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G. Herglotz's treatment of acceleration waves in his lectures on "Mechanik der Kontinua" applied to the jump waves of Christoffel

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Following a lecture delivered by Herglotz in 1925/26, we briefly treat acceleration waves in hyperelastic materials. Our main result is a divergence equation for the squared Euclidian norm of the so-called "wave vector." We then apply Herglotz's method (devised for acceleration waves) to the propagation of such first order discontinuities in elastic bodies as were treated by Christoffel in [1].

Introduction

In the previous article we briefly referred to the contents of the middle part of an unpublished three-part lecture "Mechanik der Kontinua" that Herglotz gave at Göttingen in 1925/26. It was prepared for the Göttinger Mathematische Institut by Frau Ph. Salié. I am grateful to her for the knowledge of this lecture. Now, we are remembering the one-hundredth birthday of Herglotz (b. 2.2.1881, d. 22.3.1953). As far as I can see, after Christoffel, Herglotz was the first to carry out the treatment of the spreading of discontinuities to such an extent that the divergence relation for the norm of the wave vectors emerges as the orthogonality condition for the determination of the higher (i.e., third) order jumps. This gives the meaning of the rays as the shadow boundary of geometric optics. At the conclusion of this article, we show how the computations of Herglotz for the second order discontinuities can be derived from Christoffel's first order discontinuities without his symbolism.

1. Herglotz's treatment of acceleration waves

In the following, let $D \subset \mathbf{R}^4$ be an open set that decomposes into two open subsets D_1, D_2 by means of a portion of a sufficiently flat, oriented hypersurface in $\mathbf{R}^4$:

$$\Sigma = \{(x, t) \mid x \in \mathbf{R}^3,\ t = \tau(x) \in \mathbf{R}\};$$

in this, we let $\tau: G \to \mathbf{R}$, $G \to \mathbf{R}^3$ be a sufficiently many times continuously partial differentiable function.

[1]) This work was commissioned by the Deutschen Forschungsgemeinschaft. My collaborator is Herr Dipl.-Math. Hans-Jürgen Böttger.

We now consider a motion of a hyperelastic continuum in Σ whose first partial derivatives of the Eulerian position coordinates and whose second partial derivatives of these coordinates are discontinuous on Σ. The discontinuity surface Σ will also be called the "wave surface" in what follows, and the jump across Σ becomes a function g that is continuous in the complement of Σ, which, since the time of Christoffel, has been denoted $[g]$.

The Eulerian coordinates of a particle at time t will be denoted by $x'^i = x'^i(t, x)$, $i = 1, 2, 3$, in which $x = (x^1, x^2, x^3)$ mean the (Lagrangian) position coordinates of the particle at an arbitrary, but fixed, time $t = t_0$ (a number). Let the functions x'^i, $i = 1, 2, 3$ belong to $C^1(D) \cap C^2(D_1 \cup D_2)$, and its second partial derivatives with respect to t, x^a, $a = 1, 2, 3$ shall possess regular, finite limiting values on D_1 (D_2, resp.) under one-sided approximation; the second partial derivatives of the Eulerian position coordinates will therefore have a jump discontinuity on Σ, as required.

On both sides of Σ the x'^i, $i = 1, 2, 3$ satisfy the equations of motion:

$$\rho \ddot{x}'^i = X^i + \frac{dW_i^\beta}{dx^\beta}, \qquad i = 1, 2, 3, \qquad \text{with} \qquad \ddot{x}'^i := \frac{\partial^2}{\partial t^2} x'^i ;$$

ρ is the density in a fixed relative configuration, X^i is the i^{th} component of the external force, and W_i^β is one component of the (first) Piola-Kirchhoff tensor. We set:

$$p_\beta^i := \frac{\partial x'^i}{\partial x^\beta}, \quad i, \beta = 1, 2, 3, \quad \text{and demand that} \qquad W_i^\beta = \frac{\partial W}{\partial p_\beta^i}$$

with an internal energy function $W = W(p_\beta^i(t, x), x)$ (i.e., hyperelasticity) that shall be sufficiently many times continuously differentiable.

