# The existence conditions for the generalized kinetic potential 

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Helmholtz introduced the concept of the kinetic potential in his treatise "Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung" (*), and at the conclusion of the first part of it, he gave the characteristic relations that must exist for the forces in order for them to possess a kinetic potential. L. Koenigsberger suggested a proof that those relations were, in fact, characteristic of such a force system (Helmholtz himself did not provide such a thing) in a recently-appearing paper $\left({ }^{* *}\right)$ in which the principles of mechanics were generalized by establishing an extension of the concept of a kinetic potential. Finally, A. Mayer produced a direct proof of the validity of Helmholtz's assertion ( ${ }^{* * *}$ ).

Now, in what follows the question of the conditions under which a system offorces will possess a kinetic potential in the sense of Koenigsberger's extension will be addressed in full generality, and at the same time, Helmholtz's result, which is so meaningful from the physical standpoint, will experience a comprehensive formulation that might perhaps seem suitable for making the interrelationships and reciprocity behavior in a force system that depends upon a kinetic potential emerge especially clearly.

In terms of its mathematical content, the question that was posed overlaps essentially with the problem of investigating the structural behavior of the differential equations that appear in a class of problems in the calculus of variations. One understands $y_{1}, y_{2}, \ldots, y_{n}$ to mean undetermined functions of $x$ that will enter, along with their derivatives with respect to $x$ of to orders $r_{1}, r_{2}, \ldots$, $r_{n}$, resp., as arguments in a function:

$$
f\left(x ; y_{1}, y_{1}^{\prime}, \ldots, y_{1}^{\left(r_{1}\right)} ; y_{2}, y_{2}^{\prime}, \ldots, y_{2}^{\left(r_{2}\right)} ; \cdots ; y_{n}, y_{n}^{\prime}, \ldots, y_{n}^{\left(r_{n}\right)}\right),
$$

and one deals with the problem of determining the quantities $y_{1}, y_{2}, \ldots, y_{n}$ as functions of $x$ in such a way that the first variation of the integral:

[^0]$$
J=\int_{x_{0}}^{x_{1}} f \cdot d x
$$
vanishes. To that end, we give the quantities $y_{1}, y_{2}, \ldots, y_{n}$ the increments $u_{1}, u_{2}, \ldots, u_{n}$, resp., which represent arbitrary functions of $x$, with the one restriction that the derivatives:
$$
u_{i}, u_{i}^{\prime}, u_{i}^{\prime \prime}, \ldots, u_{i}^{\left(r_{i}-1\right)} \quad(i=1,2, \ldots, n)
$$
should be zero at the fixed limits of the integral $x=x_{0}$ and $x=x_{1}$. If one denotes:
\[

$$
\begin{align*}
\delta_{\kappa} f & \equiv \sum_{\lambda=0}^{r_{\kappa}} \frac{\partial f}{\partial y_{\kappa}^{(\lambda)}} u_{\kappa}^{(\lambda)}  \tag{1}\\
\delta f & \equiv \sum_{\kappa=1}^{n} \delta_{\kappa} f \tag{2}
\end{align*}
$$
\]

then one must set:

$$
\begin{equation*}
\delta J=\delta \int_{x_{0}}^{x_{1}} f \cdot d x=\int_{x_{0}}^{x_{1}} \delta f \cdot d x=\int_{x_{0}}^{x_{1}} \sum_{\kappa=1}^{n} \delta_{\kappa} f \cdot d x=0 \tag{3}
\end{equation*}
$$

If we appeal to the abbreviated notation:

$$
\Phi\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right) \sim \Psi\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)
$$

in order to express the idea that the difference of the functions $\Phi$ and $\Psi$ can be expressed by an exact differential quotient:

$$
\frac{d}{d x} X\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)
$$

for undetermined functions $y_{1}, y_{2}, \ldots, y_{n}$ then that will give:

$$
\begin{equation*}
\delta_{\kappa} f \equiv \sum_{\lambda=0}^{r_{\kappa}} \frac{\partial f}{\partial y_{\kappa}^{(\lambda)}} u_{\kappa}^{(\lambda)} \sim u_{\kappa} \cdot F_{\kappa}\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right) \tag{4}
\end{equation*}
$$

by repeated partial conversions, in which $F_{\kappa}$ represents the differential expression:

$$
\begin{gather*}
F_{\kappa}\left(x ; y_{1}, y_{1}^{\prime}, \ldots y_{1}^{\left(r_{1}+r_{\kappa}\right)} ; \ldots, y_{n}, y_{n}^{\prime}, \ldots, y_{1}^{\left(r_{n}+r_{\kappa}\right)}\right) \equiv \sum_{\lambda=0}^{r_{\kappa}}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{\kappa}^{(\lambda)}}\right)  \tag{5}\\
(\kappa=1,2, \ldots, n),
\end{gather*}
$$

