"Die infinitesimalen Berührungstransformationen der Variationsrechnung," Jber. Deutsche Math.-Verein **49** (1939), 162-178.

The infinitesimal contact transformations of the variational calculus. [1])

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With 1 figure

1. In the following, I would like speak on the implications that the concept of a one-parameter group of contact transformations, as well as their infinitesimal transformations, has in the calculus of variations – and also for the multiple extremal integrals with many desired functions. For one-dimensional extremal integrals, the relation to the geometry of contact transformations – which is already implicit in Hamilton's [2]) optical works – are well-known, if they are, however, perhaps not always sufficiently discussed in the textbooks.

Lie [3]), without referring to Hamilton, has stated several times that the simplest example of a one-parameter group of contact transformations was given by the wave motions, and that the group property of all dilatations was intimately connected with Huygens's principle. In a similar way, the images of a surface under an *arbitrary one-parameter group of contact transformations* can be regarded as originating in a wave process in a permanent regime that satisfies Huygens's principle of ray optics. An initial wave surface Σ_0 , which after a time Θ becomes a certain wave surface $\Sigma = T_\Theta \Sigma_0$ (by means of a contact transformation) has, at the time $\Theta + \Theta'$, the position $T_{\Theta+\Theta'} \Sigma_0 = T_{\Theta'} \Sigma$, which originates from the new initial location Σ after the time Θ': $T_{\Theta+\Theta'} = T_\Theta T_{\Theta'}$; the time Θ is the canonical parameter.

The *partial differential equation* of first order for the wave process is obtained from the assumption that the infinitesimal contact transformation, by way of its Lie characteristic function, (essentially) gives the normal velocity of the wave for each direction of the wave normal at every point. If one goes a distance from the origin that is equal to the normal velocity at a certain point for variable normal direction, as well as the plane that it is normal to it, then this envelops a *point* structure: the *ray surface* at the point considered. From this, one obtains, by a similar reduction of 1 to $\delta\Theta$ in the time increment $\delta\Theta$, the "*elementary wave*" that is produced at each of the individual points of the surface elements and, as they vary, gives the envelope of the infinitesimally close

[1]) A somewhat extended version of a presentation that was given to the Baden-Baden meeting of the D. M. V. (Sept., 1938).

[2]) W. R. Hamilton, Third Supplement to an Essay on the Theory of Systems of Rays (1832). In particular, articles 2, 26, Math. Papers I, Cambridge 1931 (on this, cf., also the remarks of the eds. A. W. Conway and J. L. Synge, pp. XXI, 189), as well in the survey edited by G. Prange: Über W. R. Hamiltons Abhandlungen zur Strahlenoptik, Leipzig 1933, as well as the footnote on this, in particular, pp. 168, et seq. cf., Prange, Nova Acta (?) Acad. (?) (1923), No. 1. Enzykl. d. math. Wiss. IV, 12 and 13, No. 13.

[3]) Cf., Lie and Scheffers, Geometrie der Berührungstransformationen, Bd. I, Leipzig 1896, pp. 966. (?), as well as Lie, Die infinitesimalen Berührungstransformationen der Optik, Ges. Abh., Bd. 6, pp. 615-617.

wave surface. With this envelope construction (which is likewise also valid for finite contact transformations), one has outlined the scope of *Huygens's principle.*

By means of this wave picture, the notion of a one-parameter group of transformations resolves to a "particle picture." This double aspect represents, in Hamilton's theory, a bridge across the dualism of Huygens's wave theory and Newton's emission theory that led Hamilton to make the transition from applying his method to optics to applying it to mechanics, and which was the stimulus for Schrödinger [4]) a hundred years later that led up to the new physical synthesis of wave mechanics.

In this particle picture one focuses on the *paths* of the individual surface elements under the transformations of the group, which are the *rays*, optically speaking. They lead from the contact point of the elementary wave to the envelope and are given by certain *ordinary differential equations* whose right-hand side is derived from Lie's characteristic function of the infinitesimal contact transformation.

There now exists the fundamental connection that the paths of the group are, at the same time, *extremals* (minimals) of a variational problem – the one in which the indicatrix is given by the ray surface: The rays satisfy *Fermat's principle* of shortest time. Correspondingly, in mechanics the paths satisfy the principle of least action (in the Jacobi form) when the energy constant is fixed.

I would like to briefly derive this connection anew on the basis of the very penetrating examination of Vessiot [5]) (which is independent of the optical aspects), simply from Lie's notion of a one-parameter group of contact transformations. Thus, I will use the inhomogeneous formulation by singling out an axis, as opposed to the most commonly used homogeneous representation that is often suitable in the beginning – particularly, when one goes to the multi-dimensional variational calculus.

By singling out a *t*-axis, we thus consider transformations of a space of coordinates (t, x^i) that take the *surface element* (t, x^i, P^i) to another surface element, and that take an *n*-dimensional *union* of surface elements $dt + P_i\, dx_i = 0$ into another such union. The position coordinates P_i are thus $-\partial t / \partial x_i = P_i$.

We now treat a *one-parameter group* $\mathfrak{G}$ *of contact transformations:*

$$(1) \qquad t' = g(t, x_j, P_j, \Theta), \qquad x_i' = g_i(t, x_j, P_j, \Theta), \qquad P_i' = h_i(t, x_j, P_j, \Theta),$$

$$\frac{\partial(t', x_i', P_i')}{\partial(t, x_j, P_j)} \neq 0.$$

[4]) E. Schrödinger, Abhandlungen zur Wellenmechanik, 2nd ed., Leipzig 1928, pp. 489 et seq.

[5]) E. Vessiot. a) Sur l'interpretation des transformations de contact infinitésimales, Bull. Soc. math. de France **34** (1906), pp. 320-269. Vessiot also treated a time-varying medium, b) Essai sur la propagation par ondes, Annales de l'Éc. Normale sup. (3) **26** (1909), pp. 405-448. For the corresponding questions for the Lagrange problem, cf., Vessiot, c) Sur la théorie des multiplicités et le Calcul de Variations, Bull. Soc. math. de France **40** (1912), pp. 68 to 139; d) Sur la propagation par ondes et sur le problème de Mayer, Journal de Math. (6) **9** (1913), pp. 39-76.

