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# Historical overview of the mathematical theory of discontinuity waves since Riemann and Christoffel

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We give a brief historical account of the development of the mathematical theory of the propagation of discontinuities in gases, fluids, or elastic materials. The theory was initiated by Riemann, who investigated the propagation of shocks in one-dimensional isentropic gas flow. Riemann's methods were used by Christoffel to treat, more generally, the propagation of (first order) discontinuity surfaces in three-dimensional flows of perfect fluids. Subsequently, Christoffel applied his general theory to first order waves in certain elastic materials. Independently of Riemann and Christoffel, significant contributions were made by Hugoniot. The theory was completed in Hadamard's celebrated monograph [31] where, among many other things, acceleration waves in hyperelastic bodies were treated correctly. Later, Prandtl, A. Busemann, et al., attached the problem of discontinuous flow from the more practical point of view of the engineer and obtained many important results. In the final section of our report, we briefly survey some recent global weak existence theorems for Riemann and general Cauchy initial value problems of general strictly hyperbolic conservation laws.

## 1. The pioneering work of Riemann, Christoffel, Hugoniot, and Hadamard

B. Riemann [63] was one of the first to treat the spreading of discontinuity waves in gases. Along with his effort, one must point out the almost simultaneous work of Earnshaw [27], which generally did not go as far as in the Riemann treatise. Riemann investigated the system of partial differential equations in the Eulerian variables *x*, *t:*

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \tag{$C$}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{$E$}$$

for an unsteady one-dimensional gas flow, in which $u$ is the flow velocity, $\rho$ is the density, and $p$ is the pressure of the flowing gas. The pressure $p$ will also be assumed to be a function $\phi$ of only the density, such that one has:

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$$\phi'(\rho) := \frac{\partial \phi}{\partial \rho}(\rho) > 0.$$

With $a^2 := \phi'(\rho)$ and the function $v := \int_{\rho_0}^{\rho} a\, d \log \rho$, that is conjugate to $u$, along with $r := (u + v)/2$, $s := (u - v)/2$, Riemann obtained from ($C$), ($E$) the hyperbolic system (in characteristic form):

$$r_1 + (u + a)\, r_s = 0, \qquad s_1 + (u - a)\, s_s = 0.$$

In [34], I was interested in finding implicit notions from Riemann's differential geometry and function theory in his derivation of these differential equations. The resulting differential equations of one-dimensional unsteady gas flow are then precisely the (hyperbolic) Cauchy-Riemann differential equations:

$$(C - R) \qquad \frac{du}{dl} = \frac{dv}{dn},$$

with respect to the indefinite Riemannian metric $ds^2 = (a^2 - u^2)\, dt^2 + 2u\, dt\, dx - dx^2$ with two arbitrary directions $l$, $n$ that are orthogonal with respect to $ds^2$ and have opposite norms. Each of the two null directions of the metric (i.e., characteristics = Mach waves) $dx$: $dt = u \pm a$ is perpendicular to itself, and therefore the Cauchy-Riemann equation ($C - R$) has:

$$r = \frac{1}{2}(u + v) = \text{const.}, \qquad s = \frac{1}{2}(u - v) = \text{const.},$$

as integrals. Riemann introduced these "invariants" $r$, $s$ as new independent variables in place of $x$, $t$ (this is possible as long as one has $\partial(x, t)/\partial(r, s) = 2r_x\, s_x \not\equiv 0$), and thus obtained a system of linear equations, for which he then developed his well-known method of integration.

One can easily bring Riemann's considerations into agreement with the curvilinear coordinates on the solution surface $z = \psi(x, y)$ of the flow function that H. Lewy introduced in the theory of characteristics, and which always remain useful, at least in the small; they are simply the parameters $\alpha$, $\beta$ of the intersection point of the two characteristics that go through a point with the starting curve, which becomes $\alpha = \beta$. The novelty is Lewy's difference method, which gives the solution of the differential equations that emerge from the characteristic equations and the proof that for vanishing mesh width it converges to the solution of the initial value problem of the hyperbolic differential equation. I will not go into Lewy's difference method, which is important in both theory and practice, nor will I go into the first order system with several desired functions that was treated by Friedrichs and Lewy. Such systems occur immediately in gas dynamics when one must determine the entropy variable $s$, perhaps, in response to a condensation jump (one must then have that $s$ is known from its initial value). As characteristics, one then has not only the condensation waves, but the streamlines. Further initial and boundary-value problems were treated by Beckert [3], [4]. As a result, two more auxiliary functions were introduced.

We return from this excursion to Riemann's work. Under the assumption that $\phi(\rho) = C^2\rho^\kappa$ with a constant $C \in \mathbf{R}$, Riemann proved that a solution of the Cauchy initial value problem for the above hyperbolic system also includes a discontinuity in general for continuous initial values, from which a shock wave ("condensation jump") emerges. With the help of a simple physical principle (conservation of impulse for a volume element of the gas that goes through the discontinuity), Riemann obtained the velocity with which the discontinuity expanded through the gas, and further obtained a relation between the values of the density and the velocity of the gas on both sides of the condensation jump. These results enabled Riemann to prove existence of a solution of the initial value problem for the above hyperbolic system ($C$), ($E$) with the initial values:

$$(u, \rho)(x, 0) = \begin{cases} (u_1, \rho_1) & \text{for } x < 0, \\ (u_2, \rho_2) & \text{for } x > 0, \end{cases}$$

with constants $u_1$, $u_2$, $\rho_1$, $\rho_2$ , and to describe its structure precisely.

The equation of state that Riemann assumed $p = C^2\rho^\kappa$, with a pre-established constant $C$, then leads to the fact that in some particular cases of motion through a gas a condensation jump can introduce an energy loss. Rayleigh [62] criticized the work of Riemann on this basis. The contradiction with the theorem of the conservation of energy that Rayleigh spoke of as following as a consequence of Riemann's work can – as he showed – be satisfactorily removed by the thermodynamic considerations of Hugoniot. Hugoniot demanded that the quantity $C$ in the state equation not have different values on the two sides of the condensation jump. The (specific) entropy $s$ of the gas is then no longer constant across the condensation jump, as it was for Riemann, whereas the theorem of energy conservation remains valid along the path lines; therefore, on a condensation jump there will be, e.g., a conversion of kinetic energy into heat. Riemann did not arrive at such consequences.

