"Die Bedeutung des d'Alembertschen Prinzipes für starre Systeme und Gelenkmechanismen," Archiv der Math. und Phys. (3) **2** (1901), 57-77.

The meaning of d'Alembert's principle for rigid systems and link mechanisms

By KARL HEUN in Berlin

Translated by D. H. Delphenich

The following considerations in regard to d'Alembert's principle are restricted to systems that possess a finite number of degrees of freedom. For that reason, the "condition equations" that extend the kinetic differential equations will be mainly excluded. Namely, corresponding to **Clifford**'s way of thinking ("Elements of Dynamics"), the *complete systems of velocities* will be placed at the forefront of the developments, and in that way, at the same time, the meaning of the **Lagrangian** systems in kinetics will emerge in an especially clear way.

Of course, the author of *Mécanique analytique* introduced precisely those condition equations into the general developments, but with the additions to the second edition of his work, it emerged clearly that in the development of his own ideas, he started from that formal manner of representation and he recognized that the essential problem of the mechanics of bound systems was the analytical formulation of the *systems of velocities*.

It is only when one actually works through that basic kinematical fact that **d'Alembert**'s principle, in conjunction with the principle of virtual work, will take on a far-reaching efficacy, whereas without it, a schema would remain that has been achieved, as such, only in those cases where the system constraints lie clearly in view, so the systems of velocities would be known from the outset.

Moreover, the problem of mechanics for bound systems is by no means resolved with the exhibition of the explicit equations of motion; the determination of the reactions in arbitrary cross-sections of the subsystems, the links, and the supporting bearings will still remain to be treated as a no-less-important group of problems (kinetostatics).

That second aspect of d'Alembert's problem will be emphasized in the present paper, while the presentations that general mechanics gives to it will be mostly just touched upon superficially. However, the engineer is often in a position to be required to assign great value to the stress equations over the precise investigation of the motion of the system.

Throughout history, the *realm of applications* has exerted a clearly-recognizable influence on rational mechanics. At first, it was astronomy that had especially required the most fruitful way of treating the kinetics of free systems of points, and then physics, which had already influenced the kinetics of variable systems so broadly and in such a singular way, that one could very well speak of "physical mechanics," and in recent

times, a host of interesting problems in the theory of machines unmistakably point to a deeper analysis of **d'Alembert**'s principle as the natural foundation for kineticstatics.

In the following presentation of d'Alembert's principle and its consequences, we have, without exception, regarded the kinematical and dynamical quantities as vectors and the *inner* and *outer* product of two vectors are also employed correspondingly for the calculations in the algorithms. Those operations have already been naturalized to such an extent that we have only mentioned the notation here, and can refer to the numerous papers (e.g., **A. Föppls**, "Einführung in die Maxwellsche Theorie der Elektrizität").

We denote every vector by an overbar on the relevant symbol. Hence, the *inner* product will be defined by the equation:

$$\bar{a}\,\bar{b} = a\,b\cos\,(\bar{b}\,|\,\bar{a}).$$

The *outer* product is denoted by a continuous overbar, since it represents a new vector. If we set:

$$\overline{a\,b} = \bar{c}$$

then $\bar{c}$ will be perpendicular to $\bar{a}$ and $\bar{b}$, and its magnitude will be determined by:

$$c = ab\,\sin\,(\bar{b}\,|\,\bar{a}).$$

It also follows from this that one must have:

$$\overline{b\,a} = -\,\overline{a\,b},$$

since $\sin\,(\bar{b}\,|\,\bar{a}) = -\sin\,(\bar{a}\,|\,\bar{b})$.

For triple products, one has the relations (cf., **Föppl**, *Einführ.*, pp. 25)

$$\bar{a}\,\overline{b\,c} = \bar{c}\,\overline{a\,b} = \bar{b}\,\overline{c\,a}$$

and (*ibid.*, pp. 27):

$$\overline{a\,(b\,c)} = (\bar{a}\,\bar{c})\cdot\bar{b} - (\bar{a}\,\bar{b})\cdot\bar{c}\,.$$

Quadruple products will appear here only the following combination:

$$\overline{a\,b}\;\overline{c\,d} = (\bar{a}\,\bar{c})\cdot(\bar{b}\,\bar{d}) - (\bar{b}\,\bar{c})\cdot(\bar{a}\,\bar{d}).$$

Calculating with these simple tools has the advantage over the usual analytical coordinate methods (which is not appreciated nearly well enough) that it gives an uninterrupted, intuitive insight into the natural process of problem solving in which a mostly constructive idea enters in place of the schematic calculations with arithmetic quantities that follows the undivided basic geometrical and mechanical concepts step-by-step.

A. Formulation of the principle.

1. The history of the principle. – **Christian Huygens** was the first to successfully treat a difficult problem in the mechanics of bound systems in his *Horologium oscillatorium* (1673), in which he determined the center of oscillation for the compound pendulum in a novel way by applying the principle of the conservation of energy. His indirect method of solution of this problem, which was probably first posed by Pater **Mersenne**, was not to the taste of his contemporaries, and that started a scientific dispute that lasted from then until the Eighteenth Century. The history of d'Alembert's problem falls within that span of time (1681-1703). **Jacob Bernoulli** was the first to suggest a direct solution when he came to the fortunate idea of decomposing the vector of the applied force (viz., gravity) for any material point of the pendulum into two components, the first of which produces the effective acceleration of the mass, while the other one represents the reaction of the system constraints. The latter vectors collectively maintain equilibrium, since they cannot influence the actual state of motion of the pendulum. However, one still did not see the greater scope of that fundamental argument in that period of time, but only believed that one must apply it to each special case of a system problem with some idiosyncratic artifice in order to arrive at the equations of motions.

Perhaps, one can regard the fact that the systematization of kinetics did not happen sooner as more fortunate for the development of that theory. In any event, one starts with the drawback that a premature choice and definition of a general method has been carried out for the free unfolding of the original thoughts and construction of a lively intuition in regard to the processes of motion in concrete cases only so often.

