rectilinear.

If the orbit is initially elliptical, so $0 < \varepsilon_0 < 1$, $\mathcal{E}$ will diminish in time, a will likewise diminish, and formula (22) will show that ε diminishes, so the trajectory will tend to a circular form. That will have the consequence that the circular motion is stable, and that the rectilinear motion that is envisioned in the case, will be, by contrast, unstable.

On the other hand, the result of the calculations will permit us to correct a very pervasive notion that is reproduced very often in theoretical treatises. One sometimes says: When the electron is on a quantized orbit, it does not radiate, and the Lorentz formulas that express the radiation are not valid, but when it leaves a quantized orbit to go to another one, the Lorentz formulas will be applicable during the transition. Now, since the radiation will essentially obey classical laws, the passage from one quantized orbit to another will be possible only if one has, from (21):

$$\frac{\mathcal{E}}{L}+\frac{me^4}{2}\frac{1}{L^3}=\frac{\mathcal{E}'}{L'}+\frac{me^4}{2}\frac{1}{L'^3},$$

the left-hand side of which expresses the quantities that relate to the starting orbit, and the right-hand side of whichpertains to the final orbit. If one expresses the energy $\mathcal{E}$ of a hydrogen atom as a function of a principal quantum number n and its moment L as a function of Sommerfeld's azimuthal quantum number k then the radiative transition, à la Lorentz, will be possible only if:

$$\frac{1}{k'^2}-\frac{1}{k^3}=\frac{1}{n'^2k'}-\frac{1}{n^2k}. \tag{23}$$

Now, there are cases in which some possible quantum transitions do not satisfy that equation. For example, from (23), the transition $(n = 3) \to (n' = 2)$, $(k = 2) \to (k' = 1)$, which is possible as a quantum transition, will not be of the radiation takes place according to the Lorentz laws.

On the other hand, if one introduces the Lorentz vector potentials while using formula (2), which gives the impulse moments, then one will get a zero mean value for the impulse moment that is found between r and $r + dr$. The problem has the second kind, and a consideration of the impulse moment will not permit one to solve it.

8. Rotation of an electrified sphere. – This problem has a certain historical importance: Indeed, it was upon calculating the magnetic moment and electric moment of a rotating, electrified sphere that Goudsmit gave a first interpretation of the spin of the electron. We envision two problems in what follows:

Problem a. – The sphere carries a surface density of electricity σ and turns with an angular velocity of ω around the z axis.

Problem b. – A sphere of density σ turns with an angular velocity ω while in immediate contact with an immobile sphere of the same radius that carries the surface density – σ. This corresponds to the case of a superconductor that carries immobile positive charges and electrons that circulate on the surface. Such a case can also give one a picture of the spin of the neutron that is probably rather far-fetched.

In what follows, **S** and **m** will denote the mechanical moment and magnetic moment, resp., while a will denote the radius of the sphere. In the two problems, the magnetic moment and the magnetic field have the same value:

$$m_x = m_y = 0, \quad m_z = m = \frac{1}{3}\frac{ea^2\omega}{c},$$

$$\mathfrak{H}_x^{(i)} = \mathfrak{H}_y^{(i)} = 0, \qquad \mathfrak{H}_z^{(i)} = \frac{2m}{a^3}.$$

The magnetic field is uniform inside (i). Externally (e), it is a doublet field:

$$\mathfrak{H}_x^{(e)} = \frac{3m}{r^3}x\,z, \qquad \mathfrak{H}_y^{(e)} = \frac{3m}{r^3}y\,z, \qquad \mathfrak{H}_z^{(e)} = -\frac{m}{r^3}\left(1 - \frac{3z^2}{r^2}\right).$$

Problems (a) and (b) can be separated if one writes the electric field and vector potential as:

$$(a)\quad \begin{cases} h^{(i)} = 0, & h_x^{(e)} = \dfrac{ex}{cr^3}, \quad h_y^{(e)} = \dfrac{ey}{cr^3}, \quad h_z^{(e)} = \dfrac{ez}{cr^3}, \\ F_x^{(i)} = -\dfrac{my}{a^2}, & F_y^{(i)} = \dfrac{mx}{a^2}, \quad F_z^{(i)} = 0, \quad F_u^{(i)} = \dfrac{ie}{ca}, \\ F_x^{(e)} = -\dfrac{my}{r^2}, & F_y^{(e)} = \dfrac{mx}{r^2}, \quad F_y^{(e)} = 0, \quad F_y^{(e)} = \dfrac{ie}{cr}. \end{cases}$$

Δiv **F** = 0, and **F** is continuous on the sphere, but its derivatives, which give the field, will be discontinuous:

$$(b)\quad \begin{cases} h^{(i)} = h^{(e)} = 0. \\ \text{The components } x, y, z, \text{ and } F \text{ are the same as in case (a), but } F_u = 0 \text{ and } \Delta\text{iv } F = 0. \end{cases}$$

The magnetic energy corresponds to the kinetic energy of rotation:

$$\int \frac{\mathfrak{H}^2}{8\pi}\,dv = \tfrac{1}{2} I\omega^2,$$

in which the integral is taken over all space – inside and outside – and I represents the moment of inertia. The same thing will be true in both cases, so one will get the same value for the moment of inertia I that one can deduce from an initial evaluation of S:

(24) $$S = I\omega = \frac{2e^2 a\omega}{9c^2}.$$

A second method consists of calculating an electromagnetic impulse moment density by means of the formula $\lambda_z = [\mathbf{r} \cdot \boldsymbol{g}]_z$ and integrating over all space. When applied to problem (*a*), that method will give the result of formula (24), as opposed to when it is applied to problem (*b*), which will give 0, so the application of that formula will give results that are hardly coherent. A third method consists of considering the problem to be a problem of the second kind and imagining the momentor. Meanwhile, we find ourselves in a particular case that we must treat directly because in our general formulas, we have supposed that the true charge density was zero. Now, it will be non-zero on the surface of the sphere, here.

Take the expression for the antisymmetric tensor:

$$4\pi\gamma = [\mathbf{b} \cdot \mathbf{h}] + [\mathfrak{B} \cdot \mathfrak{H}],$$

and write down the real components. Then, replace **h** and $\mathfrak{B}$ with their expressions as functions **F** (§ 4) in the expressions for those real components. If one takes into account Maxwell's equations with an electric current:

$$C_x = \rho\dot{\xi}, \qquad C_y = \rho\dot{\eta},$$

$$\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} - \frac{\partial i b_x}{\partial u} = \frac{4\pi}{c}\rho\dot{\xi}, \qquad \frac{\partial i h_z}{\partial y} - \frac{\partial i h_y}{\partial z} - \frac{\partial \mathcal{H}_x}{\partial u} = 0,$$

$$\dots\dots\dots \qquad \dots\dots\dots$$

then an easy calculation will give:

(25) $$\begin{aligned} 4\pi\gamma_{xy} = {} & \frac{\partial}{\partial x}(F_z\,\mathfrak{H}_x - ib_y\,F_u + F_x\,\mathfrak{H}_z) + \frac{\partial}{\partial y}(F_z\,\mathfrak{H}_y + ib_y\,F_u + F_y\,\mathfrak{H}_z) \\ & + \frac{\partial}{\partial z}(-F_x\,\mathfrak{H}_x - F_y\,\mathfrak{H}_y + F_z\,\mathfrak{H}_z) + \frac{\partial}{\partial y}(-ib_x\,F_z + ib_y\,F_x + F_u\,\mathfrak{H}_z) \\ & + \frac{4\pi}{c}[F_x\,C_y - F_y\,C_x]. \end{aligned}$$

Outside the current sheet **C** = 0, and the momentor density along the z axis will be μ_{xy} will be:

$$M_{xuy} = i\,e\,\mu_{xy},$$

so

(26′) $$4\pi\mu_{xy} = b_x\,F_y - b_y\,F_x - \frac{F_u}{i}\mathfrak{H}_z.$$

In the case of problem (*a*), it suffices to set **b** = **h** in the last formula and to replace **F** with the values that were given above in regard to that case, and one will find that outside of the sphere:

$$4\pi\,\mu_{xy}^{(e)} = \frac{2em}{cr^4}(1 - 2\cos^2\theta),$$

in which θ is the colatitude. If one integrates this over all external space then that expression will give the total momentor that is localized in the exterior as:

$$4\pi\,S_{xy}^{(e)} = -\frac{8\pi}{3}\frac{me}{a}.$$

Inside of the sphere, only the term in F_u will exist, and one will find by a simple calculation that:

$$4\pi\,S_{xy}^{(i)} = +\frac{8\pi}{3}\frac{me}{a}.$$

The momentor density is not zero either inside or outside the sphere, but when it is integrated over all space, it will give a total momentor of zero.