Christoffel recognized the significance of Riemann's treatise [64] and spoke of it during (1859/60) in the "Fortschritten der Physik." In the first work [12], Christoffel then investigated, with the help of the methods that Riemann applied, the three-dimensional spreading of discontinuity surfaces in ideal fluids (these considerations are also valid for the motion of a discontinuity through a gas at rest).

If $\Sigma(X, t)$ is a surface that expands in space ($X$ = spatial coordinate, $t$ = time), on which a density $\rho$, pressure $p$, or flow velocity $\dot{x}$ of the fluid is discontinuous (a so-called first-order discontinuity) then Christoffel next deduced from the continuity of the flow on the discontinuity surface $\Sigma(X, t)$, the jump relation $[\rho U] = 0$, with $U := \dot{x}_n - u_n$ , in which $u_n$ means the velocity in the normal direction to the advancing surface, while $\dot{x}_n$ is the component of the flow velocity of the fluid in the normal direction; the square brackets denote – since Christoffel onward – the difference of the "left" and "right" limiting values of $\rho U$ on the discontinuity surface. Under the assumption that one is dealing with an ideal fluid (i.e., no shear stresses or stresses due to force couples appear), Christoffel immediately obtained on $\Sigma(X, t)$ the further relation $[p]\boldsymbol{n} + \rho^\pm U^\pm \dot{x} = 0$, where $\boldsymbol{n}$ is the normal vector and $\rho^+$, $\rho^-$, $U^+$, $U^-$ mean the limiting values of $\rho$ ($U$, resp.) on $\Sigma(X, t)$ when one approaches the surface in the direction of the "positive" ("negative," resp.) normal.

Christoffel derived these "impulse equations" (or also, the "dynamical compatibility equations") immediately from the Riemann model, where such equations were proved for one-dimensional gas flow. As a corollary to the equations, one immediately finds, for instance, that the pressure $p$ must be continuous on $\Sigma(X, t)$ when $\Sigma(X, t)$ is not a condensation jump (i.e., when $[\dot{x}] \cdot \boldsymbol{n} = 0$ is true), and that in ideal fluids jump waves are always longitudinal, moreover (when $\rho^{\pm} U^{\pm} \neq 0$).

Christoffel (deviating from Riemann) now linearized the density $\rho$ and the pressure $p$, for which he demanded that $\rho = \rho_0 + \rho_0 S$, $p = p_0 + a^2 S$, with a variable $S$ and constants $p_0$, $\rho_0$, and substituted this into the existing relations. Along with that, he demanded that the initial state of the fluid be one of rest. From this, Christoffel then deduced that a fluid cell that overtakes $\Sigma(X, t)$ includes a jump that results in the direction of the normal to $\Sigma(X, t)$ at the point $(X, t)$ where the cell is found at the time point of the jump. Christoffel further concluded that each tangent plane to $(X, t)$ moves in the direction of its normal with the constant velocity $\pm\, a$, and $\Sigma(X, t)$ is an orthogonal trajectory of a fixed system of normals for all $t > 0$, which is determined from the position of $\Sigma(X, 0)$. Finally, Christoffel determined the jump $[S] = S^{+} - S^{-}$ from the linearized basic hydrodynamical equations that were valid on both sides of $\Sigma(X, t)$. Therefore, he did not need to solve these equations, but he used what he called "phoronomic" discontinuity conditions on $\Sigma(X, t)$, which he derived for the first partial derivatives of an arbitrary function that was differentiable on both sides of the surface $\Sigma$, but whose first derivative was discontinuous on $\Sigma$ itself. Christoffel's phornomic discontinuity conditions essentially corresponded to compatibility conditions (kinematical compatibility conditions of first order) that Hugoniot and Hadamard introduced, but they are more unwieldy. The thought of proposing such conditions had already been found only implicitly in the work of Riemann, so its discovery was Christoffel's own achievement. With his phoronomic conditions, Christoffel obtained an expression for the normal derivative $(d/d\boldsymbol{n})[S]$, and from this, he concluded that for an "infinitely thin" bundle of normals on $\Sigma(X, t)$ with a certain surface element $\Delta\Sigma(X, t)$ the product $[S]\sqrt{\Delta\Sigma(X, t)}$ is constant as long as $\Delta\Sigma(X, t)$ propagates along the normal bundle. In the last part of the treatise [12], Christoffel treated bounded fluids and showed that a discontinuity surface will be reflected by an encounter with a boundary that is impenetrable for the fluid, and that the surface that results from the reflection with the boundary is likewise a discontinuity surface whose normals are created by reflection of the normals to the original discontinuity surface on the boundary (with the usual law of reflection).

The results that Christoffel proved are also so beautiful and interesting that one must not overlook the fact that his conclusions are, above all, quite particular to the consequent linearization. Correspondingly, Hadamard [31], no. 69, pp. 82, et seq., expressed the opinion that "…he (i.e., Christoffel) is limited to very exceptional waves, the shock waves (waves of first order) whose existence was discovered by Riemann, and, moreover, since the study of these waves presents special difficulties, he is forced to consider only a limiting case, the one where the discontinuities are infinitely small… (Hugoniot) brought to light a fundamental notion: that of compatibility…whose necessity seems not to have dawned on Christoffel, as was pointed out by Riemann in the case of rectilinear motion…" Hadamard therefore seemed not to appreciate Christoffel's phoronomic discontinuity conditions, which indeed – as we remarked above – are compatibility conditions for discontinuities of order 1 (perhaps it has to do with the peculiar name that Christoffel gave to them).