Newton appreciated the kinetics of bound systems, whose first developmental phase was known to him from **Huygens**'s work, but he made no mention of that fact in his *Principia* (1687). When he confined himself to free systems of points, the dynamical notion of system reactions was what mostly stood in his way. By contrast, he did not recognize the correct meaning of his *lex tertia* for the mechanics of machines (cf., *Principia*, Leges motus, Scholium), in general. However, since he had by no means anticipated **Bernoulli**'s principle for the decomposition of vectors for bound systems, the attempts of his great fellow countrymen, the English scholars (e.g., **Thomson and Tait, Perry**), must be regarded as lacking. In assessing **Newton**'s contributions to mechanics, one should, above all, always keep in mind that he already had enough to do with the formal foundation and specialized research into the laws of motion for free point systems (e.g., planetary problems), and that those problems of physical astronomy that related to bound systems (e.g., precession of the equinoxes) demanded tools for dealing with them that were not at all available to him.

2. The general conception of the principle by d'Alembert. – The Eighteenth Century defined an entirely distinct chapter in the history of the intellectual progress of civilized people. The naïve basic ideas lay behind them, which had been promoted and disseminated so far that the driving spirit now took on the responsibility of liberating those ideas from the restrictive basis that arose from, exploring their scope in all directions, systematizing them, and thus erecting an edifice for the exact sciences upon pre-existing foundations that seemed humble in its beginnings (when looking back from

some future epoch). Yet, that imaginary architecture is still quite marvelous in the Eighteenth Century.

The builder has seen excellent growth in his advanced problems, exceptionally equipped with the rapidly-expanding mathematics as he is, prudently and farsightedly grateful for the numerous important discoveries in astronomy and physics and imbued with a love and enthusiasm for its facts that has imprinted the character of a lively classicism upon its services for all time.

D'Alembert stands in the middle of the bold works of that century. As a philosopher, mathematician, and physicist, he had enriched science with numerous treatises and with astonishing energy, but his most significant achievements were in the field of mechanics. In *Traité de Dynamique* (1743), he laid the foundations of the kinetics of bound systems, and therefore the evolution of ideas that **Galileo** had so fruitfully inaugurated was brought to a certain systematic conclusion. The unrestricted validity of **Bernoulli**'s ideas on the decomposition of dynamical vectors was not recognized, and its basis took the form of a specific kinetic principle from the general theory of motion that was an infallible tool. In **d'Alembert**'s version of things ([1]) (*Dynamik*, pp. 58), that main recipe read:

> "One decomposes the motions *a*, *b*, *c*, etc., that are imposed upon each body (i.e., mass-point) into two other ones *a*, α ; *b*, β ; *c*, γ, etc., in such a way that when one imposes only the motions *a*, *b*, *c*, etc., upon the bodies, those motions can be preserved without hindering each other, and that when one has imposed only the motions α, β, γ, etc., upon them, the system would remain at rest."

If an external impulse $\overline{h}$, which provokes the quantity of motion $m\overline{v}$ starting the state of rest, acts upon the mass-point *m* of the system then the reaction $\overline{r}$ will arise as a consequence of the system constraints. One will then have:

$$\overline{h} = m\overline{v} + \overline{r} \tag{1}$$

for each mass-point of the system, and the reaction impulses $\overline{r'}$, $\overline{r''}$, $\overline{r'''}$ will collectively maintain equilibrium. If the system already possessed a velocity, which will be denoted by the symbol $\overline{v}_0$, before the impulse $\overline{h}$ appeared then:

$$\overline{h} = m(\overline{v} - \overline{v}_0) + \overline{r} \tag{2}$$

for each individual material point, and the resultant of all reactions $\overline{r}$ will be equal to zero, as before. The last equation mediates the transition from the kinetics of impulses to the kinetics of continuously-acting forces. The impulse $\overline{h}$ must be infinitely-small of the same order for the time element *dt*, and thus equal to $d\overline{h}$. Correspondingly, $\overline{r}$ must also

([1]) For the sake of greater accessibility, we shall cite the German edition by **Arthur Korn** in Ostwald's Klassikersammlung. Leipzig 1899.

be replaced with $d\bar{r}$. The associated velocity increase $\bar{v} - \bar{v}_0$ will be equal to $d\bar{v}$, and we will get the relation:

$$d\bar{h} = m\,d\bar{v} + d\bar{r}\,.$$

We further set:

$$d\bar{h} = \bar{k}\,dt\,, \qquad d\bar{r} = \bar{s}\,dt\,,$$

so the continuously-acting forces $\bar{k}$ and $\bar{s}$ will relate to a unit time, and one will thus arrive at the basic equation for the lasting dynamical effect in the form:

$$\bar{k} = m\frac{d\bar{v}}{dt} + \bar{s}\,. \tag{3}$$

The reactions will again maintain equilibrium in regard to the complete system. **D'Alembert** regarded that transition as entirely obvious, and for that reason he did not emphasize it especially. One cannot blame him for that. The advanced nature of the concept of impulse has become modern now, and also has the generally-unknown advantage of being more intuitive than the direct consideration of continuous motions.

Our way of writing out equations (1) and (3) is obviously based upon the parallelogram law for impulses and forces. **D'Alembert** employed the law of the lever as a static principle, as well, and, in passing, the principle of virtual velocities, in a context that can be clearly recognized, since he, unlike **Jacob Bernoulli** (1717), did not regard the inertia of the latter even once within that scope. For that reason, his numerous examples have a somewhat relentlessly elementary character, even though their significance for the state of affairs at the time could not be underestimated. The most interesting part of *Traité* – namely, the mathematical treatment of the laws of collisions – must be excluded from the present considerations, since its complete implementation would require more physical hypotheses.