which includes the derivatives of $y_{i}$ up to order at most $\left(r_{i}+r_{k}\right)$. If we recall our assumption in regard to the vanishing of the $u_{i}$ at the limits of the integrals then equation (3) will go to the following one by means of (4):

$$
\delta J=\int_{x_{0}}^{x_{1}} \sum_{\kappa=1}^{n} u_{\kappa} \cdot F_{\kappa} \cdot d x=0
$$

and due to the arbitrariness in the increments $u_{\kappa}$, it has the equations:

$$
F_{\kappa}\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)=0 \quad(\kappa=1,2, \ldots, n)
$$

as a consequence, which are the differential equations of the problem.
If we interpret $x$ as the time coordinate, the $n$ quantities $y_{1}, y_{2}, \ldots, y_{n}$ as general coordinates of a moving system of material points, and further regard the function $f\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)$ as a generalized kinetic potential then the functions $F_{\kappa}$ that are defined by (5) will be the Lagrange expressions for the motive forces that act upon the system.

Our problem shall now be to characterize the system of functions $F_{1}, F_{2}, \ldots, F_{n}$ that appear here exhaustively, in other words, to look for the conditions that $n$ given functions:

$$
F_{\kappa}\left(x ; y_{1}, y_{1}^{\prime}, \ldots y_{1}^{\left(r_{1}+r_{\kappa}\right)} ; \ldots, y_{n}, y_{n}^{\prime}, \ldots, y_{1}^{\left(r_{n}+r_{\kappa}\right)}\right) \quad(\kappa=1,2, \ldots, n)
$$

must fulfill in order for there to exist a function:

$$
f\left(x ; y_{1}, y_{1}^{\prime}, \ldots, y^{\left(r_{1}\right)} ; y_{n}, y_{n}^{\prime}, \ldots, y_{n}^{\left(r_{n}\right)}\right)
$$

by means of which the $F_{\kappa}$ can be represented in the form:

$$
F_{\kappa}=\sum_{\lambda=0}^{r_{\kappa}}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{\kappa}^{(\lambda)}}\right) \quad(\kappa=1,2, \ldots, n) .
$$

We have already treated the aforementioned problem for the special case of $n=1$ in a paper "Ueber eine charakterische Eigenschaft der Differentialgleichungen der Variationsrechnung" ("), whose methods and results will also be employed in what follows and which shall be cited as V.

Before we turn to the topic itself, in § 1, we shall preface some remarks about systems of linear differential equations that are adjoint to each other. In § 2, we will then derive a certain property of the potential forces that will prove to characteristic of them in § 3. Finally, § $\mathbf{4}$ will include the proof that the property of a system of functions that we speak of is essentially invariant under general point transformations.

[^1]
## § 1.

Let:

$$
P(u) \equiv \sum_{\lambda=0}^{r} p_{\lambda}(x) \frac{d^{\lambda} u}{d x^{\lambda}}
$$

be a linear homogeneous differential expression, so the linear differential expression that is adjoint to it:

$$
P^{\prime}(u) \equiv \sum_{\lambda=0}^{r}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left\{p_{\lambda}(x) \cdot u\right\}
$$

is known to be characterized completely by the property that:

$$
v \cdot P(u) \sim u \cdot P^{\prime}(v) .
$$

We now let the symbols $P_{i \kappa}(u)$ denote $n^{2}$ linear homogeneous differential expressions:

$$
P_{i \kappa}(u) \equiv \sum_{\lambda=0}^{r_{i \kappa}} p_{i \kappa \lambda}(x) u^{(\lambda)} \quad(i, \kappa=1,2, \ldots, n),
$$

to abbreviate, from which we compose the system of $n$ linear homogeneous differential equations in the $n$ arguments $u_{1}, u_{2}, \ldots, u_{n}$, which depend upon $x$ :

$$
\begin{equation*}
\sum_{\kappa=1}^{n} P_{i \kappa}\left(u_{\kappa}\right)=0 \quad(i=1,2, \ldots, n) \tag{6}
\end{equation*}
$$

A system of multipliers $v_{1}, v_{2}, \ldots, v_{n}$, which are likewise thought of as functions of $x$, shall be determined for this system in such a way that the sum of the left-hand sides of equations (6), once one has multiplied them by $v_{1}, v_{2}, \ldots, v_{n}$, in succession, will become an exact differential quotient, so one will have:

$$
\begin{equation*}
\sum_{i=1}^{n} \sum_{\kappa=1}^{n} v_{i} P_{i \kappa}\left(u_{\kappa}\right) \sim 0 . \tag{7}
\end{equation*}
$$