In his second treatise [13], Christoffel examined the first order discontinuities in the theory of elasticity, where linearization is commonplace. In another place, I showed that the general theory of second order discontinuities in the nonlinear mechanics of continua, which I presented after a lecture of Herglotz [33], also delivers the corresponding results of Christoffel, but without his "symbolic decompositions." This gives the foundation (if one holds it to even be necessary) for Hadamard's assessment ([31], no. 262, pp. 262, pp. 244, et seq.): "…such waves (i.e., of the first order) have been studied by Christoffel. Thanks to the hypothesis that the motions are infinitely small, this savant obtained results that were, moreover, identical, at their basis, to the ones that furnished the study of acceleration waves." What seems significant to me, above all, is the divergence relation that Christoffel found in [13] between the ray vector and the norm of the jump vector; it represents something that was indeed recognized by Herglotz, but not in the literature, and gives the physical interpretation of the rays as the shadow boundary.

Whereas Riemann's aforementioned work was very well-known, this did not apply to Christoffel's treatises. Merely Christoffel's treatment of the discontinuous motion of a string that is defined by an arc was mentioned in Helmholtz's famous "Lehre von den Tonempfindungen" (see also Riemann-Weber [64]).

Independently of Riemann and Christoffel, Hugoniot [39], [40], [41] treated the spreading of discontinuity waves in gases. Above all, Hugoniot examined the propagation of second order discontinuities in a moving gas in space (so-called acceleration waves). Then he proved, among other things, that for the propagation velocity $a$ of a non-stationary wave (second order) relative to a moving gas one has the formula $a = (\partial p/\partial\rho)^{1/2}$ ($p$ = pressure, $\rho$ = density) for a general three-dimensional barotropic gas flow; this was previously known only for the linearized theory.

Hugoniot further treated the phenomenon that Hadamard named after him, and even discussed in his "Cours d'Analyse II" [30]. From a constant state, hence, from a planar piece of the flow surface $z = \psi(x, t)$, one may – as Hugoniot showed – connect up with a destination that lies in one of the characteristics. It is determined in, say, a particular case, by the (possibly also jerky) motion of a boundary (e.g., a piston). These consequences relate to the case that was already considered by Riemann of a one-dimensional unsteady gas motion. One then has a Riemann invariant, perhaps $s = (u - v)/2 = s_0$ = const., for the flow surface $z = \psi(x, y)$, $y = t$ (cf., *supra*), a first order partial differential that depends only on $\psi_x$, $\psi_y$, an "integral" of the second order partial differential equation for $\psi(x, y)$. The solutions of the first order differential equations are the desired destinations, and this gives rise to the Hugoniot phenomenon. Riemann dealt with it on a cone whose tangent planes are, e.g., the so-called rarefaction waves.

From the outset, however, Hugoniot first founded gas dynamics on thermodynamics, so as a consequence he required the conservation of energy along each pathline, which (cf., *supra*) contradicts Riemann's ("static") adiabatic law that goes through the (first order) compression jump. The jump itself converts mechanical energy into heat, which yields an increase in the entropy. Already, Rankine [61] had seen this in 1867, thus twenty years before Hugoniot, without his making note of that fact. Also, on thermodynamic grounds the spreading of (theoretically conceivable) rarefaction jumps through a gas is not possible, since when a rarefaction jump goes through a volume of gas the entropy of the gas must go down, which contradicts the second law of thermodynamics. Hugoniot proved, moreover, a noteworthy, simple, purely thermodynamic relation, which is also called a "dynamical" state equation," between the pressure $p$ and density $\rho$ (the specific volume $\bar{v} = \rho^{-1}$, resp.) of the gas, the so-called Rankine-Hugoniot curve:

$$H(\bar{v}, p) = e(\bar{v}, p) - e(\bar{v}_0, p_0) + \frac{1}{2}(\bar{v} - \bar{v}_0)(p + p_0) = 0,$$

in which the constant values $\bar{v}_0$, $p_0$ relate to a fixed initial state and $e$ means the specific internal energy of the gas. This curve also governs detonation.

Hadamard gave a summary and clear exposition of the results discovered by Riemann, Christoffel, and Hugoniot in his famous book [31], but his work also gave a precise classification of waves by their order (in Lagrange variables), the representation of the theory of characteristics of differential equations with several independent variables, and several functions that had been sought after (by Beudon), the bicharacteristics and, above all, existence theorems (by using Cauchy-Kowalewsky or even by using developments in fractional exponents at the edge of regression). In addition, Hadamard contributed numerous important particular results relative to the mathematical theory of the spreading of discontinuity waves. As far as that is concerned, here we will only treat acceleration waves in hyperelastic media, generalizing the results and suggestions of Christoffel and Hugoniot. An acceleration wave in an elastic body is well-known to be a discontinuity surface that spreads through the medium in such a manner that the deformation gradient and the velocity vector $\boldsymbol{u}$ of a point (of the medium) ranges continuously over this surface, while the acceleration vector $\dot{\boldsymbol{u}}$ of this point experiences a finite jump $[\dot{\boldsymbol{u}}]$ when the wave passes through the point. One obtains a spreading condition of the form:

$$\det (Q(\boldsymbol{n}) - \rho U^2 E) = 0,$$

where $\rho$ is the density of the material and $U$ means the ("intrinsic") velocity of the wave front relative to the material; $\boldsymbol{n}$ is the normal to the wave front, $Q(\boldsymbol{n})$ is a non-singular $3\times3$ matrix (the so-called acoustic tensor) and $E$ is the $3\times3$ identity matrix. In general, the acoustic tensor depends upon the deformation gradient, the material, and the direction of propagation $\boldsymbol{n}$ of the acceleration wave at arbitrary points of the elastic medium. The possible spreading velocities of the wave for a given $\boldsymbol{n}$ are given by assuming that $\rho U^2$ must be a real eigenvalue of the acoustic tensor; the direction of the jump in the acceleration vector $[\dot{\boldsymbol{u}}]$ will be given by the corresponding (normalized) eigenvector of $Q(\boldsymbol{n})$. For the spreading of the acceleration waves in hyperelastic media, in which the acoustic tensor is well-known to be symmetric, Hadamard proved the existence of three orthogonal directions for each $\boldsymbol{n}$ in which the discontinuity of the acceleration can spread. Later, in [70] Truesdell represented this fact more generally with remarkable clarity and precision. One implicitly finds such results already in Christoffel's treatise [13], but they are generally not very clearly formulated. The isotropic case was treated by Hugoniot in [39], who recognized the significance of this case.