Further representations are given for the case of an *n*-dimensional ray surface by T. Levi-Civita and U. Amaldi, Lezzioni di Meccanica razionale II, pp. 456-469 (Bologna 1927), L. P. Eisenhart, Continuous groups of tansfromations (Princeton 1933), p. 263-273 and G. Juvet, Mécanique analytique et mécanique ondulatoire, Mémorial Sci. Math. Fasc. **83** (Paris 1937).

This has the function $F(t, x_j, P_j) \neq 0$ as the *Lie characteristic function of the infinitesimal transformation;* it makes $F\ \delta\Theta$ the infinitesimal displacement of the surface element in the direction of the t-axis if Θ is the canonical parameter of the group. The paths of the group:

$$t = g(t^0, x_j^0, P_j^0, \Theta), \qquad x_i = g_i(t^0, x_j^0, P_j^0, \Theta), \qquad P_i = h_i(t^0, x_j^0, P_j^0, \Theta), \tag{2}$$

obey the differential relations [6]):

$$dt + P_i\, dx_i = F\, d\Theta + G_h\, dc_h, \qquad \frac{\partial G_h}{\partial \Theta} = F_t \cdot G_h, \tag{3}$$

in which c_h means an arbitrary parameter upon which the initial values t^0, x_j^0, P_j^0 depend; perhaps one can set $c_h = x_j^0$ and fix x_j^0 and P_j^0. Just like t, x_i, P_i, F and G_h then depend upon Θ, c_1, c_2, … One then has:

$$\frac{\partial F}{\partial \Theta} = F_t \cdot F. \tag{4}$$

Conversely, a $2n$-parameter family:

$$t = t(c_1, c_2, \ldots, c_{2n}; \Theta), \qquad x_i = x_i(c_1, c_2, \ldots c_{2n}; \Theta), \qquad P_i = P_i(c_1, c_2, \ldots c_{2n}; \Theta), \tag{5}$$

with:

$$\frac{\partial(x_i, P_i)}{\partial(c_1, \cdots, c_{2n})} \neq 0 \tag{6}$$

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[6]) From the system of differential equations for the paths:

$$\begin{cases} \dfrac{dt}{d\Theta} = & F - P_i F_{P_i} & = -\Phi \\ \dfrac{dx_i}{d\Theta} = & F_{P_i} & = \Pi_i \\ \dfrac{dP_i}{d\Theta} = & -F_{x_i} + P_i F_t & \end{cases} \tag{3a}$$

that are associated with the infinitesimal contact transformation and the canonical parameter Θ, it follows that there is agreement between the coefficients of $d\Theta$ on the left-hand and right-hand sides of the differential relation $(3)_1$, which thus defines the quantities G_h; in order to do this, one then calculates the derivative $(3)_2$.

Herglotz, in particular, treated the differential relations (3) in his seminar on continuum mechanics, Göttingen 1925/26. – There, one will also find the basic facts of ray optics derived from the second-order differential equations of continuum mechanics. He also treats the general case of variable regimes, which leads into the Mayer problem; cf., Vessiot, loc. cit. [5]) b) The specialization to permanent regimes produced the ordinary variational problem in homogeneous form. Heglotz has treated a one-parameter group of contact transformations in the plane in his seminar on differential equations, Göttingen Summer 1928, in which the paths were treated as extremals in a variational problem, and are denoted by the same independent variable x as the transversals in inhomogeneous form.

is characterized by the differential relation $(3)_1$, along with $(3)_2$, as the family of paths of a one-parameter group of contact transformations.

In order to go from the group of contact transformations to the associated family of canonical transformations, one writes:

$$\begin{aligned} \frac{P_i}{F} &= \pi_i \\ -\frac{1}{F} &= \varphi(t, x_j, \pi_j); \end{aligned} \tag{7}$$

from the first equations, under the assumption that $\Phi \neq 0$, the P_i may be represented as expressions in the new variables (impulses) π_i [7]), which will then be substituted in $-F^{-1}$. When one substitutes Θ for t by means of $(5)_1$ and substitutes in $(5)_{2,3}$, under the same assumption that $\Phi \neq 0$, formula (5) now gives the family:

$$x_i = \xi_i(c_1, \ldots, c_{2n}; t), \quad \pi_i = \eta_i(c_1, \ldots, c_{2n}; t), \tag{8}$$

with:

$$\frac{\partial(x_i, \pi_i)}{\partial(c_1, \cdots, c_{2n})} = \frac{\partial(x_i, \pi_i)}{\partial(x_j, P_j)} \frac{\partial(x_i, P_j)}{\partial(c_1, \cdots, c_{2n})} \neq 0, \tag{9}$$

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[7]) They are, in fact:

$$\frac{\partial \pi_i}{\partial P_j} = \frac{1}{F^2}(\delta_{ij} F - P_i \Pi_j), \quad \det\left(\frac{\partial \pi_i}{\partial P_j}\right) = \frac{F^{n-1}}{F^{2n}}(F - P_i \Pi_i) = -\frac{\Phi}{F^{n+1}}.$$

I then compute the differential:

$$d\varphi = \frac{1}{F^2}\left[F_t dt + F_{x_i} dx_i + F_{P_i}\left(Fd\frac{P_i}{F} - FP_i d\frac{1}{F}\right)\right],$$

i.e.:

$$-\Phi\, d\varphi = (F - P_i \Pi_i)\, d\varphi = \frac{F_t}{F^{2n}} dt + \frac{F_{x_i}}{F} dx_i + \Pi_i d\pi_i .$$

