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On the principles of Hamilton and Maupertius

By

O. Hölder in Tübingen

Translated by D. H. Delphenich

(Submitted by **F. Klein**)

In the introduction to his *Mechanik,* **Heinrich Hertz** ([1]) said that **Hamilton**'s principle often yields results that are physically false. In order to document that, he cited a case in which one could, as he himself remarked, assess the motions that can be performed, as well as the ones that would correspond to **Hamilton**'s principle, through a mere consideration without calculation. **Hertz** added that the result would not change if one employed the **Maupertuisian** principle of least action, instead of **Hamilton**'s principle. Let us consider an example. It consists of a ball whose entire inertia rolls without slipping on a fixed horizontal plane ([2]). According to **Hertz**, the motions that correspond to **Hamilton**'s principle here are the ones that arrive at the given goal in shortest time for a given constant *vis viva*, which would imply that passing from any initial point to any final point would have to be possible without the action of a force. That conclusion, which is more closely connected with the principle of least action than it is with that of **Hamilton**, was reached approximately. If one chooses the initial and final positions of the ball arbitrarily then there will always be a pure rolling passage ([3]) from the one to the other. Among those transitions, each of which should come about at constant *vis viva* and all of which should have the same constant *vis viva*, there will be one of them that takes the least time ([4]). In **Hertz**'s opinion, that will correspond to **Hamilton**'s principle and the principle of least action. **Hertz** contrasted that result with the fact that in reality, despite the arbitrariness that the initial velocity is stuck with, no natural transition from any position to any other one is possible without forces being involved.

([1]) *Gesammelte Werke*, 1894, Bd. III, pp. 23

([2]) The ball does not need to be homogeneous. If it were homogeneous then a uniform motion of the center of the ball, combined with a uniform rotation of the ball around a fixed axis that goes through the center, would occur.

([3]) In regard to the existence of that passage, one should cf. the final remark of § **12**. The fact that the passage, which will be contrived here for the proof, is also one of rolling without slipping was not stated expressly by **Hertz** at that point. However, one could not arrive at the same conclusion without establishing that. The fact that I have echoed **Hertz**'s opinion correctly by adding the proof of the latter will emerge from numbers 347, 358, 112, 111.

([4]) That might be added here, although it could still be challenged from a rigorous mathematical standpoint.

The aforementioned truth, which was self-explanatory for **Hertz**, can, in fact, be inferred by mere observation. To that end, we need only to consider that the motion must be determined by the initial state of the ball. Other than the initial position, all that one needs to determine the initial state are the instantaneous axis of rotation (which we shall assume goes through the center of the ball), the associated angular velocity, and a displacement velocity. However, the magnitude and direction of the displacement velocity will then be determined, because rolling with slipping must take place on the horizontal plane. Nothing can be said about the magnitude of the initial rotational velocity. However, since the initial rotational axis can be chosen in a double infinitude of ways, and each choice of axis will lead to a simply-infinite manifold of positions for the ball, one can arrive at only a triple infinitude of positions when one starts from a given position. By contrast, the totality of all positions of the ball define a five-fold manifold, because the center can be placed in a double infinitude of ways, while the ball can be positioned about its center in a triple infinitude of ways. That implies the impossibility of going from a given position to any other one without the action of forces.

The attempt to clarify the contradiction that comes about from the fact that, strictly speaking, no rolling occurs in nature that is not coupled with at least a small amount of slipping did not satisfy **Hertz**. It also emerges from the foregoing well enough that here one is not dealing with a contradiction in which ordinary mechanics should take the advice of experiments as much as with contradictory conclusions of the different arguments. The contradiction must then be removed from the theory.

The thorough developments in **Hertz**'s book contain such a solution. In order to understand that, one must focus on the condition equations by which the motion of a material system can be constrained. **Hertz** allowed only condition equations that did not contain time. However, the coordinates of the points of the system could also appear in the form of differentials. More precisely, the condition equations are assumed to have the form ([1]):

$$(1) \qquad \sum_{(\nu)} (\varphi_{i\nu}\, dx_\nu + \psi_{i\nu}\, dy_\nu + \chi_{i\nu}\, dz_\nu) = 0 \qquad (i = 1, 2, \ldots),$$

in which the symbols φ, ψ, χ denote dimensionless functions of the coordinates:

$$x_1, y_1, z_1, \qquad x_2, y_2, z_2, \qquad x_3, y_3, z_3, \qquad \ldots$$

of the material points. Now, there is a special case in which the totality of conditions (1) is equivalent to a complex of conditions of the form:

$$(2) \qquad d\Phi_1 = 0, \qquad d\Phi_1 = 0, \qquad \ldots;$$

i.e., one that is "completely integrable." In that case, **Hertz** called the material system *holonomic* ([2]). His solution to the previous contradiction is this: The basic laws of mechanics that he presented were true in general for both holonomic systems and non-holonomic systems, but he arrived at **Hamilton**'s principle and the principle of least