3. The general, formal elimination of the reactions by Lagrange. – In his *Traité*, **d'Alembert** said (cf., 57):

> "Here, I shall be content to treat the motion of bodies that collide with each other in an arbitrary way, or ones that exert strains upon each other by means of strings or inflexible rods. I shall devote more attention to this topic, since up to now, the greatest geometers have given only a very small number of problems of that kind, and I hope to put myself in a position to solve all of the most difficult problems of that kind that are known from arithmetic and the principles of mechanics by the general method that I will present."

That daring statement can quite easily give rise to misunderstandings in regard to the efficacy of **d'Alembert**'s method. For all simpler (i.e., easy to assess) system constraints, it is obvious that the reactions $\bar{r}$ or $\bar{s}$ can be eliminated, since the elementary principles of statics suffice in those cases, and one will arrive at the differential equations of kinetics in that way. However, if one were considering finite-

dimensional rigid systems instead of the discrete mass-points (*corps*) with their string and rod constraints then one would come to know the limits of the efficacy of the tools of statics that one has at one's command only to soon. However, the most important first step towards founding the kinetics of bound systems has already been taken. It is precisely because one has generally reduced the presentation of the equations of motion of such systems to a *problem of pure statics* that one will have posed a well-defined goal for the mechanics of one's time whose full scope one does not know, namely, the *systematic construction of the statics* of bound systems that **Lagrange** addressed, along with his younger contemporaries, with truly astonishing success.

Building further upon **Johann Bernoulli**'s idea, **Lagrange** created the *principle of virtual velocities* as the more general formal foundation of the statics of all material systems. That brilliant promoter of mechanics devoted himself to working out that fundamental principle in the context of rigid systems, link systems of rigid structures, systems of strings, fixed, elastic, and fluid continua from the twenty-fourth year of his life up to the end (1913). Of course, that accomplishment occupies a relatively small space in his collected works, but it shined like a priceless gemstone amidst all the rich treasures that he had created. *Mécanique analytique* (1788) was a "scientific poem" for **Hamilton** and it will remain so as long as one considers mechanics to be a "mathematical paradise," as **Leonardo da Vinci** did.

The crux of the Lagrangian system of mechanics is the strong and purposeful conception of the notion of a *possible velocity* (or the virtual element of velocity, which is equivalent to it) of an arbitrary mass element of well-defined material system. As long as one can succeed in *mathematically formulating* such systems of possible velocities, mechanics will have achieved a positive result; beyond that, all is darkness. If one knows a unique, complete expression for the element of virtual motion $\delta\overline{x}$ of a system point of mass *m* whose position in space is determined by the vector $\overline{x}$ then, from the principle of virtual displacements, a system of impulses $\overline{h}$ or forces $\overline{k}$ the equation ([1]):

$$\sum \overline{h}_\nu \, \delta\overline{x}_\nu = 0 \qquad \text{or} \qquad \sum \overline{k}_\nu \, \delta\overline{x}_\nu = 0$$

will exist for equilibrium, in which the summations extend over the **entire** system. Upon taking the "inner" vector product, each of these equations will decompose into a system of just as many independent static relations as the number of mutually-independent parameters (system coordinates, resp.) that enter into $\delta\overline{x}$. We have then assumed that $\delta\overline{x}$ admits a unique mathematical formulation, in the sense of **Lagrange.**

If the number of those parameters is *infinite* when it is take over the entire system then the virtual work will be dealt with *explicitly* only for a well-defined *spatial element* of the system, and one will generally get partial differential equations as the condition equations of equilibrium for a force-system that must be established for each spatial element. The internal stresses will no longer drop out of the kinetic equations any more.

In general, **Lagrange** did not always have the *complete* expression for $\delta\overline{x}$ in mind in his static developments, but he seemed to have a special fondness for employing multiple *condition equations* in his general Ansätzen. However, as was remarked before, all holonomic conditions are excluded from this, while non-holonomic restrictions on the

([1]) The case of inequality will be excluded from now on.

motions (despite their great significance for the kinetics of real processes of motion) will be excluded from the present consideration of **d'Alembert**'s principle.

In the context of this narrower view of things, the kinetic equations will assume the form:

$$\sum \overline{r}\,\overline{\delta x} = 0 \qquad\qquad \sum \overline{s}\,\overline{\delta x} = 0,$$

and due to equations (1) and (3), one can also write these in the form:

(4) $$\sum \overline{(h - mv)}\,\overline{\delta x} = 0,$$

(5) $$\sum \left(\overline{k - m\frac{dv}{dt}} \right) \overline{\delta x} = 0.$$

These equations define the foundations for the kinetics of impulse effects and the effects of continuous forces in **Lagrange**'s system. **Lagrange**'s specific achievement – namely, working out these fundamental equations for those groups of problems that shall be considered here next – will find a more detailed discussion in Section C.

4. D'Alembert's principle for Poisson. – **D'Alembert**, as well as **Lagrange**, had the derivation of the *differential equations of motion* in mind as the next and most important goal, while the determinations of the *reactions* in a problem was of subordinate significance to them, although they certainly did not neglect those determinations in all cases, but still pursued them no further from the general viewpoint. However, the ever-expanding volume of applications of the rational mechanics of systems must eventually direct the attention of the mathematician to the fact that this initially-neglected aspect of **d'Alembert**'s principle (viz., kinetostatics) is just as important in practice as the phenomena of motion as such. Now, **Lagrange** especially preferred the applications of mechanics to astronomical problems, while the engineering applications were distant to him, as his complete neglect of surface friction in *Mécanique analytique* shows well enough.

Poisson likewise had a great interest in astronomical problems (e.g., perturbation theory, precession and nutation), but he still considered the construction of *mechanical physics* as the main problem of his life and in that way, he was already prematurely distracted from systematically advancing the construction of general rational mechanics. Apart from that, he threw himself into some specific practical problems with great success, and along those lines he laid the groundwork for *Mécanique analytique* that soon found its cleverest and most zealous proponent in **Poncelet**. We would also like to ponder **Poisson**'s beautiful work on the effects of shooting a cannon on the different parts of its carriage (J. Polyt., cah. 21) in order to suggest how his tendencies (in contrast to those of **Lagrange**) had already noticeably taken a different direction.