Combining this with the Legendre transformation (13) gives:

$$p_i = -\frac{\Pi_i}{\Phi}, \qquad f = \frac{1}{F} - \frac{\Pi_i}{\Phi}\frac{P_i}{F} = -\frac{1}{\Phi},$$

hence, the birational involutory contact transformation $(F, P_i, \Pi_i) \to (f, p_i, \pi_i)$:

$$f = -\frac{1}{\Phi}, \qquad \varphi = -\frac{1}{F}, \qquad p_i = -\frac{\Pi_i}{\Phi}, \qquad \pi_i = \frac{P_i}{F}, \tag{7a}$$

that Haar presented (in another connection: Über adjungierte Variationsprobleme und adjungierte Extremalflächen. Math. Ann. **100** (1928), pp. 487 et seq.) and Carathéodory, loc. cit. [11]) d) pp. 194 et seq. has used in a definitive formulation of his generalized Legendre transformation; we shall discuss this in no. 2. The formulas with one independent variable that one subsequently needs are naturally much easier to prove.

for which, (3), after dividing by F, yields:

$$(10) \qquad -\varphi\, dt + \pi_i\, dx_i = d\Theta + C_h\, dc_h\,, \qquad \frac{\partial C_h}{\partial t} = 0.$$

However, the differential relation (10) characterizes (8), with (9), as the family of solutions of the canonical system:

$$(11) \qquad \frac{dx_i}{dt} = \varphi_{\pi_i}\,, \qquad \frac{d\pi_i}{dt} = -\varphi_{x_i}$$

with the Hamilton function $\varphi(t, x_i, \pi_i)$.

With no further restrictions, the family of canonical transformations is then also given by:

$$x_i = x_i(x_j^0, \pi_j^0; t)\,, \qquad \pi_i = \pi_i(x_j^0, \pi_j^0; t)\,.$$

With this, we have the bridge to the variational problem:

$$(12) \qquad \int f\, dt = \min \quad \text{for the curve } x_i = x_i(t)$$

(for given endpoints), whose extremals are the paths of the group. Its basic Lagrange function $f(t, x_i, p_i)$, with $p_i = dx_i\,/dt$, goes over, in a well-known way, by using the Legendre transformation:

$$(13) \qquad p_i = \varphi_{\pi_i}\,, \qquad f = -\varphi + p_i\,\pi_i\,,$$

to the Hamilton function φ and thus to the Lie characteristic function F [8]).

The value of the extremal integral along a path segment is equal to the associated canonical parameter increment Θ.

Our representation allows us to immediately recognize that, conversely, the entire path of the variational problem can also be obtained from the family of canonical transformations as one runs through the one-parameter group of contact transformations. The transformation of the desired functions is now (under the assumption that $f \cdot \varphi \neq 0$), from (7):

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[8]) Here, we restrict ourselves the case in which the Hessian determinant satisfies:

$$(13a) \qquad \left|\varphi_{\pi_i \pi_j}\right| = \frac{F^{n+2}}{(-\Phi)^{n+2}} \left|F_{P_i P_j}\right| \neq 0.$$

In other cases, one is led to a Lagrange problem, cf., Vessiot, loc. cit. [5]), pp. 81, 107, as well as more recently in the textbook of Carathéodory, loc. cit. [11]), a), pp. 354 et seq., and also Boerner [11]), pp. 201, second formula from the top, where the first two factors on the right must be $(fF)^{2(n+\mu)-n\mu}$.

(14) $$\begin{cases} P_i = -\dfrac{\pi_i}{\varphi} \\ F = -\dfrac{1}{\varphi}, \end{cases}$$

in which (similar to $(7)_1$ in rem. [7])), under the assumption that $f \neq 0$, the first equation $(14)_1$ can be solved for the π_i (on this, cf., also Carathéodory, loc. cit. [11]), pp. 358) and its expressions in t, x_i, P_i can be substituted in $(14)_2$; as the independent variable, one introduces $\int f\,dt = \Theta$ along the extremal. If (8) and the differential relations (10), as well as (9), are true for this situation then the differential relation (3), as well as (6), follows for (5), which characterizes (5) as the path of a one-parameter group of contact transformations.

In the variational calculus, one says that a surface element (t, x_i, P_i) intersects its path (with the line element (t, x_i, p_i)) *transversally*. With the addition of the impulse π_i the transversality is expressed by (7), ((14, resp.).

If one then takes an initial surface (union) M_n^0 and subjects it to the contact transformation T_Θ of the group $\mathfrak{G}$ then on any image surface (union) M_n the canonical parameter will, in a certain neighborhood, describe a function of position [9]):

(15) $$\Theta = S(t, x_i).$$

The family of ∞^1 M_n: $S(t, x_i) = \Theta = \text{const.}$ is called a *geodetic field*; it intersects the paths *transversally* (and together with them defines a complete figure in the sense of Carathéodory).

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[9]) This is true under the assumption that $F \neq 0$, which we have already made. $F = 0$ represents another first-order partial differential equation, namely:

$$F\left(t, x_i, -\frac{\partial t}{\partial x_i}\right) = 0,$$

for *one* surface $t = t(x_i)$ in the same t, x_i space by which it is determined that it includes the surface element with $F = 0$ that lies on an $n-1$-dimensional manifold. This surface has the property that its surface elements are displaced into *themselves* under the one-parameter group of contact transformations, so any surface element with $F = 0$ will be displaced to an infinitely close element that is *united* with it on the characteristic strip that is determined by the initial element. Cf., S. Lie, Ges. Abh. IV, pp. 287, as well as pp. 591; VI, pp. 636, as well as footnote pp. 905; furthermore, see the footnotes of Engels in Bd. III, pp. 615, and Theorie der Transformationsgruppen II, pp. 256 (Leipzig 1890). In recent representations, in the construction of the integral surface as the characteristic strip, it is mostly not emphasized that it can be described by a one-parameter group of contact transformations on the entire space of integral elements.