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([1]) Cf., no. **124**. **Voß** already treated such conditions before; cf., Math. Ann., v. **25** , pp. 258 *et seq.*
([2]) Cf., no. **123**, **132**, **133**.

action only by adding the restriction to holonomic systems. The ball that rolls on the plane presents a non-holonomic system, which the tendency to slip would destroy.

If that solution were satisfied then that would not contradict the general belief that **Hamilton**'s principle is merely another form of **d'Alembert**'s, and that this would be true in general. The deviation from the usual picture of **Hertz**'s theory can also not explain the fact that it has placed a new law at the foundations, since his basic law is equivalent to **d'Alembert**'s principle in the cases that he considered ([1]). That raises the basically-mathematical question: Does the usual derivation of **Hamilton**'s principle from **d'Alembert**'s require a restricting condition? The present article will serve to answer that question. That answer that it will give is that when **d'Alembert**'s principle is true in general, **Hamilton**'s must also be generally true in its most complete formulation. However, if one chooses the formulation that **Hertz** assumed then restriction that he pointed out will, in fact, enter into it. In this paper, I will explain yet another point, and in more detail that it has been given up to now: First of all, the concept of the variation of a motion itself will be discussed, and then the forms that the principle of least action can take, along with the relationship of that principle to **Hamilton**'s, which can encompass both principles with a more general integral principle. At the same time, it will be shown that the principle of least action can also be formulated in such a way that it will remain valid when time enters into the condition equations.

Whenever both principles are at issue, above all, I would like to at least suggest that here by once more considering the motions of the ball. During its actual motion, which is one of pure rolling, the ball will assume a continuous succession of positions. The application of the aforementioned principles will demand only a small change in the motion. In order to accomplish that, we will first displace each of the original running positions of the ball slightly, such that a second continuous succession of positions will

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([1]) Confer **Hertz**'s no. 394. As far as his basic law is concerned, which he unnecessarily restricted to *free* systems (nos. 309, 122, 117), it includes two statements: One of them determines the constancy of the differential quotients with respect to time *ds* / *dt*. The quantity *s* is the defined by the equation:

$$ds^2 \sum_{(\nu)} m_\nu = \sum_{(\nu)} m_\nu (dx_\nu^2 + dy_\nu^2 + dz_\nu^2),$$

in which $m_1, m_2, \ldots$ mean the masses of the system points. Obviously, that part of the basic law is nothing but the law of conservation of *vis viva*. The other part is derived from the fact that the quantity:

$$\sum_{(\nu)} m_\nu \left\{\left(\frac{d^2 x_\nu}{ds^2}\right)^2 + \left(\frac{d^2 y_\nu}{ds^2}\right)^2 + \left(\frac{d^2 z_\nu}{ds^2}\right)^2\right\}$$

is continually a minimum under the motion. If one sets $s = \text{const.} \times t$ in the last sum then one will obtain essentially the same expression that was supposed to be a minimum in **Gauss**'s principle of least pressure, in which all of the forces that **Hertz** excluded from the foundations are set to zero. One should compare this to my own presentation of his basic law in **Hertz** nos. 309, 266, 263, 55, 100, 106, 151, 152, 153. I have once more introduced the usual notation of rectangular coordinates in place of his notation for coordinates.

In my opinion, the significance of **Hertz**'s book does not take the form of a basic law, but the fact that the forces can be nonetheless constructed mathematically from a basic law that does not contain forces, as it is now formulated. I shall go no further into that construction, which first appears in the later parts of his book and lies beyond the scope of the present study.

arise. At the same time, the positions of this new sequence can be related to the positions of the first sequence. The second motion is still not determined completely in that way, since it has not been stated that the corresponding positions of the two motions will be passed through at the same time. That is required by **Hamilton**'s principle, while the principle of least action requires something else. However, both principles will be applied here in such a way that aforementioned small displacement of the ball should result from a simple rolling motion, while **Hertz**, in contrast to that, employed the condition that the second motion – i.e., the varied one – should also itself exhibit rolling without slipping. If one performs the variations in the correct way then that will imply the rolling motion of the ball in precisely the way that **Hertz** said corresponds to the facts. It is not distinguished *from the motions of its kind*, even when it, in fact, requires less time. However, we have lived upon a different basis for a long time that conceptualizes the principle of least action and **Hamilton**'s principle only to the extent that the variation of an integral or an integral that contains a variation should be set to zero. Of course, the name "principle of least action" will no longer be appropriate then.