The equations of motions on both sides of the wave surface then read:

$$\text{(B)} \qquad \rho \ddot{x}'^i = X^i + \frac{\partial W}{\partial p_\beta^i \, \partial x^\beta} + \frac{\partial W}{\partial p_\beta^i \, \partial p_\alpha^k} \frac{\partial p_\alpha^k}{\partial x^\beta}, \quad i = 1, 2, 3;$$

in this – and everywhere in the sequel – we sum over all indices that appear twice. One sets:

$$p_{\beta\alpha}^i := \frac{\partial p_\beta^i}{\partial x^a} \quad \text{and} \quad P_\alpha := -\frac{\partial \tau}{\partial x^\alpha}, \qquad i, \alpha, \beta = 1, 2, 3.$$

With the help of a well-known lemma of Hadamard (cf. [2], § 72 or [3], § 174), one derives the following jump relation for the functions $p_{\beta\alpha}^i$ on Σ:

$$[p_{\beta\alpha}^i] = [\ddot{x}'^i] P_\alpha P_\beta .$$

If one takes this into account, along with $x'^i \in C^1(D)$, then it follows, with the assumption that $[X^i] = 0$, $i = 1, 2, 3$, and when one further introduces the notation $\eta^i := [\ddot{x}'^i]$, $i = 1, 2, 3$, that:

$$\rho\eta^i = \frac{\partial^2 W}{\partial p^i_\beta \, \partial p^k_\alpha} P_\alpha P_\beta \eta^k, \qquad i = 1, 2, 3.$$

We write:

$$\eta_i = X_{ij}\,\eta^j, \qquad \text{with} \qquad X_{ij} := \frac{1}{\rho}\frac{\partial^2 W}{\partial p^i_\beta \, \partial p^k_\alpha} P_\alpha P_\beta .$$

This gives:

$$0 = (X_{ij} - \delta_{ij})\,\eta^j, \qquad i = 1, 2, 3,$$

in which δ_{ij} refers to the Kronecker symbol. The vector $\eta = (\eta^1, \eta^2, \eta^3)$ will be referred to as the "wave vector." Due to the discontinuity of $\ddot{x}'^i$ on Σ, one has $\eta \neq 0$, and it follows that:

$$0 = \det\{X_{ij}(t, x, P) - \delta_{ij}\} =: F(t, x, P) \qquad \text{with} \qquad P := (P_1, P_2, P_3).$$

We call the surface in $\mathbf{R}^3$ that is defined by $F(t, x, P) = 0$ (with the running variable P) the "normal surface." F is a polynomial in P of the form $F = F_6 + F_4 + F_2 - 1$, in which the F_k, $k = 2, 4, 6$ are homogeneous polynomials of degree k in P.

An interesting interpretation for F derives from the fact that the wave motion can be described by a one-parameter group of contact transformations of the surface elements (t, x, P), whose generating function is $F(t, x, P)$ precisely. With the canonical parameter Θ, one obtains the following differential equations for the individual pathlines of the surface elements of the wave surface, which are displaced into each other by the transformations of the group in such a way that $F = 0$:

$$\frac{dx^i}{d\Theta} = F_{P_i} =: \Pi^i, \qquad \frac{dP_i}{d\Theta} = -F_{x^i} + F_t\,P_i, \qquad i = 1, 2, 3, \qquad -\frac{dt}{d\Theta} = P_i\,\Pi^i =: \Phi.$$

These are obviously the equations for the characteristics of the first order partial differential equation:

$$0 = F(t, x, p'), \quad t = \tau(x), \quad p' = (p'_1, p'_2, p'_3), \quad p'_i = -P_i = \frac{\partial \tau}{\partial x^i}, \quad i = 1, 2, 3.$$

Hadamard called these differential equations for the rays the "bicharacteristics of the equations of motion" for the characteristic wave $t = \tau(x)$.

Along with the wave vector η, one also introduces the "ray vector" $p = (p^1, p^2, p^3)$ through:

$$p^i := -\frac{1}{2\Omega}\frac{\partial \Omega}{\partial P_i}, \qquad i = 1, 2, 3, \qquad \text{with} \qquad \Omega := X_{ij}\,\eta^i\,\eta^j$$

(since $\eta \neq 0$, one obviously has $\Omega \neq 0$). On the grounds of the homogeneity of Ω in the P_i, $i = 1, 2, 3$, one next remarks that:

$$-1 = P_i\, p^i.$$

In the case where the matrix $(X_{ij} - \delta_{ij})$ has rank two, one further has:

$$F_{P_i} = kp^i, \qquad k \in \mathbf{R} - \{0\}, \qquad i = 1, 2, 3.$$

One sees this perhaps in the following way: Let F^{hk}, h, $k = 1, 2, 3$ be the adjoint that is associated with $X_{ij} - \delta_{ij}$; since $\mathrm{rank}(X_{ij} - \delta_{ij}) = 2$, one has:

$$F^{hk} = \eta^h\, \Lambda^k, \quad h, k = 1, 2, 3, \quad \text{with} \quad \Lambda^k \in \mathbf{R}.$$