I remark that the paths that appear here (as anomalous line elements) are *boundary curves*, which can be either minima or maxima of the variational problem. Cf., Vessiot, loc. cit. [5]) c), pp. 69, as well as Carathéodory, loc. cit [11]) a), pp. 283.

Different formal considerations are presented for this case by M. Herzberger, Theory of transversal curves and the connections between the calculus of variations and the theory of partial differential equations. Proc. Nat. Acad. Sciences **24** (1938), pp. 466-473.

For $S(t, x_i)$, one has the partial differential equation [10]):

(16) $$S_t F = 1.$$

The equation:

(17) $$S_{x_i} = P_i\, S_t$$

then exhibits P_i as an expression in the derivatives of S. By means of (7), this also makes:

(18) $$S_t + \varphi = 0,$$

with:

(19) $$\pi_i = S_{x_i},$$

which is the first-order differential equation of Hamilton-Jacobi.

The extremal integral over an *arbitrary* comparison curve that runs through the geodetic field is:

(20) $$\int f\, dt = \Theta + \int \mathcal{E}\, dt,$$

where Θ is the difference between the S-value at the endpoint of the arc and at the starting point. If the $\mathcal{E}$-function > 0 here then one obtains the minimizing property of the extremals (paths).

We have derived the complete connection between the one-parameter group of contact transformations and the variational problem in a somewhat different manner from that of Vessiot, and in the (inhomogeneous) formulation throughout, which represents a one-dimensional case of the general formulas discussed by Carathéodory for multi-dimensional variational calculus. In the stated special case, we added the interpretation of Carathéodory's F as the Lie characteristic function.

In the new representation that Carathéodory [11]) gave in his textbook on the variational calculus, as well as in his Geometrische Optik, for the form of variational calculus – I am speaking, at the moment, of a line integral – will, in any case, from the outset, be regarded as a certain embodiment of both the principles of Fermat and Huygens; thus, the selfsame origin in the group viewpoint is not completely realized here. The representation – without the apparatus of the contact transformations – will therefore be

[10]) On M_n, one has:

(15a) $$S_t\, dt + S_{x_i}\, dx_i = 0, \qquad dt + P_i\, dx_i = 0.$$

Furthermore, one has:

(15b) $$S(-\Phi) + S_{x_i}\, \Pi_i = 1.$$

From (15a), one deduces (17), and then from (15b), by means of $(3a)_1$, also (16).

[11]) C. Carathéodory, a) Variationsrechnung und partielle Differentialgleichungen erster Ordnung. Leipzig 1935. b) Geometrische Optik, Erg. d. Math. IV_5. Berlin 1937. Cf., above all, also C. Carathéodory, c) Les transformations canoniques de glissment et leur application a l'optique géométriques, Rom. Linc. Rend. (6) **12** $(1930)_2$, pp. 353-360, in particular, pp. 357 et seq. Die mehrdimensionale Variationsrechnung bei mehrfacher Integralen, Acta Szeged **4** (1928-29), pp. 193-216. Cf., also the representation of H. Boerner, Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale. Math. Annalen **112** (1936), pp. 187-220.

briefly unsurpassed, and, what is extremely important beyond the didactic advantage, it is suitable for the generalization to multiple extremal integrals (with many unknown functions) that Carathéodory has based his theory on.

2. If we now consider a variational problem for a multiple integral:

$$\int f\, dt_1 \ldots dt_\mu = \text{min.}, \tag{21}$$

in order to define the basic function:

$$f = f(t_\alpha, x_i, p_{i\alpha})$$

for a μ-dimensional surface:

$$x_i = x_i(t_\alpha), \tag{22}$$

that lies in the space $R_{n+\mu}$ of the variables t_α, x_i ($\alpha = 1, \ldots, \mu$; $i = 1, \ldots, n$), while we define $p_{i\alpha} = \partial x_i / \partial t_\alpha$ to be its surface element. This is to be integrated over a region G_t in the t-space, and the comparison functions shall be given on the boundary of G_t. Let the desired extremal surface be $\mathcal{E}_\mu$: $x_i = x_i(t_\alpha)$.

Carathéodory now takes a family of n-dimensional surfaces that depend upon μ parameters $\Theta_1, \ldots, \Theta_\mu$ thus:

$$\infty^\mu M_n : \qquad S_\alpha(t_\beta, x_j) = \Theta_\alpha = \text{const.} \tag{23}$$

(which will then be the family of surfaces that are transversal to the geodetic field) and, with the help of the basic function f, converts to an *equivalent* $f - \Delta$, which is associated with the same extremal surface $\mathcal{E}_\mu$. Therefore, the integral over Δ must depend only upon the boundary of the comparison surface segment; Carathéodory defines Δ to be the determinant:

$$\Delta = \left|\frac{\partial S_\alpha}{\partial t_\beta}\right| = |\, S_{i\alpha} + S_{i\alpha}\, p_{i\beta}\,| = \Delta(t_\alpha, x_i, p_{i\alpha}), \tag{24}$$

$$S_{\alpha\beta} = S_{\alpha t_\beta}, \qquad S_{i\alpha} = S_{\alpha x_i}.$$

The family of M_n shall now be chosen in such a way that at *one* particular point (t_α, x_i) the difference $f - \Delta$, which is regarded as a function of the $p_{i\alpha}$, possesses a null:

$$f - \Delta \geq 0; \tag{25}$$

thus, the equality symbol shall obtain for a certain surface element (t_α, x_i, $p_{i\alpha}$), which will "*transversally intersect the geodetic family* (23) *at the point in question.*"

A family that is geodetic at *any* point of a certain region in the space $R_{n+\mu}$ is called a *geodetic field*. That is the fundamental notion that Carathéodory introduced. The family that is geodetic at one point is only an auxiliary construction that I introduce in order to later on realize the covariance of the notion of transversality simply and independently of the (yet to be constructed) geodetic field.

