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The meaning of d'Alembert's principle for rigid systems and link mechanisms

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(Continuation)

16. Euler's equations of motion for rotating bodies. – From equations (3) and (3′) in Table V, the basic kinetic equations of a rotating rigid system are:

$$\bar{M}_h = \bar{M}_v \qquad \text{and} \qquad \bar{M}_k = \frac{d\bar{M}_v}{dt},$$

since the total moments of the reactions vanish. In the formula:

$$\bar{M}_v = \sum m \overline{x\dot{x}},$$

one needs only to replace $\bar{\dot{x}}$ with the expression $\overline{\sigma x}$ and sum over all mass-points of the body in order to obtain the impulse equation in explicit form. However, one has (cf., the introduction):

$$\overline{x(\sigma x)} = x^2 \cdot \bar{\sigma} - (\bar{x}\,\bar{\sigma})\cdot\bar{x}\,.$$

Ordinarily, one decomposes $\bar{x}$ along three rectangular axes that are rigidly linked with the system when one sets:

$$\bar{x} = \bar{a}_1 + \bar{a}_2 + \bar{a}_3\,.$$

The corresponding components of the vector $\bar{M}_v$ follow in that way, namely:

$$\begin{aligned}
M_{v,1} &= \sum m(a_2^2 + a_3^2)\cdot\sigma_1 - \sum m\,a_1\,a_2\cdot\sigma_2 - \sum m\,a_3\,a_1\cdot\sigma_3\,,\\
M_{v,2} &= \sum m(a_2^2 + a_3^2)\cdot\sigma_2 - \sum m\,a_2\,a_3\cdot\sigma_3 - \sum m\,a_1\,a_2\cdot\sigma_1\,,\\
M_{v,3} &= \sum m(a_2^2 + a_3^2)\cdot\sigma_3 - \sum m\,a_3\,a_1\cdot\sigma_1 - \sum m\,a_2\,a_3\cdot\sigma_2\,,
\end{aligned}$$

in which, we have set:

$$\sum m(a_2^2 + a_3^2) = A_1, \quad \sum m(a_3^2 + a_1^2) = A_2, \quad \sum m(a_1^2 + a_2^2) = A_3,$$

to abbreviate, and have set:

$$\sum m\, a_2\, a_3 = D_1, \qquad \sum m\, a_3\, a_1 = D_2, \qquad \sum m\, a_1\, a_2 = D_3,$$

and have referred to those quantities as the moments of inertia and moments of deviation. The kinetic *impulse equations* then take on the usual form:

$$\left\{ \begin{array}{l} M_{h,1} = A_1 \cdot \sigma_1 - D_3 \cdot \sigma_2 - D_2 \cdot \sigma_3, \\ M_{h,2} = A_2 \cdot \sigma_2 - D_1 \cdot \sigma_3 - D_3 \cdot \sigma_1, \\ M_{h,3} = A_3 \cdot \sigma_3 - D_2 \cdot \sigma_1 - D_1 \cdot \sigma_2. \end{array} \right. \tag{18}$$

In order to construct the **Euler** equations, we must now form the differential quotients $d\bar{M}_\nu / dt$. Instead of doing that, one can also differentiate the elementary vector:

$$\bar{M}_\nu = (\bar{x}\,\bar{x}) \cdot \bar{\sigma} - (\bar{x}\,\bar{\sigma}) \cdot \bar{x}$$

with respect to time and get:

$$\frac{d\bar{M}_\nu}{dt} = 2(\bar{x}\,\dot{\bar{x}}) \cdot \bar{\sigma} + (\bar{x}\,\bar{x}) \cdot \dot{\bar{\sigma}} - (\bar{x}\,\dot{\bar{\sigma}}) \cdot \bar{x} - (\dot{\bar{x}}\,\bar{\sigma}) \cdot \bar{x} - (\bar{x}\,\bar{\sigma}) \cdot \dot{\bar{x}}.$$

However, one obviously has $\bar{x}\,\dot{\bar{x}} = 0$ and $\dot{\bar{x}}\,\bar{\sigma} = 0$, since the vectors in question are perpendicular to each other. As a result, one will have:

$$\frac{d\bar{M}_\nu}{dt} = 2(\bar{x}\,\bar{x}) \cdot \dot{\bar{\sigma}} + (\bar{x}\,\dot{\bar{\sigma}}) \cdot \bar{x} - (\bar{x}\,\bar{\sigma}) \cdot \dot{\bar{x}} = (\bar{x}\,\bar{x}) \cdot \dot{\sigma} + (\bar{x}\,\dot{\bar{\sigma}}) \cdot x + \overline{\sigma M_\nu},$$

and correspondingly:

$$\frac{d\bar{M}_\nu}{dt} = \left(\frac{d\bar{M}_\nu}{dt} \right) + \overline{\sigma M_\nu}, \tag{19}$$

in which the brackets around the derivative of $\bar{M}_\nu$ suggest that one must consider only the quantity $\bar{\sigma}$ to be variable in that differentiation. In our notations, the **Euler** equations will then read:

$$\bar{M}_k = \left(\frac{d\bar{M}_\nu}{dt} \right) + \overline{\sigma M_\nu}. \tag{20}$$

They were first published in that form (naturally, without the symbolism of vector analysis) by **Lagrange** in his *Mécan. anal.* 2nd ed., t. **2**, pp. 239, in which he derived them

from the kinetic principle of virtual displacements (cf., no. **13** of that book). In that book, **Lagrange** employed the kinetic energy E of rotating systems, which will have the form:

$$E = \tfrac{1}{2}\sum m(\overline{\sigma x}\cdot\overline{\sigma x}) = \tfrac{1}{2}(A_1\sigma_1^2 + A_2\sigma_2^2 + A_3\sigma_3^2) - D_1\,\sigma_2\,\sigma_3 - D_2\,\sigma_3\,\sigma_1 - D_3\,\sigma_1\,\sigma_2\,,$$

due to the equation $\bar{\dot{x}} = \overline{\sigma x}$. From equations (18), one will then have:

$$M_{v,1} = \frac{\partial E}{\partial \sigma_1}, \quad M_{v,2} = \frac{\partial E}{\partial \sigma_2}, \quad M_{v,3} = \frac{\partial E}{\partial \sigma_3}.$$

As a result (or really, due to the essential difference between the derivations), for **Lagrange**, the quantities $\frac{d}{dt}\frac{\partial E}{\partial \sigma_1}$, $\frac{d}{dt}\frac{\partial E}{\partial \sigma_2}$, and $\frac{d}{dt}\frac{\partial E}{\partial \sigma_3}$ appeared in placed of the components of the relative velocity $\left(\frac{d\bar{M}_v}{dt}\right)$.

If one refers the vector $\bar{\sigma}$ to the principle axes then the moments of deviation in the expression for $\bar{M}_v$ will vanish, and one will get the usual **Euler** equations:

(21)
$$\begin{cases} A_1\dfrac{d\sigma_1}{dt} + (A_3 - A_2)\,\sigma_2\,\sigma_3 = M_{k,1}, \\ A_2\dfrac{d\sigma_2}{dt} + (A_1 - A_3)\,\sigma_3\,\sigma_1 = M_{k,2}, \\ A_3\dfrac{d\sigma_3}{dt} + (A_2 - A_1)\,\sigma_1\,\sigma_2 = M_{k,3} \end{cases}$$

from equation (20).

One can write down equation (19) directly, since it follows immediately from the principle of relative motion. $\left(\frac{d\bar{M}_v}{dt}\right)$ is obviously the vector of the relative change in velocity of $\bar{M}_v$ with respect to the rotating system, while $\overline{\sigma M_v}$ is the vector of the associated guiding velocity (*Führungsgeschwindigkeit*). Strictly speaking, **Euler** employed the same line of reasoning for the derivation of his equation, but without clothing it a definite analytical form.

17. Lagrange's transitivity equations for rigid systems. – **D'Alembert**'s principle, in the integral form that **Lagrange** and **Hamilton** employed:

(22)
$$\left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} (\delta E + \delta' A_k)\,dt\,,$$

gives rise to a remarkable difficulty when one employs a system of velocities that is expressed by kinematical parameters that are not, at the same time, the time derivatives of coordinates, and which **Lagrange** was the first to clearly recognize and overcome with his own skill (for the case of rotating, rigid systems). Namely, the variations $\delta\sigma_i$ in the expression:

$$\delta E = \frac{\partial E}{\partial \sigma_1}\delta\sigma_1 + \frac{\partial E}{\partial \sigma_2}\delta\sigma_2 + \frac{\partial E}{\partial \sigma_3}\delta\sigma_3 \tag{23}$$

must be transformed in such a way that they will contain only the $\delta\theta_i$ and the complete time derivatives of those quantities. **Lagrange** (*Mécan. anal.*, 2nd ed., t. **2**, pp. 229) arrived at that by employing the relations that exist between the nine direction cosines. Instead of that, we shall pursue a more direct and convenient path by starting immediately with the concept of the systems of possible velocities. As a result of the equation $\dot{\overline{x}} = \overline{\sigma x}$, we will have:

$$d\overline{x} = \overline{d\theta \cdot x} \quad \text{and} \quad \delta\overline{x} = \overline{\delta\theta \cdot x}.$$

By varying and differentiating this, one will get:

$$\delta d\overline{x} = \overline{\delta d\theta \cdot x} + \overline{d\theta \cdot \delta x} \quad \text{and} \quad d\delta\overline{x} = \overline{d\delta\theta \cdot x} + \overline{\delta\theta \cdot dx}.$$

Now, since one obviously has $\delta d\overline{x} = d\delta\overline{x}$, it will follow by subtracting the foregoing equations that:

$$\overline{(\delta d\theta - d\delta\theta)\cdot x} = \overline{d\theta(\delta\theta \cdot x)} - \overline{\delta\theta \cdot (d\theta \cdot x)} = \overline{(d\theta \cdot \delta\theta)\cdot x},$$

or, since $\overline{x}$ is completely arbitrary:

$$\delta\overline{d\theta} - d\overline{\delta\theta} = \overline{d\theta \cdot \delta\theta}. \tag{24}$$

That is **Lagrange**'s *transitivity equation* for a rotating, rigid system. It is an immediate consequence of the kinematical expression for the system of velocity. We conclude from this that every characteristic form for a system of velocities as a function of *essentially-kinematical* parameters must correspond to a special transitivity equation, which is likewise characteristic of the material system.

