"Zur Variationsrechnung," Math. Ann. **62** (1906), 351-370.

On the calculus of variations (*)

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Contents.

Necessity of the existence of the Lagrange differential equations.

The question of the necessity of the Lagrange criterion, i.e., the existence of the differential equations that are required by the vanishing of the first variation, was treated by **A. Mayer** (**) and **A. Kneser** (***), in particular. Here, I would like to give a rigorous, and at the same time very simple, path that leads to the desired proof of the necessity of the Lagrange criterion.

For brevity, I have assumed that the given functions and differential relations are analytic everywhere in the present communication, and in that way, the analytic character of the solutions that will come to be employed will likewise be ensured.

Furthermore, for sake of a more convenient representation (which will not restrict the generality of the method), we will choose the case of three desired functions $y(x)$, $z(x)$, $s(x)$ of the independent variable x. Let them and their first derivatives with respect to x:

(*) Reproduced essentially unchanged from the Göttinger Nachrichten 1905.

(**) Math. Ann., Bd. 26, and Leipziger Berichte 1895. In the latter note, **A. Mayer** extended his foundation of the Lagrange differential equations to the most general problem.

(***) *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, § 56-58. The problem was likewise addressed in full generality there.

$$\frac{dy}{dx} = y'(x), \quad \frac{dz}{dx} = z'(x), \quad \frac{ds}{dx} = s'(x)$$

will be subject to two conditions of the form:

$$(1) \qquad \begin{gathered} f(y', z', s', y, z, s; x) = 0, \\ g(y', z', s', y, z, s; x) = 0. \end{gathered}$$

$y(x)$, $z(x)$, $s(x)$ might denote three particular function that satisfy the conditions (1) and have the following character: For all values of x that lie between $x = a_1$ and $x = a_2$, one has:

$$(2) \qquad \begin{vmatrix} \dfrac{\partial f}{\partial y'} & \dfrac{\partial f}{\partial z'} \\ \dfrac{\partial g}{\partial y'} & \dfrac{\partial g}{\partial z'} \end{vmatrix} \neq 0 .$$

If we choose any other three functions $Y(x)$, $Z(x)$, $S(x)$ that likewise satisfy the conditions (1) and for which we have:

$$\begin{aligned} Y(a_1) &= y(a_1), & Y(a_2) &= y(a_2), \\ Z(a_1) &= z(a_1), & Z(a_2) &= z(a_2), \\ S(a_1) &= s(a_1), & S(a_2) &= s(a_2) \end{aligned}$$

then we suppose that we constantly have:

$$(3) \qquad Y(a_3) \geq y(a_3)$$

[assuming that the functions $Y(x)$, $Z(x)$, $S(x)$, along with their derivatives, or those particular functions $y(x)$, $z(x)$, $s(x)$, and their derivatives, resp., differ sufficiently little]. *If that minimal requirement is fulfilled then there will necessarily be two functions $\lambda(x)$, $\mu(x)$ that do not both vanish identically for all x and that will fulfill the differential equations that arise by setting the first variation of the integral:*

$$\int_{a_1}^{a_2} \{\lambda\, f(y', z', s', y, z, s; x) + \mu\, g(y', z', s', y, z, s; x)\}\, dx$$

equal to zero, namely, the Lagrange equations:

$$(4) \qquad \frac{d}{dx}\frac{\partial(\lambda f + \mu g)}{\partial y'} - \frac{\partial(\lambda f + \mu g)}{\partial y} = 0,$$

(5) $$\frac{d}{dx}\frac{\partial(\lambda f + \mu g)}{\partial z'} - \frac{\partial(\lambda f + \mu g)}{\partial z} = 0 ,$$

(6) $$\frac{d}{dx}\frac{\partial(\lambda f + \mu g)}{\partial s'} - \frac{\partial(\lambda f + \mu g)}{\partial s} = 0 ,$$

together with the functions y (x), z (x), s (x).

In order to prove that, we take any two well-defined functions σ_1 (x), σ_2 (x) that vanish for $x = a_1$ and $x = a_2$ and replace y, z, s in (1) with:

$$Y = Y\,(x, \varepsilon_1, \varepsilon_2) ,$$
$$Z = Z\,(x, \varepsilon_1, \varepsilon_2) ,$$
$$S = s\,(x) + \varepsilon_1\,\sigma_1\,(x) + \varepsilon_2\,\sigma_2\,(x) ,$$

resp., in which ε_1, ε_2 mean two parameters. We regard the two equations that arise in that way:

(7) $$f\,(Y', Z', S', Y, Z, S; x) = 0,$$
$$g\,(Y', Z', S', Y, Z, S; x) = 0$$

as a system of two differential equations for determining the two functions Y, Z. As the theory of differential equations teaches us (*), due to the assumption (2), for sufficiently-small values of ε_1, ε_2, there is certainly a system of two functions:

$$Y\,(x, \varepsilon_1, \varepsilon_2) \qquad \text{and} \qquad Z\,(x, \varepsilon_1, \varepsilon_2)$$

that fulfill those equations identically in x, ε_1, ε_2, which go to y (x), z (x), resp., for $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and further assume the values y (a_1), z (a_1), resp. for $x = a_1$ and arbitrary ε_1, ε_2 .

Since our requirement of a minimum (3) says that Y (x, ε_1, ε_2), as a function of ε_1, ε_2, must certainly have a minimum for $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, while the equation:

$$Z\,(x, \varepsilon_1, \varepsilon_2) = z(a_2)$$

exists between ε_1, ε_2, the theory of the relative minimum of a function of two variables will say that there must necessarily be two non-zero constants l, m for which:

(*) Cf., É. Picard, *Traité d'Analyse*, t. III, chap. VIII.