In the case of problem (*b*), **b** and F_u will both be zero inside and outside, so the same thing will be true for μ, and the total momentor will be zero, as in problem (*a*). It remains to calculate the momentor that is localized on the current sheet. The term that provides it will be:

(26) $$4\pi\,[\mathbf{F}\cdot\mathbf{C}]_{xy} = 4\pi\,(F_x\rho\dot{\eta} - F_y\rho\dot{\xi}).$$

One can put this into the form of a divergence, and in order to see that, it will suffice to replace the expressions of the form $\rho\dot{\xi}$ with the corresponding d'Alembertian, because:

$$-\Box\mathbf{F} = 4\pi\,\rho\dot{\xi},$$

and to take into account some identities of the form:

$$F_x\frac{\partial^2 F_y}{\partial x^2} - F_y\frac{\partial^2 F_x}{\partial x^2} = \frac{\partial}{\partial x}\left(F_x\frac{\partial F_y}{\partial x} - F_y\frac{\partial F_x}{\partial x}\right).$$

However, that way of writing things is not in the spirit of the problem, and those formulas are not applicable to the present case, since **F** is not differentiable with respect to any of the four variables on the sphere. Nonetheless, **F** is continuous on the sphere, and the expressions that were given above will give one the value of **F** on the surface. F_x and F_x have the same value in problems (*a*) and (*b*), so the expression (26) will give the same result in both cases, and the calculated moment will again be the same. It is convenient to use spherical coordinates such that $r = a$, φ is longitude, and θ is colatitude. If $\gamma^{(s)}$ is the surface density of torque, $C_\varphi^{(s)}$ is the line density of moment flux in a direction that is tangent to a parallel, and $\mu^{(s)}$ is the surface density of momentor along the z axis, then the balance of moments that are applied to an infinitesimal spherical rectangle is that found between (θ, $\theta + d\theta$), (φ, $\varphi + d\varphi$) can be written:

$$\gamma^{(s)} = -\frac{\partial C_\varphi^{(s)}}{\partial a\varphi} - \frac{\partial \mu^{(s)}}{\partial t}. \tag{27}$$

Here, since the phenomenon has axial symmetry, $\partial / \partial\varphi \equiv 0$, and the same thing will be true for the acceleration that the sphere can experience. On the other hand, if one introduces the expressions for F_x and **F** into (26):

$$4\pi\,\gamma^{(s)} = 4\pi\,[\mathbf{F}\cdot\mathbf{C}]_{xy} = -\frac{4\pi\sigma m}{ca}(x\dot{\xi} + y\dot{\eta}) = 0,$$

in which σ is the surface density of electricity, then $\gamma^{(s)}$ can be put into the form of a partial derivative with respect to time.

Consider the functions that depend upon the spatial coordinates:

$$x = a \sin\theta \cos\varphi,$$
$$y = a \sin\theta \sin\varphi,$$

and the functions of space and time:

$$\xi = a \sin\theta \cos \omega t = a \sin\theta \cos\Phi,$$
$$\eta = a \sin\theta \cos \omega t = a \sin\theta \sin\Phi,$$

$$\dot{\xi} = \frac{d\xi}{dt}, \qquad \dot{\eta} = \frac{d\eta}{dt},$$

$$x\dot{\xi} + y\dot{\eta} = \frac{\partial}{\partial t}[a^2 \sin^2\theta \cos(\varphi - \omega t)],$$

in which the derivative is taken at the instant $t = \varphi / \omega$.

The function:

$$\frac{4\pi\sigma m}{ca}\sin^2\theta\cos(\varphi - \omega t)$$

is such that its partial derivatives with respect to time will change sign and take the value 4π times the surface density of torque at the instant $t = \varphi / \omega$. It is natural to take the surface density of momentor to be the function:

$$\mu^{(s)} = \frac{\sigma m}{ca}\sin^2\theta,$$

which will give a total momentor of:

$$S = \frac{2em}{3ac}$$

when it is integrated over the entire sphere.

When one replaces the magnetic moment m in this expression with its value, one will find that:

$$S = \frac{2e^2 a}{9c^2}\omega .$$

The moment is the same in the two cases, as the result of an energetic calculation that involves only the magnetic field would suggest. The anomaly that is introduced by the use of formula (2) that yields a moment of zero in case (*b*) will then be discarded. I shall not elaborate further upon that problem, so I shall pass over everything that one might say about imaginary torque and momentor components. Similarly, for reasons of brevity, I shall give the last part of the calculation a form whose rapidity can be subjected to some criticisms, as I see it. I will return to these various points in other publications.

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CHAPTER IV

APPLICATION TO THE CASE OF A POLARIZABLE SPACE

In this chapter, I will confine myself to some applications of the general formulas to some simple cases for which the complementary torque θ is zero.

9. Birefringent medium. – We shall study the exchanges of momentor between a plane, elliptic wave that propagates in a principal direction of a crystal and the matter in it. We assume that the medium is not absorbent and is referred to its principal directions:

$$b_x = n_x^2\, h_x, \qquad b_y = n_y^2\, h_y, \qquad b_z = n_z^2\, h_z,$$

in which the n are the three principal axes. Let a and b be the semi-axes of the ellipse, and let z be the direction of propagation.

$$h_x = a \cos \omega\left(t - \frac{n_x z}{c}\right), \qquad h_y = b \sin \omega\left(t - \frac{n_y z}{c}\right),$$

$$\mathfrak{B}_x = \mathfrak{H}_x = -\, b\, n_y \sin \omega\left(t - \frac{n_y z}{c}\right), \qquad \mathfrak{B}_y = \mathfrak{H}_y = b\, n_x \cos \omega\left(t - \frac{n_x z}{c}\right),$$

$$F_x = -\frac{ca}{\omega} \sin \omega\left(t - \frac{n_x z}{c}\right), \qquad F_y = \frac{cb}{\omega} \cos \omega\left(t - \frac{n_y z}{c}\right),$$

$$F_x = 0, \qquad F_u = 0.$$

The complementary terms θ to the torque are annulled because:

$$F_u = 0, \qquad \operatorname{div} \mathfrak{H} = 0.$$

The momentor flux along the z-axis that crosses a unit surface normal to z per unit time is M_{xzy}, and:

$$4\pi\, M_{xzy} = F_x\, \mathfrak{H}_x + F_y\, \mathfrak{H}_y,$$

$$4\pi\, M_{xzy} = \frac{cab}{\omega}\left\{\frac{n_x + n_y}{2} \cos\left[\omega(n_x - n_y)\frac{z}{c}\right] - \frac{n_x - n_y}{2} \cos\left[\omega\left(2t - \frac{n_x + n_y}{c}\right) z\right]\right\},$$

whose mean value in time will be:

$$4\pi\overline{M_{xzy}} = \frac{cab}{\omega}\frac{n_x + n_y}{2}\cos\left(\omega\frac{n_x - n_y}{c}z\right)$$

when it is taken at a point whose abscissa is z. The density of the torque that is exerted by the wave on the matter will be:

$$\gamma_{xy} = -\frac{\partial}{\partial z}M_{xzy} - \frac{\partial}{\partial u}M_{xuy},$$

and its mean value in time will be:

$$\overline{\gamma_{xy}} = ab\,\frac{n_x^2 - n_y^2}{2}\sin\left[\omega\frac{n_x - n_y}{c}z\right].$$

There are local, periodic exchanges of torque between the wave and the matter. The mean density of energy flux is given by the Poynting vector Σ_z :

$$4\pi\bar{\Sigma}_z = \frac{c}{2}(a^2\,n_x + b^2\,n_y),$$

and the mean energy density $\bar{w}$ is:

$$4\pi\,\bar{w} = \tfrac{1}{2}(a^2 n_x^2 + b^2 n_y^2).$$

When the elliptic wave penetrates the layer, the momentor flux divided by the energy flux will be, for $z = 0$:

$$\frac{(\bar{\Phi}_z)_0}{(\bar{\Sigma}_z)_0} = \frac{2ab(n_x + n_y)}{a^2 n_x + b^2 n_y}\frac{1}{\omega}.$$

That ratio is not the same as it is for the elliptic wave before it arrives at the layer, namely:

$$\frac{(\bar{\Phi}'_z)_0}{(\bar{\Sigma}'_z)_0} = \frac{2a'b'}{a'^2 + b'^2}\frac{1}{\omega}.$$

The counterpart of the difference exists in the momentor and energy of the elliptic wave that is reflected when it enters the layer. When the substance is slightly birefringent, n_x will be slightly different from n_y, and these expressions will become the same:

$$\frac{(\bar{\Phi}_z)_0}{(\bar{\Sigma}_z)_0} = \frac{2ab}{a^2 + b^2}\frac{1}{\omega}.$$

For example, imagine the case of an incident circular light wave that falls on a quarter wave, $a = b$:

$$(\bar{\Phi}_z)_0 = \frac{(\bar{\Sigma}_z)}{\omega}.$$

After an interval e such that e $(n_x - n_y) = \lambda / 4$, the wave will become rectilinear, and correspondingly, M_{xyz} will become zero, and the wave will give up all of its momentor. The couple that the layer experiences must be equal to the momentum flux:

$$\int_0^{\lambda/4} \bar{\gamma}_{xy}\, dz = \bar{M}_{xzy},$$

and one verifies that one indeed has:

$$j = \frac{\bar{\Sigma}_z}{\omega},$$

in which j is the couple.

