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On the concept of the class of a differential equation (*)

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We shall base our investigation upon a differential equation for *two* functions y and z of *one* variable x. It possesses the form:

$$F\left(\frac{d^n y}{dx^n}, \ldots, \frac{dy}{dx}, y, \frac{d^n z}{dx^n}, \ldots, \frac{dz}{dx}, z; x\right) = 0 , \tag{1}$$

and we would like to assume that this differential equation cannot be obtained by differentiating a differential equation of the same form and lower order one or more times and linearly combining the differential equations that arise in that way.

We now set:

$$\left\{\begin{aligned} \xi &= \varphi\left(x; y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}, \ldots, z, \frac{dz}{dx}, \ldots, \frac{d^n z}{dx^n}, \ldots\right), \\ \eta &= \psi\left(x; y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}, \ldots, z, \frac{dz}{dx}, \ldots, \frac{d^n z}{dx^n}, \ldots\right), \\ \zeta &= \chi\left(x; y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}, \ldots, z, \frac{dz}{dx}, \ldots, \frac{d^n z}{dx^n}, \ldots\right), \end{aligned}\right. \tag{2}$$

in which the variable x and the functions y, z appear as arguments on the right, along with their derivatives up to a certain order, and express:

(*) Reproduced from the *Festschrift Heinrich Weber* (Leipzig, Teubner, 1912).

$$\frac{d\eta}{d\xi} = \frac{\frac{d\psi}{dx}}{\frac{d\varphi}{dx}}, \quad \frac{d^2\eta}{d\xi^2} = \frac{\frac{d\varphi}{dx}\frac{d^2\psi}{dx^2} - \frac{d\psi}{dx}\frac{d^2\varphi}{dx^2}}{\left(\frac{d\varphi}{dx}\right)^3}, \quad \ldots,$$

$$\frac{d\zeta}{d\xi} = \frac{\frac{d\chi}{dx}}{\frac{d\varphi}{dx}}, \quad \frac{d^2\zeta}{d\xi^2} = \frac{\frac{d\varphi}{dx}\frac{d^2\chi}{dx^2} - \frac{d\chi}{dx}\frac{d^2\varphi}{dx^2}}{\left(\frac{d\varphi}{dx}\right)^3}, \quad \ldots$$

in terms of x and y, z, along with their derivatives with respect to x. Therefore, in general (i.e., when the functions φ, ψ, χ do not satisfy any special condition equations), the quantities x, y, z can be expressed in terms of ξ and η, ζ, along with their derivatives with respect to ξ, as follows:

$$(3) \qquad \left\{ \begin{aligned} x &= g\left(\xi;\eta,\frac{d\eta}{d\xi},\frac{d^2\xi}{d\xi^2},\ldots,\zeta,\frac{d\zeta}{d\xi},\frac{d^2\zeta}{d\xi^2},\ldots\right), \\ y &= h\left(\xi;\eta,\frac{d\eta}{d\xi},\frac{d^2\eta}{d\xi^2},\ldots,\zeta,\frac{d\zeta}{d\xi},\frac{d^2\zeta}{d\xi^2},\ldots\right), \\ z &= k\left(\xi;\eta,\frac{d\eta}{d\xi},\frac{d^2\eta}{d\xi^2},\ldots,\zeta,\frac{d\zeta}{d\xi},\frac{d^2\zeta}{d\xi^2},\ldots\right). \end{aligned} \right.$$

In that way, (1) will go to a differential equation for η, ζ as functions of ξ that has the form:

$$(4) \qquad \Phi\left(\frac{d^\nu\eta}{d\xi^\nu},\ldots,\frac{d\eta}{d\xi},\eta,\frac{d^2\zeta}{d\xi^2},\ldots,\frac{d\zeta}{d\xi},\zeta;\xi\right) = 0 .$$

We say of the transformation (2) [(3), resp.] that it transforms the differential equations (1) and (4) into each other *invertibly without integration*. We shall classify all differential equations like (4) that can be converted into (1) invertibly without integration in the same *class of differential equations*.