Since for at least one ordered pair (h_i, k_j), $h_i, k_j \in \{1, 2, 3\}$ one must have $F^{hk} \neq 0$, one deduces, because $F^{hk} = F^{kh}$:

$$\Lambda^k = \frac{\eta^k}{\lambda}, \qquad \lambda \in \mathbf{R} - \{0\}, \quad k = 1, 2, 3, \qquad \text{i.e.,} \qquad F^{hk} = \frac{1}{\lambda}\eta^h\, \eta^k .$$

It follows that:

$$F_{P_i} = F^{hk}\,\frac{\partial X_{hk}}{\partial P_i} = \frac{1}{\lambda}\eta^h\, \eta^k\,\frac{\partial X_{hk}}{\partial P_i} = \frac{1}{\lambda}\frac{\partial \Omega}{\partial P_i} = -\frac{2\Omega}{\lambda}p^i.$$

Hence $-2\Omega/\lambda = k = -\Phi$:

$$p^i = \frac{\Pi^i}{-\Phi}, \qquad i = 1, 2, 3.$$

The vectors $p = (p^1, p^2, p^3)$, $p^i = \dfrac{\Pi^i}{-\Phi}$ that emanate from the null point define the so-called "ray surface." If one takes into account the equations $P_i\, p^i = -1$ that we just proved and the fact that $F_{P_i} = kp^i$, $k \neq 0$, $i = 1, 2, 3$ then one easily sees that the same relationship exists between the rays and the normal surface $F(t, x, p') = 0$, with $p' = (p'_1, p'_2, p'_3)$, $p'_i := -P_i$, $i = 1, 2, 3$ through the reciprocal transformation – namely, the polar relationship in the real unit sphere – which, according to Minkowski, defines an ordinary variational problem between the indicatrix and the figuratrix.

If $\mathrm{rank}(X_{ij} - \delta_{ij}) < 2$ then the foregoing conclusions are not valid; in the sequel, this case shall be omitted.

2. Herglotz's proof of a divergence equations for the wave and ray vectors

In the following, let $D \subset \mathbf{R}^4$ again be an open set that decomposes through the wave surface $\Sigma = \{(x, t) \mid x \in \mathbf{R}^3, t = \tau(x) \in \mathbf{R}\}$ into two open subsets D_1, D_2: $D = D_1 \cup \Sigma \cup D_2$.

Let the functions x'^i, i = 1, 2, 3 belong to $C^3(D_1 \cap D_2) \cap C^1(D)$. For a multi-index $\mu = (\mu_0, \mu_1, \mu_2, \mu_3)$, $\mu_0, \mu_1, \mu_2, \mu_3 \in \mathbf{N} - \{0\}$ with $\mu_0 + \mu_1 + \mu_2 + \mu_3$, one sets:

$$D^\mu := \frac{\partial^{\mu_0+\mu_1+\mu_2+\mu_3}}{(\partial t)^{\mu_0}(\partial x^1)^{\mu_1}(\partial x^2)^{\mu_2}(\partial x^3)^{\mu_3}};$$

one has for all such multi-indices and all points $(x, t(x)) \in \Sigma$:

$$0 = \lim_{\varepsilon\to 0} \sup\{|\, D^\mu x^n(y) - D^\mu x^n(z)\,|;\ y, z \in D_k,\ |\, y - (x, \tau(x))\,|_{E^4} + |\, z - (x, \tau(x))|_{E^4} < e\}$$

i = 1, 2, 3, k = 1, 2.

We introduce the following notations:

$$\dot p^i_\beta := \frac{\partial^2 x'^i}{\partial t\,\partial x^\beta}, \quad \dot p^i_{\beta\gamma} := \frac{\partial^3 x'^i}{\partial t\,\partial x^\beta \partial x^\gamma}, \qquad \ddot p^i_\beta := \frac{\partial^3 x'^i}{\partial t^2\,\partial x^\beta}, \quad \zeta^i := [\dddot x'^i], \quad \dddot x'^i := \frac{\partial \ddot x'^i}{\partial t},$$

and:

$$\eta^i_\alpha := \frac{d}{dx^\alpha}[\ddot x'^i].$$

A simple calculation gives the jump relations on Σ:

(1) $[\dot p^i_\beta] = \eta^i P_\beta$,

(2) $[p^i_{\beta\gamma}] = [\dot p^i_\beta] P_\beta = \eta^i P_\beta P_\gamma$,

(3) $[\ddot p^i_\beta] = \eta^i_\beta + \zeta^i P_\beta$,

(4) $[\dot p^i_{\beta\gamma}] = \eta^i P_{\beta\gamma} + \eta^i_\gamma P_\beta + \eta^i_\beta P_\gamma + \zeta^i P_\beta P_\gamma$;

one proves this, say, by repeated application of the aforementioned lemma of Hadamard.