(8)
$$\left[\frac{\partial(l\,Y(a_2,\varepsilon_1,\varepsilon_2)+m\,Z\,(a_2,\varepsilon_1,\varepsilon_2))}{\partial\varepsilon_1}\right]_0=0,$$
$$\left[\frac{\partial(l\,Y(a_2,\varepsilon_1,\varepsilon_2)+m\,Z\,(a_2,\varepsilon_1,\varepsilon_2))}{\partial\varepsilon_2}\right]_0=0,$$

in which the subscript 0 means that both parameters ε_1, ε_2 are set equal to zero.

We now determine [and due to (2), this is certainly possible] two functions $\lambda(x)$, $\mu(x)$ of the variable x that both satisfy linear homogeneous differential equations (4), (5) and for which, the boundary conditions:

(9)
$$\left[\frac{\partial(\lambda\,f+\mu\,g)}{\partial y'}\right]_{x=a_2}=l,$$
$$\left[\frac{\partial(\lambda\,f+\mu\,g)}{\partial z'}\right]_{x=a_2}=m$$

at the location $x = a_2$. Since l, m are not both zero, the two functions $\lambda(x)$, $\mu(x)$ that are determined in that way will certainly not both vanish identically.

By differentiating equations (7) with respect to ε_1, ε_2 and then setting the two parameters equal to zero, we will get the equations:

$$\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0\frac{\partial f}{\partial y'}+\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0\frac{\partial f}{\partial y}+\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0\frac{\partial f}{\partial z'}+\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0\frac{\partial f}{\partial z}+\sigma_1'\frac{\partial f}{\partial s'}+\sigma_1\frac{\partial f}{\partial s}=0,$$
$$\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0\frac{\partial g}{\partial y'}+\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0\frac{\partial g}{\partial y}+\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0\frac{\partial g}{\partial z'}+\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0\frac{\partial g}{\partial z}+\sigma_1'\frac{\partial g}{\partial s'}+\sigma_1\frac{\partial g}{\partial s}=0,$$
$$\left[\frac{\partial Y'}{\partial\varepsilon_2}\right]_0\frac{\partial f}{\partial y'}+\left[\frac{\partial Y'}{\partial\varepsilon_2}\right]_0\frac{\partial f}{\partial y}+\left[\frac{\partial Z'}{\partial\varepsilon_2}\right]_0\frac{\partial f}{\partial z'}+\left[\frac{\partial Z'}{\partial\varepsilon_2}\right]_0\frac{\partial f}{\partial z}+\sigma_2'\frac{\partial f}{\partial s'}+\sigma_2\frac{\partial f}{\partial s}=0,$$
$$\left[\frac{\partial Y'}{\partial\varepsilon_2}\right]_0\frac{\partial g}{\partial y'}+\left[\frac{\partial Y'}{\partial\varepsilon_2}\right]_0\frac{\partial g}{\partial y}+\left[\frac{\partial Z'}{\partial\varepsilon_2}\right]_0\frac{\partial g}{\partial z'}+\left[\frac{\partial Z'}{\partial\varepsilon_2}\right]_0\frac{\partial g}{\partial z}+\sigma_2'\frac{\partial g}{\partial s'}+\sigma_2\frac{\partial g}{\partial s}=0,$$

in which, in turn, the subscript 0 means that both parameters ε_1, ε_2 are set equal to 0 in each case. Of those equations, on the one hand, the first one (second one, resp.) will be multiplied by λ, μ and the equations that result will be added together and then integrated between the limits $x = a_1$, $x = a_2$. On the other hand, the third and fourth equation will be multiplied by λ, μ, resp., and the resulting equations added together and then integrated between the limits $x = a_1$ and $x = a_2$. In that way, we will get:

$$\int_{a_1}^{a_2}\left\{\frac{\partial(\lambda\,f+\mu\,g)}{\partial y'}\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0+\frac{\partial(\lambda\,f+\mu\,g)}{\partial y}\left[\frac{\partial Y'}{\partial\varepsilon_1}\right]_0+\frac{\partial(\lambda\,f+\mu\,g)}{\partial z'}\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0+\frac{\partial(\lambda\,f+\mu\,g)}{\partial z}\left[\frac{\partial Z'}{\partial\varepsilon_1}\right]_0\right.$$

$$+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s'}\,\sigma_1'+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s}\,\sigma_1'\Bigg\}\,dx = 0\,,$$

(10)

$$\int_{a_1}^{a_2}\left\{\frac{\partial(\lambda\, f+\mu\, g)}{\partial y'}\left[\frac{\partial Y'}{\partial \varepsilon_2}\right]_0+\frac{\partial(\lambda\, f+\mu\, g)}{\partial y}\left[\frac{\partial Y'}{\partial \varepsilon_2}\right]_0+\frac{\partial(\lambda\, f+\mu\, g)}{\partial z'}\left[\frac{\partial Z'}{\partial \varepsilon_2}\right]_0+\frac{\partial(\lambda\, f+\mu\, g)}{\partial z}\left[\frac{\partial Z'}{\partial \varepsilon_2}\right]_0\right.$$

$$\left.+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s'}\,\sigma_2'+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s}\,\sigma_2'\right\}dx = 0\,.$$