In the theory of differential relations between two functions y (x) and z (x), the concepts of an invertible transformation without integration and that of class that were just introduced are analogous to the concepts in the theory of algebraic functions of one variable that are known as an invertible single-valued (viz., birational) transformation in the Riemannian picture and the Riemannian concept of the class of an algebraic function, respectively.

On the other hand, we shall now set:

$$(5)\qquad \begin{cases} x = \varphi(t, w, w_1, \ldots, w_r), \\ y = \psi(t, w, w_1, \ldots, w_r), \\ z = \chi(t, w, w_1, \ldots, w_r), \end{cases}$$

where the functions φ, ψ, χ are not precisely of the special type that all of them depend upon *only one* coupling of their arguments t, w, w_1, …, w_r, and we further understand w to mean an arbitrary function of the variable t, while:

$$w_1 = \frac{dw}{dt}, \qquad \ldots, \qquad w_r = \frac{d^r w}{dt^r}$$

mean its derivatives with respect to t, and define:

$$(6)\qquad \begin{cases} \dfrac{dy}{dx} = \dfrac{\psi'}{\varphi'}, \quad \dfrac{d^2 y}{dx^2} = \dfrac{\varphi'\psi'' - \psi'\varphi''}{\varphi'^3}, \quad \cdots, \\[2ex] \dfrac{dz}{dx} = \dfrac{\chi'}{\varphi'}, \quad \dfrac{d^2 z}{dx^2} = \dfrac{\varphi'\chi'' - \chi'\varphi''}{\varphi'^3}, \quad \cdots, \end{cases}$$

in which one sets:

$$\varphi' = \frac{d\varphi}{dt} = \varphi_t + w_1\,\varphi_w + w_2\,\varphi_{w_1} + \cdots + w_{r+1}\,\varphi_{w_r},$$

$$\psi' = \frac{d\psi}{dt} = \psi_t + w_1\,\psi_w + w_2\,\psi_{w_1} + \cdots + w_{r+1}\,\psi_{w_r},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

while here, in turn, the lower indices t, w, w_1, …, w_r mean partial derivatives with respect to those quantities. Therefore, if the differential equation (1) is fulfilled by any arbitrary function $w(t)$ (i.e., identically in t, w, w_1, w_2, …), after taking (5), (6) into account, then we will say that the differential equation (1) possesses the *solution without integration* (5). That shows that the following theorem is valid:

All differential equations that can be solved without integration define one and the same class of differential equations.

According to Monge, the first-order differential equations of the form (1) (i.e., $n = 1$, $m = 1$) can be solved without integration. As a result of our general assertion, all first-order differential equations must then be capable of being transformed into each other invertibly without integration.

In fact, according to Monge, for any given first-order differential equation:

$$(7)\qquad F\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) = 0,$$

one can find a function $J(x, y, z, \xi)$ of the variables x, y, z, and one parameter ξ such that one will once more arrive at the differential equation (7) by eliminating the parameter ξ from the equations:

$$\frac{\partial J}{\partial x}+\frac{\partial J}{\partial y}\frac{dy}{dx}+\frac{\partial J}{\partial z}\frac{dz}{dx} = 0\,, \tag{8}$$

$$\frac{\partial^2 J}{\partial x\,\partial\xi}+\frac{\partial^2 J}{\partial y\,\partial\xi}\frac{dy}{dx}+\frac{\partial^2 J}{\partial z\,\partial\xi}\frac{dz}{dx} = 0\,. \tag{9}$$

