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The foundations of physics

By

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What follows is essentially a correction of the two older papers ([1]) that I published on the “foundations of physics” and the remarks regarding them that F. Klein published in his notice ([2]) “Zu Hilbert’s erster Note über die Grundlagen der Physik,” with only minor editorial revisions and alterations that should make things more understandable.

The unifying mechanistic ideal in physics that was created by the great researchers of the previous generation and was established during the reign of classical electrodynamics must ultimately be abandoned today. The introduction and development of the concept of a field gradually exhibited a new possibility for our perception of the physical world. Mie was the first to point to a path, along which, this newly-emerged “unifying field-theoretic ideal,” as I would like to call it, can be made accessible to a general mathematical treatment. While the older mechanistic conception immediately took matter itself to be its starting point, and determined it by means of a finite number of discrete parameters, the new field-theoretic ideal of the physical continuum – viz., the so-called space-time manifold – served as its foundation. If the form of the laws of the universe were previously differential equations with one independent variable then now they would necessarily be expressed in terms of partial differential equations.

As I showed in my first notice, the profound problem statement and mental picture of Einstein’s general theory of relativity now finds its simplest and most natural expression along the path that was embarked upon by Mie, and, at the same time, a systematic extension and rounding-off in a formal context.

Some meaningful treatises on this state of affairs have appeared since the publication of my first notice. I shall mention only the deep and brilliant investigations of Weyl and the publications of Einstein, which are rich in even newer Ansätzen and ideas. However, Weyl later altered his path of research in such a way that he likewise arrived at the equations that I had proposed, while Einstein, on the other hand, ultimately returned directly to the equations of my theory in his later publications, although he repeatedly started from difference and mutually distinct Ansätzen.

I certainly believe that the theory that I shall develop here contains a residual nucleus and creates a framework, inside of which there is sufficient room for the future

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([1]) Göttinger Nachr.: First notice, submitted on 20 Nov. 1915, second notice, submitted on 23 Dec. 1916.

([2]) Göttinger Nachr.: Submitted in 25 Jan. 1918.

construction of physics in the spirit of a unifying field-theoretic ideal. In each case, it is also of epistemological interest to see how the small number of simple assumption that I posed in axioms I, II, III, IV will suffice for the construction of the entire theory.

Admittedly, whether or not the purely field-theoretic unifying ideal (some extensions and modifications of which might possibly be necessary) is definitive enough to make it possible to address the existence of negative and positive electrons, in particular, as well as the consistent formulation of the laws that govern the atomic interior, will have to be a problem for the future.

Part I.

Let x_s (s = 1, 2, 3, 4) be any coordinates that essentially specify a world-point uniquely; viz., the so-called *world-parameters* (i.e., the most general space-time coordinates). Let the quantities that characterize the phenomena at x_s be:

1. The gravitational potentials $g_{\mu\nu}$ (μ, ν = 1, 2, 3, 4), which were first introduced by Einstein, and have a symmetric tensor character under an arbitrary transformation of the world-parameters x_s ; they define the coefficients of the invariant differential form:

$$\sum_{\mu,\nu} g_{\mu\nu}\, dx^{\mu} dx^{\nu} .$$

2. The four electrodynamical potentials q_s , which have a vector character in the same sense, and which define the invariant linear form:

$$\sum_{s} q_s dx_s .$$

Physical phenomena are not arbitrary, since one must verify the following axioms, moreover:

Axiom I (Mie's axiom of the world-function ([3])). *The law of physical phenomena is determined by a world-function H that contains the following arguments:*

(1) $$g_{\mu\nu} , \quad g_{\mu\nu l} = \frac{\partial g_{\mu\nu}}{\partial x_l} , \quad g_{\mu\nu lk} = \frac{\partial g_{\mu\nu}}{\partial x_l\, \partial x_k} ,$$

(2) $$q_s , \qquad q_{sl} = \frac{\partial q_s}{\partial x_l} , \qquad (s, l = 1, 2, 3, 4),$$

([3]) Mie's world-function does not refer to this argument precisely; in particular, Born returned to the use of argument (2). However, the introduction and employment of such a world-function into Hamilton's principle is precisely characteristic of Mie's electrodynamics.

and indeed the variation of the integral:

$$\int H\sqrt{g}\, d\omega$$

$$(g = -\,|\,g_{\mu\nu}\,|, \quad d\omega = dx_1\, d_2\, dx_3\, dx_4)$$

must vanish for each of the 14 potentials $g_{\mu\nu}$, q_s .

The arguments:

$$g^{\mu\nu}, \qquad g_l^{\mu\nu} = \frac{\partial g^{\mu\nu}}{\partial x_l}, \quad g_{lk}^{\mu\nu} = \frac{\partial g^{\mu\nu}}{\partial x_l\, \partial x_k} \tag{3}$$

can obviously appear in place of the arguments (1), in which $g^{\mu\nu}$ means the sub-determinant of the determinant $(-\,g)$ that relates to its element $g_{\mu\nu}$, divided by $(-\,g)$.

The ten Lagrangian differential equations:

$$\frac{\partial\sqrt{g}\,H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial x_k}\frac{\partial\sqrt{g}\,H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial x_k\,\partial x_l}\frac{\partial\sqrt{g}\,H}{\partial g_{kl}^{\mu\nu}} = 0 \qquad (\mu, \nu = 1, 2, 3, 4) \tag{4}$$

for the ten gravitational potentials then follow from Axiom I, and then the four Lagrangian differential equations:

$$\frac{\partial\sqrt{g}\,H}{\partial q_k} - \sum_k \frac{\partial}{\partial x_k}\frac{\partial\sqrt{g}\,H}{\partial q_{hk}} = 0 \qquad (h = 1, 2, 3, 4). \tag{5}$$

Let it be remarked, once and for all, about the differential quotients with respect to $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{lk}^{\mu\nu}$ that appear in (4) and subsequent formulas that due to the symmetry in μ, ν, on the one hand, and in k, l, on the other, the differential quotients with respect to $g^{\mu\nu}$, $g_l^{\mu\nu}$ are understood to mean that one applies a factor of 1 (1/2, resp.) to them according to whether $\mu = \nu$ or $\mu \neq \nu$, resp., and furthermore multiplies the differential quotients with respect $g_{lk}^{\mu\nu}$ by 1 (1/2, 1/4, resp.) according to whether $\mu = \nu$ and $k = l$ ($\mu = \nu$ and $k \neq l$, resp.) or $\mu \neq \nu$ and $k = l$ ($\mu \neq \nu$ and $k \neq l$, resp.).

For the sake of brevity, we shall denote the left-hand sides of equations (4), (5) by:

$$[\sqrt{g}H]_{\mu\nu}, \qquad [\sqrt{g}H]_h,$$

resp.

Equations (4) can be called the *basic equations of gravitation* and equations (5), the *basic equations of electrodynamics.*

Axiom II (Axiom of general invariance ([4])). *The world-function H is an invariant under an arbitrary transformation of the world-parameters* x_s .

Axiom II is the simplest mathematical expression of the demand that the coordinates should have no physical meaning in and of themselves, but represent only a numbering of the world-points, such that the concatenation of the potentials $g_{\mu\nu}$, q_s is completely independent of the type of numbering.

In what follows, we shall employ the easily-proved fact that if p^j (j = 1, 2, 3, 4) means an arbitrary contravariant vector then the expression:

$$p^{\mu\nu} = \sum_s (g_s^{\mu\nu} p^s - g^{\mu s} p_s^\nu - g^{\nu s} p_s^\mu) \qquad \left(p_s^j = \frac{\partial p^j}{\partial x_s} \right)$$

represents a symmetric, contravariant tensor, and the expression ([5]):

$$p_l = \sum_s (q_{ls} p^s + q_s p_l^s) .$$

Moreover, we state two mathematical theorems, which read as follows:

Theorem 1. If J is an invariant that depends upon $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{lk}^{\mu\nu}$ then one will always have:

$$\sum_{\mu,\nu,l,k} \left(\frac{\partial J}{\partial g^{\mu\nu}} \Delta g^{\mu\nu} + \frac{\partial J}{\partial g_i^{\mu\nu}} \Delta g_i^{\mu\nu} + \frac{\partial J}{\partial g_{lk}^{\mu\nu}} \Delta g_{lk}^{\mu\nu} \right) + \sum_{s,k} \left(\frac{\partial J}{\partial q_s} \Delta q_s + \frac{\partial J}{\partial q_{sk}} \Delta q_{sk} \right) = 0$$

identically in all arguments and for every arbitrary contravariant vector p^s.

In this, one has:

$$\Delta g^{\mu\nu} = \sum_m (g^{\mu m} p_m^\nu + g^{\nu m} p_m^\mu) ,$$

$$\Delta g_l^{\mu\nu} = -\sum_m g_m^{\mu\nu} p_l^m + \frac{\partial \Delta g^{\mu\nu}}{\partial x_l} ,$$

$$\Delta g_{lk}^{\mu\nu} = -\sum_m (g_m^{\mu\nu} p_{lk}^m + g_{lm}^{\mu\nu} p_k^m + g_{km}^{\mu\nu} p_l^m) + \frac{\partial^2 \Delta g^{\mu\nu}}{\partial x_l \, \partial x_k} ,$$

([4]) Mie had already posed the demand of orthogonal invariance. Einstein's fundamental concept of general invariance finds its simplest expression in Axiom II that is posed above, even if Hamilton's principle played only a supporting role for Einstein, and his functions *H* were in no way invariants, nor did they include the electric potentials.