Now, on the one hand, due to the conventions that we encountered, we have:

$$Y(a_1, \varepsilon_1, \varepsilon_2) = y(a_1)\,, \qquad Z(a_1, \varepsilon_1, \varepsilon_2) = z(a_1)\,,$$

and therefore:

$$\left[\frac{\partial Y}{\partial \varepsilon_1}\right]_0 = 0\,, \qquad \left[\frac{\partial Z}{\partial \varepsilon_1}\right]_0 = 0\,,$$

$$\left[\frac{\partial Y}{\partial \varepsilon_2}\right]_0 = 0\,, \qquad \left[\frac{\partial Z}{\partial \varepsilon_2}\right]_0 = 0$$

at the location $x = a_1$. On the other hand, we infer from equations (8) and (9) that:

$$\frac{\partial(\lambda f+\mu g)}{\partial y'}\left[\frac{\partial Y}{\partial \varepsilon_1}\right]_0+\frac{\partial(\lambda f+\mu g)}{\partial z'}\left[\frac{\partial Z}{\partial \varepsilon_1}\right]_0 = 0\,,$$

$$\frac{\partial(\lambda f+\mu g)}{\partial y'}\left[\frac{\partial Y}{\partial \varepsilon_2}\right]_0+\frac{\partial(\lambda f+\mu g)}{\partial z'}\left[\frac{\partial Z}{\partial \varepsilon_2}\right]_0 = 0$$

for the location $x = a_2$. When we keep that in mind, it will follow from (10), by means of (4), (5), while using the formula for the integration of a product (viz., partial integration), that we will have the equations:

$$\int_{a_1}^{a_2}\left\{\frac{\partial(\lambda\, f+\mu\, g)}{\partial s'}\,\sigma_1'+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s}\,\sigma_1\right\}dx = 0\,,$$

$$\int_{a_1}^{a_2}\left\{\frac{\partial(\lambda\, f+\mu\, g)}{\partial s'}\,\sigma_2'+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s}\,\sigma_2\right\}dx = 0\,.$$

If we set:

$$(\lambda\,\mu,\,\sigma) = \int_{a_1}^{a_2}\left\{\frac{\partial(\lambda\, f+\mu\, g)}{\partial s'}\,\sigma'+\frac{\partial(\lambda\, f+\mu\, g)}{\partial s}\,\sigma\right\}dx$$

to abbreviate, then we can express the result that was just obtained as follows:

For any two functions σ_1, σ_2 *that vanish at* $x = a_1$ *and* $x = a_2$, *there is always a system of solutions* λ, μ *to the differential equations* (4), (5) *that do not vanish identically and for which one will get:*

$$(\lambda\, \mu,\, \sigma_1) = 0 \quad \textit{and} \quad (\lambda\, \mu,\, \sigma_2) = 0\ .$$

If we now assume that for this system of solutions λ, μ, there is a function σ_3 such that the inequality exists:

$$(\lambda\, \mu,\, \sigma_3) \neq 0 \tag{11}$$

and we then construct any system of solutions λ', μ' to the differential equations (4), (5) that does not vanish identically and is such that:

$$(\lambda'\, \mu', \sigma_3) = 0\ . \tag{12}$$

If we assume, in turn, that there is a function σ_4 for which the inequality:

$$(\lambda'\, \mu', \sigma_4) \neq 0 \tag{13}$$

exists then we can apply our previous result to the functions σ_3, σ_4 and infer the existence of a system of solutions λ'', μ'' to (4), (5) such that the equations:

$$(\lambda''\, \mu'', \sigma_3) = 0\ , \tag{14}$$

$$(\lambda''\, \mu'', \sigma_4) = 0 \tag{15}$$

exist. Since λ, μ ; λ', μ' ; λ'', μ'' are solutions to a system of two first-order linear differential equations, two homogeneous linear relations of the form:

$$a\,\lambda + a'\,\lambda' + a''\,\lambda'' = 0\ ,$$
$$a\,\mu + a'\,\mu' + a''\,\mu'' = 0$$

must exist between them, in which a, a', a'' mean constants that are not all zero. However, from (11), (12), (14), one would necessarily have $a = 0$, and it would then follow from (13), (15) that $a' = 0$, which is not possible, since indeed one would now have $a'' \neq 0$ and the system of solutions λ'', μ'' does not vanish identically in x.

Our assumptions are inapplicable then and we conclude from this that either λ, μ or λ', μ' is a system of solutions to (4), (5) such that integral relation in question:

$$(\lambda\ \mu,\ \sigma) = 0 \qquad [\,(\lambda'\ \mu', \sigma)\ = 0\ , \text{resp.}]$$

is true for any function σ. An application of the product integration (viz., partial integration) to that relation will then show that equation (6) must necessarily be true for the system of solutions λ, μ [$(\lambda'\ \mu', \sigma)$, resp.], and the desired proof is brought to completion with that.

Independence theorem and Jacobi-Hamilton theory of the associated integration problem.

In my lectures (*) "Mathematische Probleme," I gave the following method for exhibiting the further necessary and sufficient criteria in the calculus of variations:

One deals with the simplest problem in the calculus of variations, namely, the problem of finding a function y of the variable x such that the integral:

$$J = \int_a^b F(y', y; x)\, dx \qquad \left[y' = \frac{dy}{dx} \right]$$

takes on a minimum value in comparison to the values that the integral will assume when we replace y (x) in the integral with other functions of x with the same given initial and final values.

We now consider the integral:

$$J^* = \int_a^b \{F + (y' - p) F_p\}\, dx \qquad \left[F = F(p, y; x), \quad F_p = \frac{\partial F(p.y; x)}{\partial p} \right]$$

and ask how the p in it must be taken in order for the value of that integral J^* to be independent of the path of integration in the xy-plane, i.e., the choice of the function y of the variable x. The answer is: One takes any one-parameter family of integral curves for the Lagrange differential equation:

$$\frac{d\,\dfrac{\partial F}{\partial y'}}{dx} - \frac{\partial F}{\partial y} = 0 \qquad [F = F'(y', y; x)]$$

and determines the value of the derivative y' at each point x, y of the curve of the family that goes through that point. The value of that derivative y' is a function p (x, y) with the desired behavior.