In **Poisson**'s *Traité de Mécanique* (1811), we then find a conception of **d'Alembert**'s principle that sharply emphasizes the significance of the *reactions* (internal stresses, support pressures of the moving system parts), and in particular, considered the essential influence of *friction* for practical problems.

The numerous textbooks on rational mechanics that followed **Poisson**'s *Traité* added nothing remarkable in regard to the formulation of **d'Alembert**'s principle. Only the superb, and in many ways, exceptionally profound, textbook of **Routh** (viz., *Rigid Dynamics*) once more gives the stereotype conception that is connected with an omission of that the principle by **Airy** in a letter, which did not contain anything substantial, however.

B. The basic kinematical and dynamical concepts.

5. Kinematic combinations with effective elements. – The basic concepts of the kinematics of points are: the vector that determines position, which we would like to denote by $\overline{x}$, the total time derivative of that vector $d\overline{x}/dt = \dot{\overline{x}} = \overline{v}$, which represents the analytical measure of the velocity, and the total time derivatives of the velocity vector $\dot{\overline{x}}$, $d\dot{\overline{x}}/dt = \ddot{\overline{x}} = \overline{w}$, which will be called the acceleration of the point motion. We now define the scalar functions:

$$P = \tfrac{1}{2}\overline{x}\,\overline{x} \qquad \text{and} \qquad E = \tfrac{1}{2}\dot{\overline{x}}\,\dot{\overline{x}}\,.$$

We call P the *pole function* or the determining function of the vector $\overline{x}$. E is the energy function (per unit mass) or the determining function of the vector $\dot{\overline{x}}$. We further define the three inner products and the three outer products of the quantities $\overline{x}$, $\dot{\overline{x}}$, $\ddot{\overline{x}}$. One then sees immediately that $\overline{x}\,\dot{\overline{x}} = dP/dt$, $\overline{x}\,\ddot{\overline{x}} = d^2P/dt^2 - 2E$, and $\dot{\overline{x}}\,\ddot{\overline{x}} = dE/dt$. The outer product $\overline{x\dot{x}}$ is a vector that is perpendicular to $\overline{x}$ and $\dot{\overline{x}}$, and possesses the magnitude $x\dot{x}\sin(\dot{\overline{x}}\,|\,\overline{x})$. In point kinematics, one calls this vector twice the sectoral velocity of the vector $\overline{x}$, but here (due to the way that it is used in the mechanics of bound systems), we shall prefer the name *moment of velocity*, and denote it by $\overline{M}_v$. It is obvious that $\overline{x\ddot{x}} = d\overline{M}_v/dt$. The third combination in the outer product construction is $\overline{\dot{x}\ddot{x}}$. That vector is simultaneously perpendicular to the velocity and the acceleration and has the magnitude $\dot{x}\ddot{x}\sin(\ddot{\overline{x}}\,|\,\dot{\overline{x}})$. As one will see with no further discussion, its scalar value is also equal to the quotient $v^3 : r_1$, if the radius of principal curvature at the point in question along the path is denoted by r_1. Up to now, only **Somoff** has taken that scalar into consideration kinematically. Since he still has no name, we would like to further preserve his anonymity, and in what follows we shall characterize the scalar by the symbol $\overline{B}$.

The direct combinations of the basic kinematical elements are then summarized in a table as:

Table I.

Inner product:	*Outer* product:
1) $\overline{x}\,\dot{\overline{x}} = \dfrac{dP}{dt}$,	1′) $\overline{x\dot{x}} = \bar{M}_v$,
2) $\overline{x}\,\ddot{\overline{x}} = \dfrac{d^2P}{dt^2} - 2E$,	2′) $\overline{x\ddot{x}} = \dfrac{d\bar{M}_v}{dt}$,
3) $\dot{\overline{x}}\,\ddot{\overline{x}} = \dfrac{dE}{dt}$.	3′) $\overline{\dot{x}\ddot{x}} = \bar{B}$.
$P = \frac{1}{2}\overline{x}\,\overline{x}$,	$E = \frac{1}{2}\dot{\overline{x}}\,\dot{\overline{x}}$.

6. Kinematical combinations with virtual elements. – We shall define only the corresponding kinematical combinations of the virtual path elements with $\delta\overline{x}$, but not with $\delta^2\overline{x}$, since we shall exclude *astatic kinetics* from our consideration and construct only the most rudimentary basic concepts from it. With that restriction, the scalar product $\dot{\overline{x}}\,\delta\overline{x}$ will next come into view. Its meaning is the usual one from the kinematics of systems. We set $\dot{\overline{x}}\,\delta\overline{x} = \delta' A_v$; A_v is then the function that has been called the *action* (per unit mass) of the moving point. From now on, the symbols d' and δ' will denote the differential (variational, resp.) expressions that relate to the symbols in question, and they will not generally be complete differentials (variations, resp.). One now finds the following identity:

$$\ddot{\overline{x}}\,\delta\overline{x} = \frac{d}{dt}(\delta' A_v) - \delta E$$

by a simple differentiation of the foregoing relation with respect to time.

Naturally:

$$\delta E = \dot{\overline{x}}\,\delta\dot{\overline{x}}$$

in that expression.

For the sake of completeness, we shall now add the *vector products* that correspond to the *scalar* products. The outer product $\overline{\dot{x}\,\delta x}$ is perpendicular to the velocity and the virtual path-element, and has the absolute $\dot{x}\,\delta x \sin(\delta\overline{x}\,|\,\dot{\overline{x}})$. If we set $\overline{\dot{x}\,\delta x} = \delta'\overline{S}_v$ then differentiating that with respect to time will yield the identity:

$$\overline{\ddot{x}\,\delta x} = \frac{d}{dt}(\delta'\overline{S}_v) - \overline{\dot{x}\,\delta\dot{x}}\,.$$

We then have the following table of kinematic combinations with the virtual path element $\delta\overline{x}$:

Table II.