In the case of the half-wave layer, the momentor flux that leaves it will be equal and opposite to the one that enters it, and correspondingly, the circularity will be opposite, so the preceding result will double:

$$j = \frac{2\bar{\Sigma}_z}{\omega}.$$

The origin of such a torque is obvious *a priori:* **b** does not generally have the same direction as **h** in a crystal. When the medium is *monorefringent*, **b** will have the same direction as **h**, so there will no longer any exchange of momentum, $n_x = n_y$, and:

$$4\pi \bar{\Phi}_z = \frac{cabn}{\omega}, \qquad 4\pi \bar{\Sigma}_z = \frac{c\,(a^2 + b^2)\,n}{\omega}.$$

The ratio of these two quantities has the same value as it does *in vacuo:*

$$\frac{\bar{\Phi}_z}{\bar{\Sigma}_z} = \frac{1}{\omega}\frac{2ab}{a^2 + b^2}.$$

10. Absorbent media. – We imagine the case of an elliptical plane wave that falls upon an isotropic, absorbent layer in the z direction. As is easy to do, one uses the imaginary index:

$$n^* = n - i\,\sigma,$$

in which σ is an extinction coefficient, so:

$$\mathbf{b} = n^{*2}\,\mathbf{h}, \qquad \mathfrak{B} = \mathfrak{H}.$$

The Maxwell equations:

$$\text{div}\,\mathbf{b} = 0, \qquad \text{div}\,\mathfrak{B} = 0,$$

$$\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} = \frac{1}{c}\frac{\partial b_x}{\partial t}, \qquad \frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z} = -\frac{1}{c}\frac{\partial \mathfrak{B}_x}{\partial t},$$

in the case of a wave:

$$h_x = a\, e^{i\omega\left(t - \frac{n^* x}{c}\right)}, \qquad h_y = b\, e^{i\omega\left(t - \frac{n^* x}{c}\right)},$$

will yield the value of the magnetic field. To simplify the writing, set:

$$e^{i\omega\left(t - \frac{n^* x}{c}\right)} = \text{exp.},$$

$$\mathfrak{H}_x = -\, b\, n^* \text{ exp.}, \qquad \mathfrak{H}_y = a\, n^* \text{ exp.},$$

Let A and B be the real amplitudes of a and b, and let φ_0, ψ_0 be the phases of h_x and h_y at the origin:

$$a = A e^{i\varphi_0}, \qquad b = B e^{i\psi_0}.$$

For example, take $\varphi_0 = 0$, $\psi_0 = -\pi / 2$:

$$a = A, \qquad b = -\, B\, i.$$

The values of **h** and $\mathfrak{H}$ derive from a vector potential **F**:

$$F_x = \frac{ica}{\omega}\text{exp.}, \qquad F_y = \frac{icb}{\omega}\text{exp.}, \qquad F_z = F_u = 0.$$

The complementary terms θ are annulled because div $\mathfrak{H} = 0$, $F_u = 0$. We now pass to real quantities and set:

$$\varphi = \omega\left(t - \frac{nz}{c}\right), \qquad k = \frac{\sigma\omega}{c},$$

$$h_x = A\, e^{-kz} \cos \varphi, \qquad h_y = B\, e^{-kz} \sin \varphi,$$

$$\mathfrak{H}_x = B\, e^{-kz}\, (\sigma \cos \varphi - n \sin \varphi), \qquad \mathfrak{H}_y = A\, e^{-kz}\, (n \cos \varphi + \sigma \sin \varphi),$$

$$F_x = -\, \frac{eA}{\omega}\, e^{-kz} \sin \varphi, \qquad F_y = \frac{eB}{\omega}\, e^{-kz} \cos \varphi,$$

$$b_x = A\, e^{-kz}\, [(n^2 - \sigma^2) \cos \varphi + 2n\sigma \sin \varphi],$$

$$b_y = B\, e^{-kz}\, [-\, 2n\sigma \cos \varphi + (n^2 - \sigma^2) \sin \varphi].$$

b and **h** do not have the same direction, so the wave will exert a torque on the matter whose density χ_{xy} is given by:

$$4\pi\,\chi_{xy} = b_x\,h_y - b_y\,h_x = 2AB\,e^{-2kz}\,n\sigma.$$

If the medium totally absorbs the wave then it will receive a couple of the second kind per unit area:

$$AB\frac{n\sigma}{2\pi}\int_0^\infty e^{-2kz}dz = \frac{ncAB}{4\pi\omega}.$$

In order for there to be conservation of moments, this must be equal to the momentor flux that penetrates the layer:

$$4\pi\,\Phi_z = 4\pi\,M_{xzy} = F_x\,\mathfrak{H}_x + F_y\,\mathfrak{H}_y = \frac{ncAB}{\omega}\,e^{-2kz},$$

and for $z = 0$, this is, in fact, equal to the value of the total couple.

On the other hand, the transversal flux is zero.

The momentor density:

$$4\pi\,\mu_{xy} = \frac{1}{ic}4\pi\,M_{xuy} = \frac{1}{c}(b_x\,F_y - b_y\,F_x) = \frac{AB}{\omega}(n^2 - \sigma^2)\,e^{-2kz}.$$

The ratio of the flux to the momentor density is constant, so one can speak of a speed of propagation of momentor that is equal to the quotient of one over the other:

$$V = \frac{c}{n\left(1-\frac{\sigma^2}{n^2}\right)},$$

whose value will become c / n when $\sigma = 0$; i.e., in a non-absorbent medium.

The density of energy flux Σ_z is provided by the Poynting vector:

$$\begin{aligned}4\pi\,\Sigma_z &= c\,(h_x\,\mathfrak{H}_y - h_y\,\mathfrak{H}_x)\\ &= c\,e^{-2kz}\{A^2\,(n\cos^2\varphi + \sigma\cos\varphi\sin\varphi) + B^2\,(n\cos^2\varphi + \sigma\cos\varphi\sin\varphi)\},\end{aligned}$$

whose mean value in time is:

$$4\pi\,\overline{\Sigma}_z = \frac{nc}{8\pi}(A^2 + B^2)\,e^{-2kz}.$$

If one divides the expression for the total couple, or the incident momentor, by the energy flux upon entering ($z = 0$) then one will obtain:

$$\frac{\bar{\Phi}_z}{\bar{\Sigma}_z} = \frac{1}{\omega}\frac{2AB}{A^2+B^2};$$

a similar relation will also be true for all of the other scales of *z*. If the incident wave is circular, so $A = B$, then:

$$\Phi_z = \frac{\bar{\Sigma}_z}{\omega}.$$

The wave loses all of its energy and all of its moment to the absorbent layer, while in the case of a transparent quarter-wave layer, it will lose its moment, but not its energy. A direct measurement of the couple Σ_z / ω that is exerted by circular light on an absorbent layer – a blackened one, for example – would be experimentally more difficult to put into practice than that of the couple that a quarter-wave experiences, due to the raising of the temperature that is produced in the first case from the absorption of energy. Indeed, such an increase in temperature can have some radiometric effects and convection currents that can be a source of trouble if one does not work *in vacuo*, and will mask the effect of the measurement.

11. Pseudo-isotropic medium. – Imagine the case of a rectilinear wave that crosses a medium that possesses the power to rotate. The wave is rectilinear when it enters and leaves, the moment flux is zero upon leaving, as well as entering, so one would expect that the matter would not experience a mean couple. Let c / n and c / n' be the direct circular and inverse circular speeds of propagation, respectively, and set:

$$N = \omega\left(t - \frac{nz}{c}\right), \qquad N' = \omega\left(t - \frac{n'z}{c}\right),$$

which are equal to the phases of the two circular waves, respectively.

For the direct circular wave, one will have:

$$h_x = a\cos N, \quad \mathfrak{H}_x = -\,an\sin N, \quad F_x = -\frac{ca}{\omega}\sin N,$$
$$h_y = a\sin N, \quad \mathfrak{H}_y = an\cos N, \quad F_y = \frac{ca}{\omega}\cos N,$$

and for the inverse circular wave:

$$h'_x = a'\cos N', \quad \mathfrak{H}'_x = a'n'\sin N', \quad F'_x = -\frac{ca'}{\omega}\sin N',$$
$$h'_y = -\,a\sin N, \quad \mathfrak{H}'_y = a'n'\cos N', \quad F'_y = -\frac{ca'}{\omega}\cos N'.$$

One determines the ratio of the amplitudes from the condition that the two circular waves transport the same energy flux; i.e., they have the same Poynting vector. If one uses the language of quantum theory then one can say that the number of direct and inverse photons that pass through it every second is the same. One will immediately obtain the condition:

$$n a^2 = n' \, a'^2.$$

One is once more in a situation where the complementary torque θ will be zero here:

$$F_u + F'_u = 0, \qquad \operatorname{div}(\mathfrak{H} + \mathfrak{H}') = 0.$$

The momentor flux along the z axis that crosses a unit surface normal to the z axis will be given by Φ_z :

$$4\pi \Phi_z = (F_x + F'_x)(\mathfrak{H}_x + \mathfrak{H}'_x) + (F_y + F'_y)(\mathfrak{H}_y + \mathfrak{H}'_y)$$
$$= \frac{c}{\omega} a \, a' \, (n' - n) \cos (N + N'),$$

and the mean value in time will indeed be zero.

The momentor density is:

$$4\pi \mu_{xy} = \frac{4\pi}{ic} M_{xuy} = (b_x + b'_x)(F_y + F'_y) - (b_y + b'_y)(F_x + F'_x),$$

so

$$4\pi \mu_{xy} = \frac{1}{\omega}(a^2 n^2 - a'^2 n'^2) + \frac{aa'}{\omega}(n'^2 - n^2) \cos (N + N').$$

The torque density:

$$4\pi \gamma_{xy} = [\mathbf{b}\ \mathbf{h}]_{xy} + [\mathfrak{B}\ \mathfrak{H}]_{xy}$$

will be obtained by taking:

$$\mathbf{b} = n^2 \, \mathbf{h}, \qquad \mathbf{b}' = n'^2 \, \mathbf{h}',$$

which will yield:

$$4\pi \gamma_{xy} = aa' \, (n'^2 - n^2) \sin (N + N').$$

One has conservation of moments; i.e., one can verify by means of the preceding formulas that:

$$\gamma_{xy} = -\frac{\partial \Phi_z}{\partial z} - \frac{\partial \mu_{xy}}{\partial t}.$$

The couple γ is zero only in the mean, and there will be incessant periodic exchanges between the field and matter. The existence of that couple is obvious *a priori*, so the resultant induction, whose components will be:

$$b_x + b'_x = n^2 h_x + n'^2 h'_x,$$

$$b_y + b'_y = n^2 h_y + n'^2 h'_y ,$$

will have only the same exceptional direction as the resultant field $h_x + h'_x$, $h_y + h'_y$. Now, the induction and the field are anti-parallel or parallel at points such as $N + N' = K\pi$.

The moment flux is the same upon entering and leaving, and the pseudo-isotropic substance that makes the azimuth of a rectilinear vibration turn through an arbitrary angle will not exchange any couple or moment whose mean is not zero.