The relations between the axis components:

$$\begin{cases} \delta d\theta_1 = d\delta\theta_1 + d\theta_2 \cdot \delta\theta_3 - d\theta_3 \cdot \delta\theta_1, \\ \delta d\theta_2 = d\delta\theta_2 + d\theta_3 \cdot \delta\theta_1 - d\theta_1 \cdot \delta\theta_3, \\ \delta d\theta_3 = d\delta\theta_3 + d\theta_1 \cdot \delta\theta_2 - d\theta_2 \cdot \delta\theta_1 \end{cases} \tag{25}$$

follow from equation (24), as **Lagrange** communicated in *loc. cit.*

We substitute those values, in conjunction with equation (23), in the integral expression:

(26) $$\left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} (\delta E + \delta' A_k)\, dt\,.$$

Now, one has the relation:

$$\delta E = \frac{\partial E}{\partial \sigma_1}\delta\sigma_1 + \frac{\partial E}{\partial \sigma_2}\delta\sigma_2 + \frac{\partial E}{\partial \sigma_3}\delta\sigma_3 = \bar{M}_v \cdot \overline{\delta\sigma}\,,$$

in which, from equation (24), one has set:

(27) $$\overline{\delta\sigma} = \frac{d}{dt}\overline{\delta\theta} + \overline{\sigma \cdot \delta\theta}\,.$$

As a result:

$$\delta E = \bar{M}_v \cdot \frac{d}{dt}\overline{d\theta} + \bar{M}_v \cdot \overline{\sigma \cdot \delta\theta} = \frac{d}{dt}(\bar{M}_v \cdot \overline{d\theta}) - \frac{d\bar{M}_v}{dt} \cdot \overline{\delta\theta} - \overline{\sigma \cdot M_v}\,\overline{\delta\theta}\,.$$

Equation (26) will then go to:

$$\left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} \left\{ \left(\frac{dM_v}{dt} \right) + (\sigma M_v) - M_k \right\} \cdot \delta\theta \cdot dt\,;$$

one will then have $\bar{M}_v \cdot \overline{\delta\theta} = \delta' A_v$ and $\bar{M}_v \overline{\delta\theta} = \delta' A_k$, since translations are excluded. One must then have:

$$\left(\frac{dM_v}{dt} \right) + \overline{\sigma M_v} = \overline{M_k}\,,$$

from which one will arrive at the **Lagrangian** form of the equations of motion by decomposing the latter equation into components:

(28) $$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial E}{\partial \sigma_1}\right) + \sigma_2 \dfrac{\partial E}{\partial \sigma_3} - \sigma_3 \dfrac{\partial E}{\partial \sigma_2} = M_{k,1}, \\ \dfrac{d}{dt}\left(\dfrac{\partial E}{\partial \sigma_2}\right) + \sigma_3 \dfrac{\partial E}{\partial \sigma_1} - \sigma_1 \dfrac{\partial E}{\partial \sigma_3} = M_{k,2}, \\ \dfrac{d}{dt}\left(\dfrac{\partial E}{\partial \sigma_3}\right) + \sigma_1 \dfrac{\partial E}{\partial \sigma_2} - \sigma_2 \dfrac{\partial E}{\partial \sigma_1} = M_{k,3}. \end{cases}$$

18. Lagrange's kinetic equations in general position coordinates. – We next assume an arbitrary system of possible velocities, which we suggest by the symbolic equation:

(29) $$\dot{\bar{x}} = \text{func.}\ (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_i, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_i)\,.$$

The $\bar{\varepsilon}$ in this are vectors in the usual sense, while the q are real, mutually-independent position coordinates. The number of the latter will be equal to the number of degrees of freedom, such that the motion of the systems is not restricted by any condition equations. The vectors $\bar{\varepsilon}$ are generally single-valued functions of those coordinates. It follows from the symbolic equation (29) that:

(30) $$\delta\bar{x} = \text{func. } (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_i, \delta q_1, \delta q_2, \ldots, \delta q_i) .$$

For a *free* material point, one always has:

$$\dot{\bar{x}} = \bar{\varepsilon}_1 \cdot \dot{q}_1 + \bar{\varepsilon}_2 \cdot \dot{q}_2 + \bar{\varepsilon}_3 \cdot \dot{q}_3 ,$$

and correspondingly:

$$\delta\bar{x} = \bar{\varepsilon}_1 \cdot \delta q_1 + \bar{\varepsilon}_2 \cdot \delta q_2 + \bar{\varepsilon}_3 \cdot \delta q_3 .$$

As a result:

$$\delta' A_v = \dot{\bar{x}}\, \delta\bar{x} = \sum_{i=1}^{3} \bar{\varepsilon}_i\, \dot{\bar{x}} \cdot \delta q_i ,$$

or if we set:

$$\bar{\varepsilon}_i\, \dot{\bar{x}} = p_i ,$$

as usual, to abbreviate:

$$\delta' A_v = \sum_{i=1}^{3} p_i \cdot \delta q_i .$$

The quantities p are *linear* functions of the quantities $\dot{q}$. We shall now impose the condition on the functional relation in equation (29), or equation (30), which coincides with it, that the scalar quantity $\delta' A_v$ that is derived from it must take the form:

(31) $$\delta' A_v = \sum_{i=1}^{3} p_i \cdot \delta q_i ,$$

and that the p must be *linear* functions of the $\dot{q}$.

With that assumption, one can always set $\delta' A_v = \sum_{i=1}^{3} h_i \cdot \delta q_i$, and in that way the basic equation for *impulsive* effects:

(I) $$\delta' A_v = \delta' A_h$$

will take the simple form:

(32) $$p_i = h_i \qquad (i = 1, 2, 3, \ldots, i).$$

In the equation for time-varying forces:

(II) $$\left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} (\delta E + \delta' A_k)\, dt ,$$

the kinetic energy E of the total system, which is, by assumption, expressed by equation (31), is a quadratic function of the $\dot{q}$. That equation must also remain valid when δq is replaced with dq. As a result:

(33) $$2E = \sum_{i=1}^{3} \frac{\partial E}{\partial \dot{q}_i} \cdot \dot{q}_i .$$

Without that, one would have:

$$\delta E = \sum_{i=1}^{3} \frac{\partial E}{\partial \dot{q}_i} \cdot \delta \dot{q}_i + \sum_{i=1}^{3} \frac{\partial E}{\partial q_i} \cdot \delta q_i .$$

In complete analogy with the equation $\delta' A_h = \sum_{i=1}^{3} h_i \cdot \delta q_i$, and also for time-varying forces, we further set:

(34) $$\delta' A_k = \sum \bar{k} \cdot \delta \bar{x} = \sum_{i=1}^{3} h_i \cdot \delta q_i ,$$

and, from the precedent set by **Hertz** (*Prinzipien*, pp. 218), call the quantities k_1, k_2, …, k_i the components of the **Lagrangian** force, which we would like to symbolically denote by k. As is known, **Hertz** called the symbol k a "vector relative to the total system." Now, since the *q coordinates* – i.e., the $\dot{q}$ – are *complete* derivatives with respect to time, one will always have the equation:

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i ,$$

and the basic equation (II) will go to:

$$\left[\sum_{i=1}^{3} p_i \cdot \delta q_i \right]_{t_0}^{t} = \left[\sum_{i=1}^{3} \frac{dE}{d\dot{q}_i} \cdot \delta q_i \right]_{t_0}^{t} + \int_{t_0}^{t} \sum_{i=1}^{3} \left\{ -\frac{d}{dt} \frac{\partial E}{\partial \dot{q}_i} + \frac{\partial E}{\partial q_i} + k_i \right\} \delta q_i \cdot dt .$$

That equation can be fulfilled identically for arbitrary values of the δq only when one has:

(35) $$p_i = \frac{\partial E}{\partial \dot{q}_i}$$

and

(36) $$\frac{dp_i}{dt} - \frac{\partial E}{\partial q_i} = k_i .$$

These are the **Lagrange** equations. I would like to expressly point out that the impulse equations that one gets from combining formulas (32) and (35), namely:

(37) $$\frac{\partial E}{\partial \dot{q}_i} = h_i ,$$

go back to **Lagrange** (*Méc. anal.*, 2nd ed., t. **2**, pp. 183), and not **Niven**, as **Routh** remarked in his *Rigid Dynamics*, and indeed, one finds them in the cited place in precisely the form that **Routh** gave to them. Namely, the existence of a function **Ω** is assumed there that will yield the components:

$$h_i = \frac{\partial \Omega}{\partial q_i}.$$

Clifford gave a derivation of the **Lagrangian** equations in his *Elements of Dynamic* (v. 2 of the posthumous publication, pp. 81) whose basic ideas we shall repeat here. Let the system of velocities be dependent upon only two coordinates q_1 and q_2 . With that assumption, one will have:

$$\overline{v} = \overline{\varepsilon}_1 \cdot \dot{q}_1 + \overline{\varepsilon}_2 \cdot \dot{q}_2 .$$

Clifford then first set $\dot{q}_1 = 1$ and $\dot{q}_2 = 0$ and then $\dot{q}_1 = 0$ and $\dot{q}_2 = 1$. The corresponding values of $\overline{v}$ are: $\overline{v}_1$ and $\overline{v}_2$. Now, he proved that $\frac{d\overline{v}_1}{dq_2} = \frac{d\overline{v}_2}{dq_1}$. The energy of the system has the value:

$$E = \tfrac{1}{2}(\overline{\varepsilon}_1 \overline{\varepsilon}_1 \cdot \dot{q}_1^2 + 2\overline{\varepsilon}_1 \overline{\varepsilon}_2 \cdot \dot{q}_1 \dot{q}_2 + \overline{\varepsilon}_2 \overline{\varepsilon}_2 \cdot \dot{q}_2^2) .$$