Duhem continued Hadamard's treatment of elasticity in [24], [25], [26]. The conclusions of Hadamard's do not take into account, e.g., the thermal conductivity, of elastic materials. Duhem [24], [26] examined materials in which the temperature and the entropy vary. He found that the foregoing statements about acceleration waves are also true for thermally non-conductive substances, and for substances that obey the Fourier law of thermal conductivity for a positive-definite thermal conductivity tensor. In the latter case – as Duhem showed – the acceleration wave is isothermal; i.e., the temperature gradient ranges continuously over the acceleration wave. By contrast, for thermally non-conductive substances the acceleration wave is isentropic; i.e., there is no jump in the entropy gradients across it. Truesdell proved this once more in [70, § 13], and in [71], pp. 59-79] he gave a representation of the corresponding theory of acceleration waves that was founded by Coleman-Gurtin [14], [15] in terms of the thermo-mechanical theory of thermally conductive – so-called simple – substances that was quite general.

## 2. Mathematical contributions to the problem of discontinuous gas motion in the first third of the Twentieth Century after Hadamard.

In Germany, shortly after Hadamard's aforementioned book, a progress report appeared in 1905 on the theory of the spreading of discontinuity waves in gases. This was the encyclopedia article [77] that was due to Zemplén. In his previous work, Zemplén, acting on a suggestion of Hilbert, had derived equations of motion and compatibility conditions for an arbitrary elastic medium with the help of Hamilton's variational principle.

In the pause of more than twenty-five years that then followed, as Cabannes documented in his recent Handbuch article [8], there were only mathematical inverstigations, especially Vessiot [72], [73], [74], and Kotchine [43], and the work of my Leipzig teacher Herglotz and Lichtenstein. At the onset of relativity theory, Herglotz [32] likewise treated the mechanics of continua with these acceleration waves. Later, in Göttingen, he presented his beautiful lectures on the "Mechanik der Kontinua" in general, whose publication had been so wished for. In a special lecture that is required after the middle part of this discussion (?), I will, in another place, give a sketch of the treatment of acceleration waves in the manner that Christoffel treated first order discontinuity waves in the linear case (quadratic energy density).

One finds Lichtenstein's mathematical representation of the theory of discontinuity waves (in Euler variables), to which he devoted much trouble, in his "Hydrodynamik" [46]. Like Zemplén, he based this work on Hamilton's principle – but only for gases, in general. Further corollaries to the conditions he derived (such as existence theorems) were not deduced. Lichtenstein did not treat hyperbolic problems at all, any more than he discussed thermodynamics.

## 3. The practical contributions; above all, those of Prandtl and A. Busemann

Along with the encyclopedia article of Zemplén, in the same encyclopedia from the year 1905, one should not overlook the article "Technische Mechanik" [60] by Schröter and Prandtl. Gas dynamics in Göttingen began with (the strongly influenced by Felix

Klein) Prandtl and the Göttinger dissertation (1908) of Th. Meyer [54] that emerged from him (Meyer expansion waves, flow around a corner). Beautiful analytical jump formulas were also presented and later, graphical-numerical methods were developed (1929, Prandtl-Busemann).

The renaissance of more engineering-oriented investigations is marked by the articles on gas dynamics by A. Busemann [7] and J. Ackert [1]; in addition, there are the Atti of the fifth Convegno Volta (Rome, 1935) [16]. At that start of that era, important themes in practice were the high-speed aerodynamics of the leading wave of a projectile (circular cone) and its resistance, as well as corresponding investigations on biplanes. The Volta conference was perhaps the last time that the leading contributors of the various nations, including Prandtl, G. I. Taylor, von Kármán, Pistolesi, A. Busemann could peacefully exchange their secrets together.

It was ten years after the appearance of the cited article of A. Busemann when it was brought to the attention of H. Billharz, Hantzsche & Wendt, Guderley, A. von Baranoff, and myself at the Technische Hochschule Braunschweig by a lecture of Busemann, and it inspired me to pursue the variational principle for two-dimensional problems (two-dimensional in the stationary case and one-dimensional in the non-stationary case). I will never forget the peaceful, morning lecture hours – before the work began – when A. Busemann, after lengthy, troublesome calculations and miscalculations with the exponent $\kappa$; arrived at his actual geometric idea, and from the wooden models of his pressure jump, with a serious expression, he transferred a direction onto the enormous blackboard on rollers that served as the flow plane by means of a ruler.

These engineering-intuitive, graphical-numerical procedures have allowed me to recognize that Busemann's pressure jump is the (Blaschke) figuratrix of the variational problem of stationary plane gas dynamics – N.B., also for flows that are not (as in the variational principal of Bateman) potential flows, but flows with rotation, which must be described by the stream function $z = \psi(x, y)$. My variational principle is therefore not only another formulation of Bateman's principle, as Serrin [66, pp. 204, rem. 1] suggested. I have presented my variational principle in the first part of my (unpublished) Lilienthal-Arbeit [35], and later in the Mathematischen Nachrichten [36]. The starting point of the variational problem (also in the reaction gas dynamics of equilibrium) is obtained when one writes down the state equation in the form of a Mollier diagram pressure:

$$(A) \qquad \zeta = p = p(i, s),$$

with specific enthalpy $i$ and specific entropy $s$. With:

$$(B) \qquad \frac{1}{2}\eta^2 + i + \xi = 0, \qquad u = -\eta$$

($u$ = velocity) is already the figuratrix surface $\zeta = \bar{\varphi}(\xi, \eta)$ of the variational problem for the stream function $z = y(x, y)$, $y = t$ = time that follows from the equation of continuity ($C$) (cf., *supra*).