In Chapters III and IV, I have envisioned only some cases in which the complementary torque was annulled in order to show how the use of the momentor is a simple and coherent method of calculation for an entire series of problems that one cannot solve by the use of impulse moments. In the following chapter, I will recall the general equations (7) and envision the case in which complementary torque θ is no longer zero.

CHAPTER V

RETURN TO THE GENERAL EQUATIONS

12. Case in which the complementary term is not annulled. – Recall our general equations (9) and (10) (Chap. II.4), which show that one can distinguish two kinds of terms in the principal torque γ, one of which can be immediately expressed as the divergence of a three-index tensor, and the other of which constitutes the complementary torque, which cannot be put into that form naturally, and we have:

(28) $$4\pi\,\gamma_{ij} = \Delta\text{iv}_{ij}\,(\tau + \sigma) - 4\pi\,\theta_{ij}\,.$$

One can also put the θ terms into the form of a divergence of a three-index tensor, but in a less natural manner. From (10):

$$-4\pi\,\theta_{yz} = \left(F_u\,\frac{\partial^2 F_x^*}{\partial x^2} - F_x\,\frac{\partial^2 F_u^*}{\partial x^2}\right) + \left(F_u\,\frac{\partial^2 F_x^*}{\partial y^2} - F_x\,\frac{\partial^2 F_u^*}{\partial y^2}\right)$$
$$+\left(F_u\,\frac{\partial^2 F_x^*}{\partial z^2} - F_x\,\frac{\partial^2 F_u^*}{\partial z^2}\right) + \left(F_u\,\frac{\partial^2 F_x^*}{\partial u^2} - F_x\,\frac{\partial^2 F_u^*}{\partial u^2}\right).$$

The quantities **F** and $\mathbf{F}^*$ are functions of x, y, z, u that are given for the problem. Set:

$$\beta_{i^* jk^*} = \int\left(F_i\,\frac{\partial^2 F_k^*}{\partial j^2} - F_k\,\frac{\partial^2 F_i^*}{\partial j^2}\right)dj,$$

in which i, j, k are one of the four letters x, y, z, u, and in which y^*z^* correspond to xu, z^*x^*, to yu, and x^*y^*, to zu; for example:

(28′) $$\beta_{yxz} = \int\left(F_u\,\frac{\partial^2 F_u^*}{\partial x^2} - F_x\,\frac{\partial^2 F_u^*}{\partial x^2}\right)dx.$$

These symbols have the following meaning: Integration will be performed over x, while y, z, u are regarded as constants during the integration. The result will be defined only up to a function $K(x, z, u)$. There will then be indeterminacy in the value of the corresponding momentor; that should be obvious *a priori*. If one obtains an initial value for the three-index tensor that gives the momentor then one will get another one that is equivalent from standpoint of torque by adding a second three-index tensor with zero divergence to the first tensor.

In fact, that indeterminacy in the search for momentors is analogous to the indeterminacy that hangs over the symmetric impulse-energy tensor, which is defined by the condition that its divergence is equal to the force density:

$$f = -\,\Delta\text{iv}\;T.$$

If one has an initial determination of T that gives a certain value to the flux and densities of impulse and energy then one will get an entirely different – but still valid – one by taking the components $T + T'$, where T' is defined by the condition that:

$$\Delta\text{iv}\, T' = 0.$$

The tensor T is chosen from all of the tensors $T + T'$ for reasons of simplicity and coherence. In fact, the same thing will be true for the momentor, and indeterminacy will arise in each particular case. We can write:

$$4\pi\, \gamma_{ij} = \Delta\text{iv}_{ij}\,(\sigma + \tau + \beta),$$

and the momentor will be provided by:

$$4\pi\, M_{ijk} = -(\sigma + \tau + \beta)_{ijk}\,. \tag{28″}$$

13. Birefringent medium. Rectilinear wave. – Take the case of a rectilinearly-polarized plane wave, and propagate it in an arbitrary, but non-principal, direction. In general, **b** is in the plane of the wave, and **h** is not, so a certain torque density whose axis is directed along $\mathfrak{H}$ will result.

Let (g_x, g_y, g_z, g_u) be the components of a quadri-vectorial quantity that propagates by plane wave in a crystal, with a wave normal that has the direction cosines α, β, γ, and a velocity of v. Denote the amplitude of g_i by the notation $\overline{g}_i$, set $V = iv / c$, and let $p\varphi$ denote the phase:

$$p\varphi = \frac{2\pi}{\lambda}(\alpha x + \beta y + \gamma z + Vu),$$

with $p = 2\pi / \lambda$.

The propagation of the vector potentials that were defined by (8) and (8′) in paragraph 4 can be written:

$$(F_x, F_y, F_z, F_u) = (\overline{F}_x, \overline{F}_y, \overline{F}_z, \overline{F}_u)\ \sin p\varphi,$$
$$(F_x^*, F_y^*, F_z^*, F_u^*) = (\overline{F}_x^*, \overline{F}_y^*, \overline{F}_z^*, \overline{F}_u^*)\sin p\varphi,$$
$$-i\, h_x = \frac{\partial F_u}{\partial x} - \frac{\partial F_x}{\partial u}.$$

That gives:

$$\begin{cases} h_x = ip\,(\overline{F}_u\alpha - \overline{F}_x V)\cos p\varphi, \\ h_y = ip\,(\overline{F}_u\beta - \overline{F}_y V)\cos p\varphi, \\ h_z = ip\,(\overline{F}_u\gamma - \overline{F}_z V)\cos p\varphi, \end{cases} \tag{29}$$

in which V is imaginary, h_x is real, F_x, F_y, F_z are real, and F_u is pure imaginary.

$$\mathfrak{B}_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

gives:

$$(29') \qquad \begin{cases} \mathfrak{B}_x = p\,(\bar{F}_z\beta - \bar{F}_y\gamma)\cos p\varphi, \\ \mathfrak{B}_y = p\,(\bar{F}_x\gamma - \bar{F}_z\alpha)\cos p\varphi, \\ \mathfrak{B}_z = p\,(\bar{F}_y\alpha - \bar{F}_x\beta)\cos p\varphi; \end{cases}$$

on the other hand, Δiv **F** = 0 will give:

$$(29'') \qquad \bar{F}_x\alpha + \bar{F}_y\beta + \bar{F}_z\gamma + \bar{F}_u V = 0.$$

One will get analogous equations upon expressing **b** and $\mathfrak{H}$ in terms of $\mathbf{F}^*$ from (8′):

$$(30) \qquad b_x = ip\;(\bar{F}_z^*\beta - \bar{F}_y^*\gamma)\cos p\varphi, \ldots$$

$$(30') \qquad \mathfrak{H}_x = ip\;(\bar{F}_u^*\alpha - \bar{F}_x^*\,V)\cos p\varphi, \ldots$$

$\bar{F}_x^*$, $\bar{F}_y^*$, $\bar{F}_z^*$ are pure imaginary, while $\bar{F}_u^*$ is real.

Div $\mathbf{F}^*$ = 0 yields:

$$(30'') \qquad \bar{F}_x^*\alpha + \bar{F}_y^*\beta + \bar{F}_z^*\gamma + \bar{F}_u^* V = 0.$$

On the other hand, div $\mathfrak{H}$ = div $\mathfrak{B}$ = 0 is written:

$$(31) \qquad V(\bar{F}_x^*\alpha + \bar{F}_y^*\beta + \bar{F}_z^*\gamma) - \bar{F}_u^* = 0,$$

and upon combining (30″) and (31):

$$\bar{F}_u^* = 0, \qquad \bar{F}_x^*\alpha + \bar{F}_y^*\beta + \bar{F}_z^*\gamma = 0.$$

$\mathbf{F}^*$ is transversal and situated on the wave, but **F** is not. We now write the compatibility relations by first equating $\mathfrak{B}$ and $\mathfrak{H}$, as they are inferred from (29′) and (30′):

$$(32) \qquad \begin{cases} -V\,\bar{F}_x^* = \bar{F}_z\beta - \bar{F}_y\gamma, \\ -V\,\bar{F}_y^* = \bar{F}_x\gamma - \bar{F}_z\alpha, \\ -V\,\bar{F}_z^* = \bar{F}_y\alpha - \bar{F}_x\beta. \end{cases}$$

Suppose that the chosen system of axes is principal; i.e., that:

$$b_x = n_x^2\,h_x = \frac{h_x}{v_x^2}, \qquad b_y = n_y^2\,h_y = \frac{h_y}{v_y^2}, \qquad b_z = n_z^2\,h_z = \frac{h_z}{v_z^2}.$$

Upon using the **b** and **h** that one infers from (29) and (30), resp., one will then get:

(33) $$\bar{F}_z^*\beta - \bar{F}_y^*\gamma = \frac{1}{v_x^2}(F_u\,\alpha - F_x\,V), \ldots$$

A simple combination of the preceding equations will give:

$$\frac{\alpha^2}{v_x^2+V^2}+\frac{\beta^2}{v_y^2+V^2}+\frac{\gamma^2}{v_z^2+V^2} = 0,$$

which is the equation for the normal velocities.