Since the linear differential expression $P_{i \kappa}^{\prime}(u)$ that is adjoint to $P_{i \kappa}(u)$ fulfills the relation:

$$
v P_{i \kappa}(u) \sim u P_{i \kappa}^{\prime}(v),
$$

the condition (7) will go to the following one:

$$
\sum_{\kappa=1}^{n} u_{\kappa} \sum_{i=1}^{n} P_{i \kappa}^{\prime}\left(v_{i}\right) \sim 0
$$

and due to the independence of the $u_{\kappa}$, that can be satisfied only if:

$$
\sum_{i=1}^{n} P_{i \kappa}^{\prime}\left(v_{i}\right)=0 \quad(\kappa=1,2, \ldots, n)
$$

The multipliers $v_{1}, v_{2}, \ldots, v_{n}$ must then satisfy the system of $n$ linear homogeneous differential equations:

$$
\begin{equation*}
\sum_{\kappa=1}^{n} P_{\kappa i}^{\prime}\left(v_{\kappa}\right)=0 \quad(i=1,2, \ldots, n), \tag{8}
\end{equation*}
$$

which we call the system adjoint to (6). Clearly, a reciprocity exists between those two systems of differential equations, such that the system (6) will be adjoint to the system (8). Moreover, since the expression that is adjoint to the differential expression $P(u)+Q(u)$ is given by $P^{\prime}(u)+Q^{\prime}(u)$ , the system that is adjoint to the system:

$$
\sum_{\kappa=1}^{n} P_{i \kappa}\left(u_{\kappa}\right)+\sum_{\kappa=1}^{n} Q_{i \kappa}\left(u_{\kappa}\right)=0 \quad(i=1,2, \ldots, n)
$$

will be represented in the form:

$$
\sum_{\kappa=1}^{n} P_{\kappa i}^{\prime}\left(u_{\kappa}\right)+\sum_{\kappa=1}^{n} Q_{\kappa i}^{\prime}\left(u_{\kappa}\right)=0 \quad(i=1,2, \ldots, n) .
$$

In what follows, we shall be interested in the special case in which the left-hand sides of the adjoint systems coincide as soon as one identifies the symbols $u_{\kappa}$ and $v_{\kappa}$, which will then happen when one has:

$$
P_{\kappa i}^{\prime}(u) \equiv P_{i \kappa}(u),
$$

so when the differential expressions $P_{i \kappa}(u)$ and $P_{\kappa i}(u)$ are adjoint to each other, and the $P_{i i}(u)$ are self-adjoint. In that case, we say that the system of differential expressions:

$$
\sum_{\kappa=1}^{n} P_{i \kappa}\left(u_{\kappa}\right) \quad(i=1,2, \ldots, n)
$$

is self-adjoint. It is characterized by the relations:

$$
\begin{equation*}
v \cdot P_{i \kappa}(u) \sim u \cdot P_{i \kappa}(v) \quad(i, \kappa=1,2, \ldots, n), \tag{9}
\end{equation*}
$$

and one must then have $r_{i \kappa}=r_{\kappa i}$, and $r_{i i}$ must be an even number.

If the systems:

$$
\sum_{\kappa} P_{i \kappa}\left(u_{\kappa}\right) \quad \text { and } \quad \sum_{\kappa} Q_{i \kappa}\left(u_{\kappa}\right)
$$

are self-adjoint then the system:

$$
\sum_{\kappa} P_{i \kappa}\left(u_{\kappa}\right)+\sum_{\kappa} Q_{i \kappa}\left(u_{\kappa}\right)
$$

will likewise be self-adjoint.
If we consider, for example, the system of first-order linear differential equations:

$$
\begin{equation*}
\sum_{\kappa=1}^{n} p_{i \kappa}(x) \cdot u_{\kappa}-u_{i}^{\prime}=0 \quad(i=1,2, \ldots, n) \tag{1}
\end{equation*}
$$

then the system adjoint to it will read:

$$
\begin{equation*}
\sum_{\kappa=1}^{n} p_{\kappa i}(x) \cdot v_{\kappa}+v_{i}^{\prime}=0 \quad(i=1,2, \ldots, n) \tag{2}
\end{equation*}
$$

Both systems of equations will coincide when:

$$
p_{i \kappa}(x) \equiv-p_{\kappa i}(x), \quad p_{i i}(x) \equiv 0,
$$

but without the system of differential expressions on the left-hand side of $\left(6_{1}\right)$ having to be selfadjoint in the sense that we defined, since the left-hand side in ( 81 ) possesses the opposite sign.