If we now set:

$$J(x, y, z, \xi) = \eta\,, \tag{10}$$

$$\frac{\partial J(x, y, z, \xi)}{\partial\xi} = \zeta\,, \tag{11}$$

and calculate the quantities ξ, η, ζ as functions of $x, y, z, dy / dx, dz / dx$ using (8), (10), (11) then we will have:

$$\frac{d\eta}{d\xi} = \frac{dJ}{d\xi} = \left(\frac{\partial J}{\partial x}+\frac{\partial J}{\partial y}\frac{dy}{dx}+\frac{\partial J}{\partial z}\frac{dz}{dx}\right)\frac{dx}{d\xi}+\frac{\partial J}{\partial\xi} = \zeta\,,$$

and we will have then obtained a transformation of the differential equation (7) into the special form:

$$\frac{d\eta}{d\xi} = \zeta\,.$$

Furthermore, that transformation is invertible without integration, because when one recalls (9), it will follow from (11) by differentiation that:

$$\frac{\partial^2 J(x, y, z, \xi)}{\partial\xi^2} = \frac{d\zeta}{d\xi}\,, \tag{12}$$

and it will follow from (10), (11), (12) that x, y, z can then be expressed as functions of $\xi, \eta, \zeta, d\zeta / d\xi$.

The differential equation:

$$\frac{dz}{dx} = \left(\frac{dy}{dx}\right)^2$$

will serve as an example. One has:

$$J = \xi^2 x + 2\xi y + z$$

for it, and one will get the following equations for determining the transformation without integration above and its inverse:

$$\eta = \xi^2 x + 2\xi y + z \,,$$
$$\zeta = 2\xi x + 2y \,,$$
$$\frac{d\zeta}{d\xi} = 2x \,,$$
$$\xi^2 + 2\xi \frac{dy}{dx} + \frac{dz}{dx} = 0 \,,$$
$$\xi + \frac{dy}{dx} = 0 \,.$$

In the theory of algebraic functions, the differential equations that can be solved without integration correspond to the class of rationally-soluble equations in two variables, i.e., the algebraic objects of genus zero.

In what follows, I would like to prove that there are, in any event, already second-order differential equations that *do not* belong to the class of differential equations that can be solved without integration.

To that end, we shall first examine the special differential equation:

$$\frac{dz}{dx} = \left(\frac{d^2 y}{dx^2}\right)^2 \tag{13}$$

and assume (in contrast to our assertion) that it possesses the solution without integration:

$$\begin{aligned} x &= \varphi(t, w, w_1, \dots, w_r) \,, \\ y &= \psi(t, w, w_1, \dots, w_r) \,, \\ z &= \chi(t, w, w_1, \dots, w_r) \,, \end{aligned} \tag{14}$$

in which, as in (5), w means the arbitrary function of the variable t, and we have set:

$$w_1 = \frac{dw}{dt}, \qquad \dots, \qquad w_r = \frac{d^r w}{dt^r} \,.$$

Moreover, as before, we shall use lower indices t, w, w_1, w_2, … to denote the partial derivatives with respect to those quantities, and if κ means any function of t, w, w_1, w_2, … then we shall generally set:

$$\kappa' = \frac{d\kappa}{dt} = \kappa_t + w_1 \kappa_w + w_2 \kappa_{w_1} + \cdots,$$

to abbreviate. Since the case in which one of the functions φ, ψ, χ is constant has obviously been overlooked, none of the expressions φ', ψ', χ' will be equal to zero identically in all arguments.

From now on, let w_r be the highest derivative of the arbitrary function w that actually occurs in the right-hand side of the solution without integration (14), and accordingly let the expressions φ_{w_r}, ψ_{w_r}, χ_{w_r} be not all zero identically in all arguments.