It then follows from this that:

$$\frac{\partial E}{\partial \dot{q}_1} = \overline{\varepsilon}_1 \overline{v} , \qquad \frac{\partial E}{\partial \dot{q}_2} = \overline{\varepsilon}_2 \overline{v} .$$

Now, one has the following equations, with no further conditions:

$$\frac{\partial \overline{v}}{\partial q_1} = \frac{d\overline{\varepsilon}_1}{dt} \qquad \text{and} \qquad \frac{\partial \overline{v}}{\partial q_2} = \frac{d\overline{\varepsilon}_2}{dt} .$$

It will then follow from the energy equation:

$$E = \tfrac{1}{2}\overline{v}\,\overline{v}$$

that:

$$\frac{\partial E}{\partial q_1} = \frac{\partial \overline{v}}{\partial q_1} \cdot \overline{v} , \quad \frac{\partial E}{\partial q_2} = \frac{\partial \overline{v}}{\partial q_2} \cdot \overline{v} ,$$

or

$$\frac{\partial E}{\partial q_1} = \frac{d\overline{\varepsilon}_1}{dt} \cdot \overline{v} , \quad \frac{\partial E}{\partial q_2} = \frac{d\overline{\varepsilon}_2}{dt} \cdot \overline{v} .$$

However, one has:

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{q}_1} = \frac{d\overline{\varepsilon}_1}{dt} \cdot \overline{v} + \overline{\varepsilon}_1 \cdot \frac{d\overline{v}}{dt} , \qquad \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_2} = \frac{d\overline{\varepsilon}_2}{dt} \cdot \overline{v} + \overline{\varepsilon}_2 \cdot \frac{d\overline{v}}{dt} .$$

One will get the **Lagrange** equations from this directly:

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{q}_1} - \frac{\partial E}{\partial q_1} = \bar{\varepsilon}_1 \cdot \frac{d\bar{v}}{dt}, \qquad \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_2} - \frac{\partial E}{\partial q_2} = \bar{\varepsilon}_2 \cdot \frac{d\bar{v}}{dt}.$$

One can rigorously derive the **Lagrange** equations for a free point *kinematically*. In that simple case, $\bar{v}$ will then have the form:

$$\bar{v} = \bar{\varepsilon}_1 \cdot \dot{q}_1 + \bar{\varepsilon}_2 \cdot \dot{q}_2 + \bar{\varepsilon}_3 \cdot \dot{q}_3 .$$

It will then follow that $\bar{\varepsilon}_\iota \bar{v} = p_\iota$, and a differentiation of this with respect to time will give:

$$\bar{\varepsilon}_\iota \cdot \frac{d\bar{v}}{dt} = \frac{dp_\iota}{dt} - \frac{d\bar{\varepsilon}_\iota}{dt}\bar{v} .$$

Since *v* is a complete derivative with respect to time, due to the integrability conditions, one will have the equations:

$$\frac{\partial \bar{\varepsilon}_\kappa}{\partial q_\iota} = \frac{\partial \bar{\varepsilon}_\iota}{\partial q_\kappa} .$$

With that:

$$\frac{\partial \bar{v}}{\partial q_\iota} = \frac{\partial \bar{\varepsilon}_1}{\partial q_\iota} \cdot \dot{q}_1 + \frac{\partial \bar{\varepsilon}_2}{\partial q_\iota} \cdot \dot{q}_2 + \frac{\partial \bar{\varepsilon}_3}{\partial q_\iota} \cdot \dot{q}_3 = \frac{\partial \bar{\varepsilon}_\iota}{\partial q_1} \cdot \dot{q}_1 + \frac{\partial \bar{\varepsilon}_\iota}{\partial q_2} \cdot \dot{q}_2 + \frac{\partial \bar{\varepsilon}_\iota}{\partial q_3} \cdot \dot{q}_3 ,$$

or:

$$\frac{\partial \bar{v}}{\partial q_\iota} = \frac{d\bar{\varepsilon}_\iota}{dt} .$$

One immediately obtains the **Lagrange** equations from this in the kinematical form:

$$\bar{\varepsilon}_\iota \frac{d\bar{v}}{dt} = \frac{dp_\iota}{dt} - \frac{\partial E}{\partial q_\iota} .$$

The "conceptual meaning" of the **Lagrange** equations has already been the subject of repeated investigations. However, they seem to have yielded no satisfying results. One therefore essentially addresses the question of how they emerge from the impulse equations $p_\iota = h_\iota$. If one differentiates this with respect to time then the equations that one obtains, viz.:

$$Dp_\iota = Dh_\iota ,$$

must be identical to the equations:

$$dp_\iota - \frac{\partial E}{\partial q_\iota} dt = k_\iota \cdot dt.$$

Now, one has:

$$p_\iota = \sum_\kappa \varepsilon_{\iota\kappa} \cdot \dot{q}_\kappa ,$$

$$dp_\iota = \sum_\kappa \varepsilon_{\iota\kappa} \cdot \ddot{q}_\kappa \cdot dt + \sum_\iota \sum_\kappa \frac{\partial \varepsilon_{\iota\kappa}}{\partial q_\kappa} \dot{q}_\iota \, \dot{q}_\kappa \cdot dt$$

or, if one considers the second term to be $-\partial E / \partial q_\iota \cdot dt$:

$$dp_\iota = \sum_\kappa \varepsilon_{\iota\kappa} \cdot \ddot{q}_\kappa \cdot dt + \sum_\lambda \sum_\kappa \gamma^{(\iota)}_{\lambda\kappa} \dot{q}_\lambda \, \dot{q}_\kappa \cdot dt \, .$$

We can denote the first term on the right-hand side of this equation by $(dp_\iota) \cdot dt$, as in the derivation of the **Euler** equations, if the parentheses suggest a *pure* impulse differentiation by which the coordinates will remain unchanged, which is consistent with the concept of impulse. We will then get:

$$Dp_\iota = (dp_\iota) + C_\iota = k_\iota \cdot dt.$$

The entire difficulty now comes down to the interpretation of the functions:

$$C_\iota = \sum_\lambda \sum_\kappa \gamma^{(\iota)}_{\lambda\kappa} \dot{q}_\lambda \, \dot{q}_\kappa \, .$$

In all likelihood, these functions C_ι, in which the coefficients $\gamma^{(\iota)}_{\lambda\kappa}$ are identical with the **Christoffel** symbols $\begin{bmatrix} \lambda, \kappa \\ \iota \end{bmatrix}$, are the components (or rather, simple combinations of components) of a centrifugal acceleration. However, up to now, I have not succeeded in proving that, and in that way, clarifying the special nature of things completely. Perhaps, those remarks will serve to stimulate further investigations into this situation, which is not at all inessential in kinetics.

19. Explicit form of Lagrange's equations. – As is known, Hamilton employed, not only the function $E = \frac{1}{2} \sum_\iota \sum_\kappa \varepsilon_{\iota,\kappa} \dot{q}_\iota \, \dot{q}_\kappa$, but also the reciprocal function $E = \frac{1}{2} \sum_\iota \sum_\kappa \eta_{\iota,\kappa} \, p_\iota \, p_\kappa$, which is linked with the latter by the linear relations $\overline{\varepsilon}_\iota \, \overline{v} = p_\iota$. We shall now employ the equations:

$$\frac{\partial E}{\partial \dot{q}_\iota} = p_\iota \, , \qquad \frac{\partial F}{\partial p_\iota} = \dot{q}_\iota \, , \qquad \frac{\partial F}{\partial q_\iota} = - \frac{\partial E}{\partial q_\iota}$$

in order to get an explicit representation of the **Lagrange** equations of motion, which we shall use as the basis for our study of the kinetostatics of systems of links. We next have:

$$\ddot{q}_\iota = \frac{d}{dt} \frac{\partial F}{\partial p_\iota} = \sum_\kappa \frac{\partial^2 F}{\partial p_\iota \, \partial p_\kappa} \frac{dp_\kappa}{dt} + \sum_\kappa \frac{\partial^2 F}{\partial p_\iota \, \partial p_\kappa} \cdot \dot{q}_k \, .$$

It follows from the usual form of the **Lagrange** equations:

$$\frac{dp_\iota}{dt} = k_\iota = \frac{\partial F}{\partial q_\iota}.$$

With that, the previous equation will go to:

(38) $$\frac{dq_\iota}{dt} = \sum_\kappa \left[\frac{\partial^2 F}{\partial p_\iota \partial q_\kappa} \frac{\partial F}{\partial p_\kappa} - \frac{\partial^2 F}{\partial p_\iota \partial p_\kappa} \frac{\partial F}{\partial q_\kappa} \right] + \sum_\kappa \frac{\partial^2 F}{\partial p_\iota \partial p_\kappa} \cdot k_\kappa .$$

The first term on the right-hand side of this equation is a homogeneous function of degree two in the quantities p_1, p_2, …, p_i, and thus also the quantities $\dot{q}_1$, $\dot{q}_2$, …, $\dot{q}_\iota$, as well. The following term contains only the coordinates q_1, q_2, …, q_i, in addition to the generalized **Lagrangian** force components k_1, k_2, …, k_i. For that reason, we can also write equation (38) in the following form:

(39) $$\frac{dq_\iota}{dt} = \sum_\lambda \sum_\kappa \alpha^{(\iota)}_{\lambda,\kappa} \dot{q}_\lambda \dot{q}_\kappa + \sum_\kappa \eta_{\iota,\kappa} \cdot k_\kappa .$$

The coefficients $\alpha^{(\iota)}_{\lambda,\kappa}$ and $\eta_{\iota,\kappa}$ are known functions of the coordinates in this. The quantities $\alpha^{(\iota)}_{\lambda,\kappa}$ can be expressed immediately in terms of the **Christoffel** symbols of the second kind, which are denoted by $\left\{ \begin{matrix} \lambda, \kappa \\ \iota \end{matrix} \right\}$. However, it does not seem necessary to go further into those relationships at the moment. The $\eta_{\iota,\kappa}$ are the coefficients in the reciprocal function *F*.