In order to obtain a system of first-order linear differential expressions that is self-adjoint, we assume that the number of variables is even and divide them into two groups: $x_{1}, x_{2}, \ldots, x_{n}, y_{1}, y_{2}$, $\ldots, y_{n}$. The two systems:
(62)

$$
\left\{\begin{array}{l}
\sum_{\kappa} p_{i \kappa}(t) \cdot x_{\kappa}+\sum_{\kappa} q_{i \kappa}(t) \cdot y_{\kappa}+\frac{d y_{i}}{d t}, \\
\sum_{\kappa} r_{i \kappa}(t) \cdot x_{\kappa}+\sum_{\kappa} s_{i \kappa}(t) \cdot y_{\kappa}-\frac{d y_{i}}{d t},
\end{array} \quad(i, \kappa=1,2, \ldots, n)\right.
$$

and

$$
\left\{\begin{array}{l}
\sum_{\kappa} p_{\kappa i}(t) \cdot \xi_{\kappa}+\sum_{\kappa} r_{\kappa i}(t) \cdot \eta_{\kappa}+\frac{d \eta_{i}}{d t},  \tag{2}\\
\sum_{\kappa} q_{\kappa i}(t) \cdot \xi_{\kappa}+\sum_{\kappa} s_{\kappa i}(t) \cdot \eta_{\kappa}-\frac{d \xi_{i}}{d t},
\end{array} \quad(i, \kappa=1,2, \ldots, n)\right.
$$

will then be adjoint to each other, as one easily convinces oneself, and the system ( $6_{2}$ ) will then be self-adjoint when:

$$
p_{i \kappa}(t) \equiv p_{\kappa i}(t), \quad q_{i \kappa}(t) \equiv r_{\kappa i}(t), \quad s_{i \kappa}(t) \equiv s_{\kappa i}(t)
$$

Finally, we use a system of second-order linear differential expressions as our basis:
(63)

$$
\sum_{\kappa=1}^{n} p_{i \kappa}(x) \cdot u_{\kappa}-u_{i}^{\prime \prime}=0 \quad(i=1,2, \ldots, n)
$$

that is adjoint to:

$$
\begin{equation*}
\sum_{\kappa=1}^{n} p_{\kappa i}(x) \cdot v_{\kappa}-v_{i}^{\prime \prime}=0 \quad(i=1,2, \ldots, n) \tag{3}
\end{equation*}
$$

so it will be self-adjoint when we have:

$$
p_{i \kappa}(x) \equiv p_{\kappa i}(x) .
$$

## § 2.

Now let:

$$
f\left(x ; y_{1}, y_{1}^{\prime}, \ldots, y_{1}^{\left(r_{1}\right)} ; y_{2}, y_{2}^{\prime}, \ldots, y_{2}^{\left(r_{2}\right)} ; \cdots ; y_{n}, y_{n}^{\prime}, \ldots, y_{n}^{\left(r_{n}\right)}\right)
$$

be a kinetic potential, so the forces derived from them:

$$
\begin{equation*}
F_{i}=\sum_{\lambda=0}^{r_{i}}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{i}^{(\lambda)}}\right) \quad(i=1,2, \ldots, n) \tag{5}
\end{equation*}
$$

are defined by the relations (4):

$$
\begin{equation*}
\delta_{i} f \sim u_{i} \cdot F_{i} . \tag{4}
\end{equation*}
$$

If we apply the $\delta_{\kappa}$-process to the latter then when we recall that the operations $d$ and $\delta_{\kappa}$ commute, that will give:

$$
\delta_{\kappa}\left(\delta_{i} f\right) \sim u_{i} \cdot \delta_{\kappa}\left(F_{i}\right) .
$$

We likewise have:

$$
\delta_{i}\left(\delta_{\kappa} f\right) \sim u_{\kappa} \cdot \delta_{i}\left(F_{\kappa}\right),
$$

and since:

$$
\delta_{i}\left(\delta_{\kappa} f\right)=\delta_{\kappa}\left(\delta_{i} f\right),
$$

it will follow that:

$$
\begin{equation*}
u_{i} \cdot \delta_{\kappa}\left(F_{i}\right) \sim u_{\kappa} \cdot \delta_{i}\left(F_{K}\right) . \tag{10}
\end{equation*}
$$

If we further let $\bar{\delta}_{i} f$ denote the differentiation process:

$$
\bar{\delta}_{i} f \equiv \sum_{\lambda} \frac{\partial f}{\partial y_{i}^{(\lambda)}}\left(\bar{u}_{i}\right)^{(\lambda)}
$$

then applying it to (4) will yield:

$$
\bar{\delta}_{i}\left(\delta_{i} f\right) \sim u_{i} \cdot \bar{\delta}_{i}\left(F_{i}\right) .
$$

One likewise has:

$$
\delta_{i}\left(\bar{\delta}_{i} f\right) \sim \bar{u}_{i} \cdot \delta_{i}\left(F_{i}\right),
$$

so:

$$
\begin{equation*}
u_{i} \cdot \bar{\delta}_{i}\left(F_{i}\right) \sim \bar{u}_{i} \cdot \delta_{i}\left(F_{i}\right) . \tag{11}
\end{equation*}
$$