1)	$\bar{\dot{x}}\,\delta\bar{x} = \delta' A_v$,	1′)	$\overline{\dot{x}\,\delta x} = \delta'\bar{S}_v$
2)	$\bar{\ddot{x}}\,\delta\bar{x} = \frac{d}{dt}(\delta' A_v) - \delta E.$	2′)	$\overline{\ddot{x}\,\delta x} = \frac{d}{dt}(\delta'\bar{S}_v) - \overline{\dot{x}\,\delta\dot{x}}.$

7. Dynamical combinations with effective elements. – The corresponding basic *dynamical* concepts are the impulse $\bar{h}$ and the continuously-acting force $\bar{k}$. The inner product $\bar{x}\,\bar{h}$ can then be referred to as the elementary *virial of the impulse*, where we are borrowing the nomenclature of **Clausius** with a meaningless alteration from instantaneous forces (impulses, resp.). We set $\bar{x}\,\bar{h} = V_h$. The analogous concept for time-varying forces – namely, $\bar{x}\,\bar{k}$ – is germane to mechanics. If we set $\bar{x}\,\bar{k} = V_k$ and refer to V_k as the *virial of the force* $\bar{k}$ then, since it is already commonplace, we will change only the sign of the quantity that **Clausius** called by that name and drop the factor $\frac{1}{2}$ that were introduced.

Consistent with the usual terminology, the products $\bar{\dot{x}}\,\bar{h} = \bar{L}_h$ and $\bar{\dot{x}}\,\bar{k} = \bar{L}_k$ mean the *power* of the impulse $\bar{h}$ and the time-varying force $\bar{k}$. The outer products $\overline{x\,h} = \bar{M}_h$ and $\overline{x\,k} = \bar{M}_k$ are the *moments* of $\bar{h}$ and $\bar{k}$ relative to the starting point of the vector $\bar{x}$.

The *outer* vector products $\overline{\dot{x}\,h} = \bar{N}_h$ and $\overline{\dot{x}\,k} = \bar{N}_k$ shall be considered only occasionally in what follows, since their mechanical meanings have still not been sufficiently investigated. However, it is precisely on that basis that we recommend looking into those meanings here. In the development of the mechanics of systems, it has already been shown repeatedly that some formal concepts that were initially regarded as pointless can later take on great meaning. In that regard, I shall recall only the *centrifugal moment* that was treated in detail by **Schwein** and was since then brought to general attention by **Clausius** and **Yvon Villarceau.**

A summary of the dynamical combinations with effective kinematical elements will then yield the following overview:

Table III.

	For instantaneous forces:		For time-varying forces:
1)	$\bar{x}\,\bar{h} = V_h$,	1′)	$\bar{x}\,\bar{k} = V_k$,
2)	$\bar{\dot{x}}\,\bar{h} = L_h$,	2′)	$\bar{\dot{x}}\,\tilde{k} = L_k$,
3)	$\overline{x\,h} = \bar{M}_h$,	3′)	$\overline{x\,k} = \bar{M}_k$,
4)	$\overline{\dot{x}\,h} = \bar{N}_h$.	4′)	$\overline{\dot{x}\,k} = \bar{N}_k$.

8. Dynamical combinations with the virtual path-element. – The *inner* products $\bar{h} \cdot \bar{\delta} x = \delta' A_h$ and $\bar{k} \cdot \bar{\delta} x = \delta' A_k$ are the most important ones under that rubric, since they define the virtual *work* for instantaneous forces and time-varying forces, resp. However, along with that, we would also like to work with the corresponding outer products $\overline{h \cdot \delta x} = \delta' \bar{S}_h$ and $\overline{k \cdot \delta x} = \delta' \bar{S}_k$ here ([1]), since they are of use in the *astatic* consideration of system mechanics.

We then get the following summary:

Table IV.

For instantaneous forces:		For time-varying forces:	
1)	$\bar{h} \cdot \delta \bar{x} = \delta' A_h$,	1′)	$\bar{k} \cdot \delta \bar{x} = \delta' A_k$,
2)	$\overline{h \cdot \delta x} = \delta' \bar{S}_h$.	2′)	$\overline{k \cdot \delta x} = \delta' \bar{S}_k$.

The kinematical and dynamical combinations that were quoted in the foregoing define a formal skeleton for the mechanics of systems of points, within certain limits. Naturally, one cannot later wish that mechanics should actually be developed by means of such a schematic program. That would then imply the pedantry of making it the guiding principle for progress, which would mostly run contrary to the healthy development of mechanics. On the other hand, now that statics and kinetics lie before in a highly sophisticated phase of development, we probably have the right to pose the question of how we can organize the basic concepts that emerged from many (and frequently more-or-less random) demands into a synthetic schema. Even apparently-isolated general laws of mechanics, such the theorem of **Yvon Villarceau** for free systems of points, have their place in such a schema. The theorem in question is nothing but formula 2) of Table I, once the summation over the entire system of points has been performed.

For us, the presentation of the schema above has a well-defined purpose: Namely, we would like to apply it directly to **d'Alembert**'s basic equation, which expresses the decomposition of the dynamical vector, and in that way, exhibit links between the fundamental kinetic relations that seem to present themselves in a not-so-casual way in other respects.