20. Rodrigues-Cayley position coordinates for rigid systems. – In order to shorten the derivation as much as possible, I shall now appeal to the *Theorie des Kreisels* by **F. Klein** and **A. Sommerfeld**. In that book (pp. 21 and 43), the components σ_1, σ_2, σ_3 of the vector of rotational velocity are expressed in terms of the four quaternion components *A*, *B*, *C*, *D* in the following way:

(40) $$\begin{cases} \frac{1}{2}\sigma_1 = D\dot{A} - A\dot{D} - (B\dot{C} - C\dot{B}), \\ \frac{1}{2}\sigma_2 = D\dot{B} - B\dot{D} - (C\dot{A} - A\dot{C}), \\ \frac{1}{2}\sigma_3 = D\dot{C} - C\dot{D} - (A\dot{B} - B\dot{A}). \end{cases}$$

Actually, there are complex combinations of the σ on pp. 43 of that book, but they will imply equations (40) with no further assumption. We shall now assume that *A*, *B*, *C* are the components of a vector $\bar{\lambda}$. Equations (40) can then be combined into a single vector equation, namely:

(41) $$\tfrac{1}{2}\bar{\sigma} = \mu^2 \frac{d\bar{\lambda}}{dt\mu} - \overline{\lambda\dot{\lambda}},$$

in which we have set:

$$1 - \bar{\lambda}\,\bar{\lambda} = 1 - \lambda^2 = \mu^2 ,$$

to abbreviate. If we introduce yet another vector $\bar{\kappa}$ by way of the equation:

$$\bar{\lambda} = \mu\,\bar{\kappa}$$

then we will have $\mu^2 + \mu^2\,\kappa^2 = 1$, and:

(42) $$\bar{\sigma} = \frac{1}{1+\kappa^2}(\dot{\bar{\kappa}} - \overline{\kappa\dot{\kappa}}) .$$

Cayley published that beautiful equation [Camb. and Dublin J. **1** (1846)], which expresses $\bar{\sigma}$ in terms of the necessary and sufficient number of coordinates, and then constructed a very elegant theory of the rotation of rigid bodies. Although **Somoff**'s kinematics refers to that book, it has still not found the attention that it deserves, in our opinion. Equation (42) implies the components of the rotational velocity in the clear and symmetric form:

(43) $$\begin{cases} \sigma_1 = \dfrac{2}{1+\kappa^2}[\dot{\kappa}_1 - (\kappa_2\,\dot{\kappa}_3 - \kappa_3\,\dot{\kappa}_2)], \\ \sigma_2 = \dfrac{2}{1+\kappa^2}[\dot{\kappa}_2 - (\kappa_3\,\dot{\kappa}_1 - \kappa_1\,\dot{\kappa}_3)], \\ \sigma_3 = \dfrac{2}{1+\kappa^2}[\dot{\kappa}_3 - (\kappa_1\,\dot{\kappa}_2 - \kappa_2\,\dot{\kappa}_1)]. \end{cases}$$

If one substitutes these quantities into the value for the kinetic energy E then one can derive **Lagrange**'s equations of motion from the expression thus-obtained with no further discussion, since the κ_1, κ_2, κ_3 are independent position coordinates.

21. The vector $\bar{B}$. – Just as we could represent the energy E of a system with a finite number of degrees of freedom in the general **Lagrangian** coordinates q_1, q_2, …, q_i, that can also be done for the system vector $\bar{B}$. However, we would like to restrict ourselves here to the elementary vector $\bar{B}$ for a free material point (m = 1). In the defining equation:

$$\bar{B} = \overline{\dot{x}\,\ddot{x}} ,$$

one then sets:

$$\dot{\bar{x}} = \bar{\varepsilon}_1\,\dot{q}_1 + \bar{\varepsilon}_2\,\dot{q}_2 + \bar{\varepsilon}_3\,\dot{q}_3 = \bar{v} ,$$

and correspondingly:

$$\ddot{\bar{x}} = \bar{\varepsilon}_1\,\ddot{q}_1 + \bar{\varepsilon}_2\,\ddot{q}_2 + \bar{\varepsilon}_3\,\ddot{q}_3 + \dot{\bar{\varepsilon}}_1\,\dot{q}_1 + \dot{\bar{\varepsilon}}_2\,\dot{q}_2 + \dot{\bar{\varepsilon}}_3\,\dot{q}_3 .$$

Carrying out that substitution will immediately yield:

$$\begin{aligned}\bar{B} = {} & \overline{\varepsilon_1 \varepsilon_2}\,(\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1) + \overline{\varepsilon_2 \varepsilon_3}\,(\dot{q}_2 \ddot{q}_3 - \dot{q}_3 \ddot{q}_2) + \overline{\varepsilon_3 \varepsilon_1}\,(\dot{q}_3 \ddot{q}_1 - \dot{q}_1 \ddot{q}_3) \\ & + (\overline{\varepsilon_1 \dot{\varepsilon}_2} + \overline{\varepsilon_2 \dot{\varepsilon}_1})\,\dot{q}_1 \ddot{q}_3 + \overline{\varepsilon_1 \dot{\varepsilon}_1}\,\dot{q}_1^2 \\ & + (\overline{\varepsilon_2 \dot{\varepsilon}_3} + \overline{\varepsilon_3 \dot{\varepsilon}_2})\,\dot{q}_2 \ddot{q}_3 + \overline{\varepsilon_2 \dot{\varepsilon}_2}\,\dot{q}_2^2 \\ & + (\overline{\varepsilon_3 \dot{\varepsilon}_1} + \overline{\varepsilon_1 \dot{\varepsilon}_3})\,\dot{q}_3 \ddot{q}_1 + \overline{\varepsilon_3 \dot{\varepsilon}_3}\,\dot{q}_3^2 .\end{aligned}$$

If one now imagines that the equations:

$$\bar{\dot{\varepsilon}}_1 = \frac{\partial \bar{v}}{\partial q_1}, \qquad \bar{\dot{\varepsilon}}_2 = \frac{\partial \bar{v}}{\partial q_2}, \qquad \bar{\dot{\varepsilon}}_3 = \frac{\partial \bar{v}}{\partial q_3}$$

are valid then one will see with no further assumptions that $\bar{B}$ can be put into the following form:

$$\bar{B} = \overline{\varepsilon_2 \varepsilon_3}\,(\dot{q}_2 \ddot{q}_3 - \dot{q}_3 \ddot{q}_2) + \overline{\varepsilon_3 \varepsilon_1}\,(\dot{q}_3 \ddot{q}_1 - \dot{q}_1 \ddot{q}_3) + \overline{\varepsilon_1 \varepsilon_2}\,(\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1) + H,$$

in which H means a homogeneous function of degree *three* in the velocity components $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$. If one then sets:

$$\overline{\varepsilon_2 \varepsilon_3} = \bar{\varepsilon}_1', \qquad \overline{\varepsilon_3 \varepsilon_1} = \bar{\varepsilon}_2', \qquad \overline{\varepsilon_1 \varepsilon_2} = \bar{\varepsilon}_3',$$

then one will have:

$$\bar{v} = \bar{\varepsilon}_1' p_1 + \bar{\varepsilon}_2' p_2 + \bar{\varepsilon}_3' p_3,$$

such that one can also express $\bar{B}$ in terms of the quantities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ and q_1, q_2, q_3. The quantities $\bar{\varepsilon}_\iota'$ and $\partial \bar{v} / \partial q_\iota$ that enter into that expression are functions of the q_ι.

We would like to determine $\bar{B}$ as the *system vector* only for rotating rigid bodies. In order to do that, we substitute the values of $\bar{\dot{x}}$ and $\bar{\ddot{x}}$ into:

$$\bar{B} = \sum m \overline{\dot{x}\,\ddot{x}},$$

namely:

$$\bar{\dot{x}} = \overline{\sigma x} \qquad \text{and} \qquad \bar{\ddot{x}} = \overline{\dot{\sigma} x} + (\bar{\sigma}\,\bar{x}) \cdot \bar{\sigma} - \sigma^2 \cdot \bar{x},$$

and obtain:

$$\bar{B} = \sum m (\overline{\sigma x} \cdot \bar{\sigma}) \cdot \bar{x} - \sum m (\bar{\sigma}\,\bar{x})^2 + \sum m (\sigma^2 x^2) \cdot \bar{\sigma},$$

after some reductions. However, we have:

$$\sum m (\sigma^2 x^2) - \sum m (\bar{\sigma}\,\bar{x})^2 = \sum m \overline{\sigma x}\,\overline{\sigma x} = 2E.$$

Hence, we will have:

$$\bar{B} = 2E \cdot \bar{\sigma} + \bar{G}$$

when we set:

$$\bar{G} = \sum m(\overline{\dot{\sigma}\sigma} \cdot \bar{x})\,\bar{x}\,,$$

to abbreviate.

The components of that vector are:

$$\begin{cases} \bar{G}_1 = (\dot{\sigma}_2\sigma_3 - \dot{\sigma}_3\sigma_2)T_{11} + (\dot{\sigma}_3\sigma_1 - \dot{\sigma}_1\sigma_3)T_{12} + (\dot{\sigma}_1\sigma_2 - \dot{\sigma}_2\sigma_1)T_{13}, \\ \bar{G}_2 = (\dot{\sigma}_2\sigma_3 - \dot{\sigma}_3\sigma_2)T_{21} + (\dot{\sigma}_3\sigma_1 - \dot{\sigma}_1\sigma_3)T_{22} + (\dot{\sigma}_1\sigma_2 - \dot{\sigma}_2\sigma_1)T_{23}, \\ \bar{G}_3 = (\dot{\sigma}_2\sigma_3 - \dot{\sigma}_3\sigma_2)T_{31} + (\dot{\sigma}_3\sigma_1 - \dot{\sigma}_1\sigma_3)T_{32} + (\dot{\sigma}_1\sigma_2 - \dot{\sigma}_2\sigma_1)T_{33}, \end{cases}$$

in which:

$$T_{\lambda\mu} = \sum m\,x_\lambda\,x_\mu\,.$$

If the angular acceleration is zero, or if its vector lies in the same direction as the angular velocity then the vector $\bar{G}$ will vanish, and $\bar{B}$ will then contain the direction of the momentary axis. $\bar{B}$ will then be proportional to the kinetic energy, as well as the angular velocity of the system. The components of $\bar{B}$ will then be homogeneous functions of degree three in the components of the angular velocity in this special case.