From (28″), the momentor that is derived from $(\sigma + \tau)$ has a zero mean, and:

$$4\pi\,\gamma = \Delta\text{iv}\,\beta.$$

On the other hand, from (28′):

$$\beta_{y^*xz^*} = p^2\alpha^2(\bar{F}_u\bar{F}_x^* - \bar{F}_x\bar{F}_u^*)\int \sin^2 p\varphi\,dx$$

will give:

$$\beta_{y^*xz^*} = \frac{p^2\alpha^2}{V}\bar{F}_u(\bar{F}_z\beta - \bar{F}_y\gamma)\left[\frac{x}{2}+\frac{1}{4p\alpha}\sin 2p\varphi\right],$$

$$\beta_{y^*yz^*} = \frac{p^2\alpha^2}{V}\bar{F}_u(\bar{F}_z\beta - \bar{F}_y\gamma)\left[\frac{y}{2}+\frac{1}{4p\beta}\sin 2p\varphi\right],$$

$$\beta_{y^*zz^*} = \frac{p^2\alpha^2}{V}\bar{F}_u(\bar{F}_z\beta - \bar{F}_y\gamma)\left[\frac{y}{2}+\frac{1}{4p\gamma}\sin 2p\varphi\right],$$

$$\beta_{y^*uz^*} = \frac{p^2\alpha^2}{V}\bar{F}_u(\bar{F}_z\beta - \bar{F}_y\gamma)\left[\frac{u}{2}+\frac{1}{4\pi V}\sin 2p\varphi\right],$$

and

$$4\pi\,\gamma_{yz} = \frac{\partial}{\partial x}\beta_{y^*xz^*}+\frac{\partial}{\partial y}\beta_{y^*yz^*}+\frac{\partial}{\partial z}\beta_{y^*zz^*}+\frac{\partial}{\partial u}\beta_{y^*uz^*}.$$

If one expresses these quantities as functions of the components of the field and induction then one will get:

$$4\pi\,\gamma_{yz} = -p^2\left(\frac{1+V^2}{V}\right)\bar{F}_u(\bar{F}_z\beta - \bar{F}_y\gamma)\cos 2p\varphi = [\mathbf{b}\cdot\mathbf{h}]_{yz}.$$

That is what we had predicted. In the case of a rectilinear wave, the principal torque is provided by just the complementary term. However, that way of proceeding is hardly natural. Properly speaking, there is no propagation of momentor β, and we would like to avoid the introduction of the factors x, y, z, u that enter into the expression for β and cannot be transposed to the general case.

On the other hand, if one envisions an elliptical wave, instead of a rectilinear wave, then it will propagate a real momentor that is provided by the $(\sigma + \tau)$ terms in (28″), and β will not correspond to the propagation of momentor. It is then much less natural and conforms to the spirit of these calculations to refer θ in the left-hand side to equation (28) and write:

$$4\pi(\gamma + \theta) = \Delta\text{iv}\,(\sigma + \tau),$$

and to take:

$$4\pi M = -(\sigma + \tau)$$

to be the momentor that is transported by the wave.

14. Dragging of waves by a moving monorefringent medium. – To fix ideas, we consider a plane wave that propagates in the z direction and is rectilinearly polarized, and initially place ourselves in a system of axis in which the refringent medium is at rest. In this system of axes:

$$F_u = 0, \qquad \text{div}\,\mathfrak{H} = 0,$$

and the complementary torque will be annulled. The three real components of the principal torque are real, and the wave does not transport real momentor. Meanwhile, the three imaginary components of the principal torques are not zero, and if one places oneself in a new system of axes S' that moves with respect to the preceding one with velocity v then those three imaginary components will provide a real contribution to the real components of the torque γ' in the system S' in the course of a change of axes. In the present case, the annihilation of the complementary terms does not have an intrinsic or invariant character, as it does in the vacuum case, and the torque γ will provide the complementary terms in the system of axes S'. The change of axes that permits one to pass from the system S into S' will transform $\mathbf{F}$ and γ in the following manner if one uses the formulas of paragraph 3:

$$F_{x'} = F_x \cos\varphi + F_u \sin\varphi, \qquad F_{y'} = F_y,$$
$$F_{u'} = -F_x \sin\varphi + F_u \cos\varphi, \qquad F_{z'} = F_z,$$

$$\gamma_{y'z'} = \gamma_{yz}, \qquad \gamma_{z'u'} = \gamma_{xu},$$
$$\gamma_{z'x'} = \gamma_{yx} \cos\varphi + \gamma_{zu} \sin\varphi,$$
$$\gamma_{x'y'} = \gamma_{xy} \cos\varphi - \gamma_{yu} \sin\varphi,$$
$$\dots\dots\dots\dots\dots$$

We specialize the problem by supposing that the wave vibrates rectilinearly in the x direction in the system S; i.e.:

$$h_x \neq 0, \qquad h_y = 0, \qquad h_z = 0.$$

In the course of a change of axes, the 6-vectors $\mathfrak{M}$ and $\mathfrak{N}$ transform in the following manner:

$$b_{x'} = b_x, \qquad h_{x'} = h_x, \qquad b_{x'} = n^2\, h_{x'}\,,$$

$$h_{y'} = h_y = 0, \quad b_{y'} = b_y = 0,$$

$$b_{z'} = h_{z'} = \frac{\mathfrak{H}_y \beta}{\sqrt{1-\beta^2}}\,, \quad \mathfrak{H}_{y'} = \frac{\mathfrak{H}_y}{\sqrt{1-\beta^2}}\,.$$

$\mathbf{b}'$ does not have the same direction as $\mathbf{h}'$ in the system S', and a real torque will result. The problem is very analogous to the problem that we treated previously – namely, the birefringent medium – in which the torque was provided by only the complementary terms. If one writes the imaginary components of γ in the system S:

$$4\pi\,\gamma_{xu} = i\,\{[\mathbf{b}\cdot\mathfrak{B}] - [\mathbf{h}\cdot\mathfrak{H}]\}_x\,,$$

and if one uses the transformation formulas:

$$4\pi\,\gamma_{z'u'} = \gamma_{zu}\sin\varphi + \gamma_{zx}\cos\varphi,$$

in which $\gamma_{zx} = 0$ and $\gamma_{zu} \neq 0$ then one will indeed find a real torque, and one will be easily assured that its value is:

$$4\pi\,\gamma_{z'x'} = [\mathbf{b}'\cdot\mathbf{h}']_{z'x'}\,.$$

If one would like to obtain the expression for the momentors then one must conclude the calculation as in the preceding paragraph that related to a birefringent medium. One writes the expression for $\mathbf{F}$ and $\mathbf{F}^*$ in the system S, and one subjects them to the Lorentz transformation in order to obtain their values in the system S', and one calculates the terms $\sigma + \tau$ and β. Here again, the torque is provided by the complementary terms uniquely, and it is natural to set:

$$4\pi\,M = (\sigma + \tau),$$

which will give a mean value of zero for the momentor that is propagated by the wave. The preceding calculation indeed shows the necessity for the existence of the imaginary components of γ.

15. Expressing for the moment in a form that uses F and $\mathbf{F}^*$ symmetrically. – Recall the expression for the principal torque:

$$4\pi\,\gamma_{xu} = i\,\{[\mathbf{b}\cdot\mathfrak{B}]_{yz} - [\mathbf{h}\cdot\mathfrak{H}]_{yz}\},$$
$$4\pi\,\gamma_{yz} = i\,\{[\mathbf{b}\cdot\mathbf{h}]_{yz} + [\mathfrak{B}\cdot\mathfrak{H}]_{yz}\},$$

and once one has made each of the vector products explicit, replace **b** and $\mathfrak{H}$ with their expressions as functions of the vector potential $\mathbf{F}^*$ by using (8′) of paragraph 4.

Then set:

$$-\tau^*_{ijk} = F^*_i\,\mathfrak{M}^*_{jk} + F^*_j\,\mathfrak{M}^*_{ik} + F^*_k\,\mathfrak{M}^*_{ij},$$
$$-\mathbf{J}^* = [\mathbf{F}^*\,\mathfrak{M}];$$

i.e.:

$$-J^*_x = F^*_x\,\mathfrak{M}^*_{xx} + F^*_y\,\mathfrak{M}^*_{xy} + F^*_z\,\mathfrak{M}^*_{xz} + F^*_u\,\mathfrak{M}^*_{xu},$$

so one will have:

$$\sigma^*_{xyy} = -\sigma^*_{zzx} = \sigma^*_{uux} = J^*_x,$$
$$\sigma^*_{yzz} = -\sigma^*_{xxy} = \sigma^*_{yzz} = J^*_y,$$
$$\sigma^*_{zuu} = -\sigma^*_{yyz} = \sigma^*_{zxx} = J^*_z,$$
$$-\sigma^*_{xxu} = -\sigma^*_{yyu} = -\sigma^*_{zzu} = J^*_u,$$

$$4\pi\,\gamma_{yz} + 4\pi\,\theta^*_{yz} = \Delta\text{iv}\,(\sigma^* + \tau^*),$$

with

$$-4\pi\,\theta^*_{yz} = -F^*_x\Delta\text{iv}_u\mathfrak{M} + F^*_u\Delta\text{iv}_x\mathfrak{M} = F^*_x\Box F_u - F^*_u\Box F_x.$$

Hence:

(34) $$4\pi\,(\gamma + \theta^*)_{ij} = -4\pi\,\Delta\text{iv}_{ij}\,M^*,$$

with:

$$M^* = -\frac{\sigma^* + \tau^*}{4\pi}.$$

It is possible to form a more symmetric expression by taking the momentor to be $\frac{M + M^*}{2}$, so the complementary torque will be $\frac{\theta + \theta^*}{2}$. In order to apply this to a concrete case, recall the problem of paragraph 6 and calculate the moment flux along the *z* axis by taking:

$$C_z = \tfrac{1}{2}(M + M^*)_{xyz} = \frac{1}{8\pi}[F_y\,\mathfrak{H}_y + F_x\,\mathfrak{H}_x + i\,(F^*_x h_x + F^*_y h_y)] = \frac{ab}{4\pi\omega}c.$$

The result is therefore unmodified. In a general manner, the complementary torque can be put into a less abstract form as follows:

Take the 4-vector **q** that is defined by:

$$4\pi\,\mathbf{q} = c\,\Delta\text{iv}\,\mathfrak{M}.$$

If one denotes Hertz's free charge density by ε (as opposed to ρ, which denotes the true charge density) then its *u* component will be equal to $4\pi\,q_u = ic$ div $\mathbf{h} = 4\pi\,ic\varepsilon$. We can refer to the four components:

$$q_x\,,\ q_y\,,\ q_z\,,\ q_u = ic\ \varepsilon$$

by the name of *free electric current density*. It is easy to see that Δiv **q** = 0, which expresses the idea that free electric charge is conserved, just like true electric charge.