([5]) p_l should not be confused with the covariant vector $\sum_s g_{ls} p^s$ that is associated with p^s.

$$\Delta q_s = -\sum_m q_m p_s^m ,$$

$$\Delta q_{sk} = -\sum_m q_{sm} p_k^m + \frac{\partial \Delta q_s}{\partial x_s} .$$

Theorem 1 can also be expressed as follows:

If J is an invariant, and p_s is an arbitrary vector, as before, then one will have the identity:

$$\sum_s \frac{\partial J}{\partial x_s} p^s = P(J), \tag{6}$$

in which one sets:

$$P = P_g + P_q ,$$

$$P_g = \sum_{\mu,\nu,l,k} \left(p^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} + p_l^{\mu\nu} \frac{\partial}{\partial g_l^{\mu\nu}} + p_{lk}^{\mu\nu} \frac{\partial}{\partial g_{lk}^{\mu\nu}} \right),$$

$$P_q = \sum_{l,k} \left(p_l \frac{\partial}{\partial q_l} + p_{lk} \frac{\partial}{\partial q_{lk}} \right),$$

and one has the abbreviations:

$$p_k^{\mu\nu} = \frac{\partial p^{\mu\nu}}{\partial x_k}, \qquad p_{kl}^{\mu\nu} = \frac{\partial^2 p^{\mu\nu}}{\partial x_k \, \partial x_l}, \qquad p_{lk} = \frac{\partial p_l}{\partial x_k} .$$

The proof of (6) is simple to obtain. This identity is obviously correct when p^s is a constant vector, and its invariance in general will follow from that.

Theorem 2. If J is an invariant that depends upon $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{lk}^{\mu\nu}$, q_s, q_{sk} as in Theorem 1, and the variational equations of $\sqrt{g}\,J$ with respect to $g^{\mu\nu}$ (with respect to q_μ, resp.) are denoted by $[\sqrt{g}\,J]_{\mu\nu}$ ($[\sqrt{g}\,J]_\mu$, resp.) , and if one further sets:

$$i_s = \sum_{\mu,\nu} \left(\left[\sqrt{g}\,J\right]_{\mu\nu} g_s^{\mu\nu} + \left[\sqrt{g}\,J\right]_\mu q_{\mu s} \right),$$

$$i_s^l = -2 \sum_\mu \left[\sqrt{g}\,J\right]_{\mu s} g^{\mu l} + \left[\sqrt{g}\,J\right]_l q_s$$

to abbreviate then the identities:

$$(7) \qquad i_s = \sum_l \frac{\partial i_s^l}{\partial x_l} \qquad (s = 1, 2, 3, 4)$$

will be true.

Theorem 2 contains a general mathematical theorem ([6]) as its essential core that was my guiding principle for the construction of the theory, and which is expressed as follows:

If F is a function of n quantities (that are functions of x_1, x_2, x_3, x_4) and their derivatives, and if the integral:

$$\int F\,d\omega$$

is invariant under arbitrary transformations of the four world-parameters x_1, x_2, x_3, x_4 then four of the equations in the system of n Lagrangian differential equations that belong to the variational problem:

$$\delta \int F\,d\omega = 0$$

will always be a consequence of the remaining n – 4, in the sense that four linearly-independent relations between the n Lagrangian derivatives of F with respect to each of the n quantities and their total differential quotients with respect to x_1, x_2, x_3, x_4 will always be fulfilled identically.

In order to prove Theorem 2, we consider a finite piece of the four-dimensional universe. Furthermore, let p^s be a vector that vanishes, along with its derivatives, on the three-dimensional outer surface of that region. According to the definition of P, one will have:

$$P(\sqrt{g}J) = \sqrt{g}P(J) + J\sum_{\mu,\nu} \frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} p^{\mu\nu} = \sqrt{g}P(J) + J\sum_s \left(\frac{\partial\sqrt{g}}{\partial x_s} p^s + \sqrt{g}\,p_s^s\right),$$

$$P(\sqrt{g}J) = \sqrt{g}\sum_{\mu,\nu} \frac{\partial J}{\partial x_s} p^s + J\sum_s \left(\frac{\partial\sqrt{g}}{\partial x_s} p^s + \sqrt{g}\,p_s^s\right) = \sum_s \frac{\partial\sqrt{g}\,Jp^s}{\partial x_s}.$$

If we integrate this equation over the world-region in question then due to the form of the divergence on the right-hand side and the assumption on p^s that will give:

$$\int P(\sqrt{g}J)\,d\omega = 0.$$

([6]) Emmy Noether gave the general proof of this theorem ("Invariante Variationsprobleme," Göttinger Nachr., 1918, Heft 2). Indeed, in my first notice, the identities that were given in Theorem 2 were stated only for the case in which the invariant depended upon $g^{\mu\nu}$ and its derivatives; however, the method of proof that was set down there and reproduced in this article is just as true for our general invariant J, as well. The identities were first derived in their general form by F. Klein, on the basis of the method of infinitesimal transformations ("Zu Hilbert's erster Note über die Grundlagen der Physik," Gött. Nachr., 1917, Heft 3).

Due to the way that the Lagrangian derivative is defined, one also has:

$$\int \left\{ \sum_{\mu,\nu} \left[\sqrt{g}\, J \right]_{\mu\nu} p^{\eta\nu} + \sum_{\mu} \left[\sqrt{g}\, J \right]_{\mu} p_{\mu} \right\} d\omega = 0.$$

The integrand can be written in the form:

$$\sum_{k,l} (i_s p^s + i_s^l p_l^s)$$

here. From the formula that then arises:

$$\int \sum_{k,l} (i_s p^s + i_s^l p_l^s)\, d\omega = 0,$$

we obtain:

$$\sum_s \int \left(i_s - \sum_l \frac{\partial i_s^l}{\partial x_l} \right) p^s d\omega = 0,$$

and with it, the statement of Theorem 2, as well.

Some further axioms are required in order to determine the world-function H. Should the basic equations (4), (5) of gravitation and electrodynamics contain only second derivatives of the $g^{\mu\nu}$ then H would have to be composed additively of a linear function with constant coefficients of the invariant:

$$K = \sum_{\mu,\nu} g^{\mu\nu} K_{\mu\nu} ,$$

in which $K_{\mu\nu}$ means the Riemann curvature tensor:

$$K_{\mu\nu} = \sum_{\kappa} \left(\frac{\partial}{\partial x_\nu} \begin{Bmatrix} \mu\,\kappa \\ \kappa \end{Bmatrix} - \frac{\partial}{\partial x_\kappa} \begin{Bmatrix} \mu\,\nu \\ \kappa \end{Bmatrix} \right) + \sum_{\kappa,\lambda} \left(\begin{Bmatrix} \mu\,\kappa \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\,\nu \\ \kappa \end{Bmatrix} - \begin{Bmatrix} \mu\,\nu \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\,\kappa \\ \kappa \end{Bmatrix} \right),$$

and an invariant L that depends upon only $g^{\mu\nu}$, $g_l^{\mu\nu}$, q_s, q_{sk} . We make the following special assumptions:

Axiom III (Axiom of gravitation and electricity). *The world-function H has the form:*

$$H = K + L,$$

in which K is the invariant that arises from the Riemann tensor – i.e., the curvature – and L depends upon only $g^{\mu\nu}$, q_s, q_{sk} .

Thus, the gravitational equations will assume the form:

(8) $$\left[\sqrt{g}\,K\right]_{\mu\nu} = -\frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}} \qquad (\mu,\ \nu = 1, 2, 3, 4),$$

and the electrodynamical equations will assume the form:

(9) $$\frac{1}{\sqrt{g}}\sum_k \frac{\partial}{\partial x_k}\frac{\partial\sqrt{g}\,L}{\partial q_{hk}} = \frac{\partial L}{\partial q_k} \qquad (h = 1, 2, 3, 4).$$

In order to determine the expression for $\left[\sqrt{g}\,K\right]_{\mu\nu}$, one next specializes the coordinate system in such a way that all of the $g_s^{\mu\nu}$ vanish for the world-point in question. In that way, one will find that:

$$\left[\sqrt{g}\,K\right]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}K\right).$$

If we introduce the notation $T_{\mu\nu}$ for the tensor:

$$-\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}}$$

then the gravitational equations will read:

$$K_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\,L = T_{\mu\nu}\,.$$

On the other hand, we apply Theorem 1 to the invariant 1 and thus obtain:

(10) $$\sum_{\mu,\nu,m}\frac{\partial L}{\partial g^{\mu\nu}}(g^{\mu\nu}p_m^{\nu} + g^{\nu m}p_m^{\mu}) - \sum_{s,m}\frac{\partial L}{\partial q_s}q_m p_s^m - \sum_{s,k,m}\frac{\partial L}{\partial q_{sk}}(q_{sm}p_l^m + q_{mk}p_s^m + q_m p_{sk}^m) = 0.$$

Setting the coefficients of p_{sk}^m equal to zero in the left-hand side of this will yield the equation:

$$\left(\frac{\partial L}{\partial q_{sk}} + \frac{\partial L}{\partial q_{ks}}\right)q_m = 0,$$

or

(11) $$\frac{\partial L}{\partial q_{sk}} + \frac{\partial L}{\partial q_{ks}} = 0;$$

i.e., the derivatives of the electrodynamical potentials q_s appear only in the combinations:

$$M_{ks} = q_{sk} - q_{ks} .$$

With that, we recognize that by our assumptions, the invariant L will depend upon merely the components of the skew-symmetric invariant tensor:

$$M = (M_{ks}) = \mathrm{Rot}\,(q_s),$$

in addition to the potentials $g^{\mu\nu}$, q_s ; i.e., the so-called electromagnetic six-vector. It follows further from this that:

$$\frac{\partial L}{\partial q_{sk}} = \frac{\partial L}{\partial M_{ks}} = H^{ks}$$

is a skew-symmetric, contravariant tensor, as well as the fact that:

$$\frac{\partial L}{\partial q_k} = r^k$$

is a contravariant vector.