The analytical condition for the family (23) to be geodetic at a point is obtained by the same considerations that Carathéodory has applied to the geodetic field, if they indeed always relate to just *one* point. We write M_n in the form $t_\alpha = t_\alpha(x_i\,;\Theta_\beta)$ and set:

$$(26)\qquad -\frac{\partial t_\alpha}{\partial x_i} = P_{i\alpha}\,, \qquad\qquad S_{\alpha i} = S_{\alpha\rho}\,P_{i\rho}\,,$$

in other words, such that it expresses, in the event that the family (23) in $(t_\alpha\,, x_i)$ is geodetic, the surface element $(t_\alpha\,, x_i\,, P_{i\alpha})$ in terms of only the transversally intersecting $(t_\alpha\,, x_i\,, p_{i\alpha})$ (in term of only $t_\alpha\,, x_i\,, \pi_{i\alpha}$, resp., where $\pi_{i\alpha} = f_{p_{i\alpha}}$) [12]):

$$(27)\qquad P_{i\alpha} = \frac{\bar{a}_{\alpha\beta}}{a}\,\pi_{i\beta}\,.$$

In this condition for the μ-dimensional surface element $(t_\alpha\,, x_i\,, p_{i\alpha})$ to be transversally intersected by the n-dimensional surface element $(t_\alpha\,, x_i\,, P_{i\alpha})$, one similarly defines the Hamilton function to be:

$$(28)\qquad \varphi(t_\alpha\,, x_i\,, p_{i\alpha}) = -f + p_{i\alpha}\,\pi_{i\alpha} \quad \text{with } \pi_{i\alpha} = f_{p_{i\alpha}}\,,$$

$$(29)\qquad -a_{\alpha\beta} = -f\,\delta_{\alpha\beta} + p_{i\alpha}\,\pi_{i\beta}\,, \qquad a = \det(a_{\alpha\beta})$$

and $\bar{a}_{\alpha\beta}$ is the algebraic complement of $a_{\alpha\beta}$ in $(a_{\alpha\beta})$.

Carathéodory then introduced a construction of F that is similar to the one-dimensional case, namely:

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[12]) From the property (23), it follows for the minimum, with the introduction of the algebraic complement $\bar{c}_{\alpha\beta}$ to $c_{\alpha\beta} \equiv S_{\alpha\beta} - S_{\alpha i}\,p_{i\beta}$, that:

$$f = \Delta \equiv |\,c_{\alpha\beta}\,|, \qquad \pi_{i\alpha} \equiv f_{p_{i\alpha}} = \Delta_{p_{i\alpha}}\,,$$

from which:

$$\pi_{i\alpha} = S_{\rho i}\;\bar{c}_{\rho\beta}$$

$$a_{\alpha\beta} = f\,\delta_{\alpha\beta} - p_{i\alpha}\,\pi_{i\beta} = c_{\rho\alpha}\bar{c}_{\rho\beta} - p_{i\alpha}S_{\rho i}\bar{c}_{\rho\beta} = S_{\alpha\beta}\bar{c}_{\rho\beta}\,.$$

If $P_{i\alpha}$ were introduced by way of $(26)_2$ then this would make:

$$\pi_{i\beta} = P_{i\alpha}\,a_{\alpha\beta}\,;$$

moreover, one has $a = |\,S_{\alpha\beta}\,|\,f^{\mu-1}$, hence, with F, (30), one also has $|\,S_{\alpha\beta}\,|\,F = 1$. By generalizing the considerations that pertained to (14) one sees: Equations (27) are soluble in terms of the $\pi_{i\beta}$; the expressions for the $\pi_{i\beta}$ in terms of the $t_\alpha\,, x_i\,, P_{i\alpha}$ will be substituted in (30) on the right. Cf., Boerner, loc. cit. [11]), pp. 200 et seq. We shall not require the detailed formulation of the Carathéodory transformation here.

(30) $$F = \frac{f^{\mu-1}}{a},$$

which proves to be a function $F(t_\alpha, x_i, P_{i\alpha})$, when one expresses the $p_{i\alpha}$ in terms of the $\pi_{i\alpha}$, and these, in turn, in terms of the $P_{i\alpha}$, by means of the transversality condition (27).

One then has the further condition [12]):

(31) $$|S_{\alpha\beta}| \cdot F = 1$$

for the geodetic field. Thus, by way of:

(32) $$S_{\alpha\rho} P_{i\rho} = S_{\alpha i},$$

the $P_{i\alpha}$ are expressed in terms of the partial derivatives $S_{\alpha\beta}$, $S_{\alpha i}$ of the S_α (under the assumption that $|S_{\alpha\beta}| \neq 0$) and substituted into F. One thus obtain *one* first-order partial differential equation for the S_α; this characterizes the geodetic field [13]).

With the help of the geodetic field, Carathéodory presented the "Legendre condition" and the "Weierstrass $\mathcal{E}$-function" for multiple integrals; they do not appear as one would presume. The most important thing is the fact that the functions S_α of the geodetic field drop out: All that remains are the $p_{i\alpha}$ or the $P_{i\alpha}$. Nonetheless, it is important to the construction of the theory, as well as the establishment of a strong minimum (for a positive $\mathcal{E}$-function), for a given extremal $\mathcal{E}_\mu$ to be embedded in a geodetic field that transversally intersects it.

Before I go into that, I remark that above all the notion of the geodetic field, and likewise that of being transversally intersected, is originally defined by (25) in manner that is *independent of the choice of variables* – that is only meaningful relative to the extremal integral that was given *a priori*.