We find from (14) that:

$$(15)\qquad \frac{dz}{dx} = \frac{\chi'}{\varphi'} = \frac{\chi_t + w_1\,\chi_w + \cdots + w_{r+1}\,\chi_{w_r}}{\varphi_t + w_1\,\varphi_w + \cdots + w_{r+1}\,\varphi_{w_r}},$$

$$(16)\qquad \frac{dy}{dx} = \frac{\psi'}{\varphi'} = \frac{\psi_t + w_1\,\psi_w + \cdots + w_{r+1}\,\psi_{w_r}}{\varphi_t + w_1\,\varphi_w + \cdots + w_{r+1}\,\varphi_{w_r}},$$

and if we set:

$$\mu = \frac{\psi'}{\varphi'},$$

to abbreviate, then:

$$(17)\qquad \frac{d^2y}{dx^2} = \frac{\mu'}{\varphi'} = \frac{\mu_t + w_1\,\mu_w + \cdots + w_{r+1}\,\mu_{w_r} + w_{r+2}\,\mu_{w_{r+1}}}{\varphi_t + w_1\,\varphi_w + \cdots + w_{r+1}\,\varphi_{w_r}}.$$

Equation (13) must be fulfilled identically in the quantities t, w, w_1, …, w_{r+2} after substituting (15) and (17). However, since the quantity w_{r+2} does not enter into the left-hand in χ'/φ', the right-hand side, i.e., μ'/φ', must also be free of w_{r+2}, so it will then follow identically that:

$$\mu_{w_{r+1}} = 0,$$

i.e., μ is independent of w_{r+1}. As a result, due to (17), μ'/φ' will take the form of an entire or fractional linear function of w_{r+1}, and with our substitution the right-hand side of (13) will then take the form of a fractional quadratic function of w_{r+1}, while the function (15) that is linear in w_{r+1} will appear on the left-hand side. Both sides can then prove to be identical to each other only when each of the two expressions:

$$\frac{\chi'}{\varphi'} \quad \text{and} \quad \frac{\mu'}{\varphi'}$$

proves to be independent of w_{r+1}. It will then follow immediately from (15), (17) that:

$$\varphi_{w_r}\,\frac{\chi'}{\varphi'} = \chi_{w_r},$$

$$\varphi_{w_r}\,\frac{\mu'}{\varphi'} = \mu_{w_r},$$

and since μ is also independent of w_{r+1}, from the above, due to (16), we will also have:

$$\varphi_{w_r}\,\mu = \psi_{w_r}.$$

Now, if one of the quantities φ_{w_r}, ψ_{w_r}, χ_{w_r} is zero identically then each of them must vanish identically as a result of those relations, i.e., φ, ψ, χ must all be independent of w_r, which contradicts our original assumption.

We can write the relations that we just found in the form:

$$\mu = \frac{\psi'}{\varphi'} = \frac{\psi_{w_r}}{\varphi_{w_r}}, \tag{18}$$

$$\frac{\mu'}{\varphi'} = \frac{\mu_{w_r}}{\varphi_{w_r}}, \tag{19}$$

$$\frac{\chi'}{\varphi'} = \frac{\chi_{w_r}}{\varphi_{w_r}}. \tag{20}$$

If the functions φ, ψ, χ include only the arguments t, w then (18), (20) would imply the equations:

$$\psi_t\,\varphi_w - \psi_w\,\varphi_t = 0\,,$$
$$\chi_t\,\varphi_w - \chi_w\,\varphi_t = 0\,,$$

and since $\varphi_w \neq 0$, the functions φ, ψ, χ would then be of the special type that is such that they depend upon *only one* coupling of the arguments t, w, which is a case that was excluded from the outset.

Due to that argument, we can assume that the highest order of the differential quotients that occur in (14) is $r \geq 1$.

We shall now calculate the quantity w_r in terms of $t, w, w_1, \ldots, w_{r-1}, x$ by means of:

$$x = \varphi\,(t, w, w_1, \ldots, w_r)$$

and introduce the expression for w_r thus-obtained into ψ and χ. We denote the functions that arise in that way by:

$$f\,(t, w, w_1, \ldots, w_{r-1}, x), \qquad g\,(t, w, w_1, \ldots, w_{r-1}, x), \qquad \text{resp.}$$

Furthermore, in what follows, the symbol $\equiv$ will always mean that both sides will become identical to each other in $t, w, w_1, \ldots, w_r$ as soon as we introduce $x = \varphi$. We will certainly have:

$$\psi \equiv f \tag{21}$$

and

(22) $$\chi \equiv g$$

then. Finally, if k is any function of t, w, w_1, …, w_{r-1}, x then we shall set:

$$k' = k_t + w_1\, k_w + w_2\, k_{w_1} + \cdots + w_r\, k_{w_{r-1}},$$

to abbreviate.