In order to emphasize the dependency of the expressions $\delta_{\kappa} F_{i}$ on the $u_{\kappa}^{(\lambda)}$, we now introduce the symbol $P_{i \kappa}\left(u_{\kappa}\right)$ for $\delta_{\kappa} F_{i}$, with the use of which the relations (10) and (11) can be written as follows:

$$
\begin{gather*}
u_{i} \cdot P_{i \kappa}\left(u_{\kappa}\right) \sim u_{\kappa} \cdot P_{\kappa l}\left(u_{i}\right),  \tag{10'}\\
u_{i} \cdot P_{i i}\left(\bar{u}_{i}\right) \sim \bar{u}_{i} \cdot P_{i i}\left(u_{i}\right) .
\end{gather*}
$$

When we combine them into one equation:

$$
v \cdot P_{i \kappa}(u) \sim u \cdot P_{\kappa l}(v) \quad(i, \kappa=1,2, \ldots, n)
$$

we will see that based upon (9), the system of differential expressions:

$$
\sum_{\kappa=1}^{n} P_{i \kappa}\left(u_{\kappa}\right) \equiv \sum_{\kappa=1}^{n} \delta_{\kappa} F_{i} \equiv \delta F_{i} \quad(i=1,2, \ldots, n)
$$

is self-adjoint. With that, we have arrived at the result:
I. - The functions $F_{i}$ that are defined by (5) have the property that the system of linear differential expressions $\delta F_{i}$ that is derived from them is self-adjoint.

## § 3.

It shall now be proved that the theorem that was just established can be inverted in the following way:
II. - Let $F_{1}, F_{2}, \ldots, F_{n}$ be $n$ functions of $x$ and the derivatives of the functions $y_{1}, y_{2}, \ldots, y_{n}$, and indeed let $F_{i}$ include the derivatives of $y_{\kappa}$ up to order at most $\left(r_{i}+r_{\kappa}\right)$. Furthermore, the linear differential expressions that are derived from the $F_{i}$ :

$$
\delta F_{i} \equiv \sum_{\kappa} \sum_{\lambda} \frac{\partial F}{\partial y_{k}^{(\lambda)}} u_{\kappa}^{(\lambda)}
$$

might define a self-adjoint system. One can then construct a function $f\left(x ; y_{1}, y_{1}^{\prime}, \ldots, y_{1}^{\left(r_{1}\right)}\right.$; $\left.y_{n}, y_{n}^{\prime}, \ldots, y_{n}^{\left(r_{n}\right)}\right)$ with the help of quadratures by means of which the functions $F_{i}$ can be represented in the form:

$$
F_{i}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{i}^{(\lambda)}}\right) .
$$

That theorem is correct in the case $n=1$, as was proved in V , and its validity will be assumed for $(n-1)$ variables $y$. We must then show that it is also true for $n$ variables, and as a result, in general. To that end, we proceed as follows: When we first consider the way that the function $F_{1}$ depends upon the derivatives of $y_{1}$, which it includes up to order $2 r_{1}$ or one less than that, but even order in any event, we emphasize that from our assumption, $\delta_{1} F_{1}$ is a self-adjoint linear differential expression. However, it will follow from V that one can determine a function:

$$
\varphi\left(x ; y_{1}, y_{1}^{\prime}, \ldots, y_{1}^{\left(r_{1}\right)} ; y_{2}, y_{2}^{\prime}, \ldots\right)
$$

by quadratures that includes the derivatives of $y_{1}$ up to order $r_{1}$ and the derivatives of $y_{2}, y_{3}, \ldots, y_{n}$ up to any order, and by means of which $F_{1}$ can be represented in the form:

$$
F_{1}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial \varphi}{\partial y_{1}^{(\lambda)}}\right) .
$$

In general, the proof of that result that was given in V does not consider the appearance of parameters $y_{2}, y_{3}, \ldots, y_{n}$, along with the function $y_{1}$. However, one easily convinces oneself that no modification of the argument or result is required by adding them. If we now set:

$$
\Phi_{i}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial \varphi}{\partial y_{i}^{(\lambda)}}\right),
$$

to abbreviate, then according to § 2, the linear differential expressions $\delta \Phi_{i}$ will define a selfadjoint system.