We transform this to new independent variables $\bar{t}_\alpha$, which are functions of t_α and x_i:

(33) $$\bar{t}_\alpha = T_\alpha(t_\beta, x_i) \quad \text{with} \quad |T_{\alpha\beta}| \neq 0, \qquad T_{\alpha\beta} = \frac{\partial T_\alpha}{\partial t_\beta};$$

the x_i will remain the same. This transformation is arranged such that a comparison surface (lying in the neighborhood of the extremal in question) $x_i = x_i(t_\alpha)$ *intersects* the μ-

[13]) In addition, the indicated consideration shows: To given numerical values $P_{i\alpha}$ one can always very easily determine a *geodetic family at the point* (t_α, x_i), which has an M_n that goes through (t_α, x_i) with precisely the position $P_{i\alpha}$: One chooses the $S_{\alpha\beta}$ at (t_α, x_i) arbitrarily, except that the determinant satisfies:

$$|S_{\alpha\beta}| = \frac{1}{F(t_\alpha, x_i, P_{i\alpha})},$$

and then determines the $S_{\alpha i}$ by means of (32) at the point (t_α, x_i). The functions $S_\alpha(t_\beta, x_i)$ must then have the computed first derivatives $S_{\alpha\beta}$, $S_{\alpha i}$ only at (t_α, x_i).

parameter family of n-dimensional coordinate manifolds $\tilde{t}_\alpha$ = const. in such a way that for the assembled function:

(34) $$\tilde{t}_\alpha = T_\alpha(t_\beta, x_j(t_\beta))$$

the functional determinant is:

(34)₁ $$d = \frac{d(\tilde{t}_\alpha)}{d(t_\beta)} = |\, d_{\alpha\beta} \,| > 0, \qquad d_{\alpha\beta} = \frac{\partial T_\alpha}{\partial t_\beta} + \frac{\partial T_\alpha}{\partial x_j} p_{j\beta} = \frac{d\tilde{t}_\alpha}{dt_\beta}.$$

If one solves (34) for t_β then one obtains the following for the surface:

(35) $$x_i = \tilde{X}_i(\tilde{t}_\beta), \qquad \tilde{p}_{i\alpha} = \frac{d\tilde{t}_\alpha}{dt_\beta},$$

and from the identity $x_i = \tilde{X}_i(T_\alpha(t_\beta, x_j(t_\beta))$, it further follows that:

(36) $$p_{i\alpha} = \tilde{p}_{i\gamma} d_{\gamma\alpha},$$

where $d_{\gamma\alpha}$ depends only upon t_β, x_j, $p_{i\alpha}$, such that, with the algebraic complement $\bar{d}_{\alpha\beta}$ in the determinant d, the new expressions:

(37) $$\tilde{p}_{i\beta} = \frac{\bar{d}_{\beta\alpha}}{d} p_{i\alpha}$$

will be expressed in terms of only t_β, x_j, $p_{i\alpha}$.

Moreover, it follows from the required invariance of the extremal integrals that were given *a priori*, i.e., from the demand that $f\, dt_1 \dots dt_\mu = \tilde{f}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu$, which gives the transformation character of the basic function $f = \tilde{f} \cdot d$, hence:

(38) $$\tilde{f} = \tfrac{1}{d} f,$$

where on the right-hand side the t_β are expressed in terms of the $\tilde{t}_\alpha$ and the x_i, $p_{i\alpha}$ in terms of the $\tilde{t}_\alpha$, x_i, $\tilde{p}_{i\alpha}$ – simply by switching the roles of t_α and $\tilde{t}_\alpha$.

The "path of the independent integral" $\int d\Theta_1 \dots d\Theta_\mu$ also allows one to convert the $\tilde{t}_\alpha$, where the invariant integral is represented by (35). If the conversion equations for the family (23) read:

(39) $$\tilde{S}_\alpha(\tilde{t}_\beta, x_j) = \Theta_\alpha$$

then one will have:

(40) $$\int d\Theta_1 \ldots d\Theta_\mu = \int \tilde{\Delta}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu \quad \text{with} \quad \tilde{\Delta} = \left| \frac{d\tilde{S}_\alpha}{d\tilde{t}_\beta} \right| = \left| \tilde{S}_{\alpha\beta} + \tilde{S}_{\alpha i}\,\tilde{p}_{i\beta} \right|;$$

since, on the other hand, this integral is:

(41) $$\int d\Theta_1 \ldots d\Theta_\mu = \int \Delta\, dt_1 \ldots dt_\mu = \int \tilde{\Delta} \cdot \tfrac{1}{d}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu ,$$

one then has:

(42) $$\tilde{\Delta} = \tfrac{1}{d}\Delta ,$$

which one can also verify directly quite easily.

Under the transition to the new variables, one also merely multiplies the left-hand side of the fundamental relation (25) by $1/d > 0$; one has:

(43) $$\tilde{f} - \tilde{\Delta} = \tfrac{1}{d}(f - \Delta) \geq 0$$

when and only when $f - \Delta \geq 0$ is true, resp.: The geodetic field retains the property that the same is true for a family that is geodetic at a point, and thus an n-dimensional surface element that is transversal to a μ-dimensional surface element also remains transversal after the coordinate transformation – relative to transformed basic function of our (*a priori* given) extremal integral. The position coordinates and the equations of the family of surfaces (23) are naturally to be converted, but the analytic relations (27), (30), (31), (32), which are obvious consequences of the fundamental inequality (25), are *covariant*: They have the old form with regard to the unconverted basic function $\tilde{f}$.

I further remark that in the recent work of Finsler and Cartan [14]) such invariance considerations are presented in terms of the geometry of a space whose metric is based on the multiple extremal integral (with only one unknown function).

3. All that remains is the problem of embedding a given extremal $\mathcal{E}_\mu$ (at least in the small) in a geodetic field that intersects it transversally. Boerner [15]) has given a construction in the spaces of Carathéodory's theory. In conclusion, I would like to show how, when one is given an infinitesimal contact transformation of a family of $n + 1$-dimensional manifolds – which must only be transversal to $\mathcal{E}_\mu$ – the production of a geodetic field that is transversal to $\mathcal{E}_\mu$ can lead back to the aforementioned construction of line integrals.

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[14]) Cf., E. Cartan, Les espaces de Finsler, Actualités scient. et ind. no. 79. Paris 1934; Les espaces métriques fondés sur la notion d'aire, id. no. 72, Paris 1933.