22. Euler's equations for bar chains. – On pp. 154 of the first volume of *Theorie des Kreisels* by **F. Klein** and **A. Sommerfeld**, we find the following viewpoint of general kinetic interest expounded:

> **"Euler**'s equations occupy an entirely singular position in the system of mechanics and do not subordinate themselves to the general type of mechanical differential equations that **Lagrange** presented. Neither is it possible to exhibit equations for arbitrary mechanical systems that would afford advantages that are similar to the ones that **Euler**'s equations afford for rigid bodies."

For us, the question of whether kinetic equations with the typical **Euler** form can be exhibited for a given system can be answered in a definite way in connection with the general argument that has been presented up to now. Namely, if one succeeds in find the analytical expressions for the system of velocities that characterizes the system in terms of the necessary and sufficient number of *purely-kinematical* vectors (i.e., parameters) and presents the requisite transitivity equations for the latter, in addition, then **d'Alembert**'s principle, in the integral form:

$$\left[\delta' A_v\right]_{t_0}^{t} = \int_{t_0}^{t} (\delta E + \delta' A_k)\,dt\,,$$

or in the simpler form:

$$\delta' A_v = \delta' A_h$$

for impulses, will always yield the requisite number of vector equations, which will belong to the same genre as **Euler**'s equations of motion (impulse, resp.)

In general, one can exhibit **Lagrange**'s equations (in the narrow sense of the term) only when the system of velocities can be represented by the necessary and sufficient number of *coordinates* and their first derivatives with respect to time.

If one then arrives at *both* representations – viz., the kinematical and the geometrical – for a certain material system then nothing will stand in the way of exhibiting the basic kinetic equation in both forms, as long as one necessarily assumes that the transitivity equations are known. Naturally, the last requirement is superfluous for the equations of impulse.

An especially important class of systems for which equations of **Euler** type exist from the outset are the *bar chains* that are a type of engineering machine (in the general sense). The link couplings of rigid subsystems are general very restricted in practice. They essentially reduce to ball joints, cylindrical pin guides, and planar straight guides. In what follows, we shall consider only ball joints, because the other two cases can be easily reduced to that case, or lead, in any event, to kinetic equations that do not deviate from the ones that have been mainly treated here.

In order to make the conceptualization of systems more precise, we imagine our starting point to be a fixed ball joint (or even several of them, which will not complicate the treatment). A solid body of arbitrary shape with a ball pin might be an example of that. The first body couples to a second one, or other ones, in the same way, and so on. That multi-component bar chain can be *open* – i.e., the last component is not coupled to the first one – or *closed*, when one constrains it to move within a prescribed guide.

For the sake of simplicity in the following calculations, we would like to consider a bar chain that is open at the ends and consists of two rigid components, since that case is already sufficient to make the characteristics of the basic kinetic equations more intuitive.

The first link in the chain can exhibit only rotations, which can be represented by a rotation vector $\overline{\sigma'}$. The corresponding system of velocities is therefore $\overline{\dot{x}} = \overline{\sigma' a'}$, if we denote the vector that determines the position of an arbitrary material point of the subsystem by $\overline{a'}$. The reference point for $\overline{a'}$ is obviously the center of the fixed ball joint. We draw a vector $\overline{c'}$ from the same point to the center of the moving ball joint and call the vector the *velocity* of the second point $\overline{\dot{c}'}$. One then has $\overline{\dot{c}'} = \overline{\sigma' c'}$. the points of the second system component might be established in space by the equation:

$$\overline{x''} = \overline{c'} + \overline{a''}.$$

The relative vectors $\overline{a''}$ are then referred to the center of the moving ball joint. If we then denote the associated rotation vector by $\overline{\sigma''}$ then we will have the equation:

$$\overline{\dot{x}''} = \overline{\sigma' c'} + \overline{\sigma'' a''},$$

and we will get the associated elementary motion in the form:

$$\delta\overline{x'} = \overline{\delta\theta' \cdot a'}, \qquad \delta\overline{x''} = \overline{\delta\theta' \cdot c'} + \overline{\delta\theta'' a''}.$$

The kinetic *impulse equations* for the combined system can be presented with the help of those relations. For a material point of the first subsystem (with a unit mass), one has:

$$\delta' A'_v = \overline{\dot{x}'}\,\overline{\delta x'} = \overline{\sigma' a'} \cdot \overline{\delta\theta' \cdot a'}$$

or

$$\delta' A'_v = (\overline{\sigma'}\,\overline{\delta\theta'})(\overline{a'}\,\overline{a'}) - (\overline{a'}\,\overline{\delta\theta'})(\overline{a'}\,\overline{\sigma'}) .$$

However, the moment of the velocity of that system of points is:

$$\overline{M'_a} = \overline{a'(\sigma' a')} = \overline{\sigma'} \cdot (\overline{a'}\,\overline{a'}) - \overline{a'} \cdot (\overline{a'}\,\overline{\sigma'}) .$$

As a result, one will have:

(45) $$\delta' A'_v = \overline{M'_v} \cdot \overline{\delta\theta'} .$$

For a point (m = 1) of the second subsystem, we have:

$$\delta' A''_v = \overline{\dot{x}''}\,\overline{\delta x''} = [\overline{(\sigma' c') + (\sigma'' a'')}][\overline{(\delta\theta \cdot c') + (\delta\theta'' \cdot a'')}] ,$$

or when this is developed:

$$\delta' A''_v = \overline{\sigma' c'}\,\overline{\delta\theta' \cdot c'} + \overline{\sigma' c'}\,\overline{\delta\theta'' \cdot a''} + \overline{\sigma'' a''}\,\overline{\delta\theta' \cdot c'} + \overline{\sigma'' a''}\,\overline{\delta\theta'' \cdot a''} .$$

The first and fourth term in this expression for $\delta' A''_v$ can also be written:

$$\begin{aligned} \overline{\sigma' c'}\,\overline{\delta\theta' \cdot c'} &= (\overline{\sigma'}\,\overline{\delta\theta'})(\overline{c'}\,\overline{c'}) - (\overline{c'}\,\overline{\delta\theta'})(\overline{\sigma'}\,\overline{c'}) \\ &= \overline{c'}\,(\overline{\sigma' c'})\,\overline{\delta\theta'} = \overline{M'_c}\;\overline{\delta\theta'} , \end{aligned}$$

$$\begin{aligned} \overline{\sigma'' a''}\,\overline{\delta\theta'' \cdot a''} &= (\overline{\sigma''}\,\overline{\delta\theta''})(\overline{a''}\,\overline{a''}) - (\overline{a''}\,\overline{\delta\theta''})(\overline{\sigma''}\;\overline{a''}) \\ &= \overline{a''(\sigma'' c'')} \cdot \overline{\delta\theta''} = \overline{M''_a}\;\overline{\delta\theta''} , \end{aligned}$$

in which $\overline{M''_a}$ and $\overline{M''_a}$ are sufficiently well-defined velocity moments. Interpreting the middle two terms in the expression for $\delta' A''_v$ is not as simple. Here, we would like to introduce two new vectors $\overline{A''}$ and $\overline{C''}$ whose magnitudes and directions can be deduced from the equations:

$$\overline{\sigma' c'}\,\overline{\delta\theta'' \cdot a''} = \overline{A''}\,\overline{\delta\theta''} , \qquad \overline{\sigma'' a''}\,\overline{\delta\theta' \cdot c'} = \overline{C''}\,\overline{\delta\theta'} .$$

In that way, we will get the expression:

(46) $$\delta'' A''_v = \overline{M'_c}\;\overline{\delta\theta'} + \overline{M''_a}\;\overline{\delta\theta''} + \overline{C''}\;\overline{\delta\theta'} + \overline{A''}\;\overline{\delta\theta''} .$$

Now, we can go from equations (45) and (46) to the corresponding system equations by summing over the elementary quantities. We set:

$$\sum' m' \overline{M'_a} = \overline{M'_a}, \qquad \sum'' m'' \overline{M''_a} = \overline{M''_a}, \qquad \sum'' m'' \overline{A''} = \overline{A''}, \qquad \sum'' m'' \overline{C''} = \overline{C''},$$

and get:

(47) $$\delta' A_v = \overline{M'_a + M'_c + M''} \cdot \overline{\delta\theta'} + \overline{M'_a + A''} \cdot \overline{\delta\theta''}.$$

Furthermore, when one considers the system of velocities for the applied impulse, one will have:

$$\delta' A'_h = \overline{h'}\,\overline{\delta\theta' a'} = \overline{a' h'}\,\overline{\delta\theta'},$$

$$\begin{aligned} \delta' A''_h &= \overline{h''}\,\overline{\delta\theta' a'} + \overline{h''}\,\overline{\delta\theta'' a''} \\ &= \overline{c' h''} \cdot \overline{\delta\theta'} + \overline{a'' h''} \cdot \overline{\delta\theta''}, \end{aligned}$$

and when one goes over to the subsystems:

$$\delta' A'_h = \sum' \overline{a' h'} \cdot \overline{\delta\theta'} = \overline{M'_h} \cdot \overline{\delta\theta'},$$

$$\begin{aligned} \delta' A''_h &= \sum'' \overline{c' h''} \cdot \overline{\delta\theta'} + \sum'' \overline{a'' h''} \cdot \overline{\delta\theta''} \\ &= \overline{c' h''} \cdot \overline{\delta\theta'} + \overline{M''_h} \cdot \overline{\delta\theta''}, \end{aligned}$$

in which one has set $\sum'' \overline{h''} = \overline{h''}$.