We further denote the differences $F_{i}-\Phi_{i}$ by $\Psi_{i}$, in which $\Psi_{1}$ is then equal to zero identically, and then put $F_{i}$ into the form:

$$
F_{i}=\Phi_{i}+\Psi_{i} .
$$

By assumption, the linear system:

$$
\delta F_{i}=\delta \Phi_{i}+\delta \Psi_{i}
$$

is self-adjoint, so the same thing will be true of the system $\delta \Phi_{i}$, from what was just said. Therefore, based upon a remark in § 1, that will imply that the system $\delta \Psi_{i}$ must also be self-adjoint. In particular, the differential expressions $\delta_{1} \Psi_{i}$ and $\delta_{i} \Psi_{1}$ must then be adjoint to each other, so since $\Psi_{1}$, as well as $\delta_{i} \Psi_{1}$ is identically zero:

$$
\delta_{1} \Psi_{i} \equiv \sum_{\lambda} \frac{\partial \Psi_{i}}{\partial y_{1}^{(\lambda)}} u_{\kappa}^{(\lambda)}
$$

must vanish identically, so one must have:

$$
\frac{\partial \Psi_{i}}{\partial y_{1}^{(\lambda)}} \equiv 0
$$

That is: the functions $\Psi_{2}, \Psi_{3}, \ldots, \Psi_{n}$ no longer include the function $y_{1}$ and its derivatives at all. Thus, we now have $(n-1)$ functions $\Psi_{2}, \Psi_{3}, \ldots, \Psi_{n}$ that depend upon $x$ and the derivatives of the $(n-1)$ variables $y_{2}, y_{3}, \ldots, y_{n}$ in such a way that the system:

$$
\delta \Psi_{i} \quad(i=2,3, \ldots, n)
$$

is self-adjoint. By our assumption, there will then exist a function:

$$
\psi\left(x ; y_{2}, y_{2}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)
$$

that can be calculated by quadratures, and by means of which one can put $\Psi_{i}$ into the form:

$$
\Psi_{i}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial \psi}{\partial y_{i}^{(\lambda)}}\right) \quad(i=2,3, \ldots, n)
$$

If we then set:

$$
\varphi+\psi=f
$$

then since $\psi$ is free of $y_{1}$, that will give:

$$
\begin{equation*}
F_{i}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{i}^{(\lambda)}}\right) \quad(i=1,2, \ldots, n) \tag{12}
\end{equation*}
$$

The function $f$ that is found in that way includes the derivatives of $y_{1}$ up to order $r_{1}$, but will generally include the derivatives of $y_{i}$ for $i>1$ to an order higher than $r_{i}$. Thus, it still remains to be shown that $f$ can be replaced with another function in which the derivatives of $y_{i}$ occur only to the given degree. From a known theorem in the calculus of variations, the set of all functions $g$ that make it possible to represent the $F_{i}$ as in (12) is coupled with $f$ by the relation:

$$
g \sim f
$$

As a result, one can replace $f$ with any function $g$ that differs from it by an exact differential quotient. Now, the derivatives of $y_{1}, y_{2}, \ldots, y_{n-1}$ up to orders $r_{1}, r_{2}, \ldots, r_{n-1}$, resp., might enter into
$f$, while those of the $y_{\kappa}$ up to order $\left(r_{\kappa}+1\right)$ might occur. [Should the order in question be greater than $\left(r_{\kappa}+1\right)$, then one could assign a correspondingly higher value to the symbol $r_{\kappa}$.] One will then find that:

$$
F_{\kappa}\left(\cdots y_{\kappa}^{\left(2 r_{\kappa}\right)} \cdots\right)=\sum_{\lambda=0}^{r_{k}+1}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{\kappa}^{(\lambda)}}\right)=(-1)^{r_{\kappa}+1} \frac{\partial^{2} f}{\left(\partial y_{\kappa}^{\left(r_{k}+1\right)}\right)^{2}} y_{\kappa}^{\left(2 r_{k}+2\right)}+\cdots,
$$

so since the left-hand side does not include $y_{\kappa}^{\left(2 r_{\kappa}+2\right)}$, one must have:

$$
\frac{\partial^{2} f}{\left(\partial y_{\kappa}^{\left(r_{\kappa}+1\right)}\right)^{2}} \equiv 0,
$$

and $f$ will include $y_{\kappa}^{\left(r_{\kappa}+1\right)}$ only linearly in the form:

$$
f=\varphi \cdot y_{\kappa}^{\left(v_{\kappa}+1\right)}+\psi .
$$

Furthermore, one has:

$$
\begin{gathered}
F_{i}\left(\cdots y_{\kappa}^{\left(r_{i}+r_{\kappa}\right)}, \cdots\right)=\sum_{\lambda=0}^{r_{i}}(-1)^{\lambda} \frac{d^{\lambda}}{d x^{\lambda}}\left(\frac{\partial f}{\partial y_{i}^{(\lambda)}}\right)=(-1)^{r_{i}} \frac{\partial^{2} f}{\partial y_{i}^{\left(r_{i}\right)} \cdot \partial y_{\kappa}^{\left(\left(r k_{\kappa}+1\right)\right.}} y_{\kappa}^{\left(r_{i}+r_{\kappa}+2\right)}+\cdots \\
(i=1,2, \ldots,(\kappa-1)) .
\end{gathered}
$$