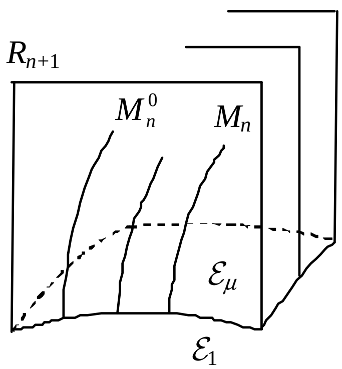
[15]) H. Boerner, loc. cit. [11]), pp. 203-213. On the basis of another definition of the geodetic field, H. Weyl gave a field construction in: Geodesic fields in the calculus of variations for multiple integrals, Annals of Math. **36** (1935), pp. 607-629. Th.-H.-J. Lepage considered the two definitions within a unified viewpoint in: Sur les champs géodesiques du calcul des variations, Bull. Acad. Roy. Belg. (5) **22** (1936), pp. 716-729 and pp. 1036-1046. Boerner has recently explained how the Carathéodory theory is indicated within this general Ansatz (talk at the Marburger Colloquium, Feb., 1939).

For the construction of a geodetic field, one must solve only *one* first-order partial differential equation (31) for *one* function S_1, in the event that $S_{\alpha'}$, $\alpha' = 2, \ldots, \mu$ is given arbitrarily; thus the $P_{i\alpha}$ in F are to be replaced with their expressions in terms of the first derivatives of the S_α that one computes from (32). However, it is, above all, necessary for the field to be transversal to the given extremals $\mathcal{E}_\mu$. Boerner [16]) thus takes the functions $S_{\alpha'}(t_\beta, x_j)$ in such a way that the:

$$(44) \qquad \infty^{\mu-1}\, R_{n+1}: \qquad S_{\alpha'}(t_\beta, x_j) = \Theta_{\alpha'} = \text{const.}$$

is transversal to $\mathcal{E}_\mu$, i.e., it includes the transversal n-directions $\bar{P}_{i\alpha}$ that are transversal to the surface element $p_{i\alpha}$ of $\mathcal{E}_\mu$.

However, we immediately convert this R_{n+1} to (n+1-dimensional) *coordinate planes* $S_{\alpha'} \equiv t_{\alpha'} = \Theta_{\alpha'}$ by the introduction of new independent variables, which are again denoted by $t_{\alpha'}$ [17]); $t_1 = t$ can remain true [18]).



A family of ∞^1 M_n must be completely contained in each R_{n+1} for the n-dimensional surfaces M_n of the geodetic field to be constructed. An M_n therefore has the n-direction $P_{i1} = P_i$, $P_{i\alpha'} = 0$.

I now allow the M_n in R_{n+1} to go over to each other under a one-parameter group of contact transformations whose infinitesimal contact transformation has the following Lie characteristic function:

$$(45) \qquad F^0(t, x_i, P_i) = F(t, \Theta_2, \ldots, \Theta_\mu, x_i, P_i, 0, \ldots, 0),$$

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[16]) Cf., loc. cit. [15]), pp. 209, footnote 23.

[17]) The fact that the properties of the transformation $\tilde{t}_1 = t_1$, $\tilde{t}_{\alpha'} = S_{\alpha'}(t_\beta, x_i)$ that were required above in (33) and $(34)_1$ are satisfied can be gathered from the previously-cited footnote in Boerner's work: Let $g_{11} \neq 0$ (possibly achieved by a suitable transformation that is produced at *one* starting point of the extremal $\bar{P}_{i\alpha} = 0$), and then assume that the $s_{\alpha'}(t_\beta)$ are independent of t_1.

[18]) One can seek to carry out the suitable transformation $x_i = \dot{x}_i(t) + \bar{x}_i$, $\pi_{i\alpha} = \dot{\pi}_{i\alpha}(t) + \bar{\pi}_{i\alpha}$ that brought about a great simplification in Weyl, loc. cit. [15]) in the spaces of Carathéodory's theory, as well – with the intention of applying the contact transformation with the necessary foresight to convert the new Lagrange function f^*, which *vanishes* along the initial extremal $x_i = \dot{x}_i(t_\beta)$, $\pi_{i\alpha} = \dot{\pi}_{i\alpha}(t_\beta)$, along with its first derivatives, and convert the likewise-obtained new Hamilton function φ^* into $\bar{f} = f^* + 1$, $\bar{\varphi} = \varphi^* - 1$.

One then arrives at certain surfaces $\sigma_\alpha(t, \bar{x})$ = const., which do not, however, yield the transversal surfaces of the original problem in the original space simply by conversion; these are, moreover, other surfaces that are given by $S_\alpha(t_\beta, x_j)$ = const., where $S_\alpha(t_\beta, x_j) = S_\alpha(t_\beta, \dot{x}_j(t)) + \dot{\pi}_{i\alpha}(t)\bar{x}_i + \sigma_\alpha(t_\beta, \bar{x}_j)$. – Furthermore, it seems to me that in Weyl, pp. 621, one must add the (negative Hamilton function $H = -\varphi^*$ in formula (35) and the sum on the right-hand side of the foregoing one.

i.e., the Carathéodory function that is specialized to $t_{\alpha'} = \Theta_{\alpha'} = \text{const.}$, $P_{i\alpha'} = 0$. Thus, let a surface be chosen in each R_{n+1} to be the initial surface M_n^0, and which is transversal to the (one-dimensional) intersection curve $\mathcal{E}_1$ of $\mathcal{E}_\mu$ with R_{n+1} – relative to F^0 in R_{n+1}.