If one substitutes these values into the equation:

$$\delta' A_h = \delta' A_v$$

then one will get the impulse formulas:

(48) $$\begin{cases} \overline{M'_h} + \overline{c' h''} = \overline{M'_a} + \overline{M'_c} + \overline{C''}, \\ \overline{M''_h} \qquad\qquad = \overline{M'_a} + \overline{A''}. \end{cases}$$

The six quantities σ'_1, σ'_2, σ'_3, and σ''_1, σ''_2, σ''_3 can be determined from the equations once the impulses that act upon the system are given. Naturally, the vectors $\overline{M'_a}$, $\overline{M''_a}$, $\overline{A''}$, and $\overline{C''}$ depend upon the moments of inertia and the moments of deviation of the subsystem.

The transition to ***Euler's*** *equations of motion* is relatively simple now. We have only to define the expression for the quantity δE in the equation:

$$\left[\delta' A_v\right]_{t_a}^{t_b} = \int_{t_a}^{t_b} [\delta E + \delta A]\, dt\,.$$

Now, for the individual material points (with masses $m' = 1$ and $m'' = 1$), one has:

$$E = \tfrac{1}{2}(\overline{\sigma' a'})(\overline{\sigma' a'}) \qquad \text{and} \qquad \delta E' = \overline{M_a'} \cdot \delta\sigma',$$

as well as:

$$E'' = \tfrac{1}{2}[(\overline{\sigma' c'}) + (\overline{\sigma'' a''})][(\overline{\sigma' c'}) + (\overline{\sigma'' a''})].$$

Hence:

$$\delta E'' = \overline{M_a' + M_c' + C''} \cdot \overline{\delta\sigma'} + \overline{M_a' + A''} \cdot \overline{\delta\sigma'}\,.$$

The transitivity equations are:

$$\overline{\delta\sigma'} = \frac{d}{dt}\overline{\delta\theta'} + \overline{\sigma'}\,\overline{\delta\theta'},$$
$$\overline{\delta\sigma''} = \frac{d}{dt}\overline{\delta\theta''} + \overline{\sigma''}\,\overline{\delta\theta''}.$$

If one substitutes these values into the integral equation and considers the fact that the reduction of the time-varying forces $\overline{k'}$ and $\overline{k''}$ is the same as it is in the case of impulse then one will get **Euler**'s equation of motion for the two-component system of links in the vector form:

(49)
$$\begin{cases} \left(\dfrac{d\overline{R'}}{dt}\right) + \overline{\sigma' R'} = \overline{M_k'} + \overline{c' k''}, \\ \left(\dfrac{d\overline{R''}}{dt}\right) + \overline{\sigma'' R''} = \overline{M_k''}, \end{cases}$$

in which one sets:

$$\overline{M_a'} + \overline{M_c'} + \overline{C''} = \overline{R'}, \qquad \overline{M_a''} + \overline{A''} = \overline{R''}, \qquad \sum \overline{k''} = \overline{k''},$$

to abbreviate.

The vectors $\left(\dfrac{d\overline{A''}}{dt}\right)$ and $\left(\dfrac{d\overline{C''}}{dt}\right)$ have a certain analogy with the combined centripetal acceleration, which **Coriolis** introduced in his consideration of the relative motion of the individual mass-points.

Naturally, we can also present the **Lagrange** equations in generalized coordinates for our bar chain that consists of rigid components. In order to do that, we can exhibit the **Rodrigues-Cayley** expressions for any subsystem, define the kinetic energy E of the entire system, and arrive at the explicit equations of motion from the known prescription. In the special examples that were worked through above, the result was six **Lagrange** equations that would suffice to determine the motion of the system completely.

Equations (49) are always preferable when the motion results in the absence of external forces. That theoretically-interesting case has no meaning for engineering mechanics, since the forces of friction are never absent in that realm. However, in the treatment of kinetic machine problems, one will also be just as rarely called upon to consider general systems of velocities of the kind that we assumed in the discussion above. However, in any event, the well-defined conceptualization of general processes of motion will also have great significance when their realization in practice is quite remote.

E. Determining the reactions.

23. Introduction of the sectional reactions. – If a simple rigid body is found in its most general form of motion then the cohesion of its parts can be taken advantage of in various ways. For the ideal structures that we call *rigid systems*, those internal forces can each assume arbitrary values, since we tacitly assume that their robustness is unbounded. Moreover, if a solid body were put into a general state of motion (viz., translation and rotation) then the internal stresses would attain values so large that the cohesive forces at the individual locations or in certain surface domains would no longer suffice to preserve the connectivity of the parts. The body would shatter, and a new state of motion would arise. However, even when we ignore that catastrophe for real systems (which are assumed to be rigid) that correspond to a well-defined velocity state, an increase in the stresses would occur for increasing velocity, which would no longer be permissible if the hypothesis of the "rigidity" of the system were to be maintained, since elastic or plastic deformations of appreciable magnitudes would occur. The same thing is true to a greater degree for systems of links that consist of "rigid" components (i.e., they are assumed to be rigid in the first approximation). In that sense, the kinetics of machines, and in particular, combustion engines with parts that go back and forth, have also directed special attention to the *stresses*, along with concept of motion. The quantitative determination of the reactions in moving systems of masses is then an important chapter in engineering mechanics and, as such, deserves to be treated systematically.

In order to fix the concept of system reactions, we imagine that the entire connected material system is decomposed into two parts by a surface section without altering the force-system and the velocity state that exists in any way. If the physical connection along the separating cut surface were suddenly canceled then each subsystem would generally have to begin a new form of motion at that moment. All of the reactions of one piece would combine into a resultant system that, from **d'Alembert**'s principle or **Newton**'s basic law of the equality of action and reaction, would be equivalent to the resultant system of the reactions of the other pieces in the opposite sense. The decomposition of the system into two pieces corresponds to the decomposition of the total energy E into two parts E' and E'', such that one would have:

$$E = E' + E''.$$

We would like to further assume that the entire system is determined by coordinates, such that for the impulsive effects, the **Lagrange** equations will be:

$$p_\iota = \frac{\partial E}{\partial \dot{q}_\iota} = h_\iota \qquad (\iota = 1, 2, 3, \ldots, i).$$

We likewise derive quantities p_ι of a kinetic character from the energy components E' and E'', when we set:

$$p'_\iota = \frac{\partial E'}{\partial \dot{q}_\iota}, \qquad p''_\iota = \frac{\partial E''}{\partial \dot{q}_\iota}.$$

It will then follow directly from **d'Alembert**'s principle that:

$$p'_\iota = h'_\iota - r'_\iota, \qquad p''_\iota = h''_\iota - r''_\iota,$$

in which the quantities r'_ι, r''_ι mean components of the resultant system of reactions, when expressed in terms of the general coordinates q_ι. Since one must have $r''_\iota = -r'_\iota$, one of the foregoing systems of equations, say:

(50) $$r'_\iota = h'_\iota - p'_\iota,$$

will suffice to determine those reaction components.

One determines the impulse components h'_ι that are required to calculate the r'_ι from the formula:

$$\delta' A'_h = \sum{}' \bar{h} \cdot \delta \bar{x}$$

when one expresses the $\delta \bar{x}$ in it in terms of the q_ι and δq_ι, from which, one will obtain the equation:

$$\delta' A'_h = \sum_{\iota=1}^{i} h'_\iota \cdot \delta q_\iota.$$

If the system is subjected to the effects of time-varying forces (k) then one will employ the usual **Lagrange** equations:

$$\dot{p}_\iota - \frac{\partial E}{\partial q_\iota} = k_\iota \qquad (\iota = 1, 2, 3, \ldots, i)$$

for the determination of the components, and the aforementioned splitting of the system will yield the reaction formulas:

$$\dot{p}'_\iota - \frac{\partial E'}{\partial q_\iota} = k'_\iota - s'_\iota \qquad \text{and} \qquad \dot{p}''_\iota - \frac{\partial E''}{\partial q_\iota} = k''_\iota + s''_\iota,$$

in which, one must once again set:

$$p_\iota' = \frac{\partial E'}{\partial \dot{q}_\iota} \qquad \text{and} \qquad p_\iota'' = \frac{\partial E''}{\partial \dot{q}_\iota}.$$

One can also employ **Euler**'s equations of motion in order to determine the corresponding components of the sectional reactions in an entirely similar way. In the case of a simple rigid body that is not under the influence of external forces, that path will deviate from the simple one.

24. Explicit representation of the sectional reactions. – For impulse problems, the sectional reactions depend upon only the position coordinates and the external impulses that act upon material systems. However, if time-varying forces act upon the system then the velocity state will also be crucial for the sectional reactions. In the equations for the reactions, which we expressed in terms of generalized coordinates above, in the first case we can eliminate the generalized velocity components $\dot{q}_\iota$ by means of the equations $p_\iota = h_\iota$, and in the second case the acceleration components $\ddot{q}_\iota$ can be eliminated by the use of the **Lagrangian** equations of motion, and in that way we will arrive at explicit representations for the components of the sectional reactions.

That is exceedingly simple for the impulse reactions:

$$r_\iota' = h_\iota' - p_\iota'.$$

If we transform the kinetic energy E by introducing the $h_\iota = p_\iota$ in place of the $\dot{q}_\iota$ in **Hamilton**'s reciprocal function F, which is now a homogeneous functions of degree two in the h_ι, then we will have:

$$\dot{q}_\iota = \frac{\partial F}{\partial h_\iota}.$$

One substitutes these values in the equation:

$$p_\iota' = \sum_{\kappa=1}^{i} \eta_{\iota\kappa}' \dot{q}_\kappa,$$

which is linear in the $\dot{q}_\iota$, and obtains the final equations:

(51) $$r_\iota' = h_\iota' - \sum_{\kappa=1}^{i} \eta_{\iota\kappa}' \frac{\partial F}{\partial h_\iota}$$

for the explicit representation of the sectional reactions for impulses. The coefficients $\eta_{\iota\kappa}'$ are known functions of the position coordinates $q_1, q_2, \ldots, q_i$.