If one applies the notations that we introduced then the electrodynamical equations will assume the form:

$$\text{(12)} \qquad \frac{1}{\sqrt{g}} \sum_k \frac{\partial \sqrt{g}\, H^{kh}}{\partial x_k} = r^h \qquad (h = 1, 2, 3, 4).$$

One recognizes a generalization of one of the systems of Maxwell equations in these equations; one obtains the other one from the equations:

$$M_{ks} = q_{sk} - q_{ks}$$

by differentiation and addition:

$$\text{(13)} \qquad \frac{\partial M_{ks}}{\partial x_t} + \frac{\partial M_{st}}{\partial x_k} + \frac{\partial M_{tk}}{\partial x_s} = 0 \qquad (t, k, s = 1, 2, 3, 4).$$

We then see that the form of these "generalized Maxwell equations" (12), (13) is already determined, in essence, by the requirement of general invariance, and thus, upon the basis of Axiom II. If we set the coefficients of p_m^ν equal to zero on the left-hand side of the identity (10) then, with the use of (11), we will get:

$$\text{(14)} \qquad 2\sum_\mu \frac{\partial L}{\partial g^{\mu\nu}} g^{\mu m} - \frac{\partial L}{\partial q_m} q_\nu - \sum_s \frac{\partial L}{\partial M_{ms}} M_{\nu s} = 0 \qquad (\mu = 1, 2, 3, 4),$$

so

$$2\sum_\mu \frac{\partial L}{\partial g^{\mu\nu}} g^{\mu m} = \sum_s H^{ms} M_{\nu s} + r^m q_\nu ,$$

or

$$-\frac{2}{\sqrt{g}}\sum_{\mu}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}g^{\mu m} = L\delta_{\nu}^{m} - \sum_{s} H^{ms}M_{\nu s} - r^{m}q_{\nu},$$

$$\delta_{\nu}^{m} = 0 \qquad (m \neq \nu),$$
$$\delta_{\nu}^{m} = 1.$$

With that, one gets the representation of $T_{\mu\nu}$:

$$T_{\mu\nu} = \sum_{\mu} g_{\mu m}T_{\nu}^{m},$$
$$T_{\nu}^{m} = \tfrac{1}{2}\left\{L\,\delta_{\nu}^{m} - \sum_{s} H^{ms}M_{\nu s} - r^{m}q_{\nu}\right\}.$$

The expression on the right agrees with Mie's electromagnetic energy tensor, and we then find that *Mie's energy tensor is nothing but the generally-invariant tensor that arises by differentiating the invariant L with respect to the gravitational potentials* $g_{\mu\nu}$, which is a situation that I first proved in the necessarily narrow connection between Einstein's general theory of relativity and Mie's electrodynamics, and which then convinced me that the theory that is developed here is correct.

Applying Theorem 2 to the invariant K will yield:

(15a) $$\sum_{\mu,\nu}\left[\sqrt{g}\,K\right]_{\mu\nu} g_{s}^{\mu\nu} + 2\sum_{m}\frac{\partial}{\partial x_m}\left(\sum_{\mu}\left[\sqrt{g}\,K\right]_{\mu s} g^{\mu m}\right) = 0.$$

Applying it to L will yield:

(15b) $$\sum_{\mu,\nu}\left(-\sqrt{g}\,T_{\mu\nu}\right)g_{s}^{\mu\nu} + 2\sum_{m}\frac{\partial}{\partial x_m}\left(-\sqrt{g}\,T_{s}^{m}\right)$$
$$+\sum_{\mu}\left[\sqrt{g}\;L\right]_{\mu} q_{\mu s} - \sum_{\mu}\frac{\partial}{\partial x_{\mu}}\left(\left[\sqrt{g}\;L\right]_{\mu} q_{s}\right) = 0 \qquad (s = 1, 2, 3, 4).$$

As a consequence of the electrodynamical equations, we obtain from this that:

(16) $$\sum_{\mu,\nu}\sqrt{g}\,T_{\mu\nu}g_{s}^{\mu\nu} + 2\sum_{m}\frac{\partial\sqrt{g}\,T_{s}^{m}}{\partial x_m} = 0.$$

These equations (16) can also be obtained as a consequence of the gravitational equations on the basis of (15a). They have the meaning of the basic mechanical equations. In the case of special relativity, for which the $g_{\mu\nu}$ are constants, they will go to the equations:

$$\sum \frac{\partial T_s^m}{\partial x_m} = 0,$$

which express the conservation of energy and impulse.

It follows from equations (16), on the basis of the identities (15b), that:

$$\sum_\mu \left[\sqrt{g}\,L\right]_\mu q_{\mu s} - \sum_\mu \frac{\partial}{\partial x_\mu}\left(\left[\sqrt{g}\,L\right]_\mu q_s\right) = 0,$$

or

$$\sum_\mu \left\{ M_{\mu s}\left[\sqrt{g}\,L\right]_\mu + q_s \sum_\mu \frac{\partial}{\partial x_\mu}\left(\left[\sqrt{g}\,L\right]_\mu\right)\right\} = 0\,; \tag{17}$$

i.e., four mutually-independent linear relations between the basic electrodynamical equations (5) and their first derivatives will follow from the gravitational equations (4). That is the precise mathematical expression of the connection between gravitation and electrodynamics that governs the entire theory.

Since L should not, by our assumption, depend upon the derivatives of $g^{\mu\nu}$, it must be a function of four certain invariants that correspond to Mie's special orthogonal invariants, and the two simplest of them are:

$$Q = \sum_{k,l,m,n} M_{mn} M_{kl}\, g^{mk} g^{nl}$$

and

$$q = \sum_{k,l} q_k q_l\, g^{kl}\,.$$

The simplest and (in regard to the structure of K) closest Ansatz for L is, at the same time, the one that corresponds to Mie's electrodynamics, namely:

$$L = \alpha Q + f(q) \qquad (\alpha = \text{const.}).$$

According to this Ansatz, one obtains the following relations between the quantities that appear in the generalized Maxwell equations:

$$H^{ks} = 4a\, M^{ks},$$
$$r^k = 2f'(q)\, q^k,$$

in which one sets:

$$M^k = \sum_{\mu,\nu} g^{k\mu} g^{s\nu} M_{\mu\nu}\,,$$
$$q^k = \sum_l g^{kl} q_l\,.$$

For the entirely special case of:

$$f(q) = \beta q \qquad (\beta = \text{const.}),$$

it follows that the "current vector" r^k will be proportional to the contravariant vector q^k.

Part II.

The connection between the theory and experiment shall be discussed more closely. Another axiom is required for this.

Axiom IV (Space-time axiom). *The quadratic form:*

$$G\,(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu \tag{18}$$

shall be such that in its representation as a sum of four squares of linear forms in X_s *, three of the squares will always appear with positive signs, and one of them will always have a negative sign.*

The quadratic form (18) yields the *metric of a pseudo-geometry* for our four-dimensional world of x_s . The determinant g of the $g_{\mu\nu}$ proves to be negative.

If a curve:

$$x_s = x_s(p) \qquad (s = 1, 2, 3, 4)$$

is given in this geometry, where $x_s(p)$ mean any real functions of the parameter p, then it can be divided into pieces, along each of which the expression:

$$G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right)$$

does not change its sign. A piece of the curve for which one has:

$$G\left(\frac{dx_s}{dp}\right) > 0$$

is called a *segment*, and the integral:

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)}\, dp\,,$$

when taken along this piece of the curve, is then called the *length of the segment*. A piece of the curve or which:

$$G\left(\frac{dx_s}{dp}\right) < 0$$

is called a *time line*, and the integral:

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)}\, dp ,$$

when taken along that piece of the curve, will be called the *proper time of the time line.* Finally, a piece of the curve along which one has:

$$G\left(\frac{dx_s}{dp}\right) = 0$$

will be called a *null line*.

In order to make this concept of our pseudo-geometry intuitive, we imagine that we have an ideal measuring instrument – viz., the *light-clock* – by means of which we can determine the proper time along any time line.