C. General consequences of d'Alembert's principle

9. Virial theorems for bound systems. – It follows immediately from **d'Alembert**'s impulse equation:

$$\bar{h} = m\dot{\bar{x}} + \bar{r} ,$$

([1]) The equation $\delta S_k = 0$ contains the complete and sufficient conditions for all *astatic* forms of equilibrium in its application to the rigid systems. It will then accomplish the same thing here that the equation of the virtual displacements $\delta' A_k = 0$ does for *position equilibrium.*

by the operation of forming the inner product, that:

(6) $$\bar{x}\,\bar{h} = m\,\bar{x}\,\dot{\bar{x}} + \bar{x}\,\bar{r}\,.$$

We now sum over all mass-points *m* of the system and set:

$$\sum m\,\bar{x}\,\bar{x} = P, \qquad \sum \bar{x}\,\bar{h} = V_h\,, \qquad \sum \bar{x}\,\bar{r} = V_r\,.$$

One will then have:

$$\sum m\,\bar{x}\,\dot{\bar{x}} = \frac{dP}{dt},$$

and one will get the consequence of equation (6):

(7) $$V_h - V_r = \frac{dP}{dt},$$

which is valid for all systems, or when expressed differently:

For each bound system, the difference between the system virials for the impulses and reactions is equal to the complete derivative of the pole function with respect to time.

For a system that rotates rigidly around the starting point of the vector $\bar{x}$, *P* will obviously be constant. As a result, one will have the following theorem in that case:

For the rotating rigid system on which only impulses act, the virial of all impulses is equal to the virial of all elementary reactions when both virial are referred to the same fixed point.

We can treat **d'Alembert**'s equation for time-varying forces:

$$\bar{k} = m\,\ddot{\bar{x}} + \bar{s}$$

in exactly the same way and then get:

$$\bar{x}\,\bar{k} = m\,\bar{x}\,\ddot{\bar{x}} + \bar{x}\,\bar{s}\,.$$

If we now set:

$$\sum m\,\dot{\bar{x}}\,\dot{\bar{x}} = E$$

for the entire system then if we recall equation (2) of Table I, it will follow that:

(8) $$V_k - V_s = \frac{d^2P}{dt^2} - 2E,$$

which will go over to the theorem of **Yvon Villarceau** for free systems, and thus for V_s = 0. Hence, one has the general virial theorem for bound systems:

The virial of the elementary forces that act upon a system minus the virial of the corresponding elementary reactions is equal to the second derivative of the associated pole function with respect to time minus two times the energy of the system.

P is independent for a body that rotates around a fixed point. One will then have the theorem:

Two times the kinetic energy of a rotating rigid system is always equal to the difference between the virial of the reactions that are referred to the fixed point and the virial of the external forces.

10. The power law for bound systems. – The theorems that come under consideration here were already known in the first developmental period of system mechanics, so here they will be only quoted, for the sake of logical connectivity. It will follow from the basic equations:

$$\bar{h} = m\bar{\dot{x}} + \bar{r} \quad \text{and} \quad \bar{k} = m\bar{\ddot{x}} + \bar{s},$$

when one multiplies them by $\bar{\dot{x}}$ and sums over all mass-points of the system, that:

$$\sum \bar{\dot{x}}\,\bar{h} = \sum m\,\bar{\dot{x}}\,\bar{\dot{x}} + \sum \bar{\dot{x}}\,\bar{r} \quad \text{and} \quad \sum \bar{\dot{x}}\,\bar{k} = \sum m\,\bar{\ddot{x}}\,\bar{\dot{x}} + \sum \bar{\dot{x}}\,\bar{s}\,.$$

However, from **d'Alembert**'s principle, one will have:

$$\sum \bar{\dot{x}}\,\bar{r} = 0 \quad \text{and} \quad \sum \bar{\dot{x}}\,\bar{s} = 0.$$

If we then set:

$$\sum \bar{\dot{x}}\,\bar{h} = L_h \quad \text{and} \quad \sum \bar{\dot{x}}\,\bar{k} = L_k$$

for the entire system, which is consistent with the previous notations, then we will get the well-known power formulas:

$$L_h = 2E \quad \text{and} \quad L_k = \frac{dE}{dt}\,.$$

The power in a system of impulses that acts upon a bound system at rest is equal to twice the kinetic energy that is generated, and the power in a system of time-varying forces will be measured by the change in the kinetic energy per unit time.

11. The moment theorem. – We multiply the equations:

$$\bar{h} = m\bar{\dot{x}} + \bar{r} \quad \text{and} \quad \bar{k} = m\bar{\ddot{x}} + \bar{s}$$

by the vector $\bar{x}$ and get:

$$\overline{xh} = m\overline{x\dot{x}} + \overline{xr} \qquad \text{and} \qquad \overline{xk} = m\overline{\dot{x}\ddot{x}} + \overline{xs}\,.$$

Summation over the entire system yields:

$$\bar{M}_h - \bar{M}_r = \bar{M}_v \qquad \text{and} \qquad \bar{M}_k - \bar{M}_s = \frac{d\bar{M}_v}{dt}.$$

For a free, rigid system, one will have $\bar{M}_r = 0$ and $\bar{M}_s = 0$. One will then have the known basic equation for its rotational motion:

$$\bar{M}_h = \bar{M}_v \qquad \text{and} \qquad \bar{M}_k = \frac{d\bar{M}_v}{dt}.$$

These are the analytical expressions for the areal principle.

12. Two analogous laws for the dynamical vector $\bar{N}$. – If one takes the outer product with the velocity vector $\bar{\dot{x}}$ then one will get the equations:

$$\sum \overline{\dot{x}h} = \sum \overline{\dot{x}r} \qquad \text{and} \qquad \sum \overline{\dot{x}k} = \sum m\overline{\ddot{x}\dot{x}} + \sum \overline{\dot{x}s}\,,$$

or, if one sets:

$$\sum m\overline{\ddot{x}\dot{x}} = \bar{B}$$

then one will get:

$$\bar{N}_h = \bar{N}_r \qquad \text{and} \qquad \bar{N}_k - \bar{N}_s = \bar{B}\,.$$

For the sake of overview, we summarize the *general* consequences in the table below:

Table V.