The analogous consideration for time-varying forces can be simplified greatly when we assume **Lagrange**'s equations of motion in the explicit form, which were represented

by equations (38) or (39) in no. **19**. That will permit us to substitute the quantities $\ddot{q}_1$, $\ddot{q}_2$, …, $\ddot{q}_i$ directly into the reaction formulas:

(52) $$s_t' = k_t' + \frac{\partial E'}{\partial q_t} - \dot{p}_t' .$$

We will then have:

$$\dot{p}_t' = \sum_{\kappa} \eta_{t\kappa}' \dot{q}_\kappa ,$$

and as a result:

$$\dot{p}_t' = \sum_{\kappa}\sum_{\lambda} \gamma_{\kappa\lambda}^{(t)} \dot{q}_\kappa \dot{q}_\lambda + \sum_{\kappa} \eta_{t\kappa}' \cdot \ddot{q}_\kappa .$$

By substituting this expression in equation (52) and combining the terms of the same type, one will get the final formula for the reaction components:

(53) $$s_t' = k_t' - \sum_{\kappa=1}^{i} \eta_\kappa^{(t)} k_\kappa + \sum_{\kappa=1}^{i}\sum_{\lambda=1}^{i} \varepsilon_{\kappa\lambda}^{(t)} \dot{q}_\kappa \dot{q}_\kappa ,$$

in which the coefficients $\eta_\kappa^{(t)}$ and $\varepsilon_{\kappa\lambda}^{(t)}$ depend upon only the position coordinates q_1, q_2, …, q_i. In words, that result reads:

*The **Lagrangian** components of the sectional reactions for a system of links upon which arbitrary external forces act is composed of two parts: The first part depends upon only the driving forces and the position coordinates, while the second one will be represented by a homogeneous function of degree two in the generalized velocity components, just like the kinetic energy of the entire system.*

We have made no special assumptions up to now about the form of the system section. If the individual parts in the link-system are cylindrically-extended bodies (which is frequently the case for machines) then one will mostly choose a planar section that is perpendicular to the longitudinal axis in order to study the cross-sectional stresses that appear in those parts. By contrast, if one would like to find the pressures in the moving links, which is extremely important in engineering, then one should make the system section along the supporting surface (*Lagerfläche*) in question.

Here, we are concerned with only giving the general Ansätze that will make it possible to determine the system reactions on the basis of **d'Alembert**'s principle, and in that way, to show that **Lagrange**'s idea about the solution of those problems will suffice completely. In any case, the scope of rational mechanics can be extended in a fruitful way by including general problems in kinetostatics, along with the specific static and kinetic ones, and in that way accommodate some entirely-justified demands of engineering.

25. The fundamental reactions of simply or multiply-coupled link-systems. – From **d'Alembert**'s basic equation for impulses:

$$\bar{r} = \bar{h} - m\bar{\dot{x}},$$

if we consider the entire link-system then we will next derive the expressions:

$$\sum \bar{r} = \sum \bar{h} - \sum m\bar{\dot{x}}$$

and

$$\sum \overline{xr} = \sum \overline{xh} - \sum m\overline{\dot{x}\,\dot{x}}$$

and write them in the form:

(54)
$$\begin{cases} \bar{r} = \bar{h} - m\bar{\dot{x}}, \\ \overline{M_v} = \overline{M_h} - \overline{M_{\dot{x}}}. \end{cases}$$

The rigid and immobile foundation, along with the link-system, defines a larger material complex. As a result, from **d'Alembert**'s principle, the resulting reaction components $\bar{r}$ and $\overline{M_r}$, which refer to only the moving parts, will not vanish, in general. They will be included in the foundation at rest as pressures and virtual rotational moments. One must observe in this that the vector $\overline{M_r}$ refers to a certain static reduction point, and its value and direction will change as long as that reference point shifts its position to the foundation. However, as in any static problem that is concerned with rigid bodies, one will also be able to determine the central axis here, and in that way obtain the vectors $\bar{r}$ and $\overline{M_r}$ in *one* direction.

If the rigid foundation supports the moving system with more than one resting link then one can pose the question of how to distribute the fundamental reactions over the individual supports in that case. One now splits the common base into as many pieces as the number of supports that are present, gives each part the corresponding virtual motion relative to the absolute coordinate system, and applies **Lagrange**'s principle of virtual work for the determination of the individual reactions.

For the total foundation reactions under time-varying forces, the following equations will enter in place of equations (54):

(55)
$$\begin{cases} \bar{s} = \bar{k} - m\bar{\ddot{x}}, \\ \bar{M}_s = \bar{M}_h - \bar{M}_{\ddot{x}}. \end{cases}$$

That case is examined in detail in the theory of machines in a special example (parallel crank mechanisms that act upon a common shaft), and for that reason, we would like to go into it somewhat deeper in what follows.

26. The problem of adjusting the effects of mass in link systems. – The quantities $\bar{s}$ and $\overline{M_s}$ in equations (55) each consist of two terms: The first one will be derived from the external forces that act upon the system, and for that reason, it will depend upon the mass distribution of the entire system. One makes the second term equal to zero (up to vanishingly-small residual contributions) in the case of multi-crank steam engines by a suitable arrangement of the system, and in that way, eliminates the effect of mass on the

foundation in practice. We would now like to examine the conditions for the vanishing of the vectors $m\bar{\ddot{x}}$ and $\bar{M}_{\ddot{x}}$ for bar chains in the general case. To that end, we shall determine the center of gravity in each rigid subsystem. Let the vectors of the individual centers of gravity be, in turn:

$$\overline{x_s'},\ \overline{x_s''},\ldots,\ \overline{x_s^{(\nu)}},\ \ldots,\ \overline{x_s^{(n)}},$$

as measured from the absolute reference point (O). If we choose them to be the relative reference points (O', O'', etc.) for the vectors $\overline{a'}$, $\overline{a''}$, … that establish the individual material points of the subsystems then the absolute vectors of those points are:

$$\overline{x^{(\nu)}} = \overline{x_s^{(\nu)}} + \overline{a^{(\nu)}}.$$

Let the vector of the center of gravity of the entire system be $\overline{x_s}$. We will then have:

$$\sum m\bar{x} = m\overline{x_s}$$

and

$$m\bar{\ddot{x}} = m\bar{\ddot{x}}_s.$$

As a result, the reaction $m\bar{\ddot{x}}$ will vanish only when the velocity of the common center of gravity of all subsystems remains unchanged during the motion.

In order to investigate the moments, we form the equation:

$$\begin{aligned}\overline{x^{(\nu)}\,\ddot{x}^{(\nu)}} &= \overline{(x_s^{(\nu)} + a^{(\nu)})(\ddot{x}_s^{(\nu)} + \ddot{a}^{(\nu)})} \\ &= \overline{x_s^{(\nu)}\,\ddot{x}_s^{(\nu)}} + \overline{\ddot{x}_s^{(\nu)}\,\ddot{a}^{(\nu)}} + \overline{a^{(\nu)}\,\ddot{x}_s^{(\nu)}} + \overline{a^{(\nu)}\,\ddot{a}^{(\nu)}},\end{aligned}$$

and upon summation, we will get:

$$\sum{}^{(\nu)} m\,\overline{x^{(\nu)}\,\ddot{x}^{(\nu)}} = \sum{}^{(\nu)} m\,\overline{x_s^{(\nu)}\,\ddot{x}_s^{(\nu)}} + \sum{}^{(\nu)} m\,\overline{a^{(\nu)}\,\ddot{a}^{(\nu)}},$$

since the quantities:

$$\sum{}^{(\nu)} m\,\overline{x_s^{(\nu)}\,\ddot{a}^{(\nu)}}, \qquad \sum{}^{(\nu)} m\,\overline{a^{(\nu)}\,x_s^{(\nu)}}$$

will vanish for rigid bodies. Therefore, we will have:

$$\bar{M}_{\ddot{x}} = \sum_{\nu=1}^{n} m_\nu\,\overline{x_s^{(\nu)}\,\ddot{x}_s^{(\nu)}} + \sum_{\nu=1}^{n} m_\nu\,\overline{a^{(\nu)}\,\ddot{a}^{(\nu)}},$$

in which one sets $\Sigma^{(\nu)}m = m_{(\nu)}$, to abbreviate. For multi-cylindered steam engines, the value of the second term in this equation will always stay within narrow limits, because it has the order of magnitude of the rotational acceleration. In that special case, however, the conditions for the reduction of the mass effects to a rigid foundation can be represented in the form:

(56) $$\frac{dx_s}{dt} = 0 \qquad \text{and} \qquad \frac{d}{dt}\sum_{\nu=1}^{n} m_\nu \overline{x_s^{(\nu)}\,\dot{x}_s^{(\nu)}} = 0.$$

The further discussion of these equations is a problem for engineering mechanics. **H. Lorenz** gave a thorough presentation of the problem of mass effects in his *Dynamik der Kurbelgetriebe mit besonderer Berücksichtigung der Schiffsmaschinen* (1901).

F. Kinetostatic requirements.

27. Normal stresses and shear stresses. – The theory of static stresses was first developed for elastic prismatic rods (i.e., beams). In the simplest case, forces act only in the direction of the longitudinal axes, which one distinguishes from each other by calling them forces of tension and compression. They generate an internal stress state for which elastic forces will be provoked along that axis. In a second case, the external forces reduce to a force-couple whose plane intersects the cross-section perpendicularly along one of the two principal axes. The elastic effect expresses itself as a bending of the beam. Tensions and bending stresses will be commonly referred to as *normal stresses*. If the axis of the force-couple lies along the longitudinal axis of the rod then angular deviations of the body elements, which are assumed to be rectangular parallelepipeds, will occur along with the extensions, and one calls those angular deviations *shears* or *slips*. A torsional stress will arise that is statically-equivalent to the deforming force-couple. Finally, we can also imagine that the force that acts on things lies completely within the plane of a cross-section of the rod and strives to make the one part of the body slide along the other one along the sectional plane. A shearing stress will now come about in the cross-section in question. Torsional stresses and shear stresses will be collectively referred to as "shear stresses." We shall now apply this elementary concept from the theory of solids to the rigid bodies and the bar chains with rigid bars. Although deformations will be excluded in that way, one can still carry out the reduction of the internal reaction forces in such a way that the components will correspond to the usual categories of influences. At the same time, by doing that, one will get an intuitive overview of the results that is not intrinsic to the general reduction that uses **Lagrangian** coordinates.