We next show that one can succeed in calculating the values of $g_{\mu\nu}$ as functions of x_s with the help of this instrument, as long as one only introduces a certain space-time coordinate system x_s . In fact, we choose any ten time-lines that all arrive at the point x_s in question from various directions, such that whenever that end point takes on the parameter value p, it will yield the equation:

$$\left(\frac{d\lambda^{(s)}}{dp}\right)^2 = G\left(\frac{dx_s^{(k)}}{dp}\right) \qquad (h = 1, 2, \ldots, 10)$$

for each of the ten time lines at the end point; in this, the left-hand side will be known as soon as we have determined the proper time $\tau^{(h)}$ by means of the clock. If we now set:

$$D(u) = \begin{vmatrix} \left(\frac{dx_1^{(1)}}{dp}\right)^2 & \frac{dx_1^{(1)}}{dp}\frac{dx_1^{(1)}}{dp} & \cdots & \left(\frac{dx_4^{(1)}}{dp}\right)^2 & \left(\frac{d\lambda^{(1)}}{dp}\right)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{dx_1^{(10)}}{dp}\right)^2 & \frac{dx_1^{(10)}}{dp}\frac{dx_1^{(10)}}{dp} & \cdots & \left(\frac{dx_4^{(10)}}{dp}\right)^2 & \left(\frac{d\lambda^{(10)}}{dp}\right) \\ X_1^2 & X_1 X_2 & \cdots & X_4^2 & u \end{vmatrix},$$

to abbreviate, then we will obviously have:

$$G\,(X_s) = -\frac{D(0)}{\frac{\partial D}{\partial u}}, \tag{19}$$

so we can, at the same time, pose the condition:

$$\frac{\partial D}{\partial u} \neq 0$$

as necessary for the directions of the ten chosen time lines at the point $x_s(p)$.

If G is calculated from (19) then the application of the process to any 11th time line that ends at $x_s(p)$ will yield:

$$\left(\frac{d\lambda^{(11)}}{dp}\right)^2 = G\left(\frac{dx^{(11)}}{dp}\right),$$

and this equation would then be a test of the validity of the instrument, as well as an experimental confirmation of the fact that the assumptions of the theory apply to the real world.

The axiomatic construction of our pseudo-geometry can be performed with no difficulty: First, one poses an axiom, upon whose basis it will then follow that length (proper time, resp.) must be an integral whose integrand is merely a function of x_s and its first derivatives with respect to the parameter; perhaps the well-known envelope theorem for geodetic lines might serve as such an axiom. Second, one needs an axiom that would make the theorems of pseudo-Euclidian geometry (i.e., the old principle of relativity at infinity) true. Here, the axiom that was posed by E. Blaschke ([7]) would be especially suitable, which says that the condition of orthogonality should be reciprocal for any two directions, whether they are segments or time lines.

Let us now briefly summarize the main facts that the Monge-Hamilton theory of differential equations teaches us about our pseudo-geometry.

Each world-point x_s belongs to a second-order cone that has its vertex at x_s and is determined by the equation:

$$G(X_1 - x_1, X_2 - x_2, X_3 - x_3, X_4 - x_4) = 0$$

in the running point coordinates X_s ; it is called the *null cone* that is associated with the point x_s . The totality of all null cones defines a four-dimensional field of cones that is associated with, on the one hand, the "Monge" differential equation:

$$G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right) = 0,$$

and, on the other hand, the "Hamilton" partial differential equation:

$$H\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}\right) = 0, \tag{20}$$

where H means the reciprocal quadratic form to G:

$$H(U_1, U_2, U_3, U_4) = \sum_{\mu\nu} g^{\mu\nu} U_\mu U_\nu \,.$$

([7]) "Räumliche Variationsprobleme mit symmetrischer Transversalitätsbedingung," Leipziger Berichte, Math.-phys. Kl. **68** (1916), pp. 50.

The characteristics of Monge's, and likewise those of Hamilton's, partial differential equation (20) are the null geodetic lines. The totality of all null geodetic lines that emanate from a certain world-point a_s ($s = 1, 2, 3, 4$) generate a three-dimensional point manifold that might be called the *time-sheath* that belongs to the world-point a_s . The time-sheath possesses a node at a_s whose tangent cone is precisely the null cone that belongs to a_s . If we bring the equation of the time-sheath into the form:

$$x_4 = \varphi(x_1, x_2, x_3)$$

then

$$f = x_4 - \varphi(x_1, x_2, x_3)$$

will be an integral of Hamilton's differential equation (20). The totality of all time-lines that emanate from the point a_s will lie completely inside of that four-dimensional subset of the universe that has the time-sheath at a_s as its boundary.

With these preparations, we turn to the problem of *causality* in the new physics.

Up to now, we have regarded all coordinate systems x_s that emerge from any one of them by way of an arbitrary transformation as equivalent. This arbitrariness must be restricted if we would like to ensure that if two world-points that lie along the same time-line can be related to each other as cause and effect then it would not then be possible to transform such world-points into simultaneous ones. If we distinguish x_4 as the *proper* time coordinate then we will propose the following definition:

A *proper* space-time coordinate system is one for which the following four inequalities are fulfilled:

$$(21) \qquad g_{11} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0, \qquad g_{44} < 0,$$

along with $g < 0$. A transformation that takes such a space-time coordinate system into another proper space-time coordinate system will be call a *proper* space-time coordinate transformation.

The four inequalities express the idea that the null cone that is associated with any world-point a_s lies completely outside of the linear space:

$$x_4 = a_4 .$$

By comparison, the line:

$$x_1 = a_1 , \qquad x_2 = a_2 , \qquad x_3 = a_3$$

is contained inside of it; the latter line will always be a time line then.

Let any time-line $x_s = x(p)$ be given, moreover; since:

$$G\left(\frac{dx}{dp}\right) < 0,$$

it will then follow that one will always have:

$$\frac{dx_4}{dp} \neq 0$$

in a proper space-time coordinate system, and as a result, the proper time coordinate x_4 must always increase (decrease, resp.) along a time-line. Since a time-line remains a time-line under any coordinate transformation, two world-points along a time-line can never take on the same value of the time coordinate x_4 by a proper space-time coordinate transformation; i.e., it is impossible to transform them so that they will be simultaneous.

On the other hand, if the points of a curve can actually be transformed into simultaneous ones then the transformation of that curve must obey:

$$x_4 = \text{const.}, \qquad \text{i.e.,} \qquad \frac{dx_4}{dp} = 0,$$

so

$$G\left(\frac{dx_s}{dp}\right) = \sum_{\mu\nu} g_{\mu\nu} \frac{dx_\mu}{dp} \frac{dx_\nu}{dp} \qquad (\mu, \nu = 1, 2, 3),$$

and the right-hand side is positive here, due to the first three of our inequalities; the curve thus-characterized will then be a segment.

We then see that the concepts of cause and effect that lie at the basis of the principle of causality will also not lead to any sort of internal contradictions in the new physics, as long we always append the inequalities (21) to our basic equations; i.e., we restrict ourselves to the use of *proper* space-time coordinates.

In place of it, let us refer to a special space-time coordinate system that will prove to be useful later on, and which I would like to call a *Gaussian coordinate system*, since it is a generalization of the geodetic polar coordinate systems that Gauss introduced into the theory of surfaces. Let any three-dimensional space be given in our four-dimensional universe that is such that every curve that runs through that space is a segment – viz., a *segment space*, as I would like to call it; let x_1, x_2, x_3 be the coordinates of any point in that space. At any point x_1, x_2, x_3 of it, we now construct the geodetic line that is orthogonal to it, which will be a time-line and will be associated with x_4 as the proper time along it. We assign the coordinates x_1, x_2, x_3, x_4 to the point of the four-dimensional universe thus-obtained. As is easy to see, one will have:

(22)
$$G(X_s) = \sum_{\mu\nu}^{1,2,3} g_{\mu\nu} X_\mu X_\nu - X_4^2$$

in these coordinates; i.e., the Gaussian coordinate system is characterized analytically by the equations:

(23) $\qquad g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0, \qquad g_{44} = 1.$

Due to the assumed behavior of the three-dimensional space $x_4 = 0$, the quadratic form in the variables X_1, X_2, X_3 that appears in the right-hand side of (22) is necessarily positive-definite; i.e., the first three of the inequalities (21) are fulfilled, and since that would also be true for the fourth one, the Gaussian coordinate system always proves to be a *proper* space-time coordinate system.

We now return to the study of the causality principle in physics. We see that its main content is the fact (which has been true of every physical theory up till now) that the values of physical quantities and their temporal derivatives in the future can be determined uniquely when one knows those quantities in the present: The laws of physics up to now have indeed, without exception, found their expression in a system of differential equation which are such that the number of functions that appear in them essentially agrees with the number of independent differential equations, and thus the known Cauchy theorem on the existence of integrals of differential equations will then immediately serve as the basis for the proof of that fact.

Now, our basic equations of physics (4) and (5) are, by no means, of the type that was just characterized; moreover, as I have shown, four of them are a consequence of the remaining ones. We can regard the electrodynamical equations (5) as consequences of the ten gravitational equations (4), and we will thus have only ten essentially mutually-independent equations (4) for the 14 potentials $g_{\mu\nu}$, q_s.

As long as we maintain the demand of general invariance for the basic equations of physics, the aforementioned situation is also essential and necessary. Namely, if there are other invariant equations for the 14 potentials that are independent of (4) then the introduction of a Gaussian coordinate system by means of (23) would yield a system of equations for the ten physical quantities:

$$g_{\mu\nu} \qquad (\mu, \nu = 1, 2, 3), \qquad\qquad q_s \qquad (s = 1, 2, 3, 4)$$

that would be, in turn, mutually-independent, and since there are more than ten of them, they would define an over-determined system.