Upon differentiating (21) with respect to t, we will get:

$$\psi' \equiv f' + f_x\, \varphi',$$

and upon differentiating with respect to w_r, we will get:

$$\psi_{w_r} \equiv f_x\, \varphi_{w_r}.$$

By means of (18), it follows from this that:

(23) $$\mu \equiv f_x$$

and

(24) $$f' \equiv 0 .$$

It likewise follows from (23), by means of (19), that:

(25) $$\frac{\mu'}{\varphi'} \equiv f_{xx}$$

and

(26) $$(f_x)' \equiv 0 ,$$

and finally, it follows from (22), by means of (20), that:

(27) $$\frac{\chi'}{\psi'} \equiv g_x$$

and

(28) $$g' \equiv 0 .$$

We will now differentiate (24) with respect to w_r, which will then give:

$$(f_x)'\varphi_{w_r} + f_{w_r-1} \equiv 0\ ,$$

and due to (26), it will follow from this that:

$$f_{w_r-1} \equiv 0\ .$$

However, since f, and as a result f_{w_r-1} as well, do not contain the quantity w_r explicitly, it will follow from this that one also has:

$$f_{w_r-1} = 0$$

identically in t, w, w_1, …, w_{r-1}, x, i.e., f does not include the quantity w_{r-1} explicitly, either. As a result of the latter situation, f' will not, in turn, include the quantity w_r explicitly, and due to (24), it will follow from this that one has:

$$f' = 0$$

identically in t, w, w_1, …, w_{r-1}, x, i.e.:

$$f_1 + w_1\, f_w + w_2\, f_{w_1} + \cdots + w_{r-1}\, f_{w_{r-2}} = 0\ .$$

We infer from that equation, in succession, that:

$$f_{w_r-2} = 0, \qquad f_{w_r-3} = 0, \qquad \ldots, \qquad f_w = 0, \qquad f_t = 0,$$

and ultimately recognize that f cannot include any of the quantities t, w, w_1, …, w_{r-1} explicitly, but it can depend upon only x.

We infer from (17) and (25) that:

$$\frac{d^2y}{dx^2} = f_{xx}\ ,$$

and from (15) and (27) that:

$$\frac{dz}{dx} = g_x\ .$$

As a result, the given differential equation (13) will go to:

$$g_x = f_{xx}^2\ .$$

That shows that g_x is always just a function of x alone, and it follows from this that:

$$g = X + W,$$

in which X depends upon only x, and W depends upon only t, w, w_1, …, w_{r-1} . It follows from (28) that:

$$W' = 0,$$

i.e., W is a constant. Hence, g also depends upon only x.

With that, we see that in any case, φ, ψ, χ depend upon *only one* coupling of the quantities t, w, w_1, …, w_{r-1}, which is an eventuality that was excluded from the outset. Our original assumption is then impossible, and we have then proved the following theorem:

The second-order differential equation:

$$\frac{dz}{dx} = \left(\frac{d^2 y}{dx^2}\right)^2$$

possesses no solution without integration.

The line of reasoning that was just applied to the special differential equation (13) is valid in precisely the same way for the more general differential equation:

$$\frac{dz}{dx} = F\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, z, x\right), \tag{29}$$

when F is not exactly an entire or fractional linear function of $d^2 y / dx^2$.