For instantaneous forces:	For time-varying forces:
1) $V_h - V_r = \frac{dP}{dt}$,	1′) $V_k - V_s = \frac{d^2P}{dt^2} - 2E$,
2) $L_h = 2E$	2′) $L_k = \frac{dE}{dt}$,
3) $\bar{M}_k - \bar{M}_r = \bar{M}_v$,	3′) $\bar{M}_k - \bar{M}_s = \frac{d\bar{M}_v}{dt}$,
4) $\bar{N}_h - \bar{N}_r = 0$.	4′) $\bar{N}_k - \bar{N}_s = \bar{B}$.

D. The differential equations of motion.

13. The Lagrange-Hamilton form of d'Alembert's principle for impulses and time-forces. – It follows from the basic equation for the decomposition of the applied impulses:

$$\bar{h} = m\dot{\bar{x}} + \bar{r},$$

when one multiplies it by the virtual path element $\delta\bar{x}$ and then sums over all mass-points of the system:

$$\sum \bar{h}\,\delta\bar{x} = \sum m\dot{\bar{x}}\,\delta\bar{x} + \sum \bar{r}\,\delta\bar{x}$$

or

$$\delta' A_h = \delta' A_v + \delta' A_r.$$

Since $\delta' A_r = 0$, from the principle of virtual displacements, one will get the basic equation for impulsive motion in the form:

(9) $$\delta' A_h = \delta' A_v.$$

For time-varying forces, one defines the expression:

$$\sum \bar{k}\cdot\delta\bar{x} = \sum m\ddot{\bar{x}}\cdot\delta\bar{x} + \sum \bar{s}\cdot\delta\bar{x},$$

or from eq. (2) of Table II:

(10) $$\delta' A_k = \frac{d}{dt}(\delta' A_v) - \delta E,$$

since $\sum \bar{s}\cdot\delta\bar{x} = 0$.

Equations (9) and (10) can be regarded as the formal analytical expression of **d'Alembert**'s principle, and both of them go back to **Lagrange**. The expression that emerges from equation (10) by integrating over time *t* was also completely familiar to **Lagrange** (*Méc. anal.*, 2nd ed., t. **1**, pp. 307-310), and he used it as the basis for the

derivative of **Euler**'s equation for the rotating body (*Méc. anal.*, t. **2**, pp. 238-240). It seems necessary to me to expressly emphasize this here, because one always refers to the integral formula:

$$(11) \qquad \left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} (\delta E + \delta' A_k)\, dt$$

as *Hamilton's principle*, even though **Hamilton**, who knew about *Mécanique analytique* very intimately, never actually claimed authorship for it. His service consisted in presenting and employing the *characteristic function* and knowing its meaning for the formal representation of canonical integrals when a force function exists.

14. Analogous vector formulas in which the reactions are not eliminated. – For impulses, upon outer multiplying the basic equation, one will get the relation:

$$\sum \overline{h \cdot \delta x} = \sum m \overline{\dot{x} \cdot \delta x} + \sum \overline{r \cdot \delta x},$$

or in our notation:

$$(12) \qquad \delta' \overline{S}_h = \delta' \overline{S}_v + \delta' \overline{S}_r .$$

When one recalls equation 2) of Table II, the equation:

$$\sum \overline{k \cdot \delta x} = \sum m \overline{\ddot{x} \cdot \delta x} + \sum \overline{s \cdot \delta x}$$

will likewise imply the expression:

$$\delta' \overline{S}_k = \frac{d}{dt}(\delta' \overline{S}_v) - \sum m \overline{\dot{x} \cdot \delta \dot{x}} + \delta' \overline{S}_s ,$$

or, upon integrating over time t :

$$(13) \qquad \left[\delta' \overline{S}_v\right]_{t_0}^{t} = \int_{t_0}^{t} [\delta' \overline{S}_k - \delta' \overline{S}_s + \sum m \overline{\dot{x} \cdot \delta \dot{x}}]\, dt .$$

The mechanical meaning of that formula, which I have examined only in some simple examples up to now, and have also transformed into generalized **Lagrangian** coordinates, might remain unmentioned here, since I hope to be able to communicate it in detail in the continuation of this paper on **d'Alembert**'s principle.

15. Systems of possible velocities. – We understand a complete system of possible velocities to mean an arbitrary material complex to mean the totality of the analytical expressions for $\overline{\dot{x}}$ or, what is equivalent to that, for the virtual displacements $\delta \overline{x}$ that are compatible with the system constraints, which one can arrive at by considering the actual elementary motions of *all* mass-points. Since we have restricted ourselves to the simplest

types of material systems (viz., rigid systems and chains of links), that system of velocities will be mostly available to us in a finished form, even though perhaps there is still much that has not been sufficiently worked through in the individual cases.

There are essentially two different paths to the representation of systems of velocities for all material systems with a finite number of degrees of freedom, and historically they have been developed in connection with the rigid-body problem. **Euler**, **Clairaut**, and **d'Alembert** likewise arrived at the kinematical concept of the momentary axis and the associated rotational velocity by an exact intuition. We shall now unite the two ways of looking at things in the usual way into the concept of a single vector $\bar{\sigma}$ that falls along the direction of the momentary axis, and its length will represent the magnitude of the momentary angular velocity $\dot{\theta}$. One's immediate intuition will show that for every point that is separated from the fixed point *O* by the vector $\bar{x}$ with a well-defined sense of direction, the equation:

(14) $$\dot{\bar{x}} = \overline{\sigma x}$$

will be true. If we refer the rotational motion to an arbitrary starting point *C* that goes from the reference point of the vector $\bar{x}$ through the vector $\bar{c}$ then we can ascribe the translational velocity $\dot{\bar{c}}$ of the system to the point *C* and obtain the following equation in place of equation (14):

(15) $$\dot{\bar{x}} = \dot{\bar{c}} + \overline{\sigma (x - c)}.$$

That equation contains the complete analytical expression for the system of velocities for a free, rigid body.