That "independence theorem" immediately implies not only the known criteria for the occurrence of the minimum, but also all essential facts in the Jacobi-Hamilton theory of the associated integration problem.

(*) Presented at the International Congress of Mathematicians in Paris 1900.

A. Mayer (*) has proved the corresponding theorem for the case of several functions by calculation, and he exhibited its connection with the Jacobi-Hamilton theory. In what follows, I will show that the independence theorem admits a more general conceptualization, and also with no expenditure of calculations, by reducing it to the one that was just given, and very simple proofs of the special cases that were resolved in my lecture can be given.

For the sake of ease of comprehension, I shall base it upon only two functions $y(x)$, $z(x)$. The variational problem consists of choosing them such that the integral:

$$J = \int_a^b F(y', z', y, z; x)\, dx \qquad \left[y' = \frac{dy}{dx}, z' = \frac{dz}{dx} \right]$$

takes on a minimum value in comparison to those values that the integral assumes when we replace $y(x)$, $z(x)$ with other functions of x with the same given initial and final values.

We now consider the integral:

$$J^* = \int_a^b \{F + (y' - p) F_p + (z' - q) F_q\}\, dx$$

$$\left[F = F(p, y; x), \quad F_p = \frac{\partial F(p.y;x)}{\partial p}, \quad F_q = \frac{\partial F(p.y;x)}{\partial q} \right]$$

and ask how the p, q in it must be chosen as functions of x, y, z in order for the value of that integral J^* *to be independent of the path of integration in xyz-space, i.e., to be independent of the choice of the function y (x), z (x).*

In order to answer that question, we choose an arbitrary surface $T(x, y, z) = 0$ in xyz-space and imagine that the functions p, q in it are determined in such a way that when we extend the integral J^* over any curve that lies on $T = 0$ and goes between two points on that surface, it will take on a value that is independent of the choice of that curve. At each point P of the surface $T = 0$, we then construct the integral curve of the Lagrange equations:

$$\frac{d \frac{\partial F}{\partial y'}}{dx} - \frac{\partial F}{\partial y} = 0, \qquad \frac{d \frac{\partial F}{\partial z'}}{dx} - \frac{\partial F}{\partial z} = 0 \qquad [F = F(y', z', y, z; x)],$$

(*) Math. Ann. Bd. 58.

that lies in *xyz*-space and for which we will have:

(16) $$y' = p\,, \qquad z' = q\,,$$

such that a two-parameter family of integral curves that fills up a spatial field arises in that way. We now imagine that this field determines the integral curve of that family that goes though each point x, y, z. *The values of the derivatives* y', z' *at that point* x, y, z *will then be functions* $p\,(x, y, z)$, $q\,(x, y, z)$ *with the desired behavior.*

In order to prove that assertion, we connect a certain point A of the surface $T = 0$ with an arbitrary point Q of the spatial field by means of a path w. We imagine that an integral curve from our two-parameter family goes through each point of that path w: The one-parameter family of integral curves will be represented by the equations:

(17) $$\begin{aligned} y &= \psi(x, \alpha), \\ z &= \chi(x, \alpha). \end{aligned}$$

Those points of the surface $T = 0$ from which the integral curves (17) start define a path w_T on the surface $T = 0$ that goes from the point A to the point P on $T = 0$ from which the integral curve of the family that runs through Q will go.

A one-parameter family of curves (17) will generate a surface whose equation:

(18) $$z = f\,(x, y)$$

will be obtained when one eliminates the parameter α from the two equations (17).

If we now introduce the function $f\,(x, y)$ into F in place of z and set:

$$F\left(y', \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\,y', y, f\,(x, y); x\right) = \Phi(y', y; x)$$

then for every curve that lies on the surface (18), we will have:

$$\int_a^b F(y', z', y, z; x)\,dx = \int_a^b \Phi(y', y, x)\,dx\,,$$

and as a result, for every curve of the family:

(19) $$y = \psi\,(x, \alpha)$$

in the *xy*-plane, the first variation of the integral:

$$(20) \qquad \int_a^b \Phi(y', y, x)\, dx$$

will also certainly vanish, i.e., the family of curves (19) in the *xy*-plane is a family of integral curves of the Lagrange differential equations that are required by the vanishing of the first variation of the integral (20). The validity of the independence theorem for *one* function *y* follows immediately from the fact that the integral:

$$(21) \qquad \int_a^b \{\Phi + (y' - p)\,\Phi_p\}\, dx \qquad [\Phi = \Phi\,(p, y\,;\, x)]$$

possesses a value that is independent of the choice of the function *y*.