If F is a fractional linear function of $d^2 y / dx^2$ then the differential equation can be put into the form:

$$\frac{dz}{dx} = \frac{\alpha \dfrac{d^2 y}{dx^2} + \beta}{\dfrac{d^2 y}{dx^2} + \gamma}, \tag{30}$$

where α, β, γ mean functions of x, y, z, dy / dx, and β cannot be equal to $\alpha\gamma$ identically. Moreover, it is also permissible to assume that β is not identically zero, since one must certainly have that $\alpha \neq 0$ in the case of $\beta = 0$, and if we then replace the function y with $y + x^2$ in:

$$\frac{dz}{dx} = \frac{\alpha \dfrac{d^2 y}{dx^2}}{\dfrac{d^2 y}{dx^2} + \gamma}$$

then we will get a differential equation of the same form as (30) with one non-zero term β.

We now apply the special (Legendre) transformation to (30):

$$(31)\qquad \begin{array}{l|l} \xi = \dfrac{dy}{dx}, & x = \dfrac{d\eta}{d\xi}, \\ \eta = x\dfrac{dy}{dx} - y, & y = \xi\dfrac{d\eta}{d\xi} - \eta, \\ \zeta = z, & z = \zeta. \end{array}$$

We will then get a differential equation of the form:

$$\frac{d\zeta}{d\xi} = \frac{d^2\eta}{d\xi^2}\frac{\alpha + \beta\dfrac{d^2\eta}{d\xi^2}}{1 + \gamma\dfrac{d^2\eta}{d\xi^2}},$$

in which a, β, γ are functions of ξ, η, ζ, $d\eta / d\xi$. Since $\beta \neq 0$, the right-hand side of this is certainly quadratic in $d^2\eta / d\xi^2$ in the numerator, and we conclude from this, while recalling our previous argument, that the differential equation (30) also possesses no solution without integration. *Therefore, the differential equation* (29) *can certainly possess a solution without integration only in the case when F is an* ***entire*** *linear function of* $d^2\eta / d\xi^2$.

Formulas (31) serve as an example of a transformation that admits an inversion without integration without having to refer to a well-defined basic differential equation. To distinguish them from the transformations that are invertible without integration that were treated up to now, transformations that can be inverted without integration that can, like (31), be applied to y (x), z(x), [η (ξ), ζ (ξ), resp.] will be called *unrestricted transformations that can be inverted without integration*. They define the analogues of the everywhere-invertible rational (i.e., Cremona) transformations of two variables that are known in algebra.

When we previously proved the existence of differential equations that are not soluble without integration, we showed that in in addition to the class of differential equations that can be solved without integration, there is, in any event, yet another class of differential equations that is different from the latter. *The fact that there are actually infinitely-many distinct classes of differential equations* can be understood in a different way that is analogous to the one above that was used to prove the existence of differential equation that cannot be solved without integration and which I would like to briefly characterize here.

Consider the two special differential equations:

$$(32)\qquad \frac{d\zeta}{d\xi} = \left(\frac{d^2\eta}{d\xi^2}\right)^2,$$

and

$$(33)\qquad \frac{dz}{dx} = \left(\frac{d^3y}{dx^3}\right)^2.$$

As was shown before, the first of these cannot be solved without integration, and when one regards the dy / dx in the second differential equation as an unknown function, it will follow that it is not soluble without integration either. We must then show that there is no transformation of the form:

$$
\begin{aligned}
x &= \varphi(\xi, \eta, \eta_1, \ldots, \eta_r, \zeta), \\
y &= \psi(\xi, \eta, \eta_1, \ldots, \eta_r, \zeta), \\
z &= \chi(\xi, \eta, \eta_1, \ldots, \eta_r, \zeta),
\end{aligned}
$$

by means of which (32) will become the differential equation (33). We have set:

$$\eta_1 = \frac{d\eta}{d\xi}, \qquad \ldots, \qquad \eta_r = \frac{d^r\eta}{d\xi^r}$$

in the above, to abbreviate.