§ 1. – Variation of a motion.

In order to make the concept of the variation of a motion clearer, we would first like to consider a free material point. Its motion should be varied in such a way that the initial position A and final position B will remain unchanged. The original motion is the one that actually takes place, while the new varied one is only an auxiliary mathematical notion. Now, one can choose the path of the new motion from A to B such that it differs slightly from the old path and runs approximately parallel to it ([1]), and arbitrarily, in general. After that, one let can the motion along the new path evolve over time in various ways. We imagine that both motions begin at A at the same time. They do not need to arrive at B at the same time, which will not be the case, e.g., when the actual motion takes less time than the varied one. Now, in order to have a precise picture of the variation in mind, one must refer each position that is assumed by the varied motion to a position that was assumed in the original motion ([2]). Without such a relation, e.g., the variation of the integral:

$$\int T\,dt\,,$$

in which T represents the *vis viva* and t represents the time, would probably be meaningful, but the equation:

$$\delta\int T\,dt = \int \delta(T\,dt)$$

would make no sense. One associates the (identical) initial positions with each other, and similarly, the final positions. In that way, it is clear that in the event that the motions do not arrive at B at the same time, the relation could not be presented in such a way that corresponding positions of both motions would be passed through simultaneously. One

([1]) Cf., the first remark in § **2**.

([2]) Naturally, this important state of affairs has already been observed in the geometric problems of the calculus of variations; **Weierstrass** always emphasized that in his own lectures.

would then produce an arbitrary point-wise relationship between the two paths, and observe that corresponding positions are separated from each other by very little distance ([1]). One might wonder if that point-wise association of the paths is physically meaningless here, and whether that association, like the variation of the motion itself, is only an auxiliary mathematical construction. We shall consider the simpler way of expressing things, so for the moment, the time-point at which both motions begin at A will be the origin of time. Now, if C and C' are associated positions of the motions then one could logically refer to the time that it takes for the original motion to flow from A to C by τ and the time that it takes for the varied motion to flow from A to C by $\tau + \delta\tau$. *The variation* $\delta\tau$ *of time is therefore nothing but the difference between the times at which corresponding positions will be passed through.* The variation of the time differential is the algebraic overshoot of the time that it takes a small part of the new motion to flow over the time that is used by the associated part of the old motion ([2]). If one compares the initial and final times for both motions along those small pieces then one will easily see that the variation of the time differential is equal to the differential of the time variation. That corresponds to the known general theorem on the commutability of the symbols d and δ.

Now, the variation of the motion of our points would be best carried out as follows: One first gives each point of the original path a displacement such that a new path will arise that related to the old one point-wise. One then determines the velocity for each point of the new path. It must differ from the velocity at the corresponding location on the old path only slightly, but it can otherwise be taken to be arbitrary. However, we shall now distinguish between two special ways of doing that determination.

The *first* kind of variation arises from the condition that *corresponding locations on both paths are passed through simultaneously;* both motions would then have to arrive at B at the same time.

The *second* kind of variation relates to the forces under whose action the original motion proceeded. If we imagine the forces here in such a way that we can speak of a "potential energy" then we can define this kind of variation as follows: *The total energy of the corresponding states of the motions being compared must be the same.* That variational condition will be formulated somewhat differently later on such that it will also be suitable for the remaining cases. The total energy is composed of the *vis viva* and the potential energy. Now, since the original motion is thought of as being given, the *vis viva* and the potential energy will also be given for a location C on its path. For the corresponding location C' on the varied path, at first, only the potential, which depends upon just the position, will be known. One will then get the *vis viva* for the location C' from the variational condition that is required here, and thus, the velocity.