However, since:

$$z' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y',$$

$$q = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, p,$$

one will have:

$$\frac{\partial f}{\partial y}(y' - p) = z' - q,$$

and as a result, one will have:

$$\begin{aligned}\Phi\,(p, y\,;\, x) + (y' - p)\,\Phi_p &= F\,(p, q, y, z\,;\, x) + (y' - p)\left(F_p + F_q\,\frac{\partial f}{\partial y}\right)\\ &= F\,(p, q, y, z\,;\, x) + (y' - p)\,F_p + (z' - q)\,F_q\,.\end{aligned}$$

The independence of the integral (21) that was just proved then brings with it the fact that our original integral:

$$J^* = \int_a^b \{F + (y' - p)F_p + (z' - q)F_q\}\, dx$$

also keeps its value when we replace *w* as an integration path with another one that lies on the surface (18) and goes from *A* to *Q*, namely, perhaps the path that is composed of the path w_T and integral curve of the family (17) that starts from *P* and goes to *Q*. When we consider the fact that equations (16) are true along the path segment *PQ*, that fact can be expressed by the fact:

$$(22) \qquad \int_{(w)} \{F + (y' - p)F_p + (z' - q)F_q\}\, dx = \int_{(w_T)} \{F + (y' - p)F_p + (z' - q)F_q\}\, dx + \int_P^Q F\, dx\;.$$

If we let $\bar{w}$ denote another path that goes from A to Q in our spatial pq-field and let $\bar{w}_T$ denote the corresponding path on the surface $T = 0$ that goes from A to P then the same argument will also imply the equation:

$$(22)\qquad \int_{(\bar{w})} \{F + (y' - p)F_p + (z' - q)F_q\}\, dx = \int_{(\bar{w}_T)} \{F + (y' - p)F_p + (z' - q)F_q\}\, dx + \int_P^Q F\, dx ,$$

and since the first integrals on the right-hand sides of (22) and (23) have equal values as a result of our assumption, since w_T and $\bar{w}_T$ lie on $T = 0$, it will follow that the left-hand integrals in (22) and (23) are equal to each other, which proves our independence theorem.

The simplest way of choosing the functions p, q on the surface $T = 0$ that meet our requirement consists of determining them from the equations:

$$(24)\qquad F - p\, F_p - q\, F_q : F_p : F_q = \frac{\partial T}{\partial x} : \frac{\partial T}{\partial y} : \frac{\partial T}{\partial z} .$$

The integrand of the integral J^* will then vanish for every path that lies on $T = 0$, and that integral will then have the value zero independently of the path on $T = 0$.

In particular, one can replace the surface $T = 0$ with a point. All of the integral curves of the Lagrange differential equations that run through that point will then define a two-parameter family of curves that one must employ in order to construct the spatial pq-field.

Since the integral J^* is independent of the path, it will represent a function of position for a variable upper limit, i.e., a function of the endpoint x, y, z in the spatial pq-field. We set:

$$(25)\qquad J(x, y, z) = \int_A^{x,y,z} \{F + (y' - p)\, F_p + (z' - q)\, F_q\}\, dx .$$

That function obviously satisfies the equations:

$$\frac{\partial J}{\partial x} = F - p\, F_p - q\, F_q ,$$
$$\frac{\partial J}{\partial y} = F_p ,$$
$$\frac{\partial J}{\partial z} = F_q .$$

If we eliminate the quantities p, q from this then that will imply the *Jacobi-Hamilton first-order partial differential equation* for J (x, y, z). In particular, if the values of p, q on $T = 0$ in the construction of the spatial pq-field were determined in such a way that the integrand of the integral

J^* vanished [i.e., (24) exists] then J (x, y, z) will be that solution of that Jacobi-Hamilton differential equation that vanishes on $T = 0$.

If we imagine that the surface $T = 0$ belongs to a two-parameter family of surface and let a, b denote the parameters of that family then the functions p, q of the spatial field, and therefore the function J (x, y, z), as well, will also depend upon those parameters. Differentiating equation (25) with respect to those parameters a, b will yield:

$$\frac{\partial J}{\partial a} = \int_A^{x,y,z} \left\{ (y' - p)\frac{\partial F_p}{\partial a} + (z' - q)\frac{\partial F_q}{\partial a} \right\} dx ,$$

$$\frac{\partial J}{\partial b} = \int_A^{x,y,z} \left\{ (y' - p)\frac{\partial F_p}{\partial b} + (z' - q)\frac{\partial F_q}{\partial b} \right\} dx ,$$

and since obviously, from (16), the integrand in the integral on the right-hand side will vanish when one advances along an integral curve, those integrals will represent functions of x, y, z that assume the same value on every individual integral curve, i.e., if c, d mean integration constants, just like a, b, then the equations:

$$\frac{\partial J}{\partial a} = c ,$$

$$\frac{\partial J}{\partial b} = d$$

will be nothing but the integrals of the Lagrange differential equations.

That proof might suffice to show how the essential theorems of Jacobi-Hamilton theory arise directly from the independence theorem.

Adapting the method of the independent integral to double integrals.

If one treats merely the question of the conditions for a minimum of an integral then one will not need to construct a spatial pq-field. Rather, it will suffice to construct a one-parameter family of integral curves (17) of the Lagrange equations in such a way that the surface that it generates includes the varied curve w. An application of the independence theorem for *one* function will then lead to that objective in the way that presented before.

That remark is useful when one would like to adapt the method of the independent integral to the problem of finding the minimum of a double integral that includes several unknown functions of several independent variables.

In order to treat such a problem, we let *z*, *t* denote two functions of the two variables *x*, *y* and seek to determine those functions in such a way that the double integral that is extended over a given region Ω in the *xy*-plane:

$$J = \int_{(\Omega)} F(z_x, z_y, t_x, t_y, z, t; x, y)\, d\omega$$

$$\left[z_x = \frac{\partial z}{\partial x}, \quad z_y = \frac{\partial z}{\partial y}, \quad t_x = \frac{\partial t}{\partial x}, \quad t_y = \frac{\partial t}{\partial y} \right]$$

will take a minimal value in comparison to the values that the integral assumes when we replace *z*, *t* with any other functions $\bar{z}$, $\bar{t}$ that possess the same prescribed values as *z*, *t* on the boundary *S* of the region Ω. The Lagrange equations that are implied by the vanishing of the first variation read:

$$\frac{d}{dx}\frac{\partial F}{\partial z_x} + \frac{d}{dy}\frac{\partial F}{\partial z_y} - \frac{\partial F}{\partial z} = 0\,,$$

$$\frac{d}{dx}\frac{\partial F}{\partial t_x} + \frac{d}{dy}\frac{\partial F}{\partial t_y} - \frac{\partial F}{\partial t} = 0\,,$$

in this case.