We now consider the equations of motion:

$$\text{(B)} \qquad \rho \ddot x'^i = X^i + \frac{\partial W_i^\beta}{\partial x^\beta} + \frac{\partial W_i^\beta}{\partial p^k_\alpha} p^k_{\alpha\beta}, \qquad i = 1, 2, 3,$$

and differentiate with respect to t:

$$\rho \dddot x'^i = \dot X^i + \frac{\partial^2 W_i^\beta}{\partial p^k_\alpha\,\partial x^\beta}\dot p^k_\alpha + \frac{\partial W_i^\beta}{\partial p^k_\alpha}\dot p^k_{\alpha\beta} + \frac{\partial^2 W_i^\beta}{\partial p^k_\alpha\,\partial p^j_\gamma} p^k_{\alpha\beta}\dot p^j_\gamma, \qquad i = 1, 2, 3.$$

The equations are true on both sides of the wave surface. It follows that:

(5) $$\rho\dot{\zeta}^i = \frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial x^\beta}\eta^k P_\alpha + \frac{\partial W_i^\beta}{\partial p_\alpha^k}\left\{\frac{d}{dx^\beta}(\eta^k P_\alpha) + \eta_\alpha^k P_\beta + \zeta^k P_\alpha P_\beta\right\} + \frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial p_\gamma^j}[p_{\alpha\beta}^k \dot{p}_\gamma^j],$$

$i = 1, 2, 3$, in which we have assumed that $[\dot{X}^i] = 0$. One sets:

$$\dot{p}_\gamma^{-j} := \lim_{\substack{y\to(x,\tau(x))\\ y\in D_1}} \frac{\partial^2 x'^j}{\partial t\,\partial x^\gamma}(y) \qquad \text{and} \quad p_{\beta\gamma}^{-j} := \lim_{\substack{y\to(x,\tau(x))\\ y\in D_1}} \frac{\partial^2 x'^j}{\partial x^\gamma\,\partial x^\beta}(y), \qquad (x,\ \tau(x)) \in \Sigma.$$

Since $x'^i \in C^1(D)$ (as one sees from the Hadamard lemma):

$$\frac{d}{dx^\beta} p_\gamma^i = p_{\gamma\beta}^{-j} - \dot{p}_\gamma^{-j} P_\beta, \qquad \beta, \gamma, j = 1, 2, 3 \text{ on } \Sigma.$$

This yields:

$$\frac{d}{dx^\beta}\left(\frac{\partial W_i^\beta}{\partial p_\alpha^k}\eta^k P_\alpha\right) = \frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial x^\beta}\eta^k P_\alpha + \frac{\partial W_i^\beta}{\partial p_\alpha^k}\frac{d}{dx^\beta}(\eta^k P_\alpha) + \frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial p_\gamma^j}(p_{\gamma\beta}^{-j} - \dot{p}_\gamma^{-j} P_\beta)\eta^k P_\alpha$$

and

(6) $$\rho\dot{\zeta}^i = \frac{d}{dx^\beta}\left(\frac{\partial W_i^\beta}{\partial p_\alpha^k}\eta^k P_\alpha\right) + \frac{\partial W_i^\beta}{\partial p_\alpha^k}(\eta_\alpha^k P_\beta + \zeta^k P_\alpha P_\beta) + \frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial p_\gamma^j}([p_{\alpha\beta}^k \dot{p}_\gamma^j] - \dot{p}_{\gamma\beta}^{-j} P_\beta \eta^k P_\alpha).$$

Now, on account of (1), (2):

$$\begin{aligned}[p_{\alpha\beta}^k \dot{p}_\gamma^j] &= (p_{\alpha\beta}^{-k} + \eta^k P_\alpha P_\beta)(\dot{p}_\gamma^{-j} + \eta^j P_\gamma) - p_{\alpha\beta}^{-k}\dot{p}_\gamma^{-j} \\ &= \dot{p}_\gamma^{-j}\eta^k P_\alpha P_\beta + \eta^k \eta^j P_\alpha P_\beta P_\gamma - p_{\alpha\beta}^{-k}\eta^j P_\gamma\end{aligned}$$

i.e.:

(7) $$\rho\dot{\zeta}^i = \frac{d}{dx^\beta}\left(\frac{\partial W_i^\beta}{\partial p_\alpha^k}\eta^k P_\alpha\right) + \frac{\partial W_i^\beta}{\partial p_\alpha^k}(\eta_\alpha^k P_\beta + \zeta^k P_\alpha P_\beta) + 2\frac{\partial^2 W_i^\beta}{\partial p_\alpha^k \partial p_\gamma^j}(\dot{p}_\gamma^{-j} + \frac{1}{2}\eta^j P_\gamma)\eta^k P_\alpha P_\beta.$$

With:

$$\dot{p}_\gamma^{+j} := \lim_{\substack{y\to(x,\tau(x))\\ y\in D_2}} \frac{\partial^2 x'^j}{\partial t\,\partial x^\gamma}(y)$$

one has, due to (1), the equation:

$$\dot{p}_\gamma^{-j} + \frac{1}{2}\eta^j P_\gamma = \frac{1}{2}(\dot{p}_\gamma^{+j} + \dot{p}_\gamma^{-j}).$$

We introduce the following abbreviations:

$$\Omega(P \mid \zeta) := \frac{1}{\rho} \frac{\partial^2 W}{\partial p^i_\beta \, \partial p^j_\alpha} P_\alpha P_\beta \, \zeta^i \, \zeta^j, \qquad \Omega^\beta_i := -\frac{1}{\rho} \frac{\partial W^\beta_i}{\partial p^k_\alpha} \eta^k P_\alpha,$$

and:

$$Q_i := \frac{1}{\rho} \frac{\partial^3 W}{\partial p^k_\alpha \, \partial p^j_\gamma \, \partial p^i_\beta} P_\beta \, \eta^j P_\alpha (\dot{p}^{+j}_\gamma + \dot{p}^{-j}_\gamma).$$

When one observes that:

$$\frac{d}{dx^\beta}\left(\frac{\partial W^\beta_i}{\partial p^k_\alpha} \eta^k P_\alpha \right) = -\frac{d}{dx^\beta}(\rho \Omega^\beta_i),$$

$$\frac{\partial W^\beta_i}{\partial p^k_\alpha} \eta^k_\alpha P_\beta = \frac{\partial W^\alpha_k}{\partial p^i_\beta} \eta^k_\alpha P_\beta = -\rho \frac{\partial \Omega^\alpha_k}{\partial \eta^i} \frac{d\eta^k}{dx^\alpha},$$

and

$$\frac{\partial W^\beta_i}{\partial p^k_\alpha} P_\alpha P_\beta \zeta^k = \frac{\rho}{2} \frac{\partial \Omega}{\partial \zeta^i} (P \mid \zeta),$$

one arrives at:

$$\rho \zeta^i = -\frac{d(\rho \Omega^\beta_i)}{dx^\beta} - \rho \frac{\partial \Omega^\beta_k}{\partial \eta^i} \frac{d\eta^k}{dx^\alpha} + \frac{\rho}{2} \frac{\partial \Omega}{\partial \zeta^i} (P \mid \zeta) + \rho Q_i$$

or

$$\rho \left\{ \frac{1}{2} \frac{\partial \Omega}{\partial \zeta^i} (P \mid \zeta) - \zeta^i \right\} = \frac{d(\rho \Omega^\beta_i)}{dx^\beta} + \rho \frac{\partial \Omega^\beta_k}{\partial \eta^i} \frac{d\eta^k}{dx^\alpha} - \rho Q_i . \tag{8}$$

The vector $\zeta = (\zeta^1, \zeta^2, \zeta^3)$ is therefore a solution of a symmetric inhomogeneous linear system of equations; the determinant of the coefficient matrix is once again precisely $F(t, x, P)$. The solution of the homogeneous linear system of equations that is associated to (8) will be generated by $\eta = (\eta^1, \eta^2, \eta^3)$. Due to the symmetry of the coefficient matrix, (8) has precisely one solution when η is orthogonal to the right-hand of the system of equations, hence, when one has:

$$\eta^i \frac{d}{dx^\beta}(\rho \Omega^\beta_i) + \rho \eta^i \frac{\partial \Omega^\alpha_i}{\partial \eta^i} \frac{d\eta^k}{dx^\alpha} - \rho \eta^i Q_i = 0 \tag{9}$$