The Euler equation of the variational problem is the Euler equation ($E$) (cf., *supra*) of the one-dimensional non-stationary motion. For planar two-dimensional stationary gas flow, in place of ($B$) one must take the Bernoulli equation for the two velocities $u = -\eta$, $v$

= $\xi$, which follows from the equations of motion, and the Euler-Lagrange equation then becomes the rotation theorem. Analogous statements are true for stationary rotationally-symmetric flows, such as for the circulation in the dynamics of the atmosphere and for other wind fields of meteorology, and ultimately for rotationally-symmetric flows of a perfect plasma in magnetogasdynamics, cf. [37], [38].

For all of the importance that I ascribe to the pressure jump as the figuratrix of my two-dimensional variational problem, it is only the metric of a Cartanian geometry that is based on it. Luckily, only one part of the associated Euclidian connection comes under consideration: the absolute derivative of the unit normal vector, indeed, only the mean curvature, the trace $\Gamma_i^{3i} = \Gamma_{ij}^3 \bar{g}^{ij}$, if one would ultimately like to swear by the Christoffel symbols. The characteristic procedure can then be imagined to be an approximate construction of the extremal surface (= surface of vanishing mean Cartan curvature) through a triangular polyhedron. The triangle inside of it has two characteristic sides, and a kink is attached to the third side, whose "curvature" brings the mean curvature of the triangle to zero.

Something crucial that comes out of Cartan geometry is to consider the pressure jumps (figuratrices) that are associated with differing entropies $s$, $s_0$, $s > s_0$ . One can then represent the density jump across a surface element *xypq* of the flow surface, in which the edge of the kink itself is represented by two colliding surface elements, by the parallel ("double-") tangent that runs through the parallel tangent plane of the figuratrix $\mathcal{F}_{s_0}$, which meanwhile includes a second tangent plane to an ("internal") figuratrix $\mathcal{F}_s$ (with $s > s_0$).

The most important work to emerge from A. Busemann's Institut für Gasdynamik was G. Gunderley's investigations into characteristic procedures (cf., G. Guderley: Characteristikentheorie, Rep. & Transl. No. 113, Rep. & Mon. List 112). His procedure for the calculation of density jumps is also an important practical example of numerical calculation. Moreover, Guderley originated the treatment of "blast waves," which are spherical density jumps, by a difficult examination of a first order differential equation with many singularities. However, above all, one must mention the foundational book [29] that Guderley wrote later. In it, he made a first advance into the realm of partial differential equations of mixed type that was opened up by Tricomi, which should be of primary interest today, in my opinion. Such differential equations describe flows from subsonic to supersonic that again end in an often-undesired density jump into subsonic. A transonic flow of this type, where the supersonic domain breaks up into the subsonic one with a density jump, raises enormous mathematical problems. On this, one confers the book of Bers [5], where the characteristic theory is applied, in a large part, in connection with the hodograph method.

## 4. Brief outlook

Up till now, since I recounted the story of Riemann's ideas on gas dynamics and Christoffel's ideas on the general mechanics of continua, which I certainly did not exactly experience, but have only heard of and have now studied in connection with Christoffel's work, I must refer completely to the Handbuch articles of Cabannes [8] and R. E. Meyer

[53] for a presentation of recent progress in the theory and practice of discontinuity waves, where in the latter article along with infinitesimal density jumps, also strong discontinuity jumps are mentioned.

From the textbooks of gas dynamics, I have once again come to appreciate the outstanding and still-fashionable work of Courant-Friedrichs [17]. It preserves the Göttingen-Braunschweig tradition (Courant and Herglotz also took part in the Prandtl seminar on flow theory in 1929) and unites it with the intensive results and calculations of the American researchers; among them are H. Weyl, J. von Neumann, R. D. Richtmyer.

## 5. Recent work on the existence of global weak solutions of strictly hyperbolic systems of conservation laws

From the treatise of Riemann that was mentioned at the onset, emerged not only an important stimulus for the development of a mathematical theory of the spreading of discontinuity waves in gases, fluids, and elastic materials, but also, *inter alia*, the significance of the fact that in that treatment one found the complete solution of an initial-value problem (with piecewise constant initial values) for a strictly hyperbolic (nonlinear) system of partial differential equations that had the form of a conservation law (with one spatial variable). Almost one hundred years would elapse after the publication of Riemann's treatise before anyone was in a position to prove corresponding (global) existence theorems for such initial value problems for general hyperbolic systems of conservation laws.

In 1957 Lax [44] gave an existence proof for the Riemann initial value problem (under restricted assumptions on the initial values) for a general system:

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}f(u(x,t))=0, \qquad x\in\mathbf{R},\ t\in[0,\infty),$$

$$u=(u_1,\ldots,u_n), \qquad n\in\mathbf{N},\ n\geq 2,$$

with a smooth nonlinear map $f$: $D \to \mathbf{R}^n$ that is defined in an open set $D \subset \mathbf{R}^n$. The Jacobi matrix $d_n f$ possesses $n$ distinct, real eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$ (strict hyperbolicity) with right eigenvectors $r_k(u)$, $k =, \ldots, n$, $u \in D$. As for the eigenvalues, it will be assumed that they are "truly nonlinear" or "degenerate linear," i.e., one has $r_k(u)\cdot\nabla_u\lambda_k(u) \neq 0$ ($r_k(u)\cdot\nabla_u\lambda_k(u) \equiv 0$, resp.), $k = 1, \ldots, n$, $u \in D$. A system is called truly nonlinear (degenerate linear, resp.) when all of the eigenvalues $\lambda_k(u)$ are truly nonlinear (degenerate linear, resp.). Examples from mathematical physics of the systems that Lax treated are, for instance, the isotropic gas equations, the general equations of gas dynamics in relativistic and non-relativistic form, the Lundquist equations of magnetogasdynamics in Lagrangian coordinates, the equations for the finite amplitude of a plane elastic wave.