Consider the 4-vector **Q** that is defined by:

$$-\,4\pi\,\mathbf{Q} = ic\ \mathrm{div}\ \mathfrak{N}^*.$$

Its component Q_u is such that:

$$4\pi\,Q_u = ic\ \mathrm{div}\ \mathfrak{H} = 4\pi\,ic\ \delta,$$

in which δ denotes the density of free magnetism (it is not true magnetism). We can refer to the four components of **Q**:

$$Q_x\,,\ Q_y\,,\ Q_z\,,\ Q_u = ic\ \delta$$

by the name of *free magnetic current density*. On the other hand, Δiv **Q** = 0, and free magnetism will be conserved. The complementary torque will then take the form:

$$(\theta + \theta^*)_{yz} = \frac{1}{c}\,[\,(q_x F_u^* - q_u F_x^*) - i\ (Q_x\ F_u - Q_u\ F_x)],$$

which is an expression that must be approached like the one that we found in the problem of spheres in which the true electric charge density was not zero in (25′) of paragraph 8, and like the one that we will find in the following chapter. These ways of writing the complementary torque are essential for the development of the theory. I shall give their interpretation in a later publication.

CHAPTER VI

MOMENTS IN LORENTZ'S THEORY

16. Conservation of moments. – In this theory, the only dielectric medium is the vacuum. If ρ is the true density, and $\rho\dot{x}$, $\rho\dot{y}$, $\rho\dot{z}$ are the current densities then:

$$C_x = \rho\dot{x}, \qquad C_y = \rho\dot{y}, \qquad C_z = \rho\dot{z}, \qquad C_u = ic\,\rho,$$

for a 4-vector. The 6-vectors $\mathfrak{M}$ and $\mathfrak{N}$ that were defined in paragraph 3 are equal to each other:

$$\mathfrak{M}_{yz} = \mathfrak{H}_x, \quad \mathfrak{M}_{zx} = \mathfrak{H}_y, \quad \mathfrak{M}_{xy} = \mathfrak{H}_z,$$
$$\mathfrak{M}_{xu} = -ih_x, \quad \mathfrak{M}_{yu} = -ih_y, \quad \mathfrak{M}_{zu} = -ih_z,$$

and the Lorentz equations are written:

$$c\,\Delta\text{iv}\,\mathfrak{M} = 4\pi\,\mathbf{C},$$
$$\Delta\text{iv}\,\mathfrak{M}^* = 0.$$

The components of the force density 4-vector **f** are given by:

$$f_x = \rho\left[h_x + \frac{1}{c}(C_y\mathfrak{H}_z - C_z\mathfrak{H}_y)\right].$$

That can be written:

$$c\,\mathbf{f} = [\mathbf{C}\cdot\mathfrak{M}];$$

i.e.:

$$c\,f_x = C_x\,\mathfrak{M}_{xx} + C_y\,\mathfrak{M}_{xy} + C_z\,\mathfrak{M}_{xz} + C_u\,\mathfrak{M}_{xu}.$$

The force density can be put into the form:

(35) $$\mathbf{f} = -\Delta\text{iv}\,T,$$

in which T is the symmetric energy-impulse tensor:

$$T_{xx} = \frac{1}{8\pi}(h_y^2 + h_z^2 - h_x^2 + \mathfrak{H}_y^2 + \mathfrak{H}_z^2 - \mathfrak{H}_x^2),$$
$$T_{xy} = T_{yx} = -\frac{1}{4\pi}(h_x\,h_y + \mathfrak{H}_x\,\mathfrak{H}_y),$$
$$T_{xu} = T_{ux} = \frac{i}{4\pi}(h_y\,\mathfrak{H}_z - h_z\,\mathfrak{H}_y),$$
$$T_{uu} = -\frac{1}{8\pi}(h^2 + \mathfrak{H}^2).$$

The physical sense of these components is as follows: If i and j are indices x, y, z then T_{ij} is the impulse flux in the direction i across a face that is normal to j. T_{iu} is the impulse density in the direction i, up to a multiplicative constant.

We construct the *force moment density* with respect to the origin; i.e., we construct the combinations:

$$l_{ij} = [\mathbf{r} \cdot \mathbf{f}]_{ij} = i\, f_j - j\, f_i\,,$$

in which i, j represent two of the four components x, y, z, u; for example:

(36)
$$\begin{cases} l_{yz} = y\, f_z - z\, f_x, \\ l_{xu} = x\, f_u - u\, f_x. \end{cases}$$

The density thus-defined is a 6-vector. If one replaces the values of $\mathbf{f}$ with the corresponding Δiv T in the expression for l then one will get:

$$4\pi\, l_{yz} = -\frac{\partial}{\partial x}(y\, T_{zx} - z\, T_{yz}) - \frac{\partial}{\partial y}(y\, T_{zy} - z\, T_{yy})$$
$$-\frac{\partial}{\partial z}(y\, T_{zu} - z\, T_{yu}) - \frac{\partial}{\partial u}(y\, T_{zu} - z\, T_{yu}).$$

The 24 parentheses that one obtains by taking the six quantities l are the components of a three-index tensor that we will refer to by the expression *impulse moment tensor*, and we will have:

(37)
$$l = -\,\Delta\text{iv}\, L,$$

with

$$4\pi\, L_{yxz} = y\, T_{zx} - z\, T_{yz}\,, \qquad 4\pi\, L_{yyz} = y\, T_{zy} - z\, T_{yy}\,.$$

The significance of its components is as follows: $4\pi\, L_{ijk}$ is the impulse moment flux along the ik axis through a surface that is normal to j.

$$\frac{L_{yuz}}{ic} = \frac{1}{4\pi\, ic}(y\, T_{zu} - z\, T_{yu}) = l_{yz}$$

is the impulse moment density along the yz axis:

$$\boldsymbol{\lambda} = [\mathbf{r} \cdot \boldsymbol{g}],$$

in which $\boldsymbol{g}$ is the impulse density. That is formula (2) of paragraph 1; indeed, up to now, it seems to be almost the only that has been used in the study of moments. Equation (37) expresses the conservation of the impulse moment. x, y, z, u figure explicitly in the expressions for l and L, which shows that the moment thus-defined is a moment of the first kind that is meaningful only when one gives the point with respect to which one calculates it.

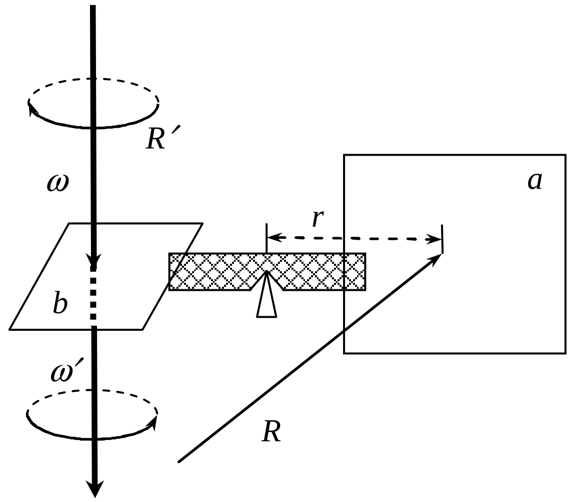

Figure 1.

It is an impulse moment of that type that enters into the experiments on the radiation pressure of light. A vertical torsion filament carries a vertical mirror (*a*) on which an unpolarized light ray *R* falls normally at a distance *r* from the filament. The radiation pressure communicates an impulse moment of the first kind to the torsion equipment. Such impulse moments, which are calculated from *L*, certainly exist physically then, but that is not an intrinsic property of an electromagnetic wave. If one confines oneself to them (and that seems to be what one often does) then one will have overlooked one-half of the terms, and in fact, the most important ones, since they belong to the moment that is intrinsic to the wave; i.e., the momentor.

Let *l* represent the force moment density. However, instead of replacing the quantities *f* with their expressions in terms of *T* in (36), express those quantities as functions of **C** and $\mathfrak{M}$:

$$(-z) \qquad c\,f_y = C_x\,\mathfrak{M}_{yx} + C_z\,\mathfrak{M}_{yz} + C_u\,\mathfrak{M}_{yu}\,,$$

$$(y) \qquad c\,f_z = C_x\,\mathfrak{M}_{zx} + C_z\,\mathfrak{M}_{zy} + C_u\,\mathfrak{M}_{zu}\,.$$

Multiply the first one by – *z* and the second one, by *y*, and then add them, while replacing the quantities **C** with their expressions as functions of $\mathfrak{M}$:

$$4\pi\,l_{yz} = y\,(\mathfrak{M}_{zx}\,\Delta\text{iv}_x\,\mathfrak{M} + \mathfrak{M}_{zy}\,\Delta\text{iv}_y\,\mathfrak{M} + \mathfrak{M}_{zu}\,\Delta\text{iv}_u\,\mathfrak{M})$$
$$- z\,(\mathfrak{M}_{yz}\,\Delta\text{iv}_x\,\mathfrak{M} + \mathfrak{M}_{yz}\,\Delta\text{iv}_z\,\mathfrak{M} + \mathfrak{M}_{zu}\,\Delta\text{iv}_u\,\mathfrak{M}).$$

Take one of the terms – for example:

$$y\,\mathfrak{M}_{zx}\,\Delta\text{iv}_x\,\mathfrak{M} = y\,\mathfrak{M}_{zx}\left(\partial\frac{\mathfrak{M}_{xy}}{\partial y} + \partial\frac{\mathfrak{M}_{xz}}{\partial z} + \partial\frac{\mathfrak{M}_{xu}}{\partial u}\right)$$

$$= \mathfrak{M}_{zx}\left[\frac{\partial}{\partial y}(y\,\mathfrak{M}_{xy}) + \frac{\partial}{\partial z}(y\mathfrak{M}_{xz}) + \frac{\partial}{\partial u}(y\,\mathfrak{M}_{xu})\right] - \mathfrak{M}_{xy}\,\mathfrak{M}_{zx}\,.$$

The first three terms are of the first kind, since y enters into them explicitly, and $-\mathfrak{M}_{xy}\,\mathfrak{M}_{zx}$ is of the second kind.