Under such circumstances then, it is in no way possible to conclude the values of physical quantities in the future uniquely from the knowledge of them in the present and the past. In order to show this intuitively with an example, let our basic equations (4) and (5) be integrated in the special case that corresponds to the presence of a single electron that is constantly at rest, such that the 14 potentials:

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x_1, x_2, x_3), \\ q_s &= q_s(x_1, x_2, x_3) \end{aligned}$$

prove to be well-defined functions of x_1, x_2, x_3 that are all independent of the time x_4, and in such a way that the first three components r^1, r^2, r^3 of the four-density might vanish, moreover. We thus apply the following coordinate transformation to the potentials:

$$\begin{cases} x_1 = x_1', & \text{for } x_4' \leq 0, \\ x_1 = x_1' + e^{-1/x_4'^2}, & \text{for } x_4' > 0, \end{cases}$$

$$\begin{aligned} x_2 &= x_2', \\ x_3 &= x_3', \end{aligned}$$

$$x_4 = x_4'.$$

For $x_4' \leq 0$, the transformed potentials $g_{\mu\nu}'$, q_s' are the same functions of x_1', x_2', x_3' as the $g_{\mu\nu}$, q_s in the original variables x_1, x_2, x_3, while for $x_4' > 0$, the $g_{\mu\nu}'$, q_s' will also depend upon the time coordinate x_4' in an essential way; i.e., the potentials $g_{\mu\nu}'$, q_s' represent an electron that is at rest up to time $x_4' = 0$, but then it is set in motion.

Therefore, I believe that only a more detailed understanding of the idea that is at the basis of the principle of general relativity ([8]) will serve to maintain the causality principle in the new physics, as well. Corresponding to the essence of the new principle of relativity, we must, in fact, require the invariance of not only the general laws of physics, but also endow every individual statement in physics with an invariant character, if it is to have any physical sense, which is harmony with the fact that any physical fact must ultimately be capable of being established by light-clocks – i.e., by instruments of an *invariant* character. Just as in the theory of curves and surfaces, a statement for which the parameter representation of the curve or surface has been chosen will have no geometric meaning for the curve or surface itself when the statement does not remain invariant under an arbitrary transformation of the parameter or cannot be brought into an invariant form, in physics, we must also say that a statement that does not remain invariant under any arbitrary transformation of the coordinate system is *physically meaningless.* For example, in the case that is considered above of the electron at rest, the statement that it is at rest at time $x_4 = 1$ has no physical meaning, since that statement is not invariant.

Now, as far as the causality principle is concerned, the physical quantities and their temporal derivatives might be known for the present in any given coordinate system. A statement would then have physical meaning only when it is invariant under all of the transformations for which present values that are assumed to be known remain unchanged. I claim that statements of this kind are all determined uniquely for the future; i.e., *the causality principle is true in this form:*

All statements about the 14 physical potentials $g_{\mu\nu}$, q_s in the future will follow necessarily and uniquely from knowing them in the present as long as they are physically meaningful.

In order to prove this assertion, we employ a Gaussian space-time coordinate system. The introduction of (23) into the basic equations (4) will produce a system of just as many partial differential equations for the ten potentials:

$$(24) \qquad g_{\mu\nu} \quad (\mu, \nu = 1, 2, 3), \qquad q_s \quad (s = 1, 2, 3, 4);$$

if we integrate it on the basis of the given initial values for $x_4 = 0$ then we will find the values of (24) for $x_4 > 0$ in a single-valued way. Since the Gaussian coordinate system is

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([8]) In his original, now-abandoned, theory (Sitzungsberichte der Akad. zu Berlin, 1914, pp. 1067), A. Einstein especially postulated four non-invariant equations for the $g_{\mu\nu}$ in order to salvage the causality principle in its older form.

itself established uniquely, all of the statements about the potentials (24) that refer to that coordinate system will have an invariant character.

The forms in which physically meaningful – i.e., invariant – statements can be expressed mathematically are quite manifold.

First. This can come about by means of an invariant coordinate system. The well-known Riemannian coordinate system is just a useful for that purpose as the previously-employed Gaussian one, and for that matter, any coordinate system in which the transformed electricity and unit density appear to be in a state of rest. If as in the conclusion to Part I, $f(q)$ denotes the function of the invariant:

$$q = \sum_{kl} q_k q_l \, g^{kl}$$

that appears in Hamilton's principle then:

$$r^s = 2f'(q) \cdot q^s = 2\, f'(q) \sum_l g^{sl} q_l$$

is the four-density of electricity. It represents a contravariant vector, and is therefore can be transformed to (0, 0, 0, 1) in a region of the universe in which $f'(q) \neq 0$ and the four-potential is nowhere-vanishing. After that transformation, the four components of the four-potential q_s will be expressible in terms of the $g_{\mu\nu}$ from the four equations:

$$\sum_l g^{sl} q_l = 0 \quad (s = 1, 2, 3), \qquad \sum_l g^{4l} q_l = \frac{1}{2\, f'(q)},$$

and any relation between the $g_{\mu\nu}$ in this coordinate system is then an invariant statement.

There can be special coordinate systems for particular solutions of the basic equations; e.g., in the case that is treated below of a centrally-symmetric gravitational field, r, ϑ, φ, t define a coordinate system that is invariant up to rotations.

Second. The statement that *a coordinate system can be found* in which the 14 potentials $g_{\mu\nu}$, q_s will have certain well-defined values in the future or fulfill certain well-defined relations is always an invariant, and therefore, physically meaningful, statement. The mathematically-invariant expressions for such a statement will be obtained eliminating the coordinates from each relation. The case above of the electron at rest will serve as an example: The essential and physically-meaningful content of the causality principle is expressed here in the statement that *for a suitable choice of space-time coordinate system*, an electron that is at rest for time $x_4 \leq 0$ will also be continually at rest, in all of its parts, for the future $x_4 > 0$.

Third. A statement will also be invariant, and will therefore always have physical meaning, when it is valid for any arbitrary coordinate system, since otherwise the expressions that appear would need to possess a formally-invariant character.

According to my way of explaining things, physics is a four-dimensional pseudo-geometry whose metric $g_{\mu\nu}$ is coupled to the electromagnetic quantities – i.e., to matter – by the basic equations (4) and (5). With that knowledge, an old geometric question will now become ripe for solution, namely, the question of whether, and in what sense, Euclidian geometry – about which, we only know from mathematics that it is a logically consistent structure – also possesses any validity in reality.

The old physics, with its concept of absolute time, subsumed the theorems of Euclidian geometry and put them at the foundations of any particular physical theory from the outset. Even Gauss proceeded in an only slightly different way: He hypothetically constructed a non-Euclidian physics in which he dropped only the parallel axiom from the theorems of Euclidian geometry, while preserving absolute time. The measurement of the angle of a triangle with large dimensions then showed him the invalidity of this non-Euclidian physics.

The new physics of Einstein's general principle of relativity assumes a completely different position with respect to geometry. It is based upon either Euclidian or some other well-defined geometry from the outset in order to deduce the actual physical laws from it, since otherwise the new theory of physics would yield the geometrical and physical laws, at a single blow, from one and the same Hamilton principle, namely, the basic equations (4) and (5), which teach us how the metric $g_{\mu\nu}$ – and at the same time, the mathematical expression for the physical phenomenon of gravitation – is concatenated with the values q_s of the electrodynamical potentials.

Euclidian geometry is *a doctrine that is remote and foreign to modern physics:* Since the theory of relativity rejects Euclidian geometry as a general assumption for physics, it teaches us moreover that geometry and physics have an equivalent character and rest upon a common foundation as *one* science.

The aforementioned geometric question comes down to the examination of whether, and under which assumptions, the four-dimensional Euclidian pseudo-geometry:

$$\begin{aligned} &g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{44} = -1, \\ &g_{\mu\nu} = 0, \quad (\mu \neq \nu) \end{aligned} \tag{25}$$

is a solution of the gravitational equations (the only regular solution of it, resp.)

The gravitational equations (8) read:

$$\left[\sqrt{g}\,K\right]_{\mu\nu} + \frac{\partial \sqrt{g}\,L}{\partial g^{\mu\nu}} = 0,$$

in which:

$$\left[\sqrt{g}\,K\right]_{\mu\nu} = \sqrt{g}\,(K_{\mu\nu} - \tfrac{1}{2} K\,g_{\mu\nu}).$$

By substituting the values (25), one will get:

$$\left[\sqrt{g}\,K\right]_{\mu\nu} = 0 \tag{26}$$

and for:

$$q_s = 0 \qquad (s = 1, 2, 3, 4)$$

one will have:

$$\frac{\partial \sqrt{g}\, L}{\partial g^{\mu\nu}} = 0;$$

i.e., the pseudo-Euclidian geometry will be possible when all of the electricity is at a distance. The question of whether this is also necessary in this case – i.e., of whether (under what additional conditions, resp.) the values (25) and the values of $g_{\mu\nu}$ that emerge from a coordinate transformation are the only regular solutions of equations (26) – is a mathematical problem that will not be discussed here in general.