We now use a well-defined solution *z*, *t* of the Lagrange equations as a basis and let $\bar{z}$, $\bar{t}$ be any system of varied functions that fulfills the same boundary condition as *z*, *t*. We then determine a function *S* (*x*, *y*) of the variables *x*, *y* such that the equation *S* (*x*, *y*) = 0 represents the boundary curve of Ω in the *xy*-plane, while *S* (*x*, *y*) = 1 is fulfilled by only the coordinates of a single point inside of Ω. Finally, the equation *S* (*x*, *y*) = α should represent a family of curves that fills up the interior of the region Ω simply and with no gaps when the value of α runs between 0 and 1. We then determine the functions:

$$z = \psi(x, y, \alpha)\,, \qquad t = \chi(x, y, \alpha) \tag{26}$$

that satisfy the Lagrange differential equations and possess the same prescribed values on the curve *S* (*x* ¸*y*) = α as the system of varied functions $\bar{z}(x, y)$, $\bar{t}\,(x, y)$ such that the functions (26) will go to the basic solution *z*, *t* for $\alpha = 0$. The functions (26) will obviously define a one-parameter family of systems of solutions to the Lagrange equations then for which the equations:

$$\bar{z}\,(x, y) = \psi\,(x, y, S\,(x, y))\,,$$

$$\bar{t}\,(x, y) = \chi\,(x, y, S\,(x, y))$$

are fulfilled identically in *x*, *y*.

If we interpret the basic solution *z*, *t* to the Lagrange equations as a two-dimensional surface in the four-dimensional *xyzt*-space and likewise interpret the varied system of functions $\bar{z}$, $\bar{t}$ then the two-dimensional integral surfaces in that *xyzt*-space will generate a one-parameter family (26) of a three-dimensional space whose equation is given by eliminating α from (26). Let the equation of that three-dimensional space have the form:

$$t = f(x, y, z) .$$

We assume that the one-parameter family (26) fills up that three-dimensional space simply and without gaps.

If we substitute the function $f(x, y, z)$ for t in F and set:

$$F\left(z_x, z_y, \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} z_x, \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} z_y, z, f(x, y, z); x, y\right) = \Phi(z_x, z_y, z; x, y)$$

then it will only be necessary for us to apply the independence theorem that I proved in the aforementioned lectures for *one* unknown function, and the argument that is linked with it, to the integral:

$$\int_{(\Omega)} \Phi(z_x, z_y, z; x, y)\, d\omega$$

in order to see that the integral J actually assumes a minimum value under the assumption that the E-function is positive for the given system of functions $z(x, y)$, $t(x, y)$. The occurrence of a minimum is then linked with the two following requirements:

1. The constructability of the family (28). That requirement is certainly fulfilled when the Lagrange partial differential equations always possess systems of solutions z, t that possess any sort of prescribed values along a closed curve K that lies inside of Ω, while they are regular functions of x, y in the region that is bounded by K.

2. A simple and gapless covering of three-dimensional space by the family (26). That requirement is certainly fulfilled when any system of solutions z, t to the Lagrange equation is determined *uniquely* by its boundary values on any arbitrary closed curve K that lies inside of Ω.

We can summarize the result briefly as follows:

Our criterion for the occurrence of a minimum demands that the boundary-value problem for the Lagrange differential equations relative to every closed curve K that lies inside of Ω is ***uniquely*** *soluble for arbitrary boundary values. Our consideration shows that this criterion is certainly sufficient.*

In particular, when the given function F under the integral sign proves to have degree two in z_x, z_y, t_x, t_y, z, t, the Lagrange differential equation will be linear in those quantities, and in that case the boundary-value problem that is required for the application of our criterion can be treated completely with the help of my theory of integral equations.

In order to develop the argument is applied in this case more closely, we define the system of *homogeneous* linear differential equations that arises from the Lagrange equations by dropping the terms that are free of z, t. We would like to refer to that system of equations as the *Jacobi equations.*

In is now immediately clear that the boundary-value problem for a curve K will admit several systems of solutions only when the Jacobi equations possess a system of solutions z, t that are zero on a curve K, but not everywhere inside of the region that is bounded by K. Now, the theory of integral equation shows that the latter case is, at the same time, the only one in which the boundary-value problem *is not* soluble for certain prescribed boundary values.

In the case of a quadratic F, our criterion for the occurrence of a minimum then will emerge from the demand that the Jacobi equations should admit no system of solutions besides zero that are zero on the boundary S or a closed curve that lies inside of Ω. (The fulfillment of the criterion is also necessary in that case.)

In the general case when the given function F under the integral sign is not quadratic, in particular, but depends upon the functions z, t to be determined and their derivatives arbitrarily, we must apply the criterion that was just expressed to the second variation of the integral J and thus arrive at a criterion that is precisely analogous to the known Jacobi criterion in the case of *one* independent variable or *one* function of several independent variables to be determined, and might therefore be briefly referred to as the *Jacobi criterion*.

Minimum of the sum of a double integral and a simple boundary integral.