or, due to the homogeneity of Ω^α_k in η:

$$\eta^i \frac{d}{dx^\beta}(\rho \Omega^\beta_i) + \rho \Omega^\alpha_k \frac{d\eta^k}{dx^\alpha} - \rho \eta^i Q_i = 0,$$

hence:

$$\frac{d}{dx^\beta}(\rho \eta^i \Omega^\beta_i) = \rho \eta^i Q_i . \tag{10}$$

The term $\rho\eta^i\Omega_i^\beta$ can, with the help of the ray vector $p = (p^1, p^2, p^3)$, be expressed as:

$$\Omega_i^\beta\eta^i = -\frac{1}{\rho}\frac{\partial^2 W}{\partial p_\alpha^k\,\partial p_\beta^i}\eta^k\eta^i P_\alpha = -\frac{1}{2}\frac{\partial\Omega}{\partial P_\beta}(P\,|\,\eta) = \Omega(P\mid\eta)\,p^\beta .$$

With $E := (1/2)\rho\Omega\,(P \mid \eta) = (\rho/2)\eta^2$, one obtains $2Ep^\beta = \rho\Omega(P \mid \eta)\,p^\beta$, and the solvability condition (10) becomes:

$$\frac{d}{dx^\beta}(Ep^\beta) = Q$$

with:

(11) $$Q := \frac{\rho}{2}\,\eta^j\,Q_i = \frac{\partial^3 W}{\partial p_\alpha^k\,\partial p_\delta^j\,\partial p_\beta^i}P_\alpha P_\beta\eta^i\eta^k\,\frac{1}{2}(\dot p_\gamma^{+j} + \dot p_\gamma^{-j}),$$

which ultimately delivers the divergence equation:

(12) $$\text{grad}\,E := \left(\frac{dE}{dx^1},\frac{dE}{dx^2},\frac{dE}{dx^3}\right) \quad\text{and}\quad \text{div}\,p := \sum_{a=1}^{3}\frac{dp^a}{dx^a}.$$

In order to better understand the meaning of the divergence relation, we consider the "ray lines" or "rays" of the wave motion. They are the solutions of the following system of ordinary differential equations:

$$\frac{dx^i}{ds} = \frac{p^i}{v_s}, \qquad i = 1, 2, 3 \qquad\text{with}\qquad v_s := \sqrt{p\cdot p}\,.$$

These differential equations give the change in E with the arc length s along the ray. From (12), it follows that:

(13) $$v_s\frac{dE}{ds} + E\,\text{div}\,p = Q\,.$$

We now make the assumption that one has $F = 0$ along the piece of a ray in question, while $F^{hk} = 0$ does not have to be true for all determinants F^{hk}, $h, k = 1, 2, 3$ (cf., *supra*) of the two-rowed submatrices of the coefficient matrix of the system (8); likewise, let $v_s \neq 0$.

There then exists a normalized solution $\eta_0 = (\eta_0^1, \eta_0^2, \eta_0^3)$ to the homogeneous linear system of equations:

$$\frac{1}{2}\frac{\partial\Omega}{\partial\eta^i}(P\,|\,\eta) - \eta^i = 0, \qquad i = 1, 2, 3,$$

which will be true at no place on the ray where all of the components are null. The actual wave vector is then given by:

$$\eta(s) = \lambda(s)\;\eta_0\,.$$

From this, it follows that:

$$E = \lambda^2 E_0 , \qquad E_0 := \frac{\rho}{2}(\eta_0)^2 > 0 \qquad \text{and} \qquad Q = R\lambda^2 + S\lambda^3$$

with known coefficients R and S. Equation (13) then gives:

$$A\frac{d\lambda}{ds} + B\lambda + C\lambda^2 = 0, \qquad A = v_s E_0 \neq 0, \tag{14}$$

in which the expressions B and C are known functions. One then determines λ, and therefore η, along the ray by the Ricatti differential equation (14).

In particular, if $\lambda = 0$ at any location $s = s_0$ then it follows from the uniqueness theorem that $\lambda \equiv 0$ along the entire ray that goes through this location. However, with $\lambda \equiv 0$, one has $\eta \equiv 0$. Thus, if the length of the wave vector η that measures the intensity of the wave is equal to zero at any location then it has the value zero all along the entire ray that runs through this location.