As a result, one also has:

$$
\frac{\partial^{2} f}{\partial y_{i}^{\left(r_{i}\right)} \cdot \partial y_{\kappa}^{\left(r_{k}+1\right)}}=\frac{\partial \varphi}{\partial y_{i}^{\left(r_{i}\right)}} \equiv 0 .
$$

That is: $\varphi$ includes the derivatives of $y_{1}, y_{2}, \ldots, y_{k-1}$ only up to orders:

$$
\left(r_{1}-1\right), \quad\left(r_{2}-1\right), \quad \ldots, \quad\left(r_{\kappa-1}-1\right) ; r_{\kappa} .
$$

If we now set:

$$
\int \varphi \cdot d y_{\kappa}^{\left(r_{\kappa}\right)}=U
$$

then

$$
\begin{gathered}
\frac{\partial U}{\partial y_{\kappa}^{\left(r_{\kappa}\right)}}=\varphi, \\
\frac{d U}{d x}=\varphi \cdot y_{\kappa}^{\left(r_{i}+r_{\kappa}\right)}+V
\end{gathered}
$$

in which $V$, just like $\psi$, includes the derivatives of $y_{1}, y_{2}, \ldots, y_{\kappa}$ up to orders $r_{1}, r_{2}, \ldots, r_{\kappa}$, resp. Finally, if we denote the difference $\psi-V$ by $\bar{f}$ then it will follow that:

$$
f=\frac{d U}{d x}+\bar{f}
$$

or

$$
f \sim \bar{f}
$$

If we then replace $f$ with $\bar{f}$ then the order of the derivatives of $y_{\kappa}$ will be lowered by one. When we continue that reduction process as far as is necessary, we will obviously always get a function $f$ that corresponds to the aforementioned requirement. After that, we can formulate the result of our investigation in the following theorem:
III. - The forces that possess a kinetic potential are characterized completely by the fact that the variations that they experience under a virtual displacement of the material system define a self-adjoint system of linear differential expressions.

## § 4.

To conclude these considerations, the following theorem shall be proved:
IV. - Suppose that one has a system of $n$ differential equations in $n$ variables:

$$
F_{i}\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right)=0 \quad(i=1,2, \ldots, n)
$$

whose left-hand sides have the property that the variations $\delta F_{i}$ define a self-adjoint system. If one performs a point transformation on it then one will get an equivalent system of differential equations whose left-hand sides enjoy the stated property.

In order to prove that theorem, we remark that on the basis of our assumption, the $F_{i}$ can be represented in terms of a function $f$ in the form that was given in (12). If, for the moment, we interpret the symbols $F_{i}$ in the system of equations (12) as arbitrary constants then we can combine them into one equation:

$$
\begin{equation*}
\delta J=0, \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
J=\int_{x_{0}}^{x_{1}}\left\{f-\sum_{i=1}^{n} F_{i} \cdot y_{i}\right\} d x \tag{14}
\end{equation*}
$$

That integral shall now be transformed by the substitution:

$$
\left\{\begin{array}{rl}
x & =\varphi\left(\xi, \eta_{1}, \eta_{2}, \ldots, \eta_{n}\right), \\
y_{i} & =\psi_{i}\left(\xi, \eta_{1}, \eta_{2}, \ldots, \eta_{n}\right),
\end{array} \quad(i=1,2, \ldots, n),\right.
$$

in which $\xi$ is regarded as the argument, $\eta_{1}, \eta_{2}, \ldots, \eta_{n}$ are regarded as functions of $\xi$, and their functional determinant:

$$
\Delta=\frac{d\left(x, y_{1}, y_{2}, \ldots, y_{n}\right)}{d\left(\xi, \eta_{1}, \eta_{2}, \ldots, \eta_{n}\right)}
$$

is assumed to be non-zero. If we imagine that the derivatives of the $y$ are expressed in terms of $\xi$ and the derivatives of the $\eta$, and we set:

$$
\begin{equation*}
f\left(x ; y_{1}, y_{1}^{\prime}, \ldots ; y_{n}, y_{n}^{\prime}, \ldots\right) \cdot \frac{d \varphi}{d \xi}=g\left(\xi ; \eta_{1}, \eta_{1}^{\prime}, \ldots, \eta_{n}, \eta_{n}^{\prime}, \ldots\right) \tag{15}
\end{equation*}
$$

then we will have:

$$
\begin{equation*}
J=\int_{\xi_{0}}^{\xi}\left\{g-\sum_{i=1}^{n} F_{i} \cdot \psi_{i} \cdot \frac{d \varphi}{d \xi}\right\} d \xi . \tag{16}
\end{equation*}
$$