First, the totality of all ∞^μ M_n (in all R_{n+1}) defines a geodetic field in any case. If M_n has the canonical parameter $\Theta_1 = \Theta = S(t, x_i) = S_1(t, \Theta_2, \ldots, \Theta_\mu, x_i)$ under the group F^0 then the original partial differential equation (16) is valid for the function of position $S(t, x_i)$ on R_{n+1}, which is still independent of the parameter $\Theta_{\alpha'}$, only with F^0 instead of F, hence:

$$S_i F^0 = 1 \qquad \text{with} \qquad S_t P_i = S_{x_i}. \tag{46}$$

If one then again introduces the quantities with the indices $t = t_1$, $\Theta_{\alpha'} = t_{\alpha'}$, $S = S_1$, $P_i = P_{i1}$, and observes the special form of the $S_{\alpha'}$, by means of which (32) gives $0 = P_{i\alpha'}$, then one recognizes, with no further assumptions [19]), that one can write the formula (46), just as well as the differential equation (31) in $R_{n+\mu}$.

Now, we still have to show that this geodetic field intersects the given extremal $\mathcal{E}_\mu$. As one then realizes, that already suffices in order to prove that all M_n in R_{n+1} are transversal to the intersection $\mathcal{E}_1$ (of $\mathcal{E}_\mu$ with R_{n+1}) – relative to F^0, which is therefore the curve $\mathcal{E}_1$ defined by the total evolution of a surface element of under the group F^0.

Above all, one has $\bar{P}_{i\alpha} = 0$ for the n-direction that is transversal to $\mathcal{E}_\mu$, since, by assumption, it indeed lies in a coordinate plane R_{n+1}. Hence, the system of equations (32) must be satisfied for $\alpha' = 2, \ldots, \mu$ by the special functions $S_{\alpha'} \equiv t_{\alpha'}$.

From (27), one then also concludes [20]) $\pi_{i\alpha} = 0$ on $\mathcal{E}_\mu$.

The Euler partial differential equations for $\mathcal{E}_\mu$, which are written canonically as:

$$\frac{dx_i}{dt_\alpha} = \varphi_{\pi_{i\alpha}}, \qquad \frac{d\pi_{i\alpha}}{dt_\alpha} = -\varphi_{x_i} \tag{47}$$

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[19]) Under certain assumptions relative to F that guarantee the differentiability of S_1 with respect to the parameters $\Theta_{\alpha'}$.

[20]) From (29), (27), under the assumption that:

$$g_{\alpha\beta} = \delta_{\alpha\beta} + P_{i\alpha}\, p_{i\beta} = \frac{1}{a}(a\,\delta_{\alpha\beta} + \bar{a}_{\alpha\gamma}\pi_{i\gamma}\, p_{i\beta}) = \frac{1}{a}\bar{a}_{\alpha\beta} f,$$

the relation follows:

$$f P_{i\alpha} = g_{\alpha\beta}\, \pi_{i\beta}.$$

For $P_{i\alpha'} = 0$, one has $g_{\alpha'\beta} = \delta_{\alpha'\beta}$ and $\pi_{i\alpha'} = 0$.

Moreover, from $\pi_{i\beta} = P_{i\alpha}\, a_{\rho\beta}$, one obtains:

$$\pi_{i1} = P_{i1}\, a_{11} = P_{i1}(f - p_{i1}\, \pi_{i1}) = P_{i1}(-\varphi).$$

Thus, one has:

$$-\varphi = \frac{1}{F} = \frac{a}{f^{\mu-1}} \qquad \text{since} \qquad a = |a_{\alpha\beta}| = \begin{vmatrix} -\varphi & 0 & \cdots & 0 \\ * & f & \cdots & 0 \\ * & 0 & \cdots & f \end{vmatrix}, \qquad \text{from (29).}$$

with the Hamilton function (28), yield, since $\pi_{i\alpha'} = 0$, a canonical system ordinary differential equations with independent variables $t_1 = t$ for x_i, and the canonically conjugate impulse $\pi_{i1} = \pi_i$:

$$\text{(48)} \qquad \frac{dx_i}{dt} = \varphi^0_{\pi_i}, \qquad \frac{d\pi_i}{dt} = -\ \varphi^0_{x_i};$$

thus, the Hamilton function is:

$$\text{(49)} \qquad \varphi_0(t, x_i, \pi_i) = \varphi(t, \Theta_2, \ldots, \Theta_\mu, x_i, \pi_i, 0, \ldots, 0)$$

which the general Hamilton function (28), when specialized to $\pi_{i\alpha'} = 0$.

On the grounds of the formal remarks that were made in footnote [20]) (that for $P_{i\alpha'} = 0$, $\pi_{i\alpha'} = 0$, the Carathéodory formulas (27), (30) go over to the corresponding one-dimensional formula (14)), the Hamilton function φ^0 is associated with the Lie function $F^0 = -1/\varphi^0$, in the sense of the first section, i.e., $\mathcal{E}_1$ is a path, relative to F^0, and indeed consists of those surface elements of M_n^0 that, by construction, intersect $\mathcal{E}^1$ transversally.

Under the transformations of the group F^0 in R_{n+1}, the image of this initial element, which is displaced along $\mathcal{E}_1$, is always transversal to $\mathcal{E}_1$. The formula:

$$\text{(50)} \qquad \pi_i = \frac{P_i}{F^0},$$

which is valid on any $\mathcal{E}_1$ and expresses this transversality relative F^0 in R_{n+1}, is the full content of Carathéodory's transversality condition (27), since $\mathcal{E}_\mu$ possesses the location $\pi_{i1} = \pi_i$, $\pi_{i\alpha'} = 0$, and the M_n (lying in R_{n+1}) possess the location $P_{i1} = P_i$, $P_{i\alpha'} = 0$. From footnote [20]), due to the unique solubility of the same, the surface element $P_{i\alpha}$ is therefore transversal to the surface elements $p_{i\alpha}$ of $\mathcal{E}_\mu$ along $\mathcal{E}_\mu$ in the space of all variables $P_{i\alpha} = \bar{P}_{i\alpha}$.

The property of the μ-parameter family (M_n) that is thus proved in order to construct a geodetic field – viz., that the extremal $\mathcal{E}_\mu$ intersects it transversally – does not depend upon the variables used, but has an invariant meaning for the variational problem.

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