In the case of pseudo-Euclidian geometry, we have:

$$g_{\mu\nu} = \gamma_{\mu\nu},$$

in which:

$$\gamma_{11} = 1, \qquad \gamma_{22} = 1, \qquad \gamma_{33} = 1, \qquad \gamma_{44} = 1,$$
$$\gamma_{\mu\nu} = 0 \qquad (\mu \neq \nu).$$

For any metric that is close to this pseudo-Euclidian geometry, one will have the Ansatz:

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon\, h_{\mu\nu} + \ldots, \tag{27}$$

in which ε is a quantity that converges to zero and $h_{\mu\nu}$ are functions of x. I shall make the following two assumptions about the metric (27):

I. The $h_{\mu\nu}$ might be independent of the variables x_4 .
II. The $h_{\mu\nu}$ might exhibit a certain regular behavior at infinity.

Now, should the metric fulfill the differential equations (26) for all ε, it would follow that the $h_{\mu\nu}$ must necessarily fulfill certain linear, homogeneous, second-order partial differential equations. If one, like Einstein ([9]), sets:

$$h_{\mu\nu} = k_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\sum_s k_{ss} \qquad (k_{\mu\nu} = k_{\nu\mu}), \tag{28}$$

$$\delta_{\mu\nu} = 0 \qquad (\mu \neq \nu),$$
$$\delta_{\nu\nu} = 1$$

and assumes the four relations:

$$\sum_s \frac{\partial k_{\mu s}}{\partial x_s} = 0 \qquad (\mu = 1, 2, 3, 4) \tag{29}$$

then these differential equations will read as follows:

([9]) "Näherungsweise Integration der Feldgleichungen der Gravitation," Berichte d. Akad. zu Berlin (1916), pp. 688.

(30) $$\square k_{\mu\nu} = 0,$$

in which we have set:

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

to abbreviate.

Due to the Ansatz (28), the relations (29) are restricting assumptions for the functions $h_{\mu\nu}$. I would like to show that if one performs a suitable infinitesimal transformation of the variables x_1, x_2, x_3, x_4 then these restricting assumptions will be fulfilled by the corresponding functions $h'_{\mu\nu}$ after the transformation. To that end, one determines four functions φ_1, φ_2, φ_3, φ_4 of the variables that satisfy the differential equations:

(31) $$\square \varphi_\mu = \frac{1}{2} \frac{\partial}{\partial x_\mu} \sum_\nu h_{\nu\nu} - \sum_\nu \frac{\partial h_{\mu\nu}}{\partial x_\nu},$$

resp. By means of the infinitesimal transformation:

$$x_s = x'_s + \varepsilon \varphi_s,$$

$g_{\mu\nu}$ will go to:

$$g'_{\mu\nu} = g_{\mu\nu} + \varepsilon \sum_\alpha g_{\alpha\nu} \frac{\partial \varphi_\alpha}{\partial x_\mu} + \varepsilon \sum_\alpha g_{\alpha\mu} \frac{\partial \varphi_\alpha}{\partial x_\nu} + \dots$$

or, due to (27), into:

$$g'_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon + h'_{\mu\nu} \;\dots,$$

in which one has set:

$$h'_{\mu\nu} = h_{\mu\nu} + \frac{\partial \varphi_\nu}{\partial x_\mu} + \frac{\partial \varphi_\mu}{\partial x_\nu}.$$

If we now choose:

$$k_{\mu\nu} = h'_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} \sum_s h'_{ss}$$

then, due to (31), these functions will fulfill the Einstein conditions (29), and we will get:

$$h'_{\mu\nu} = k_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} \sum_s k_{ss} \qquad (k_{\mu\nu} = k_{\nu\mu}).$$

Due to assumption I, the differential equations (30), which must be true as a result of what we did above in order to find the $k_{\mu\nu}$, will go to:

$$\frac{\partial^2 k_{\mu\nu}}{\partial x_1^2} + \frac{\partial^2 k_{\mu\nu}}{\partial x_2^2} + \frac{\partial^2 k_{\mu\nu}}{\partial x_3^2} = 0,$$

and, since the assumption II – when interpreted in a corresponding way – will allow one to conclude that the $k_{\mu\nu}$ approach constants at infinity, it will then follow that they must constant everywhere; i.e.:

By varying the metric of the pseudo-Euclidian geometry, under assumptions I and II, it is not possible to arrive at a regular metric that is not likewise pseudo-Euclidian and that will, at the same time, correspond to a universe that is free of electricity.

The integration of the partial differential equations (26) is achieved in yet another case that was first treated by Einstein ([10]) and Schwarzschild ([11]). In what follows, I will point out a path for this case that makes no assumptions at all about the gravitational potentials $g_{\mu\nu}$ at infinity, and will also be advantageous for my later investigations, as well. The assumptions on the $g_{\mu\nu}$ are the following ones:

1. The metric is referred to a Gaussian coordinate system, except that g_{44} is left arbitrary; i.e., one has:

$$g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0.$$

2. The $g_{\mu\nu}$ are independent of the time coordinate x_4 .

3. Gravitation $g_{\mu\nu}$ is centrally-symmetric with respect to the coordinate origin.

According to Schwarzschild, when one sets:

$$\begin{aligned} x_1 &= r \cos \vartheta, \\ x_2 &= r \sin \vartheta \cos \varphi, \\ x_3 &= r \sin \vartheta \sin \varphi, \\ x_4 &= t, \end{aligned}$$

the metric that corresponds to the most general of these assumptions will be represented by the following expression in polar coordinates:

$$F(r)\, dr^2 + G(r)\, (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) - H(r)\, dt^2, \tag{32}$$

in which $F(r)$, $G(r)$, $H(r)$ are arbitrary functions of r. If we set:

$$r^* = \sqrt{G(r)}$$

then we will be justified in interpreting r^*, ϑ, φ as spatial polar coordinates in the same way. If we introduce r^* in place of r into (32) and then once more drop the * sign then the following expression will arise:

$$M(r)\, dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - W(r)\, dt^2, \tag{33}$$

([10]) "Perihelbewegung des Merkur," Sitzungsber. d. Akad. zu Berlin (1915), pp. 831.
([11]) "Über das Gravitationsfeld eines Massenpunktes," Sitzungsber. d. Akad. zu Berlin (1916), pp. 189.

where $M(r)$, $W(r)$ mean two essentially arbitrary functions of r. The question is now whether, and how, they are to be determined in the most general way in order that the differential equations (26) would happen to be satisfied.

To that end, the given expressions $K_{\mu\nu}$, K, which were known in Part I, must be calculated. The first step in this process is to exhibit the differential equations of the geodetic lines by varying the integral:

$$\int\left[M\left(\frac{dr}{dp}\right)^2+r^2\left(\frac{d\vartheta}{dp}\right)^2+r^2\sin^2\left(\frac{d\varphi}{dp}\right)^2-W\left(\frac{dt}{dp}\right)^2\right]dp\,.$$

We get the following equations for its Lagrangian equations:

$$0=\frac{d^2r}{dp^2}+\frac{1}{2}\frac{M'}{M}\left(\frac{dr}{dp}\right)^2-\frac{r}{M}\left(\frac{d\vartheta}{dp}\right)^2-\frac{r}{M}\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2+\frac{1}{2}\frac{W'}{W}\left(\frac{dt}{dp}\right)^2\ ,$$

$$0=\frac{d^2\vartheta}{dp^2}+\frac{2}{r}\frac{dr}{dp}\frac{d\vartheta}{dp}-\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dp}\right)^2,$$

$$0=\frac{d^2\varphi}{dp^2}+\frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp}+2\cot\vartheta\frac{d\vartheta}{dp}\frac{d\varphi}{dp},$$

$$0=\frac{d^2t}{dp^2}+\frac{W'}{W}\frac{dr}{dp}\frac{dt}{dp}.$$

Here, in and in the calculation that follows, the prime symbol $'$ will refer to the derivative with respect to r. By comparing these with the general differential equations for geodetic lines:

$$\frac{d^2x_s}{dp^2}+\sum_{\mu\nu}\begin{Bmatrix}\mu\,\nu\\ s\end{Bmatrix}\frac{dx_\mu}{dp}\frac{dx_\nu}{dp}=0,$$

we can assign the following values to the bracket symbols $\begin{Bmatrix}\mu\,\nu\\ s\end{Bmatrix}$, in which we have not given the ones that vanish:

$$\begin{Bmatrix}1\,1\\ 1\end{Bmatrix}=\frac{1}{2}\frac{M'}{M},\qquad \begin{Bmatrix}2\,2\\ 1\end{Bmatrix}=-\frac{r}{M},\qquad \begin{Bmatrix}3\,3\\ 1\end{Bmatrix}=-\frac{r}{M}\sin^2\vartheta,$$

$$\begin{Bmatrix}4\,4\\ 1\end{Bmatrix}=\frac{1}{2}\frac{W'}{M},\qquad \begin{Bmatrix}1\,2\\ 2\end{Bmatrix}=\frac{1}{r},\qquad \begin{Bmatrix}3\,3\\ 2\end{Bmatrix}=-\sin\vartheta\cos\vartheta,$$

$$\begin{Bmatrix}1\,3\\3\end{Bmatrix}=\frac{1}{r},\qquad \begin{Bmatrix}2\,3\\3\end{Bmatrix}=\cot\vartheta,\qquad \begin{Bmatrix}1\,4\\4\end{Bmatrix}=\frac{1}{2}\frac{W'}{W}.$$