Group all of the terms of the first kind. One will get:

$$\begin{aligned}
&\mathfrak{M}_{zx}\left(\frac{\partial}{\partial y}y\,\mathfrak{M}_{xy} + \frac{\partial}{\partial z}y\,\mathfrak{M}_{zx} + \frac{\partial}{\partial u}y\,\mathfrak{M}_{xu}\right)\\
&+\mathfrak{M}_{zy}\left(\frac{\partial}{\partial x}y\,\mathfrak{M}_{yx} + \frac{\partial}{\partial y}y\,\mathfrak{M}_{yz} + \frac{\partial}{\partial u}y\,\mathfrak{M}_{xu}\right)\\
&+\mathfrak{M}_{zu}\left(\frac{\partial}{\partial x}y\,\mathfrak{M}_{ux} + \frac{\partial}{\partial y}y\,\mathfrak{M}_{uy} + \frac{\partial}{\partial z}y\,\mathfrak{M}_{uz}\right)\\
&-\mathfrak{M}_{yx}\left(\frac{\partial}{\partial y}z\,\mathfrak{M}_{xy} + \frac{\partial}{\partial z}z\,\mathfrak{M}_{xz} + \frac{\partial}{\partial u}z\,\mathfrak{M}_{xu}\right)\\
&-\mathfrak{M}_{yz}\left(\frac{\partial}{\partial x}z\,\mathfrak{M}_{zx} + \frac{\partial}{\partial y}z\,\mathfrak{M}_{zy} + \frac{\partial}{\partial u}z\,\mathfrak{M}_{zu}\right)\\
&-\mathfrak{M}_{yu}\left(\frac{\partial}{\partial x}z\,\mathfrak{M}_{ux} + \frac{\partial}{\partial y}z\,\mathfrak{M}_{uy} + \frac{\partial}{\partial u}z\,\mathfrak{M}_{uz}\right).
\end{aligned}$$

If one uses identities of the form:

$$\mathfrak{M}_{zx}\frac{\partial}{\partial y}y\,\mathfrak{M}_{xy} = \frac{\partial}{\partial y}(y\,\mathfrak{M}_{zx}\mathfrak{M}_{xy}) - y\,\mathfrak{M}_{xy}\frac{\partial}{\partial y}\mathfrak{M}_{zx}\,,$$

on the one hand, and the Lorentz equations, on the other, then the terms of the first kind can be put into the form:

$$-\,4\pi\,\Delta\text{iv}_{yz}\,L,$$

in which L is the three-index impulse moment tensor.

The terms of the second kind:

$$-\,\mathfrak{M}_{xy}\,\mathfrak{M}_{zx} - \mathfrak{M}_{zu}\,\mathfrak{M}_{uy} + \mathfrak{M}_{xz}\,\mathfrak{M}_{yx} + \mathfrak{M}_{uz}\,\mathfrak{M}_{yu} = 4\pi\,\gamma_{yz} = 0$$

give a zero sum. The principal torque is zero in Lorentz's theory, since **b** and **h**, as well as $\mathfrak{B}$ and $\mathfrak{H}$, have the same direction.

Replace the second symbol in each term by its expression as a function of **F**, and if one takes into account that Δiv **F** = 0 then:

$$0 = \frac{4\pi}{c}(C_y\,F_z - C_z\,F_y) + \frac{\partial}{\partial x}(\mathfrak{H}_y\,F_y - \mathfrak{H}_x\,F_x)$$
$$+ \frac{\partial}{\partial y}(-\,\mathfrak{H}_y\,F_x - \mathfrak{H}_x\,F_y + ih_z\,F_u) + \frac{\partial}{\partial z}(-\,\mathfrak{H}_z\,F_x - \mathfrak{H}_x\,F_z - ih_y\,F_u)$$
$$+ \frac{\partial}{\partial u}(ih_y\,F_z - ih_z\,F_y - \mathfrak{H}_x\,F_u)\,.$$

If one sets:

$$4\pi\,\theta_{yz} = \frac{4\pi}{c}(C_y\,F_z - C_z\,F_y)$$

then one will get:

$$\theta_{ij} = \frac{1}{c}[\mathbf{C}\cdot\mathbf{F}]_{ij}\,,$$

which is the complementary torque. Set:

$$-\,4\pi\,M_{yxz} = \mathfrak{H}_x\,F_x - \mathfrak{H}_y\,F_y - \mathfrak{H}_z\,F_z\,,$$
$$-\,4\pi\,M_{yyz} = \mathfrak{H}_y\,F_x + \mathfrak{H}_x\,F_y - ih_z\,F_u\,.$$

If we group all of the terms then we will obtain the equations that express the conservation of moments of the first and second kind:

(37) $$l + \Delta\text{iv}\,L = 0,$$
(38) $$\theta + \Delta\text{iv}\,M = 0.$$

Outside of matter (i.e., the electricity), $f = 0$, $l = 0$, $C = 0$, $\theta = 0$, and those equations will become:

(39) $$\begin{cases} \Delta\text{iv}\,L = 0, \\ \Delta\text{iv}\,M = 0. \end{cases}$$

The moment of the first kind is conservative, as is the moment of the second kind. Therefore, outside of matter, there is no possibility of transforming moments of one kind into moments of the other kind. On the contrary, inside of matter, equations (38) show that L and M are not conservative, since l and θ are no longer zero, so a certain quantity of moment of the first kind can be transformed into momentum of the second kind by the intermediary of matter.

17. Transformation of moments of one kind into moments of another kind. – Imagine, for example, the torsion pendulum that was described above (Fig. 1) and was intended to measure the impulse moment of the ray R. We fix a horizontal half-wave layer (b) that receives a sheaf of light rays R' that are polarized circularly in such a fashion that when they leave it, they will be inverse circular. That layer will experience a torque $2\mathfrak{P} / \omega$ and a force moment $2\Pi / c$ that directed in the opposite sense, where $\mathfrak{P}$ is the power that is transported by R', and Π is the power that is transported by R. Arrange $\mathfrak{P}$ and Π in such a manner that the pendulum is in equilibrium; i.e., take:

$$\frac{\mathfrak{P}}{\omega} = \frac{\Pi}{c}.$$

Since the moments equilibrate, there will be a continuous transformation of moment of one kind into moment of the other kind by the intermediary of the solid. Now, suppose that $\frac{\mathfrak{P}}{\omega} > \frac{\Pi}{c}$, so the equipment will turn in the same sense as the radiation pressure.

In the following chapter, we will see that the rotation of the half-wave layer will change the pulsation $\omega = 2\pi\nu$ of the ray R'; it will become $\omega' > \omega$. We assume that the operation is adiabatic – i.e., the mirror is perfectly reflecting, and the half-wave layer is perfectly transparent, such that no part of the energy will transform into heat – and we assume, on the other hand, that the motion is infinitely slow, so it will be possible to neglect the kinetic energy of the equipment, which is assumed to be a perfect solid. Replace the torsion filament with a frictionless suspension pivot, in order to eliminate the loss of potential energy from the torsion in the filament. When the mirror is moved far away from the ray R, there will be a reduction of the frequency of the reflected ray, and correspondingly, a reduction in the energy of that ray. Moreover, there will be a transfer of impulse moment to the solid at the same time that the pulsation ω of the ray R' changes with the infinitely-slow rotation of the half-wave. Since the operation is assumed to be adiabatic, the energy that is lost by R is given to R', and correspondingly, the moment of the first kind that corresponds to the energy that is lost will be transformed into a moment of the second kind in the ray R'.

Lorentz's theory is itself capable of giving the mechanism of that transformation: It is obvious that this is not so, since unless one adds quantum theory, Lorentz's theory is incapable explaining the existence of solid bodies, because by itself it is incapable of explaining the stability of the atoms that comprise the solid body. We then see that at its basis, the mechanism that permits the transformation of impulse moments into momentors, or conversely, will exceed the possibilities of Lorentz's theory, so one must introduce the constant h.

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CHAPTER VII

DOPPLER-FIZEAU EFFECT FOR ROTATION

18. Observable character of moments. – One can envision the question of the moments that are transported by light by using the principles of thermodynamics, and thus exhibit the real and observable character of the momentors that are transported by a circular wave.

For the impulse moments of the first kind, there is no doubt about their observable character. If a sheaf of light rays of power Π falls upon a mirror at a distance r from an axis of rotation then it will exert a force:

$$f = \frac{2\Pi}{c}$$

and an impulse moment that effectively conforms to this theoretical result:

$$\frac{2\Pi\, r}{c}.$$

Suppose that a mirror M receives the sheaf normally, and it displaces with the velocity v, which is regarded as positive in the sense of the incident mirror, but infinitely small. The mirror will impart a mechanical power f_v to the light:

$$f_v = w = -\,\Delta\Pi = 2\Pi\,\frac{v}{c}.$$

Correspondingly, in the reflected sheaf, the frequency will diminish by the Doppler-Fizeau effect. If $\omega = 2\pi\nu$ is the pulsation of the incident light:

$$\frac{\Delta\omega}{\omega} = -\,\frac{2v}{c}$$

and

$$w = -\,\Pi\,\frac{\Delta\omega}{\omega} = -\,\Delta\Pi. \tag{40}$$

The Doppler effect is inseparable from the mechanical power that is given to the mirror. Now, it can be exhibited indirectly by means of optical beats. For example, construct an interference apparatus of the air-wedge (*coin d'air*) type with one fixed glass layer and one moving layer. The light that is reflected from the fixed layer will have the same pulsation ω as the incident light, while the light reflected by the moving layer will have a pulsation of $\omega + \Delta\omega$, and what will result is an optical beat that is effectively realized at a fixed point in space by the displacement of the interference fringes that pass through that point.