Hence, once the new path and its point-wise relationship to the old path has been established, the varied path will be determined completely by the first variational condition, as well as the second one, and in different ways each time. *The time is varied for the second kind of variation, but not for the first one.*

([1]) More precisely, two corresponding, infinitely-small arcs of both paths must have a well-defined ratio for each location, and that ratio should differ from one only slightly. On that subject, cf., the first rem. in § **2**.

([2]) A more rigorous use of pure mathematics, that would distinguishes between differentials and changes and between variations and changes, would be impractical here.

The relationships for the motion of a material system are analogous. If we, with **Hertz**, take the concept of the "position of a system" to include the totality of all positions of the points of the system then motion will consist of a continuous succession of system positions that follow in time in a certain way. In order to vary that original motion, one will first assign a small displacement to each system position such that a new continuous succession of system positions will arise. If the original sequence goes through a position twice then one will have two overlapping motions that can naturally be displaced in different ways. The new paths of system points and the association between the locations on those paths are now established. For that reason, one can choose the velocity at all locations of the new path for a system point for the most general kind of variation. However, if one establishes that either the new positions are passed through at the same times as the associated old positions or that the two associated states of both motions should have the same energy then one would determine how the new succession of system positions is to be passed through completely in that way.

Previously, we did not take any condition equations into account. If a motion is subject to some conditions then it will not be excluded in that way that we can compare it to a varied motion that does not satisfy those conditions.

§ 2. – Derivation of the integral principles.

I shall now consider a material system that moves under the influence of forces and the simultaneous constraint of condition equations in the sense of ordinary mechanics. Time can once more enter into the condition equations. It will suffice to assume that the coordinates are rectangular. Now, when I vary the motion, I shall temporarily not concern myself with the condition equations at all. If m_1, m_2, … are the masses of material points then that will imply the variation of the *vis viva*:

$$\delta T = \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\,\delta\frac{dx_\nu}{dt} + \frac{dy_\nu}{dt}\,\delta\frac{dy_\nu}{dt} + \frac{dz_\nu}{dt}\,\delta\frac{dz_\nu}{dt} \right). \tag{3}$$

Now, one has, e.g. ([1]):

([1]) This formula from the calculus of variations must be used here, since the quantities that are based upon differentiation will be varied. If one would like to avoid that then one would have introduce yet another variable ϑ, as e.g., **v. Helmholtz** did. One then relates the positions of the original motion to the values of the parameter ϑ and associates the corresponding positions in the varied motion to the same values of ϑ. ϑ will not be varied in that way, but possibly the time t. The following picture is especially intuitive: Let τ be the time that it takes for the initial position A of the system to flow to the position C of the original motion, and let $\tau + \delta\tau$ be the time that elapses between the initial position and the corresponding position C' of the varied motion. All quantities, including $\delta\tau$, can be regarded as functions of τ. Now, $\delta(dx_\nu / dt)$ is the difference between the velocity components, taken along the x-axis, of the mass m_ν for the varied and unvaried motion. Obviously, one will then have:

$$\delta\frac{dx_\nu}{dt} = \frac{d(x_\nu + \delta x_\nu)}{d(\tau + \delta\tau)} - \frac{dx_\nu}{dt} = \frac{\dfrac{d}{d\tau}(x_\nu + \delta x_\nu)}{\dfrac{d}{d\tau}(\tau + \delta\tau)} - \frac{dx_\nu}{dt}$$

$$\delta \frac{dx_\nu}{dt} = \frac{\delta dx_\nu \cdot dt - \delta dt \cdot dx_\nu}{dt^2} = \frac{d\,\delta x_\nu \cdot dt - d\,\delta t \cdot dx_\nu}{dt^2}.$$

If one now converts the right-hand side of (3) with the help of that equation and its analogous equations then one will find that:

$$\delta T = \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\frac{d\,\delta x_\nu}{dt} + \frac{dy_\nu}{dt}\frac{d\,\delta y_\nu}{dt} + \frac{dz_\nu}{dt}\frac{d\,\delta z_\nu}{dt} \right) - 2T\frac{d\,\delta t}{dt}.$$

This equation should be multiplied by dt and integrated over the time interval $t_0 \dots t_1$ in which the original motion takes place. After a partial integration, one will get ([1]):

(4) $$\int_{t_0}^{t_1} \delta T \cdot dt = -\int_{t_0}^{t_1} \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\frac{d\,\delta x_\nu}{dt} + \frac{dy_\nu}{dt}\frac{d\,\delta y_\nu}{dt} + \frac{dz_\nu}{dt}\frac{d\,\delta z_\nu}{dt} \right) dt - 2\int_{t_0}^{t_1} T\,d\,\delta t.$$

The position of the system is thought of as being unvaried for t_0 and t_1, which will make the terms that appear before the integral as a result of the partial integration vanish.