Even for $C^\infty$-smooth initial values, one cannot, from just that, deduce that a Cauchy initial value problem for a system $u_t + f(u)_x = 0$ (with nonlinear $f$) possesses a smooth or even just continuous global solution, since due to the nonlinearity of $f$ the eigenvalue $\lambda_k$ of the matrix $d_u f$ depends upon $u$.

Lax thus examined weak solutions of the initial value problem, i.e., measurable and restricted functions $u(x, t)$ with:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}[\Phi_t \cdot u + \Phi_x \cdot f(u)]dt\,dx + \int_{-\infty}^{\infty}\Phi(x,0)\,u(x,0)\,dx = 0$$

for all $\Phi \in C_0^\infty(\mathbf{R} \times [0, \infty), \mathbf{R}^n)$. If $t \to x(t) \in \mathbf{R}$, $t \geq 0$, is a discontinuous curve for a weak solution $u$ then one has the Rankine-Hugoniot relations:

$$(RH) \qquad \dot{x}(t)\,[u(x(t) + 0, t) - u(x(t) - 0, t)] = f(u(x(t) + 0, t)) - f(u(x(t) - 0, t))$$

for all points $(x(t), t)$ on the (smooth) discontinuity curve. When one additionally has for some $k$, $1 \leq k \leq n$:

$$(L) \qquad \lambda_{k-1}(u(x(t) - 0, t)) < \dot{x}(t) < \lambda_k(u(x(t) - 0, t))$$

and

$$\lambda_k(u(x(t) - 0, t)) < \dot{x}(t) < \lambda_{k+1}(u(x(t) + 0, t)),$$

Lax called the discontinuity curve a $k$-shock. Moreover, when $\lambda_k$ is degenerate linear so-called $k$-contact discontinuities appear, which are discontinuity curves for whose points $(x(t), t)$ one has $\lambda_k(u(x(t) - 0, t)) = \lambda_k(u(x(t) + 0, t)) = \dot{x}(t)$.

Under the assumption that the "initial states" $u_1$, $u_r \in \mathbf{R}^n$ are close to each other in the Euclidian norm, Lax proved for the Riemann initial value problem:

$$(R) \qquad u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} u_1 & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

that their exists a global – i.e., defined in $\mathbf{R} \times [0, \infty)$ – weak solution when the system of that sort is strictly hyperbolic and the eigenvalues of $d_u f$ are truly nonlinear or degenerate linear. The solution $u(x, t)$ that Lax constructed depends only upon $x/t$ and is piecewise continuously differentiable. It assumes constant values $u_k \in \mathbf{R}^n$, $k = 0, \ldots, n$, $u_0 := u_1$, $u_n := u_r$ in the sectors $S_k := \{(x, t) \mid a_k t \leq x \leq b_k t\}$, $a_k, b_k \in \mathbf{R}$, $k = 0, \ldots, n$. $S_{k-1}$ and $S_k$, $k \geq 1$ follow in sequence, i.e., $b_{k-1} \leq a_k$, and there exist the following possibilities:

1. $S_{k-1}$ and $S_k$ have a common boundary $x(t) = a_k\,t$, $a_k = b_{k-1} =$ const. Then the "intermediate states" $u_{k-1}$, $u_k$ will be separated by a $k$-shock, in the event that $r_k(u)\cdot\nabla_u\lambda_k \neq 0$, $u \in D$, and one has:

$$\lambda_k(u(a_k\,t - 0, t)) > a_k > \lambda_k(u(a_k\,t + 0, t)).$$

By contrast, when one has $r_k(u)\cdot\nabla_u\lambda_k = 0$, $u \in D$ the common boundary of $S_{k-1}$ and $S_k$ is a contact discontinuity with:

$$a_k = \lambda_k(u(a_k\,t - 0, t)) = \lambda_k(u(a_k\,t + 0, t)).$$

$\partial S_{k-1} \cap \partial S_k = \{(a_k\, t, t) \mid t \geq 0\}$ is therefore a discontinuity curve along which $u$ makes a jump.

2. $S_{k-1}$ and $S_k$ have only the origin 0 as common boundary point. Then for all of the points $(\xi t, t)$ of $S_k^* := \{(x, t) \mid b_{k-1}t \leq x \leq a_k\, t\}$ there exists the equation $\lambda_k(u(\xi t, t) = \xi$. The boundary lines of $S_k^*$ are characteristics and the $k$-Riemann invariants are constant on $S_{k-1} \cup S_k^* \cup S_k$. In this case, the intermediate states $u_{k-1}$, $u_k$ will be – as Lax said – linked by $k$-rarefaction wave centered on 0.

The Lax conclusions were absent from [44], but thoroughly presented in the very beautiful survey article [45].

For the proof that was given by Lax of the existence of a global weak solution of the Riemann problem ($R$) the assumption that initial states $u_1$, $u_r$ are close to each other in the Euclidian norm is crucial.

In the case of a truly nonlinear, strictly hyperbolic system of just two equations $u_t + f(u, v)_x = 0$, $v_t + g(u, v)_x = 0$, Smoller [68], [69] arrived at existence and uniqueness theorems for the Riemann initial value problem without having to make these very restricting assumptions on the initial values. Smoller proved (under the assumptions that he made for the system) that four globally defined curves emanate from each point $(u_1, v_1)$ of the $(u, v)$-plane, which subdivide the plane into four quadrants, such that (due to the global geometric properties of these curves) for all $(u_r, v_r)$ that lie in three of these quadrants the Riemann initial value problem for the given 2×2 system with the initial values:

$$(u, v)(x, 0) = \begin{cases} (u_1, v_1) & x < 0 \\ (u_r, v_r) & x > 0 \end{cases}$$

By contrast, the “correct” initial state $(u_r, v_r)$ lies in the fourth quadrant, such the Riemann problem – as Smoller showed – then possesses precisely one restricted weak solution when the map $(u, v) \to (r, s)$, where $r$, $s$ are the classical Riemann invariants of the system, maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. In [69], Smoller proved that the solution of the Riemann problem is unique in the class of constant states that are linked by means of shocks or rarefaction waves that are centered at 0.