In optics, the rays are characterized precisely by a property that is analogous to the facts presented here. One thinks of, say, a light source that shines upon a lampshade. Of all of the rays that fall upon this shade the ones that fall upon the boundary of the shade will define a shadow. From our results, the rays are the lines along which the intensity of the light remains zero when it is once zero. This expresses the fact that the shadow boundary is indeed constructed from rays. This fact is true in wave optics only for a wavelength that converges to zero. If the wavelength is not zero, so one has diffraction phenomena, then there will be no sharp shadow boundary. By considering the acceleration waves, one obtains these characteristic properties of rays in their full strength.

Geometric meaning

One takes a closed surface in $\mathbf{R}^3$. Let N be the external normal to it and let do be the hypersurface element with the projections:

$$do_i = do \cos(N, x^i).$$

It then follows from (11) that the hypersurface integral is the volume integral:

$$\int E\, p^i\, do_i = \int Q\, dv .$$

As a closed surface, we now choose a "ray tube" with a normal cross-section ω. If we apply the formula above to this ray tube, which is bounded by the points 1 and 2 of the cross-section, then we obtain:

$$\omega E p^i \cos(N, x^i)\,\big|_1^2 = (E v_s \omega)_2 - (E v_s \omega)_1 = \int Q\, dv = \int_{s_1}^{s_2} Q w\, ds.$$

In the event that the differential equations are linear, so $Q \equiv 0$, one has:

$$E v_s \omega = \text{const.}$$

along the ray tube. E can be suitably regarded as the "energy of the wave."

3. Application of the Hadamard-Herglotz treatment of the acceleration waves to the discontinuity waves of Christoffel

Christoffel treated discontinuity waves of first order. Therefore, let:

$$\Sigma = \{(x, t) \mid x \in \mathbf{R}^3,\ t = \tau(x) \in \mathbf{R}\}$$

be a first order discontinuity surface, i.e., the solutions x'^i, $i = 1, 2, 3$ of the Euler equations of motion (B) are continuous on Σ, while their first partial derivatives possess (finite) jump discontinuities. All derivatives involved shall have equal two-sided limits on Σ. Furthermore, let all of the notations and definitions agree with the ones above, and one has $x'^i \in C^0(D) \cap C^2(D_1 \cup D_2)$, $i = 1, 2, 3$, with open sets D, D_1, D_2, as above.

For the first order discontinuities that were examined by Christoffel we require a jump relation on the wave surface Σ that one obtains with the help of Hamilton's principle of stationary action: From Zemplén [4], the vanishing of the first variation (the Hamilton integral) on the "edge of the kink" of an extremal $x'^i(t, x)$, $1 \le i \le 3$, with discontinuities of first order on Σ gives the Weierstrass-Erdmann corner conditions on Σ:

$$0 = \delta x x'^i (r\, [\dot{x}'^i]\,(dx) - [W_i^a]\, do_a), \qquad i = 1, 2, 3,$$

with $do_1 = -\, dt\, dx^2\, dx^3 = P_1(dx)$, $do_2 = dt\, dx^1\, dx^3 = P_2(dx)$, $do_3 = dt\, dx^1\, dx^2 = P_3(dx)$. This gives the jump relations on Σ:

$$\rho \xi^i - [W_i^a]\, P_a = 0, \qquad i = 1, 2, 3 \qquad \text{with} \qquad \xi^i := [\dot{x}'^i].$$

Christoffel treated classical elasticity; there, one has the following expression for the internal energy W:

$$W = \frac{1}{2} c_{ij}^{\beta\gamma} p_\beta^i p_\gamma^j\,; \qquad c_{ij}^{\beta\gamma} = c_{ji}^{\gamma\beta} = \text{const.} \qquad i, j, \gamma, \beta = 1, 2, 3.$$

Analogous to the second order discontinuities, one introduces:

$$\Omega(P \mid \xi) := \frac{1}{\rho}\, c_{ij}^{\beta\gamma}\, \xi^i\, \xi^j\, P_\alpha P_\beta\,.$$

This is a biquadratic form in the ξ^i, P_α with constant coefficients that are symmetric in i, j. With:

$$[p_\gamma^i] = \xi^i P_\gamma \quad \text{and} \quad [W_i^\beta] = c_{ij}^{\beta\gamma}[p_\gamma^j] = c_{ij}^{\beta\gamma}\,\xi^j P_\gamma,$$

this gives:

$$P_\alpha [W_i^\alpha] = c_{ij}^{\beta\gamma}\,\xi^j P_\alpha P_\beta = \frac{\rho}{2}\frac{\partial\Omega}{\partial\xi^i}(P\,|\,\xi).$$

Thus, there exist linear equations for $\xi = (\xi^1, \xi^2, \xi^3)$:

$$(*) \qquad 0 = \xi^i - \frac{1}{2}\frac{\partial\Omega}{\partial\xi^i}(P\,|\,\xi), \qquad i = 1, 2, 3.$$

Since $\xi \neq 0$, the determinant of the coefficient matrix of the homogeneous linear system (*) has the value $F(t, x, P) = 0$, just like the second order discontinuity waves.