It will be established when one is given the two kinematical vectors $\dot{\bar{c}}$ and $\bar{\sigma}$. If one sets $\bar{\sigma} = \bar{\omega} \cdot \frac{d\theta}{dt}$, such that $\bar{\omega}$ represents a unit vector that determines the momentary axis, then one will have:

$$d\bar{x} = d\bar{c} + \overline{\omega (x - c)} \cdot dt .$$

In what follows, we shall mostly write this as:

$$d\bar{x} = d\bar{c} + \overline{d\theta (x - c)},$$

in which we have set $\bar{\omega} \cdot dt = \overline{d\theta}$, and thus regarded the amplitude as a vector. The general expression for the possible elementary motions of a free, rigid system will then be:

(16) $$\delta\bar{x} = \delta\bar{c} + \overline{\delta\theta (x - c)}.$$

If only the rotation comes into consideration then one will have:

(17) $$\delta\bar{x} = \overline{\delta\theta \cdot x}$$

when one lets the reference point *O* coincide with *C*.

Lagrange was not satisfied with the derivation of that formula, which followed directly from intuition, but sought to arrive at it in another way. He applied a strange and especially remarkable method in t. 1 of *Mécan. analytique*, pp. 159-165. In it, he started from the notion that every curve of double curvature in a rigid system must remain invariable; i.e., any three infinitesimally-close sequential points of that curve must remain in a rigid orientation with respect to each other. In that way, he got a system of differential equation that the components of $\delta \bar{x}$ must satisfy.

Verdam (1864) has presented that line of reasoning thoroughly in his "Biydrage tot de toepassing van het beginsel van **d'Alembert**, overeenkomstig de rekenwijze van **Lagrange**."

It is also interesting to arrive at equation (19) by negating the elementary deformations of an infinitesimal triple of rays that is thought of as being variable. For the sake of simplicity, we take them to be rectangular and consider δx_1, δx_2, δx_3 to be the rectangular components of $\delta \bar{x}$, so it will then follow from negating the changes of lengths that one gets from the basic kinematical equations of the theory of elasticity that:

$$\frac{\partial}{\partial x_1}\delta x_1 = 0, \quad \frac{\partial}{\partial x_2}\delta x_2 = 0, \quad \frac{\partial}{\partial x_3}\delta x_3 = 0,$$

and it will follow from negating the shear deformations of the elementary body that:

$$\frac{\partial}{\partial x_2}\delta x_3 + \frac{\partial}{\partial x_3}\delta x_2 = 0, \qquad \frac{\partial}{\partial x_3}\delta x_1 + \frac{\partial}{\partial x_1}\delta x_3 = 0, \qquad \frac{\partial}{\partial x_1}\delta x_2 + \frac{\partial}{\partial x_2}\delta x_1 = 0.$$

The integration of these equations yields, with no further assumptions:

$$\begin{aligned}
\delta x_1 &= \delta c_1 + \delta\theta_2 \cdot x_3 - \delta\theta_3 \cdot x_2, \\
\delta x_2 &= \delta c_2 + \delta\theta_3 \cdot x_1 - \delta\theta_1 \cdot x_3, \\
\delta x_3 &= \delta c_3 + \delta\theta_1 \cdot x_2 - \delta\theta_2 \cdot x_1,
\end{aligned}$$

in which δc_1, δc_2, δc_3, $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ mean the integration constants. The expressions thus-obtained are identical with equation (16) for $c = 0$.

That method is passed over in most textbooks on the theory of elasticity.

It is not necessary here to comment further upon the most useful way of presenting equation (17) by differentiating the formulas for the coordinate transformations with respect to the cosines of the axis angles, since that is discussed in all presentations of mechanics and is reproduced again and again.

The characteristic element in the expression $\dot{\bar{x}} = \overline{\sigma x}$ is the vector $\bar{\sigma}$, and therefore a purely *kinematical* parameter that says nothing about the *configuration* of the system directly, and which is, in addition, unsuitable for analytically establishing the system of forces. For that reason, for the complete definition of a problem concerned with a rigid body, it is necessary to express the kinematical vector $\bar{\sigma}$ in terms of coordinates, and in that way, one will get a *second analytical representation* for the system of possible velocities.

It is known that this was already done by **Euler** (Mém. Ac. Berl., 1758) by introducing the three position angles that are named for him. **Lexel** (Nov. Com. Ac. Petrop., 1755) had also already represented the cosines of the axes in terms of three of them, and in that way, likewise achieved a position determination in terms of (independent) coordinates. These three processes have an essential flaw, since the symmetry of the formulas cannot be maintained. Completely-symmetric coordinate expressions for the vector $\bar{\sigma}$ seem to have been presented for the first time by **Cayley** (Camb. Dubl. Math. J., 1846), who employed the coordinates of **Rodrigues** [J. de Liouv. **5** (1840)] for that purpose. One can find thorough presentations of the recent investigations on this topic in **F. Klein** and **A. Sommerfeld**: *Über die Theorie des Kreisels* (Heft 1, 1897 and Heft 2, 1898) ([1]).

These considerations can now be adapted to arbitrary systems of links whose individual terms are rigid bodies with no special difficulties (of course, not always without great complications). From the principle of relative motion, in each of those cases, it must be possible to exhibit two expressions of the form:

$$\bar{\dot{x}} = \text{funct. } (\bar{k}', \bar{k}'', \ldots, \bar{x})$$

and

$$\bar{\dot{x}} = \text{funct. } (q_1, q_2, \ldots, q_i, \bar{x})$$

for all points of the systems, such that $\bar{k}', \bar{k}'', \ldots$ mean a sufficient number of kinematical vectors, and q_1, q_2, …, q_i mean the corresponding position coordinates. Every *analytical form* of the complete system of velocities $\bar{\dot{x}}$ corresponds to a special form of the general reduction of the forces and a form for the kinetic differential equations that is peculiar to that system. Hence, the special forms of the *statics* and *kinetics* of the material system in question will be characterized completely in that way.

(To be continued)

———————

([1]) Cf., also **F. Kötter**: "Bemerkungen zu F. Klein und A. Sommerfelds Theorie des Kreisels (1899). (Ed. rem.)