We now have, in general:

$$\kappa' = \frac{d\kappa}{d\xi} = \kappa_\xi + \eta_1\,\kappa_\eta + \eta_2\,\kappa_{\eta_1} + \cdots + \eta_{r+1}\,\kappa_{\eta_r} + \eta_2^2\,\kappa_\zeta .$$

We will then get:

$$
\begin{aligned}
&(34) & \frac{dz}{dx} &= \frac{\chi'}{\varphi'}, \\
& & \frac{dy}{dx} &= \frac{\psi'}{\varphi'} = \mu, \\
& & \frac{d^2y}{dx^2} &= \frac{\mu'}{\varphi'} = \nu, \\
&(35) & \frac{d^3y}{dx^3} &= \frac{\nu'}{\varphi'} .
\end{aligned}
$$

After substituting (34) and (35), in the event that the transformation (32) should go to (33), equation (33) will be fulfilled identically in:

$$\xi, \eta, \eta_1, \ldots, \eta_{r+3}, \zeta .$$

We easily conclude from this that for $r \geq 3$ the expressions:

$$\frac{\chi'}{\varphi'}, \mu, \nu, \frac{\nu'}{\varphi'}$$

must be independent of η_{r+1}, η_{r+2}, η_{r+3}, and that the identities must therefore exist:

$$\varphi_{\eta_r} \frac{\chi'}{\varphi'} = \chi_{\eta_r} ,$$
$$\varphi_{\eta_r} \mu = \psi_{\eta_r} ,$$
$$\varphi_{\eta_r} \nu = \mu_{\eta_r} ,$$
$$\varphi_{\eta_r} \frac{\nu'}{\varphi'} = \nu_{\eta_r} .$$

We can conclude the impossibility of our assumption from those identities in a manner that is entirely analogous to the argument above. In the cases $r = 1$, $r = 2$, we will require a specialized, but very simple, argument to arrive at the same conclusion. With that, the existence of three different classes is guaranteed, and at the same time it will become clear how the process can be continued so that one could prove the existence of arbitrarily-many classes of differential equations.

A deeper and more systematic study of differential equations of the form (1) and the concept of class would require one to add the methods of the calculus of variations, and indeed it then seems to me that the following definitions and conceptual picture would mainly be required.

Any pair of functions $y(x)$, $z(x)$ that satisfy the differential equation (1) identically in x is called a *solution* to (1). Now, if the differential equation (1) can be converted into the differential equation (4) by a transformation that is invertible without integration then (3) will imply that in general any solution of the transformed differential equation (4) will correspond to a solution of the original differential equation (1). However, there can be special solutions of (1) that *cannot* be represented by means of (3) in that way, i.e., as we would like to say, they are *omitted.* On the other hand, we call those special solutions of (1) for which the variation vanishes the *discriminating solutions* of (1). The discriminating solutions will become, in turn, all or part of the discriminating solutions under a transformation that is invertible without integration.

We already recognize the fundamental significance of this general concept from the example of the first-order (Monge) differential equation. Namely, it shows that *all of the discriminating solutions of the Monge differential equations are omitted solutions, and essentially only them* (*).

We would like to prove the assertion that was just made in regard to discriminating solutions of the Monge differential equation, and indeed for the sake of brevity, in the example of the special Monge differential equation:

$$\frac{dz}{dx} = \left(\frac{dy}{dx}\right)^2 . \tag{36}$$

Upon setting the first variation of the integral:

$$z = \int \left(\frac{dy}{dx}\right)^2 dx$$

(*) Cf., the following paper of W. Gross that came about at my suggestion.

equal to zero, we will get the differential equation:

$$\frac{d^2y}{dx^2} = 0 ,$$

and as a result, the discriminating solutions of (36) will read:

$$\text{(37)} \qquad \begin{aligned} y &= a\,x + b , \\ z &= a^2 x + c , \end{aligned}$$

in which *a*, *b*, *c* are understood to mean constants.