The relation (40) has the consequence that if one performs a finite adiabatic transformation then the ratio Π / ω will remain constant. It will then result that a light energy $\mathcal{E}$ of pulsation ω will possess an entropy S that is of function of only $\mathcal{E} / \omega$:

$$S = f\left(\frac{\mathcal{E}}{\omega}\right). \tag{40'}$$

Therefore, if one modifies the frequency of the light by an arbitrary adiabatic process then there will exist the following proportion between the energies and pulsations:

$$\frac{\mathcal{E}_1}{\omega_1} = \frac{\mathcal{E}_2}{\omega_2}. \tag{40''}$$

I previously indicated ([1]) an *adiabatic* process for changing the frequency of circular radiation by letting the radiation fall on a turning half-wave layer.

Now, envision such a layer that is perfectly transparent and receives a sheaf of circularly-polarized light with a pulsation of ω and a power of $\mathcal{P}$ normally. Count rotations to be positive when they are in the same sense as the incident circular polarization. Let z be the direction of the incident sheaf, let x and y be two directions that are fixed in space, normal to z, and form a right trihedron with it. Let x and y be the components of the light vibration when the circular wave falls upon the layer:

$$x = a \cos \omega t, \quad y = a \sin \omega t.$$

Suppose that the half-wave layer is such that one of its principal directions forms an angle of α with x. A simple calculation will show that when the circular wave leaves the layer, it will be inverse circular, and one can write:

$$x_1 = a \cos(-\omega t + 2\alpha),$$
$$y_1 = a \sin(-\omega t + 2\alpha).$$

Now, suppose that the layer turns around a direction that is parallel to z in the positive sense with a very small angular velocity Ω:

$$\alpha = \Omega t,$$
$$x_1 = a \cos[-(\omega t - 2\Omega t)],$$
$$y_1 = a \sin[-(\omega t - 2\Omega t)],$$

and the frequency of the emergent light is $\omega - 2\Omega$. It will be diminished if the layer turns in the same sense as the incident circular wave, and augmented if it turns in the opposite sense. It will produce a true Doppler effect, and the corresponding change in frequency

([1]) E. HENRIOT, "Les couples exercés par la lumière polarisée circulairement," C. R. Ac. Sc. **198** (1934), pp. 1146.

can be exhibited in reality by a optical beat. Indeed, if we juxtapose the turning half-wave layer with an immobile half-wave layer whose principal directions are directed along *x* and *y* then the emergent vibration will be:

$$x_2 = a \cos(-\omega t),$$
$$y_2 = a \sin(-\omega t).$$

Irradiate the set of the two layers with a circular wave, one part of which falls upon the turning layer, and the other of which falls upon the immobile layer, and let those two vibrations interfere with each other upon leaving.

$$x_1 + x_2 = 2a \cos(-\omega t + \alpha) \cos(\Omega t),$$
$$y_1 + y_2 = 2a \cos(-\omega t + \alpha) \cos(\Omega t)$$

will then give the optical beat that corresponds to the predicted change in frequency.

What will result is a continuous motion of the fringes in a well-defined way that was effectively stated by Righi ([1]). It would be even simpler to interpret it as a beat whose cancellations are produced four times per rotation that one obtains by turning a half-wave layer between two crossed Nicol prisms with an angular velocity Ω. The rectilinear wave that leaves the first Nicol can be decomposed into two inverse circular waves of frequency ω. Upon leaving the turning layer, one will have two inverse circular waves of frequency $\omega - 2\Omega$, $\omega + 2\Omega$, while the second Nicol will permit one to make them interfere and will give a beat that is a quadruple cancellation be rotation.

There are few practical experiments that are more current than that one, but at its basis, it shows the reality of the angular Doppler effect and its magnitude. Since the layer is perfectly transparent, and the rotation is infinitely slow, *if one assumes that the operation is adiabatic* then, from the relations (40″), if $\mathcal{P}_1$ and $\mathcal{P}_2$ are the powers before and after traversal of the turning layer then:

$$\frac{\mathcal{P}_1}{\omega_1} = \frac{\mathcal{P}_2}{\omega_2} = \frac{\delta\mathcal{P}}{-2\Omega}. \tag{41}$$

The reduction $\delta\mathcal{P}$ in power must have a counterpart in a mechanical power δW that is given to the layer, since there is no heat emitted:

$$\delta W = -\delta\mathcal{P}.$$

Since that mechanical power is correlated with a rotation Ω, that will demand that a couple *j* must exist such that:

$$\delta\mathcal{P} = -j\,\Omega.$$

([1]) A. RIGHI, Mem. d. accad. d. scienze di Bologna [IV], **4** (1882), pp. 247. *See* also, R. D'E. ATKINSON, "Energy and angular momentum in certain optical problems," Phys. Rev. **47** (1935), pp. 623.

That value is deduced from arguments that are solely energetic and kinematic, and is precisely the one that is provided by electromagnetic theory. There is then a convergence in the results that is mentally satisfying, but the thermodynamic argument offers us something more. The real, observable character of the change in frequency by rotating the half-wave layer permits one to consider the phenomenon of beats that was described above as an indirect measurement of the couple j and a proof of its observable, measurable character in the same way as for the case of the air wedge, so the displacement of the fringes can be considered to be an indirect measurement of the radiation pressure.

The only difficulty resides in the hypothesis that we made that the operation is truly adiabatic. The displacement of the fringes in the air wedge is limited, so one can make only a number of fringes pass through a point of the air wedge that depends upon the more or less monochromatic state of the radiation, and each fringe will preserve a well-defined order number in the course of its displacement. These circumstances do not exist for the displacement of interference fringes that are produced by the turning and immobile half-wave layers: The displacement of fringes is then unlimited, so an indefinite number of fringes can pass through a point, and none of them will preserve an order number that is always the same in the course of its motion. The difference is essential, and I will return to that difficulty in my next publication.

Furthermore, as convincing as such an indirect experiment can be for measuring the radiation couples by the change in frequency, the only experiment that would inspire complete confidence would be the direct measurement of the mechanical couple that is exerted, which I have attempted in the past, and it would be desirable that it should be realized despite the smallness of the couples: The utilization of short Hertzian radiation might possibly provide a first indication in that sense.

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CHAPTER VIII

CONCLUSION

19. The problems that remain to be solved. – In the foregoing treatise, for the sake of completeness, I was forced to raise several important questions that I passed over in order to limit the scope of the treatise, and also because there is no complete answer to them in certain aspects. Notably, there is good reason to adopt the energetic viewpoint; that is, to envision, on the one hand, the exchanges of energy that are produced between the electromagnetic field and matter by the intermediary of torque, and on the other hand, to analyze the physical significance of the imaginary components of the torque and momentor.

The solution of a problem is always provisional, but it is important, above all, to pose it clearly.

The question of radiation couples is an important problem, and it has been generally neglected in electromagnetism: The objective of this study was to pose it, along with all of the difficulties that it entails, and to put it into clear focus.

With time and criticism, one might modify some points and add some others, but it will not involve any adventurous hypotheses nor leave the realm of classical theories, in such a way that one can hope that its essence will still remain.

In conclusion, I would like to indicate the reasons that determined the organization of this survey and pertain to the nature of the question itself. The Maxwell-Hertz theory imagined a medium that was polarizable from a macroscopic standpoint, while in Lorentz's theory, it was microscopically polarizable. In the latter theory, one imagines that a sufficiently-subtle observer of the medium can follow the electrons of an atom or polarizable medium in the course of their motions. It would then follow that certain problems would appear to be of the second kind in the Maxwell-Hertz theory and of the first kind in Lorentz's theory. For example, take the case of a wave that crosses a birefringent or absorbent medium. In the Maxwell-Hertz theory, the problem is of the second kind, and in that of Lorentz, one imagines that one can conceive of the motion of the polarization electrons around the center of the atom, and envision the impulse of each of them and take its moment with respect to the center, which will amount to a moment of the first kind. That was the manner by which I proceeded in a previous article ([1]).

However, we know from the statement of the principle of indeterminacy that there is good reason to distinguish between what can be conceived and what can be observed. Now, one can conceive of the motion of electrons in an atom, but from the principle of indeterminacy, one cannot imagine a corresponding experiment that will tell us what the impulse of an electron is at a given moment. Observations of the atom will give only a mean effect that translates into a matrix expression, but the operation that consists of taking the impulse of the electron at a given instant and its moment with respect to the center is an operation that lies outside of physics, since it does not correspond to any realizable experiment. In wave mechanics, the impulse moment of the trajectory and that of the spin in an atom are both of the second kind, and the subtle Lorentz observer will

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([1]) E. HENRIOT, "Les couples mécaniques exercés par la lumière polarisée elliptiquement," Bull. Cl. Sc. Acad. Roy. Belg. **13** (1927), pp. 143.

lose his rights. Perhaps the Maxwell-Hertz theory that provides the mean observables is closer to the recent wave mechanics that that of Lorentz, and that would explain the fact that one must indicate how to choose one over the other when one attempts to show the train of ideas in a natural manner. Torque and momentor, whose existence is necessarily manifest in the first theory, are transposed into the second one by a sort of passage to the limit when the polarizable medium becomes the vacuum, and will then permit one to solve a series of important problems that would be insoluble without them.

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