We then construct:

$$\begin{aligned}
K_{11}=&\ \frac{\partial}{\partial r}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\right)-\frac{\partial}{\partial r}\begin{Bmatrix}1\,1\\1\end{Bmatrix}\\
&+\begin{Bmatrix}1\,1\\1\end{Bmatrix}\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\begin{Bmatrix}2\,1\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}\begin{Bmatrix}3\,1\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\begin{Bmatrix}4\,1\\4\end{Bmatrix}\\
&-\begin{Bmatrix}1\,1\\1\end{Bmatrix}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\right)\\
&=\frac{1}{2}\frac{W''}{W}-\frac{1}{4}\frac{W'^2}{W^2}-\frac{M'}{rM}-\frac{1}{4}\frac{M'W'}{M\,W},
\end{aligned}$$

$$\begin{aligned}
K_{22}=&\ \frac{\partial}{\partial\vartheta}\begin{Bmatrix}2\,3\\3\end{Bmatrix}-\frac{\partial}{\partial r}\begin{Bmatrix}2\,2\\1\end{Bmatrix}\\
&+\begin{Bmatrix}2\,1\\2\end{Bmatrix}\begin{Bmatrix}2\,2\\1\end{Bmatrix}+\begin{Bmatrix}2\,2\\1\end{Bmatrix}\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}2\,3\\3\end{Bmatrix}\begin{Bmatrix}3\,2\\3\end{Bmatrix}\\
&-\begin{Bmatrix}2\,2\\1\end{Bmatrix}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\right)\\
&=-1-\frac{1}{2}\frac{r\,M'}{M}+\frac{1}{M}+\frac{1}{2}\frac{r\,W'}{M\,W},
\end{aligned}$$

$$\begin{aligned}
K_{23}=&-\frac{\partial}{\partial r}\begin{Bmatrix}3\,3\\1\end{Bmatrix}-\frac{\partial}{\partial\vartheta}\begin{Bmatrix}3\,3\\2\end{Bmatrix}\\
&+\begin{Bmatrix}3\,1\\3\end{Bmatrix}\begin{Bmatrix}3\,3\\1\end{Bmatrix}+\begin{Bmatrix}3\,2\\3\end{Bmatrix}\begin{Bmatrix}3\,3\\2\end{Bmatrix}+\begin{Bmatrix}3\,3\\1\end{Bmatrix}\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}3\,3\\2\end{Bmatrix}\begin{Bmatrix}2\,3\\3\end{Bmatrix}\\
&-\begin{Bmatrix}3\,3\\1\end{Bmatrix}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\right)-\begin{Bmatrix}3\,3\\2\end{Bmatrix}\begin{Bmatrix}2\,3\\3\end{Bmatrix}\\
&=\sin^2\vartheta\left(-1-\frac{1}{2}\frac{r\,M'}{M^2}+\frac{1}{M}+\frac{1}{2}\frac{r\,W'}{M\,W}\right),
\end{aligned}$$

$$\begin{aligned}
K_{44}=&-\frac{\partial}{\partial r}\begin{Bmatrix}4\,4\\1\end{Bmatrix}+\begin{Bmatrix}4\,1\\4\end{Bmatrix}\begin{Bmatrix}4\,4\\1\end{Bmatrix}+\begin{Bmatrix}4\,4\\1\end{Bmatrix}\begin{Bmatrix}4\,1\\4\end{Bmatrix}\\
&-\begin{Bmatrix}4\,4\\1\end{Bmatrix}\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,3\\3\end{Bmatrix}+\begin{Bmatrix}1\,4\\4\end{Bmatrix}\right)
\end{aligned}$$

$$= \frac{1}{2}\frac{W''}{M} + \frac{1}{4}\frac{M'W'}{M^2} + \frac{1}{4}\frac{W'^2}{M\,W^2} - \frac{W'}{r\,M},$$

$$\begin{aligned} K &= \sum_s g^{ss} K_{ss} \\ &= \frac{W''}{M\,W} - \frac{1}{2}\frac{W'^2}{M\,W^2} - 2\frac{M'}{r\,M^2} - \frac{1}{2}\frac{M'W'}{M^2\,W} - \frac{2}{r^2} + \frac{2}{r^2 M} + 2\frac{W'}{r\,M\,W}. \end{aligned}$$

Since:

$$\sqrt{g} = \sqrt{MW}\; r^2 \sin\vartheta,$$

one will have:

$$K\sqrt{g} = \left\{\left(\frac{r^2\,W'}{\sqrt{MW}}\right)' - 2\frac{r\,M'\sqrt{W}}{M^{3/2}} - 2\sqrt{MW} + 2\sqrt{\frac{W}{M}}\right\} \sin\vartheta,$$

and if we set:

$$M = \frac{r}{r-m}, \qquad W = w^2\frac{r-m}{r},$$

in which m and w are unknown functions of r, then we will finally get:

$$K\sqrt{g} = \left\{\left(\frac{r^2\,W'}{\sqrt{MW}}\right)' - 2wm'\right\} \sin\vartheta,$$

such that the variation of the four-fold integral:

$$\iiiint K\sqrt{g}\,dr\,d\vartheta\,d\varphi\,dt$$

will be equivalent to the variation of the simple integral:

$$\int w\,m'\,dr,$$

and will lead to the Lagrangian equations:

$$(34) \qquad \begin{aligned} m' &= 0, \\ w' &= 0. \end{aligned}$$

One easily convinces oneself that these equations, in fact, demand the vanishing of all $K_{\mu\nu}$. They then represent essentially the most general solution of equations (26) under the assumptions 1, 2, 3 that were made. If we take $m = \alpha$ to be the integral of (34), where α is a constant and $w = 1$, which obviously implies no essential restriction, then (33) will imply the desired metric in the form that was first found by Schwarzschild:

(35) $$G\,(dr, d\vartheta, d\varphi, dt) = \frac{r}{r-\alpha}\,dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 - \frac{r-\alpha}{r}\,dt^2.$$

The singularity of this metric at $r = 0$ then appears only when one takes $\alpha = 0$; i.e., *with assumptions 1, 2, 3, the metric of pseudo-Euclidian geometry is the only regular metric that corresponds to a universe that is free of electricity.*

For $\alpha \neq 0$, $r = 0$, and for positive α, also $r = \alpha$, prove to be places at which the metric is not regular. Therefore, I shall call a metric or gravitational field $g_{\mu\nu}$ *regular* at a location when it is possible to introduce a coordinate system by an invertible, single-valued transformation such that in that system, the corresponding functions $g'_{\mu\nu}$ are regular at that location; i.e., they are continuous and differentiable arbitrarily often at that location and have a non-zero determinant g'.

Regardless of whether in my way of looking at things only regular solutions of the basic physical equations immediately represent reality, it is precisely the solutions with non-regular locations that are an important mathematical means of approximating characteristic regular solutions, and in that sense, from the procedures of Einstein and Schwarzschild, the metric (35) that is not regular for $r = 0$ and $r = \alpha$ can be regarded as an expression of the gravitation of a mass that is distributed centrally-symmetrically in the neighborhood of the origin ([12]). In the same sense, the mass can be regarded as the limiting case of a certain distribution of electricity around a point, so I shall foresee that one might derive the equations of motion at that point from my basic physical equations. The question of the differential equations for the motion of light is dealt with similarly.

According to Einstein, the following two axioms might serve as a substitute for the derivation of the basic equations:

The motion of a mass point in a gravitational field is represented by a geodetic line that is a time-line.

The motion of light in a gravitational field is represented by a null geodetic line ([13]).

Since the world-line that represents the motion of a mass point must be a time-line, as we can easily see, it will always be possible to bring the mass point to rest by a *proper* space-time transformation; i.e., there are *proper* space-time coordinate systems, relative to which the mass point is constantly at rest.

The differential equations of the geodetic line for the central gravitational field (35) originate in the variational problem:

$$\delta\int\left[\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\vartheta}{dp}\right)^2 + r^2\sin^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2\right]dp = 0,$$

([12]) Transforming the location $r = \alpha$ to the origin, as Schwarzschild did, is not recommended, in my opinion; furthermore, the Schwarzschild transformation is not the simplest one for that purpose.

([13]) Laue has shown how one can derive this theorem from the electrodynamical equations by passing to the limit of zero wavelength for the special case of $L = \alpha Q$.

so after a well-known process they will read:

$$\text{(36)} \qquad \frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\vartheta}{dp}\right)^2 + r^2\sin^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = A,$$

$$\text{(37)} \qquad \frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right) - r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dp}\right)^2 = 0,$$

$$\text{(38)} \qquad r^2\sin^2\vartheta\,\frac{d\varphi}{dp} = B,$$

$$\text{(39)} \qquad \frac{r-\alpha}{r}\frac{dt}{dp} = C,$$

in which A, B, C mean integration constants.