28. Determining the stress components. – It follows from **d'Alembert**'s basic equation for time-varying forces:

$$\bar{k} = m\ddot{\bar{x}} + \bar{s}$$

that

(57) $$\Sigma' \bar{s} = \Sigma' m\,\overline{x\,k} - \Sigma' m\,\overline{x\,\ddot{x}},$$

in which all summations extend over those parts of a rigid body that are virtually separated by a plane section. However, for each rigid system (without translation), one has:

$$\dot{\bar{x}} = \overline{\sigma x},$$

and it will follow from this by differentiating with respect to time that:

$$\overline{\ddot{x}} = \overline{\dot{\sigma}\, x} + \overline{\sigma\,(\sigma\, x)},$$

or upon expanding the triple vector product:

$$\overline{\ddot{x}} = \overline{\dot{\sigma}\, x} + (\bar{\sigma}\,\bar{x})\cdot\bar{\sigma} - (\bar{\sigma}\,\bar{\sigma})\cdot\bar{x}\,.$$

We decompose that acceleration along three rectangular axes that are fixed in the rigid body and denote the relevant projections of the vector $\bar{x}$ by a_1, a_2, a_3 :

$$\begin{aligned}
\overline{\ddot{x}}_1 &= \dot{\sigma}_2\, a_3 - \dot{\sigma}_3\, a_2 + (\sigma_2^2 + \sigma_3^2)\, a_1 + \sigma_1\,\sigma_2\,\sigma_2 + \sigma_1\,\sigma_3\,\sigma_3\,,\\
\overline{\ddot{x}}_2 &= \dot{\sigma}_3\, a_1 - \dot{\sigma}_1\, a_3 + (\sigma_3^2 + \sigma_1^2)\, a_2 + \sigma_2\,\sigma_3\,\sigma_3 + \sigma_2\,\sigma_1\,\sigma_1\,,\\
\overline{\ddot{x}}_3 &= \dot{\sigma}_1\, a_2 - \dot{\sigma}_2\, a_1 + (\sigma_1^2 + \sigma_2^2)\, a_3 + \sigma_3\,\sigma_1\,\sigma_1 + \sigma_3\,\sigma_2\,\sigma_2\,.
\end{aligned}$$

The components $\dot{\sigma}_1$, $\dot{\sigma}_2$, $\dot{\sigma}_3$ in these expressions must be eliminated with the help of **Euler**'s equations of rotation (21):

$$\begin{aligned}
A_1\,\dot{\sigma}_1 &= (A_2 - A_3)\;\sigma_2\;\sigma_3 + M_{h,\,1}\,,\\
A_2\,\dot{\sigma}_2 &= (A_3 - A_1)\;\sigma_3\;\sigma_1 + M_{h,\,2}\,,\\
A_3\,\dot{\sigma}_3 &= (A_1 - A_2)\;\sigma_1\;\sigma_2 + M_{h,\,3}\,.
\end{aligned}$$

That will give:

$$\begin{aligned}
A_2\,A_3\cdot\ddot{x}_1 &= A_2\,A_3\;(\sigma_2^2 + \sigma_3^2)\cdot a_1 + A_2\,(A_3 - A_1 + A_2)\;\sigma_1\;\sigma_2\cdot a_2\\
&\quad + A_3\,(A_2 - A_3 + A_1)\;\sigma_1\;\sigma_2\cdot a_3 + A_3\,M_{h,\,2}\cdot a_3 - A_2\,M_{h,\,3}\cdot a_2\,,
\end{aligned}$$

$$\begin{aligned}
A_3\,A_1\cdot\ddot{x}_2 &= A_3\,A_1\;(\sigma_3^2 + \sigma_1^2)\,a_2 + A_3\,(A_1 - A_2 + A_3)\;\sigma_2\;\sigma_3\cdot a_3\\
&\quad + A_1\,(A_3 - A_1 + A_2)\;\sigma_2\;\sigma_1\cdot a_1 + A_1\,M_{h,\,3}\cdot a_1 - A_3\,M_{h,\,1}\cdot a_3\,,
\end{aligned}$$

$$\begin{aligned}
A_1\,A_2\cdot\ddot{x}_3 &= A_1\,A_2\;(\sigma_1^2 + \sigma_2^2)\cdot a_3 + A_1\,(A_2 - A_3 + A_1)\;\sigma_3\;\sigma_1\cdot a_1\\
&\quad + A_2\,(A_1 - A_2 + A_3)\;\sigma_3\;\sigma_2\cdot a_2 + A_2\,M_{h,\,1}\cdot a_2 - A_3\,M_{h,\,2}\cdot a_3\,.
\end{aligned}$$

If we set:

$$\Sigma'\,\bar{s} = \bar{s}'\,,\qquad \Sigma'\,\bar{k} = \bar{k}'\,,\qquad \Sigma'\,m = m'\,,$$

to abbreviate, then equations (57) will yield the components of the resultant force of the internal stresses in the explicit form:

$$\begin{aligned}
A_2\,A_3\cdot s_1' &= A_2\,A_3\cdot k_1' + m'\,(A_2\,M_{k,\,3}\cdot\, a_2^* - A_3\,M_{k,\,2}\cdot\, a_3^*)\\
&\quad + m'\{A_2\,(A_1 - A_2 + A_3)\;\sigma_1\;\sigma_2\cdot\, a_2^* + A_3\,(A_3 - A_1 + A_2)\;\sigma_1\;\sigma_3\cdot\, a_3^* - A_2\,A_3\;(\sigma_2^2 + \sigma_3^2)\cdot\, a_1^*\},
\end{aligned}$$

and two analogous expressions for s_2', s_3' that will follow by cyclic permutation of the indices.

For the sake of getting a better overview, we write:

(59) $$\overline{s'} = \overline{k'} + m' \cdot \bar{u} + m' \cdot \bar{w}.$$

$\bar{u}$ then means a vector that depends essentially upon the total moment of the rotating forces, and $\bar{w}$ means a second time-varying vector that is determined mainly by the velocity state of the system. The components a_1^*, a_2^*, a_3^* of the center of gravity vector $\overline{a^*}$ of the virtually-separated pieces of the body appear as magnitudes in the components of $\bar{u}$ and $\bar{w}$, as well. If one goes from one separating plane to another then the position of that center of gravity will change, and the components of $\overline{s'}$ will be affected in a way that is easy to see.

With our notation, we can write equation (58), which determines the moment of the reaction forces relative to the fixed point, as:

$$\overline{M_s'} = \overline{M_k'} - \overline{M_{\ddot{x}}'}$$

or

$$\overline{M_s'} = \overline{M_k'} - \frac{d\overline{M_v'}}{dt}.$$

However, from no. **16**, equation (19), one has:

$$\frac{d\overline{M_v'}}{dt} = \left(\frac{d\overline{M_v'}}{dt}\right) + \overline{\sigma M_v'}.$$

Hence, one will have:

(60) $$\overline{M_s'} = \overline{M_k'} - \overline{\sigma M_v'} - \left(\frac{d\overline{M_v'}}{dt}\right),$$

or when decomposed into components:

$$\begin{aligned} M_{s,1}' &= M_{k,1}' - (A_3' - A_2')\,\sigma_2\,\sigma_3 - A_1'\,\dot{\sigma}_1, \\ M_{s,2}' &= M_{k,2}' - (A_1' - A_3')\,\sigma_3\,\sigma_1 - A_2'\,\dot{\sigma}_2, \\ M_{s,3}' &= M_{k,3}' - (A_2' - A_1')\,\sigma_1\,\sigma_2 - A_3'\,\dot{\sigma}_3. \end{aligned}$$

A_1', A_2', A_3' are the principal moments of inertia of the virtually-separated pieces of the body.

By eliminating the components of the angular acceleration with the help of **Euler**'s equations of motion, which are true for the entire system, we will get, with no further discussion:

$$(61)\qquad \begin{cases} A_1 M'_{s,1} = A_1 M'_{k,1} - A'_1 M_{k,1} + D'_1 \cdot \sigma_2 \sigma_3, \\ A_2 M'_{s,2} = A_2 M'_{k,1} - A'_2 M_{k,1} + D'_2 \cdot \sigma_3 \sigma_1, \\ A_3 M'_{s,3} = A_3 M'_{k,3} - A'_3 M_{k,3} + D'_3 \cdot \sigma_1 \sigma_2, \end{cases}$$

in which one sets:

$$\begin{aligned} D'_1 &= (A_1 A'_2 - A'_1 A_2) - (A_1 A'_3 - A'_1 A_3), \\ D'_2 &= (A_2 A'_3 - A'_2 A_3) - (A_2 A'_1 - A'_2 A_1), \\ D'_3 &= (A_3 A'_1 - A'_3 A_1) - (A_3 A'_2 - A'_3 A_2). \end{aligned}$$

The vector $\overline{M'_s}$ then has the form:

$$\overline{M'_s} = \bar{P} + \bar{Q}.$$

$\bar{P}$ depend essentially upon the external forces, while $\bar{Q}$ is constrained by the mainly the velocity state of the system. The moment of inertia of the virtually-separated parts of the body will affect both vectors.

Up to now, $\overline{M_s}$ has been referred to the fixed point. If we then take the moment relative to a point of the cross-section then the known rules of statics will be true for that transformation. However, that new reference point can be chosen in the plane of the cross-section in such a way that the resultant and moment of the reactions will lie in a plane that is perpendicular to the plane of the section. Since the locus of those reference points is a line, we will still be free to choose which of its points should be assumed to be the definitive reduction point. Once that choice has been made, we decompose the resultant and the moment into components, which will fall in the plane of the section or be perpendicular to it, resp., and thus obtain the quantities that are required by the virtually-separated part of the body in regard to tension (or compression), bending, torsion, and shearing.

Should the components of the kinetostatic requirements be determined for a *system of links*, we would proceed in a manner that is similar to what we do for an isolated rigid body. The single difference consists of the fact that we must add the known reactions of the next-lying line (or, more generally, all of the links that are on the same side of surface of the section). Since we have calculated those link reactions completely, we can also consider this general problem to be solved.

Berlin, 1 February 1901