In the cited works, Smoller made crucial use of the assumption that the strictly hyperbolic 2×2 system in question is truly nonlinear. For general 2×2 systems that do not satisfy this assumption of being truly nonlinear, Dafermos [18], [19] and Dafermos-DiPerna [20] constructed solutions for the Riemann problem with general initial values by means of the viscosity method, by which one first solves the Riemann initial value problem of the system perturbed by a viscosity term:

$$u_t + f(u, v)_x = \varepsilon t u_{xx}, \qquad v_t + g(u, v)_x = \varepsilon t v_{xx}, \quad \varepsilon > 0,$$

and finally show that it yields solutions to the Riemann problem for the system $u_t + f(u, v)_x = 0$, $v_t + g(u, v)_x = 0$ when one takes the limit as $\varepsilon \to 0^+$. The structure of the solution thus obtained is essentially more complicated than it would be for truly nonlinear systems and was examined by Dafermos [19] in detail.

Liu [47], [48] gave very far-reaching existence and uniqueness theorems for the Riemann problem for strictly hyperbolic 2×2 systems of conservation laws (in one spatial variable) that do not have to be truly nonlinear and found that these solutions satisfy an "entropy condition" ($E'$) that represents an immediate generalization of the famous "condition ($E$)" that Oleneik [59] presented for scalar equations. For truly nonlinear systems ($E'$) is equivalent to the Lax shock condition ($L$) (cf., *supra*).

Beyond that, Liu [49] proved, for the general 3×3 system of equations of gas dynamics in one spatial variable and in Euler coordinates:

$$(G) \qquad \rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (\rho u^2 + p)_x = 0, \qquad (\rho E)_t + (\rho u E + pu)_x = 0,$$

(with the density $\rho$, gas velocity $u$, total energy $E = u^2/2 + e$, $e$ = specific energy) with arbitrary initial data:

$$U(x, 0) = \begin{cases} U_1 & \text{for } x < 0 \\ U_r & \text{for } x > 0 \end{cases},$$

$$U_1 = (\rho_1, \rho_1 u_1, E_1), \qquad U_r = (\rho_r, \rho_r u_r, E_r) \in \mathbf{R}^3,$$

that belong to the class of constant states that are linked by shocks, rarefaction waves, or contact discontinuities, possesses a solution that satisfies the entropy condition ($E'$); this solution is unique. It is worthy of note that Liu did not need to assume the convexity condition $p_{vv}(v, s) > 0$ on the pressure $p = - e_v(v, s)$ ($v = 1/\rho$ = specific volume, $s$ = specific entropy). Already, Bethe had shown in 1942 that contact discontinuities must appear when $p_{vv}$ is not positive everywhere, and he suspected that in this case solutions would exist that had "stable" discontinuity lines in the class of constant states that are linked by shocks, rarefaction waves, and contact discontinuities. From Liu's conclusions it emerged that due to ($E'$) the characteristics of the system ($G$) point in the direction of the discontinuity lines or parallel to them, so the discontinuities are stable, which confirms Bethe's suspicion.

For the same system ($G$) of general gas dynamical equations, Wendroff likewise proved an existence theorem in [75], although under other assumptions and a restriction on the initial states $U_1$, $U_r$ .

Smith [67] gave an exhaustive treatment of the existence and uniqueness questions for the Riemann initial value problem of the general system ($G$). Under the assumptions $p_v < 0$, $p_{vv} > 0$, $p_s > 0$, together with a demand on the asymptotic behavior of $e$, he showed the existence of a solution to the initial value problem for arbitrary Riemann initial values. The solution is, moreover, not unique, even when it satisfies the Lax shock condition and an entropy condition. Smith proved that a necessary and sufficient conditions for the uniqueness of the solution is the inequality $pv(v, e) \leq p^2/2e(v, e > 0)$. The proof rests, *inter alia*, on a precise analysis of the Hugoniot curve (cf., *infra*) in the ($v$, $p$) plane and makes use of results of Weyl [76]. Smith also stated concrete Riemann initial data for which the Riemann problem for ($G$) possesses many (at least five) different solutions.

Along with the Riemann initial value problem with its piecewise constant initial data, the Cauchy initial value problem is naturally of interest. Glimm [28] proved a global existence theorem for this problem for general, strictly hyperbolic $n\times n$ systems of conservation laws under restricted assumptions about the initial values.

Glimm examined the Cauchy initial value problem:

$$(CP) \qquad u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x), x \in \mathbf{R}, t \geq 0$$

with given initial values $u_0$ ; $u$, $f$, $u_0$ are $\mathbf{R}^n$-valued functions, $n \geq 2$. The system shall be strictly hyperbolic and truly nonlinear, and the initial values $u_0$ are defined on $\mathbf{R}$ as restricted functions of restricted variations on $\mathbf{R}$. Furthermore, it will be assumed that for a fixed vector $v' \in \mathbf{R}^n$ the expression $\|u_0 - v'\|_{L^\infty(\mathbf{R}^n)^+}$ + T.V. $u_0$ (T. V. = total variation over $x$) is sufficiently small. Under these assumptions, Glimm proved the existence of a global weak solution of ($CP$) with a difference procedure.