I shall next prove that *the path curves in the $r\vartheta\varphi$-space always lie in a plane that goes through the center of gravitation.*

To that end, we eliminate the parameter p from the differential equations (37) and (38), in order to obtain a differential equation for ϑ as a function of φ. It is identically:

$$\text{(40)} \qquad \frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right) = \frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\cdot\frac{d\varphi}{dp}\right) = \left(2r\frac{dr}{d\varphi}\frac{d\vartheta}{d\varphi} + r^2\frac{d^2\vartheta}{d\varphi^2}\right)\left(\frac{d\varphi}{dp}\right)^2 + r^2\frac{d\vartheta}{d\varphi}\frac{d^2\varphi}{dp^2}.$$

On the other hand, differentiating (38) with respect to p will produce:

$$\left(2r\frac{dr}{d\varphi}\sin^2\vartheta + 2r^2\sin\vartheta\cos\vartheta\frac{d\vartheta}{d\varphi}\right)\left(\frac{d\varphi}{dp}\right)^2 + r^2\sin^2\frac{d^2\varphi}{dp^2} = 0,$$

and if we deduce the value of $d^2\varphi / dp^2$ from this and insert it into the right-hand side of (40) then that will give:

$$\frac{d}{dp}\left(r^2\frac{d\vartheta}{dp}\right) = \left[\frac{d^2\vartheta}{d\varphi^2} - 2\cot\vartheta\left(\frac{d\vartheta}{dp}\right)^2\right] r^2\left(\frac{d\varphi}{dp}\right)^2.$$

Equation (37) then assumes the form:

$$\frac{d^2\vartheta}{d\varphi^2} - 2\cot\vartheta\left(\frac{d\vartheta}{dp}\right)^2 = \sin\vartheta\cos\vartheta,$$

which is a differential equation whose general integral reads:

$$\sin \vartheta \cos (\varphi + a) + b \cos \vartheta = 0,$$

in which a, b mean integration constants.

The desired verification is then performed with that, and for the further discussion of the geodetic lines, it will then suffice to direct one's attention to just the value $\vartheta = \pi / 2$. The variational problem is then simplified as follows:

$$\delta \int \left[\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2 \sin^2 \left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 \right] dp = 0,$$

and the three first-order differential equations that arise from it read:

$$\text{(41)} \qquad \frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2 + r^2 \sin^2 \left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = A,$$

$$\text{(42)} \qquad r^2 \frac{d\varphi}{dp} = B,$$

$$\text{(43)} \qquad \frac{r-\alpha}{r}\frac{dt}{dp} = C.$$

The Lagrangian differential equation for r:

$$\text{(44)} \qquad \frac{d}{dp}\left(\frac{2r}{r-\alpha}\frac{dr}{dp}\right) + \frac{\alpha}{(r-\alpha)^2}\left(\frac{dr}{dp}\right)^2 - 2r\left(\frac{d\varphi}{dp}\right)^2 + \frac{\alpha}{r^2}\left(\frac{dt}{dp}\right)^2 = 0$$

must necessarily be concatenated with the previous equations, and indeed, when the left-hand sides of (41), (42), (43), (44) are denoted by [1], [2], [3], [4], resp., we will have identically:

$$\text{(45)} \qquad \frac{d[1]}{dp} - 2\frac{d\varphi}{dp}\frac{d[2]}{dp} + 2\frac{dt}{dp}\frac{d[3]}{dp} = \frac{dr}{dp}[4].$$

If we take $C = 1$, which will result from multiplying the parameter p by a constant and then eliminating p and t from (41), (42), (43), then we will arrive at the differential equation for $\rho = 1 / r$ as a function of φ that Einstein and Schwarzschild found, namely:

$$\text{(46)} \qquad \left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1+A}{B^2} - \frac{A\alpha}{B^2}\rho - \rho^2 + \alpha\rho^3.$$

This equation represents the trajectory of the mass point in polar coordinates; the Kepler motion follows from it in the first approximation for $\alpha = 0$ when $B = \sqrt{\alpha}\, b$, $A = -1 +$

αa, and the second approximation then leads to the most brilliant discovery of the era: the calculation of the precession of the perihelion of Mercury.

From the axiom above, the world-line for the motion of a mass point should be a time-line; thus, it will always follows from the definition of a time-line that $A < 0$.

We now ask, in particular, whether the circle – i.e., r = const. – can be the trajectory of a motion. The identity (45) shows that since $dr/dp = 0$ in this case, equation (14) will be in no way a consequence of (41), (42), (43); the last three equations will then be insufficient for the determination of the motion. Moreover, (42), (43), (44) are equations that must necessarily be fulfilled. It follows from (44) that:

$$-2r\left(\frac{d\varphi}{dp}\right)^2 + \frac{\alpha}{r^2}\left(\frac{dt}{dp}\right)^2 = 0, \tag{47}$$

or for the velocity v in the orbit:

$$v^2 = \left(r\frac{d\varphi}{dt}\right)^2 = \frac{\alpha}{2r}. \tag{48}$$

On the other hand, since $A < 0$, (41) will give the inequality:

$$r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 < 0, \tag{49}$$

or, with the use of (47):

$$r > \frac{3\alpha}{2}. \tag{50}$$

Due to (48), the inequality ([14]):

$$v < \frac{1}{\sqrt{3}} \tag{51}$$

for the velocity of the mass point that moves in a circle will follow from this.

The inequality (50) admits the following interpretation. From (48), the angular velocity of the orbiting mass point for $r = r_0$ is:

$$\frac{d\varphi}{dt} = \sqrt{\frac{\alpha}{2r_0^3}}.$$

If we would like to introduce the polar coordinates of coordinate system that is rotating about the origin in place of r, ϑ then we will necessarily have to replace:

([14]) The specification by Schwarzschild, *loc. cit.*, that the velocity of the mass point on the orbit should approach the limit $1/\sqrt{2}$ as the orbital radius is reduced corresponds to the inequality $r \geq \alpha$ and, from the above, that cannot be true.

$$\varphi \quad \text{with} \qquad \varphi + \sqrt{\frac{\alpha}{2r_0^3}}\,t\,.$$

Under the space-time transformation in question, the metric:

$$\frac{r}{r-\alpha}dr^2 + r^2 d\varphi^2 - \frac{r-\alpha}{r}dt^2$$

will go to:

$$\frac{r}{r-\alpha}dr^2 + r^2 d\varphi^2 \sqrt{\frac{2\alpha}{r_0^3}}r^2 d\varphi\, dt + \left(\frac{\alpha}{2r_0^3}r^2 - \frac{r-\alpha}{r}\right)dt^2\,.$$

For $r = r_0$, one gets:

$$\frac{r_0}{r_0-\alpha}dr^2 + r_0^2 d\varphi^2 + \sqrt{2\alpha r_0}\, d\varphi\, dt + \left(\frac{3\alpha}{2r_0} - 1\right)dt^2$$

from this, and since the inequality (21) will be fulfilled here, since $r_0 > 3\alpha / 2$, *the transformation of the mass point to rest is a* ***proper*** *space-time transformation in the neighborhood of the path of the orbiting mass point.*

On the other hand, the upper limit of $1/\sqrt{3}$ for the velocity of an orbiting mass point that was found above in (51) also has a simple interpretation. Namely, from the axiom for the motion of light, it will be represented by a null geodetic line. If we set $A = 0$ in (41) then that will yield the equation:

$$r^2\left(\frac{d\varphi}{dp}\right)^2 - \frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2 = 0$$

for the orbiting light motion, in place of the inequality (49). Together with (47), it will then follow from this that the radius of the light path should be:

$$r = \frac{3\alpha}{2},$$

and the upper limit of the velocity of the orbiting light that appears in (51) should have the value:

$$v = \frac{1}{\sqrt{3}}\,.$$

In general, since $A = 0$, we will obtain the following differential equation for the light path from (46):

$$\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1}{B^2} - \rho^2 + \alpha\rho^3; \tag{52}$$

for $B = \frac{3\sqrt{3}}{2}\alpha$, it will possess the circle $r = 3\alpha / 2$ as its Poincaré "cycle," which will correspond to the situation in which $\rho - \frac{2}{3\alpha}$ appears on the right as a double factor. In fact, in that case, the differential equation (52) [corresponding statements will be true for the general equation (46)] will possess infinitely many integral curves that approach any circle in spirals, which is what Poincaré's general theory of cycles would demand.

If we consider a light ray that comes in from infinity, and we take α to be small in comparison to the shortest distance from the light ray to the center of gravitation then the light ray will approach the form of a hyperbola with a focus at that center. That will yield the deflection that a light ray experiences by a gravitational center; namely, it will be equal to $2\alpha / B$.

A counterpart to motion in a circle is motion along a line that goes through the center of gravitation. We obtain the differential equation for this motion when we set $\varphi = 0$ in (44) and then eliminate p from (43) and (44); the differential equation for r as a function of t will then read:

$$\frac{d^2r}{dt^2} - \frac{3\alpha}{2r(r-\alpha)}\left(\frac{dr}{dt}\right)^2 + \frac{\alpha(r-\alpha)}{2r^3} = 0, \tag{53}$$

with the integral that follows from (41):

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r-\alpha}{r}\right)^2 + A\left(\frac{r-\alpha}{r}\right)^3. \tag{54}$$

From (53), the acceleration proves to be negative or positive; i.e., the gravitation will be attractive or repulsive according to whether the absolute value of the velocity satisfies:

$$\left|\frac{dr}{dt}\right| < \frac{1}{\sqrt{3}}\frac{r-\alpha}{r}$$

or

$$> \frac{1}{\sqrt{3}}\frac{r-\alpha}{r},$$

resp.

Due to (54), for light, one will have:

$$\left|\frac{dr}{dt}\right| = \frac{r-\alpha}{r};$$

light that is directed rectilinearly to the center will always be repelled, in agreement with the last inequality; its velocity will increase from 0 at $r = a$ to 1 at $r = \infty$.

If α, like dr / dt is small then (53) will go to the Newtonian equation:

$$\frac{d^2 r}{dt^2} = -\frac{\alpha}{2}\frac{1}{r^2},$$

approximately.

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