The solution without integration of (36) reads:

$$\text{(38)} \qquad \begin{aligned} x &= t^2\, w_{tt} - 2t\, w_t + 2w , \\ y &= t\, w_{tt} - w_t , \\ z &= w_{tt} . \end{aligned}$$

Now let:

$$y = f(x) , \qquad z = g(x)$$

be a system of solutions of the differential equation (36). In order to represent it by means of the formulas (38), it is necessary and sufficient for one to find a function $w(t)$ that satisfies the two differential equations:

$$\text{(39)} \qquad t\, w_{tt} - w_t = f(t^2\, w_{tt} - 2t\, w_t + 2w) ,$$

$$\text{(40)} \qquad w_{tt} = g(t^2\, w_{tt} - 2t\, w_t + 2w) ,$$

and for which the expression:

$$x = t^2\, w_{tt} - 2t\, w_t + 2w$$

does not prove to be constant, so:

$$\frac{dx}{dt} = t^2\, w_{ttt} \neq 0 ,$$

i.e.:

$$\text{(41)} \qquad w_{ttt} \neq 0 .$$

Upon differentiating (39), (40) with respect to *t*, one will get:

$$\text{(42)} \qquad t\, w_{ttt} = t^2\, w_{ttt}\, f' ,$$

$$\text{(43)} \qquad w_{ttt} = t^2\, w_{ttt}\, g' ,$$

or, due to (41):

$$1 = t\,f', \tag{44}$$

$$1 = t^2\,g'. \tag{45}$$

Now, if f, g is not one of the discriminating solutions (37) then f' will not prove to be constant, and we can then bring (44) into the form:

$$t^2\,w_{tt} - 2t\,w_t + 2w = h\left(\frac{1}{t}\right) \tag{46}$$

by inversion, in which h is a function of $1 / t$ that does not prove to be constant. That differential equation for w is certainly always soluble. Let w^0 be a particular solution of it.

Due to the fact that:

$$g' = (f')^2,$$

it will follow from (44) that (45) is likewise fulfilled when we replace w with w^0 in it. Therefore, (42), (43) will also be fulfilled for $w = w^0$, and since those equations arise from (39), (40) upon differentiation, we can infer the existence of two constants A, B from this such that:

$$t\,w_{tt}^0 - w_t^0 + A = f\,(t^2\,w_{tt}^0 - 2t\,w_t^0 + 2w^0), \tag{47}$$

$$w_{tt}^0 + B = g\,(t^2\,w_{tt}^0 - 2t\,w_t^0 + 2w^0). \tag{48}$$

If we set:

$$w = w^0 + \tfrac{1}{2}Bt^2 - At$$

then due to (47), (48), that function will satisfy the differential equations (39), (40), and due to (46), the expression:

$$t^2\,w_{tt} - 2t\,w_t + 2w = t^2\,w_{tt}^0 - 2t\,w_t^0 + 2w^0$$

will not prove to be equal to a constant. With that, it is show that our solution can, in fact, be represented by (38).

On the other hand, let f, g be a discriminating solution, as one might get from (37). (39) will then go to:

$$t\,w_{tt} - w_t = a(t^2\,w_{tt} - 2t\,w_t + 2w) + b.$$

Upon differentiating that with respect to t, we will get:

$$(t - a\,t^2)\,w_{ttt} = 0,$$

and as a result:

$$w_{ttt} = 0,$$

i.e., our solution cannot be represented by (38), and with that, the theorem that I posed that the discriminating solutions are omitted solutions, and only them, is proved completely.

In conclusion, let it be mentioned that the Monge differential equation likewise serves as an example of how the discriminating solutions will not, in any case, possess an invariant character with respect to the transformations that are invertible without integration, but we saw above that any Monge differential equation (7) can be transformed into the special form:

$$\frac{d\eta}{d\xi} = \zeta$$

invertibly without integration, and the latter differential equation obviously possesses no discriminating solution at all. The situation that was highlighted here has an intimate connection then with the previous theorem that all discriminating solutions are, at the same time, omitted solutions in the case of the Monge differential equation.