Now, if X_ν, Y_ν, Z_ν are the components of the force that acts upon the mass m_ν then the symbol $\delta' U$ shall be defined by the formula:

(5) $$\delta' U = \sum_{(\nu)} (X_\nu \delta x_\nu + Y_\nu \delta y_\nu + Z_\nu \delta z_\nu).$$

Equation (5) is once more multiplied by dt and integrated and then added to (4), which will yield:

(6) $$\int_{t_0}^{t_1} [2T\,d\,\delta t + (\delta T + \delta' U)\,dt]$$

$$= \int_{t_0}^{t_1} dt \sum_{(\nu)} \left\{ \left(X_\nu - m_\nu \frac{d^2 x_\nu}{dt^2} \right)\delta x_\nu + \left(Y_\nu - m_\nu \frac{d^2 y_\nu}{dt^2} \right)\delta y_\nu + \left(Z_\nu - m_\nu \frac{d^2 z_\nu}{dt^2} \right)\delta z_\nu \right\}.$$

$$= \left(\frac{dx_\nu}{d\tau} + \frac{d\,\delta x_\nu}{d\tau} \right)\left(1 + \frac{d\,\delta\tau}{d\tau} \right) - \frac{dx_\nu}{dt}.$$

If one now develops this and neglects terms of higher order in the derivatives of the variations then one will get:

$$\frac{d\,\delta x_\nu}{d\tau} - \frac{dx_\nu}{dt}\frac{d\,\delta\tau}{d\tau};$$

viz., the formula in the text. At the same time, one will see that not only must the variations be assumed to be small, but also their derivatives.

([1]) This formula is basically already in **Serret**, Comptes rendus de l'Acad. des Sciences **72** (1871), pp. 700, no. (7).

At the same time, if one performs that variation of the motion such that the quantities δx_ν, δy_ν, δz_ν, represent a *virtual* displacement of the system then the right-hand side of the last equation will be equal to zero, from **d'Alembert**'s principle. One then has the theorem:

If one compares the actual motion of a material system with a motion that deviates from it slightly and for which the starting and ending positions of the system remain unvaried, and the displacements of each position of the actual motion to the corresponding positions of the varied motion are virtual displacements then ([1]):

(7) $$\int [2T\, d\delta t + (\delta T + \delta' U)\, dt] = 0.$$

In this equation, T means the vis viva and $\delta' U$ means the work that is done by the effective forces under the aforementioned displacement, which is merely imagined.

Here, one can specialize the variations by introducing the first or second of the special kinds of variation that were presented in § **1**.

1) We demand that the corresponding positions in the actual and varied motions must be passed through at the same time (i.e., we set $\delta t = 0$) and obtain:

$$\int (\delta T + \delta' U)\, dt = 0.$$

That is **Hamilton**'s principle.

2) We generalize the previous second kind of variation by setting:

(8) $$\delta T = \delta' U.$$

We then require that the difference between the *vis viva* for corresponding states of the two motions should be equal to the work that the effective forces do a displacement that connects corresponding positions. That will determine how the continuous succession of varied positions should be traversed. One can then replace the quantity $\delta' U$ in (7) with δT and then get:

$$0 = \int (T\, d\delta t + \delta T\, dt) = \int (T\, \delta dt + \delta T\, dt) = \int \delta(T\, dt)$$

for those special variations; i.e. ([2]):

$$0 = \delta \int T\, dt\,.$$

([1]) Actually, the integral will only be infinitely-small of higher order when the quantities that were referred to as small up to now are made infinitely-small of order one.

([2]) The validity of the formula $\int \delta(T\, dt) = \delta \int T\, dt$ will not be impaired by the fact that the time interval changes under the variation. In order to see that, one subdivides $\int T\, dt$ into its elements and subtracts from each of those elements the quantity that corresponds to it in the varied motion.

That is *the principle of least action in its extended form* ([1]).
The other form of that principle will be discussed in § **4**.

§ 3. – Virtual displacements. Equivalence of the principles.