Finally, we shall treat the problem of determining a function z of the variables x, y in such a way that a double integral that is extended over a given region Ω in the xy-plane plus an integral that extends over a part S_1 of the boundary of Ω, namely, the sum of the integrals:

$$J = \int_{(\Omega)} F(z_z, z_y, z; x, y)\, d\omega + \int_{(S_1)} f(z_s, z; s)\, ds \qquad \left[z_x = \frac{\partial z}{\partial x}, \quad z_y = \frac{\partial z}{\partial y}, \quad z_s = \frac{dz}{ds} \right],$$

attains a minimum value when z has prescribed values on the remaining part S_2 of the boundary. In that way, F, f are given functions of their arguments, and s means the arc-length of the boundary curve S of Ω, which is calculated from a fixed point of it in the positive sense of traversal.

The vanishing of the first variation demands that the desired function z, as a function of x, y in the interior of Ω, must fulfill the partial differential equation:

$$\frac{d}{dx}\frac{\partial F}{\partial z_x} + \frac{d}{dy}\frac{\partial F}{\partial z_y} - \frac{\partial F}{\partial z} = 0 \tag{27}$$

while the differential relation:

$$\left(\frac{\partial F}{\partial z_y}\right)_{S_1}\frac{dx}{ds}-\left(\frac{\partial F}{\partial z_x}\right)_{S_1}\frac{dy}{ds}+\frac{d}{ds}\frac{\partial F}{\partial z_s}-\frac{\partial F}{\partial z} = 0 \tag{28}$$

has to be true on the boundary S_1. The dx / ds, dy / ds in that are understood to mean the derivatives of the functions x (s), y (s) that define the boundary curve S_1 .

We now consider the sum of the integrals:

$$J^* = \int\limits_{(\Omega)}\{F+(z_z-p)F_p+(z_y-q)F_q\}\,d\omega+\int\limits_{(S_1)}\{f+(z_s-\pi)f_\pi\}\,ds$$

$$\left[F=F(p,q,z;x,y),\quad F_p=\frac{\partial F}{\partial p},\quad F_q=\frac{\partial F}{\partial q},\quad f=f(\pi,z;s),\quad f_\pi=\frac{\partial f}{\partial \pi}\right],$$

and we would like to seek to determine the p, q in that as functions of x, y, z, and p as a function of s, z in such a way that the value of that sum of integrals is independent of the surface $z = z$ (x, y) that spans Ω, i.e., of the choice of the function z, when it has prescribed boundary values on only S_2 . The sum of the integral J^* has the form:

$$\int\limits_{(\Omega)}\{A\,z_z+B\,z_y-C\}\,d\omega+\int\limits_{(S_1)}\{a\,z_s-b\}\,ds\,,$$

in which A, B, C represent functions of x, y, z, and a, b are functions of s, z. As one easily sees, that sum of integrals will be independent of the desired sense of the surface $z = z$ (x, y) when the differential equation:

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z} = 0 \tag{29}$$

is fulfilled identically in x, y, z in the xyz-space that projects onto the region Ω and the differential equation:

$$(B)_{S_1}\frac{dx}{ds}-(A)_{S_1}\frac{dy}{ds}+\frac{\partial a}{\partial s}+\frac{\partial b}{\partial z} = 0 \tag{30}$$

is fulfilled identically in s, z in the sz-cylinder surface that projects onto the boundary curve S_1. When we set A, B, C, a, b equal to their values:

$$\begin{aligned} A &= F_p\,,\\ B &= F_q\,,\\ C &= p\,F_p + q\,F_q - F\,,\\ a &= f_p\,, \end{aligned} \tag{31}$$

$$b = \pi f_\pi - f ,$$

the two equations (29), (30) will represent partial differential equations for the functions p, q, π.

We now determine a one-parameter family of functions:

(32) $$z = \psi(x, y, \alpha)$$

that satisfy the Lagrange equations (27), (28) and set:

(33) $$z = \psi(x(s), y(s), \alpha) = \psi(s, \alpha)$$

on the boundary. We assume that this one-parameter family fills up the spatial field in a single-valued and gapless way. We then calculate α as a function of x, y, z from (32) and α as a function of s, z from (33) and define the expressions:

$$p(x, y, z) = \left[\frac{\partial \psi(x, y, \alpha)}{\partial x}\right]_{\alpha=\alpha(x,y,z)},$$

$$q(x, y, z) = \left[\frac{\partial \psi(x, y, \alpha)}{\partial y}\right]_{\alpha=\alpha(x,y,z)},$$

$$\pi(s, z) = \left[\frac{\partial \psi(s, \alpha)}{\partial s}\right]_{\alpha=\alpha(s,z)}.$$

The functions p, q of x, y, z and p of s, z are ones with the desired property.

Indeed, the fact that the functions p, q satisfy the equation (29) will follows easily when one considers the equation:

$$\frac{\partial p}{\partial y} + q\frac{\partial p}{\partial z} = \frac{\partial q}{\partial x} + p\frac{\partial q}{\partial z},$$

if one considers the fact that $\psi(x, y, \alpha)$ should fulfill the Lagrange equation identically for all values x, y, α. In order to prove the validity of (30), we set:

$$z_x = p ,$$
$$z_q = q ,$$
$$z_s = \pi ,$$

$$\frac{d^2 z}{ds^2} = \frac{\partial \pi}{\partial s} + \pi\frac{\partial \pi}{\partial z}$$

in the Lagrange equation (28), which is fulfilled identically in s, α, and it will then go to the equation:

$$(F_q)_{S_1}\frac{dx}{ds}-(F_p)_{S_1}\frac{dy}{ds}+\frac{\partial^2 f}{\partial\pi^2}\left(\frac{\partial\pi}{\partial s}+\pi\frac{\partial\pi}{\partial z}\right)+\frac{\partial^2 f}{\partial\pi\,\partial z}\pi+\frac{\partial^2 f}{\partial\pi\,\partial s}-\frac{\partial f}{\partial z}=0\,,$$

which is true identically for all s, z. We will get precisely the same equation when we substitute the expressions (31) into formula (30). With that, the proof of the independence theorem for the present problem is complete.