First, an approximate solution was constructed with the help of the solutions of a certain Riemann problem. Let $h$, $k > 0$ be the lattice spacing along the $x$ ($t$, resp.) axis, $h/k$ = const., $h/k > \max_j \sup_j \{\lambda_j(u)\}$ (Friedrich-Lewy condition). The approximate solution $u_h$ will be inductively defined in strips that are parallel to the $x$ axis. Glimm next discretized the initial values through $u_h(x, 0) : = \{u_0(jh), j \in \mathbf{Z}, j \equiv 1\ (2), (j - 1)h \leq x \leq (j + 1)h\}$. At the points $x = i$, $t = 0$, $i \equiv 0\ (2)$ the Riemann problem:

$$u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} u_0((i-1)h) & \text{for} \quad (i-1)h \leq x < ih \\ u_0((i+1)h) & \text{for} \quad ih \leq x < (i+1)h \end{cases}$$

is solved by the Lax method; in this way, one obtains an exact solution $u_h(x, t)$ to $u_t + f(u)_x = 0$ in the strips $t < k$. Now, let $u_h(x, t)$ be already defined for $t < jk$, $j \in \mathbf{N}$. Then Glimm sets:

$$u_h(x, jk) : = \{u_h((i + 1 + aj)h, jk - 0), ih \leq x \leq (i + 2)h, i \;\; \mathbf{Z}, i + j \equiv 0\ (2)\},$$

where the $a_j$ are chosen arbitrary on (–1, 1). $u_h(x, jk)$ is a piecewise constant function of $x$. At the points ($ih$, $jk$) with $i + j \equiv 0\ (2)$, one solves the Riemann problem:

$$u_t + f(u)_x = 0, \quad u(x, jk) = \begin{cases} u_h(h + a_j h, jk) & (i-1)h \leq x < ih \\ u_h((i+1+a_j)h, jk) & ih \leq x < (i+1)h \end{cases}$$

in the manner of Lax; the solution is comprised of “elementary waves” that are centered on the points ($ih$, $jk$). In this way, $u_h(x, t)$ will be defined on the strips $t < (j + 1)k$, and an induction argument then gives an approximate solution $u_h$ that is defined in $\mathbf{R} \times [0, \infty)$ (when one observes certain additional estimates.

The crucial original idea of the proof of Glimm now consists in the fact that he defined certain nonlinear functionals one the approximate solutions that, in a certain sense, measure the interaction of the solutions to the Riemann problem (that appear in the construction of $u_h$) above with the aforementioned elementary waves. With the help of the assumption that the hyperbolic system of equations is truly nonlinear and the total variation of the initial values is small, Glimm obtained estimates of these nonlinear functionals, from which one finally obtaines the inequality T.V. $u_h(x, t) \leq$ const. $u_0(x)$ with constants that are independent of $h$, $t$ and $a_j$, $j \in \mathbf{N}$. A compactness argument that rests upon the theorem of Helly then gives the $L^1(\Omega)$-convergence, $\Omega \subset \mathbf{R}^2$, of $\{u_{h_i}, i \in \mathbf{N}\}$ for a null sequence $\{h_i, i \in \mathbf{N}\} \subset \mathbf{R}^+$. Whether the limiting function even represents a global weak solution of ($CP$) depends upon the choice of sequence $\{a_j , j \in \mathbf{N}\}$, $a_j \in (-1, 1)$; in this way, the Glimm procedure takes on a probabilistic character. If one sets $A = (-1, 1)^{\mathrm{N}}$ then Glimm can at least show that for all $a \in A - Z$ the sequence $\{u_{h_i}, i \in \mathbf{N}\}$ converges to a global weak solution of ($CP$), where $Z$ is a set of Lebesgue measure 0 (defined on $A$ by the product construction). In [51], Liu managed to show that for each evenly-spaced sequence $a \in A$ in the interval $(-1, 1)$ the Glimm difference procedure gives a global solution to ($CP$).

Glimm did not prove that the solutions obtained by his procedure satisfy an entropy condition, and are therefore "physically sensible." Therefore, Chorin [10], [11], with the help of a numerical procedure that was based on the Glimm method, obtained results for systems of gas dynamics and reaction gas dynamics that gave rise to the suspicion that the Glimm difference procedure leads to physically meaningful solutions.

A modification of the conclusions of Glimm also makes an existence theorem for inhomogeneous systems possible. Under the same assumptions about the initial values and with $f$ as in Glimm, Liu [52] proved, with a modified Glimm difference procedure for the inhomogeneous, strictly hyperbolic system:

$$u_t + f(u)_x = g(x, u),$$

the existence of a global weak solution of the Cauchy initial value problem, when the $L^1$-norm of $g(x, u)$, $g_u(x, u)$ is sufficiently small.

The assumptions that Glimm made about the initial values are very restricted; thus, many authors could prove existence theorems for weal global solutions to the Cauchy initial value problem for a large class of strictly hyperbolic 2×2 systems of conservation laws, without having to assume that the oscillation or the total variation of the initial values is small, cf., Bakhvarov [2], Chang T'ung-Kuo Yu-fa [9], DiPerna [21], [22], along with Nishida [57], Nishida-Smoller [58], Johnson-Smoller [42]. For systems with more than two equations there are no comparable existence theorems with the assumptions of Liu [51]. Liu treated the system of differential equations for one-dimensional gas flow of a polytropic gas in Lagrangian variables:

$$u_t + p_x = 0, \qquad v_t - u_x = 0, \qquad E_t + (pu)_x = 0,$$

$$p(v, s) = \text{const.} \exp((\gamma - 1)s / R)\, v^{-\gamma},\ \gamma \in (1, 5/3)$$

($u$ = velocity, $v$ = specific volume, $p$ = pressure, $s$ = entropy, $E$ = total energy). Under the assumption concerning the initial values $u_0$, $v_0$, $s_0$ that $(\gamma - 1)$ max{T.V. $u_0$, T.V. $v_0$, T.V. $s_0$} is sufficiently small, Liu can show the existence of a global weak solution. The assumption $\gamma \in (1, 5/3)$ guarantees the usual shock interaction, in particular, that during the penetration of two shocks in a characteristic family, there exist a shock from the same family, a contact discontinuity, and a rarefaction wave of the opposite characteristic family (cf., J. von Neumann [56]).

It is unknown whether similar existence theorems are valid for other strictly hyperbolic $n \times n$ systems $n \geq 3$. Likewise, the question of a criterion for the uniqueness of the global weak solution of hyperbolic systems of conservation laws is open; on the question of uniqueness, one can confer DiPerna [23].

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