We now obtain a divergence relation for $|\,\xi\,|$ in the following way: From the Euler equations of motion:

$$\rho\ddot{x}'^i = X^i + \frac{dW_i^a}{dx^a}, \qquad i = 1, 2, 3,$$

it follows, under the assumption that $[\rho] = 0$, $[X^i] = 0$, and since:

$$W = \frac{1}{2}c_{ij}^{\alpha\beta}\,p_\alpha^i\,p_\beta^j;$$

that

$$\rho\eta^i = c_{ij}^{\alpha\beta}[p_{\alpha\beta}^j], \qquad i = 1, 2, 3.$$

An easy calculation (with the help of the Hadamard lemma) delivers:

$$[p_{\alpha\beta}^i] = [\dot{p}_\beta^i]P_\alpha + [\dot{p}_\alpha^i]P_\beta + \xi^i P_{\alpha\beta} - \eta^i P_\alpha P_\beta, \quad i, \alpha, \beta = 1, 2, 3$$

and

$$\xi_\beta^i := \frac{d}{dx^\beta}\xi^i = [\dot{p}_\beta^i] - \eta^i P_\beta;$$

i.e.:

$$[p_{\alpha\beta}^i] = \eta^i P_\alpha P_\beta + \xi_\beta^i P_\alpha + \xi_\alpha^i P_\beta + \xi^i P_{\alpha\beta}.$$

From this, one gets:

$$\eta^i - \frac{1}{2}\frac{\partial\Omega}{\partial\eta^i}(P\,|\,\eta) = \frac{1}{\rho}c_{ij}^{\alpha\beta}\,(\xi_\alpha^i P_\beta + \xi_\beta^i P_\alpha + \xi^i P_{\alpha\beta}).$$

Hence, with:

$$Q_i' := \frac{1}{\rho}c_{ij}^{\alpha\beta}\,(\xi_\alpha^i P_\beta + \xi_\beta^i P_\alpha + \xi^i P_{\alpha\beta}),$$

one has:

(**) $$\eta^i - \frac{1}{2}\frac{\partial\Omega}{\partial\eta^i}(P\,|\,\eta) = Q_i', \qquad i = 1, 2, 3.$$

The coefficient matrix of the inhomogeneous linear system of equations (**) agrees with the coefficient matrix of the homogeneous linear systems of equations (*), and is symmetric. $\eta = (\eta^1, \eta^2, \eta^3)$ is therefore indeed a solution of (**), as long as one has:

$$\xi^i Q_i' = 0 \qquad \text{(orthogonality condition).}$$

One sets:

$$\Omega := \Omega(P\,|\,\xi) \qquad \text{and} \qquad p^a := -\frac{1}{2\Omega}\frac{\partial\Omega}{\partial P_a}, \qquad a = 1, 2, 3.$$

One thus obtains:

$$-\rho\Omega p^a = \frac{\rho}{2}\frac{\partial\Omega}{\partial P_a} = c_{ij}^{\alpha\beta}\,\xi^i\,\xi^j P_\beta\,,$$

and furthermore, since $c_{ij}^{\alpha\beta} = c_{ji}^{\beta\alpha}$:

$$\begin{aligned}-\frac{d}{dx^a}(\rho\Omega p^a) &= c_{ij}^{\alpha\beta}(\xi_\alpha^i\xi^j P_\beta + \xi^i\xi_\alpha^j P_\beta + \xi^i\xi^j P_{\alpha\beta}) \\ &= c_{ij}^{\alpha\beta}(\xi_\beta^i P_\alpha + \xi_\alpha^j P_\beta + \xi^j P_{\alpha\beta})\xi^i = \rho Q_i'\xi^i\,.\end{aligned}$$

Together with the orthogonality conditions, this gives:

$$-\frac{d}{dx^a}(\rho\Omega p^a) = -\frac{d}{dx^a}(\rho\,|\,\xi\,|^2\;p^a) = 0.$$

This is the desired divergence relation. Christoffel [1] has already discussed the geometric corollaries of it concerning the ray tube (cf., *supra*).

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