As before, it follows from the independence theorem that:

$$\begin{aligned} E(z_x, z_y, p, q) &\equiv F(z_x, z_y) - F(p, q) - (z_x - p)\,F_p - (z_y - q)\,F_q > 0\,, \\ E(z_s, \pi) &\equiv f(z_s) - f(\pi) - (z_s - \pi)\,f_\pi > 0\,, \end{aligned}$$

such that *two Weierstrass E-functions come under consideration in the present problem: one for the interior and one for the boundary* S_1 *.*

On the other hand, in order for a one-parameter family (32) to exist that generates a field that is covered simply and in a gapless manner in the desired way, we pose *the requirement that every solution z of the Lagrange equations* (27), (28) *is determined* ***uniquely*** *by its boundary values on any arbitrary curved path K that is closed or begins and ends in* S_1 *and lies inside of* Ω*. Our consideration then shows that this criterion is certainly sufficient.*

In particular, when the given functions F, f under the integral sign in the problem being treated prove to have degree two in z_x, z_y, z (z_s, s, resp.) the Lagrange differential equations will be linear. If we then define the *homogeneous* linear differential equations that arise from the Lagrange equations by dropping the terms that are free of z and referring to them as *Jacobi equations* then it will be immediately clear that the boundary-value problem for a curve K will admit several solutions only when the Jacobi equations possess a solution z that is zero on K, but not everywhere inside of the region that is bounded by K (K and S_1, resp.).

In the case of quadratic F, f, our criterion for the occurrence of a minimum will then emerge from the requirement that the Jacobi equations must admit no solution z besides zero that is zero on the boundary S_2 *or on a curve K that lies within* Ω *and is closed or begins and ends in* S_1 *.*

In the general case when the given functions F, f are not quadratic, in particular, but depend upon the function z to be determined and its derivatives arbitrarily, we must apply the criterion that was just expressed to the second variation of the integral sum J and thus arrive at a criterion that is precisely analogous to the known Jacobi criterion and might therefore be briefly referred to as such.

When we pose the problem of making the double integral:

$$\int\limits_{(\Omega)} F(z_x, z_y, z; x, y)\,d\omega$$

a minimum, while the boundary value of the desired function z should fulfill the auxiliary condition:

$$f(z_s, z\,;\,s) = 0\,,$$

we can apply the formulas and arguments of the problem that was just treated immediately. It is necessary to add the equation $f = 0$ and replace $f(s)$ with $\lambda(s)\,f$ everywhere in the formulas, where the Lagrange factor $\lambda(s)$ is regarded as function of s that is yet to be determined.

General rule for treating variational problems and exhibiting a new criterion.

In conclusion, allow me to abstract a general rule for the treatment of variational problems in which the values of the functions to be determined are prescribed everywhere on the boundary from one of the cases that were treated above.

One first arrives at the Lagrange equations L of the variational problem by setting the first variation equal to zero. One then lets a system Z be known of solutions to those differential equations L that likewise fulfill all conditions B of the variational problem that relate to the interior, as well as the boundary.

If the Weierstrass E-functions for our system of solutions Z proves to be positive then we refer to the system of solutions Z as one *with a positive-definite character.*

We now focus on any subset T of the domain of integration and denote the boundary of the subregion T (to the extent that it belongs to the boundary of the original domain of integration) by S_T, but when it falls within the interior of the original domain of integration (so a new boundary arises), we will denote it by s_T .

Thus, when no other system of solutions to the Lagrange equation L exists that fulfills the conditions B besides the system of solutions Z, and when no other system of solutions to the Lagrange equations L for every subregion T that fulfills the conditions B_T other than the system of solutions Z inside of T, the system of solutions Z will then be said to have *an intrinsically-unique character.*

A minimum will certainly occur for the system of solutions Z when it has a positive-definite and intrinsically-unique character.

As one sees, a new requirement enters into the general statement that is then expressed along with the Weierstrass requirement of the definite character of the solution Z, namely, the requirement of the intrinsically-unique character of the solution Z. Now, the latter requirement has the same relationship to the Jacobi criterion (to the extent that it has been formulated in the calculus of variations up to now) that the Weierstrass criterion has to the Legendre criterion when one regards the Weierstrass criterion as the appropriate extension of the Legendre criterion that would be necessary. In fact, just as the Weierstrass criterion will arise from Legendre's by applying the second variation, the criterion that I posed (viz., the requirement of the intrinsically-unique character of the solution Z) will arise from Jacobi's by applying the second variation. Namely, if we construct the *homogeneous linear* Jacobi equations $[L]$ from the Lagrange equations L, by an analogy that is easy to see, and likewise construct the *homogeneous linear* conditions $[B]$ that are associated with the given conditions B, then our criterion will emerge from the requirement that this system of homogeneous linear equations and conditions cannot possess any solution besides zero, and indeed not even for any subregion T, when we also prescribe the boundary value of zero on the newly-arising boundary s_T of that subregion. However, the criterion that I posed is (by analogy with the Weierstrass criterion) valid as a sufficient condition with no restrictions even when arbitrary variations come under consideration and not merely ones in a sufficiently-close neighborhood. It is likewise applicable when, for example, the judgement of a minimum must be made for a curve between two conjugate points, which is where the Jacobi criterion breaks down.

Whether the criterion that I posed is also sufficient for boundary values that are not given as fixed (how it is to be modified then, resp.) requires more investigation in some special cases.