If we further introduce the notation:

$$
\begin{equation*}
G_{\kappa}=\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d \xi^{\lambda}}\left(\frac{\partial g}{\partial \eta_{\kappa}^{(\lambda)}}\right) \quad(\kappa=1,2, \ldots, n) \tag{17}
\end{equation*}
$$

then when the demand (13) is applied to (16), that will imply the equations:

$$
\begin{equation*}
G_{\kappa}-\sum_{i=1}^{n} F_{i}\left\{\frac{\partial}{\partial \eta_{\kappa}}\left(\psi \cdot \frac{d \varphi}{d \xi}\right)-\frac{d}{d \xi}\left[\psi_{i} \cdot \frac{\partial}{\partial \eta_{\kappa}}\left(\frac{d \varphi}{d \xi}\right)\right]\right\}=0 \quad(\kappa=1,2, \ldots, n) . \tag{18}
\end{equation*}
$$

Now since:

$$
\frac{\partial}{\partial \eta_{\kappa}}\left(\frac{d \varphi}{d \xi}\right)=\frac{d}{d \xi}\left(\frac{\partial \varphi}{\partial \eta_{\kappa}}\right)
$$

and

$$
\frac{\partial}{\partial \eta_{\kappa}^{\prime}}\left(\frac{d \varphi}{d \xi}\right)=\frac{\partial \varphi}{\partial \eta_{\kappa}},
$$

it will follow that:

$$
\frac{\partial}{\partial \eta_{\kappa}}\left(\psi \cdot \frac{d \varphi}{d \xi}\right)-\frac{d}{d \xi}\left[\psi_{i} \cdot \frac{\partial}{\partial \eta_{\kappa}}\left(\frac{d \varphi}{d \xi}\right)\right]=\frac{d \varphi}{d \xi} \cdot \frac{\partial \psi_{i}}{\partial \eta_{\kappa}}-\frac{\partial \varphi}{\partial \eta_{\kappa}} \cdot \frac{d \psi_{i}}{d \xi},
$$

and we will then obtain the following transformation formula for the functions $F_{i}$ that were characterized in (12) from (18):

$$
\begin{equation*}
G_{\kappa}=\sum_{i=1}^{n} F_{i}\left\{\frac{d \varphi}{d \xi} \cdot \frac{\partial \psi_{i}}{\partial \eta_{\kappa}}-\frac{\partial \varphi}{\partial \eta_{\kappa}} \cdot \frac{d \psi_{i}}{d \xi}\right\} \quad(\kappa=1,2, \ldots, n) . \tag{19}
\end{equation*}
$$

Since the determinant of the system (19):

$$
\left|\left(\frac{d \varphi}{d \xi} \cdot \frac{\partial \psi_{i}}{\partial \eta_{\kappa}}-\frac{\partial \varphi}{\partial \eta_{\kappa}} \cdot \frac{d \psi_{i}}{d \xi}\right)\right|=\left(\frac{d \varphi}{d \xi}\right)^{n-1} \cdot \Delta \quad(i, \kappa=1,2, \ldots, n)
$$

is non-zero then, the system of transformed equations $F_{i}=0$ will be equivalent to the system of equations $G_{\kappa}=0$, whose left-hand sides will possess the property that we speak of in any event, based upon the representation of $G_{\kappa}$ in (17). The invariance of that property under point transformations is then proved.

If one chooses $x=\xi$, in particular, then (19) will imply the transformation formula:

$$
\begin{equation*}
\sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d \xi^{\lambda}}\left(\frac{\partial f}{\partial \eta_{\kappa}^{(\lambda)}}\right)=\sum_{i} \frac{\partial y_{i}}{\partial \eta_{\kappa}} \cdot \sum_{\lambda}(-1)^{\lambda} \frac{d^{\lambda}}{d \xi^{\lambda}}\left(\frac{\partial f}{\partial \eta_{\kappa}^{(\lambda)}}\right), \tag{20}
\end{equation*}
$$

which is derived in a different way by Koenigsberger (*).
The system of linear differential equations $\delta F_{i}=0$ plays an essential role in our considerations, which appears to be coupled with every system of differential equations $F_{i}=0$. Let it then be remarked that Darboux ( ${ }^{* *}$ ) and Poincaré ( ${ }^{* * *}$ ) have referred to the significance of that connection for the general theory of differential equations.

Zurich, May 1897.

[^2]
[^0]:    (*) Journal für die reine und angewandte Mathematik, Bd. 100.
    (**) "Ueber die Principien der Mechanik," Sitz. Kgl. Preuss. Akad. Wiss. Berlin, (30 July 1896), 932-935.
    $\left(^{* * *}\right)$ "Die Existenzbedingungen eines kinetischen Potentiales," Ber. der math.-phys. Classe der Kgl. Sächs. Ges. Wiss. Leipzig (7 December 1896).

[^1]:    (*) Mathematische Annalen, Bd. 49.

[^2]:    (*) Loc. cit., page 901.
    (**) Théorie générale des surfaces, IV partie, note XI.
    (***) Les méthodes Nouvelles de la Mécanique celeste, t. I, Chap. IV.