The concept of virtual displacement is to be understood here in precisely the same way as in the analytical formulation of **d'Alembert**'s principle. According to that principle, when one takes into account the couplings between the material points that exist at the moment, the external forces will be in equilibrium at each time. For example, if the material points are constrained to move in accordance with the conditions:

$$\omega_\iota (x_1, y_1, z_1, \ldots, x_r, y_r, z_r ; t) = 0 \qquad (i = 1, 2, \ldots) \tag{9}$$

then one must introduce the momentary value for t. Those momentary couplings allow one to have displacements that satisfy the equations:

$$\frac{\partial \omega_\iota}{\partial x_1}\delta x_1 + \frac{\partial \omega_\iota}{\partial y_1}\delta y_1 + \frac{\partial \omega_\iota}{\partial z_1}\delta z_1 + \cdots + \frac{\partial \omega_\iota}{\partial x_r}\delta x_r + \frac{\partial \omega_\iota}{\partial y_r}\delta y_r + \frac{\partial \omega_\iota}{\partial z_r}\delta z_r = 0. \tag{10}$$

Those displacements are virtual, and they are introduced into the equilibrium condition for the external forces; i.e., into:

$$\sum_{(\iota)} \left\{ \left(X_\nu - m_\iota \frac{d^2 x_\nu}{dt^2} \right) \delta x_\nu + \left(Y_\nu - m_\iota \frac{d^2 y_\nu}{dt^2} \right) \delta y_\nu + \left(Z_\nu - m_\iota \frac{d^2 z_\nu}{dt^2} \right) \delta z_\nu \right\} = 0.$$

The fact that there is no term of the form $(\partial \omega_\iota / \partial t)\ \delta t$ in equation (10) will also be paraphrased by the remark that time is not varied in the application of **d'Alembert**'s principle. We will satisfy that prescription when a variation of time takes places in a different way. Taking into account what we must logically denote by $\delta\omega_\iota$ here, the equations that determine the virtual displacements:

([1]) **v. Helmholtz** discussed this form of the principle in detail in the Sitzungsberichten der Berliner Akademie for 1887 (**Helmholtz**'s *Ges. Abhandlungen*, 1895, Bd. III, pp. 249). It would probably be better to refer to the quantity F that he referred to as "potential energy" as the "negative force function." Namely, since F must include time, as well as the coordinates, the equation that motivates the term "potential energy" would break down. One can object to the presentation that **Helmholtz** gave in other ways. If one compares the equations on pp. 259 that were denoted by 1_f and 1_g then that will show that the term $(\partial F / \partial t)$ δt in the development of the variation δF will drop out and that it is only in that way that the equations of motion will be obtained correctly. **Helmholtz** based his procedure on the remark that F can also be regarded as a function of the coordinates and of ϑ, instead of t (cf., by first rem. in this paragraph). Now, ϑ will not be varied; however, it is expressly assumed that time will be varied. t will then be a different function of ϑ for the varied motion that it was for the original motion. For that reason, the process is not permissible. However, the entire presentation will become correct when one regards the quantity that **Helmholtz** referred to as δF as the work. Correspondingly, the variational condition must also be formulated differently then.

$$\delta\omega_\iota - \frac{\partial\omega_\iota}{\partial t}\,\delta t = 0. \tag{11}$$

If the motion of the system is subject to condition equations of the form (1):

$$\sum_{(\nu)} (\varphi_{\iota\nu}\, dx_\nu + \psi_{\iota\nu}\, dy_\nu + \chi_{\iota\nu}\, dz_\nu) = 0 \qquad (i = 1, 2, \ldots),$$

in which the functions φ, ψ, χ depend upon only the coordinates, then the virtual displacements will satisfy the relations ([1]):

$$\sum_{(\nu)} (\varphi_{\iota\nu}\, \delta x_\nu + \psi_{\iota\nu}\, \delta y_\nu + \chi_{\iota\nu}\, \delta z_\nu) = 0 \qquad (i = 1, 2, \ldots).$$

It should be remarked that in all cases the displacement of the individual positions of the system are independent of each other. For that reason, the displacements can also be assumed to be non-zero only for an infinitely-small part of the motion. If one links that concept with equation (6) then that will imply a known argument from the calculus of variations that the continual vanishing of the left-hand side of (6) will also bring with it the vanishing of each individual element in the integral that one finds on the right-hand side. The demand that the integral (7) should vanish for all variations will once more imply the fulfillment of **d'Alembert**'s principle. Let us consider the right-hand side of (6) in more detail. We think of the forces and actual motion of the material system as being given. Hence, the aforementioned right-hand side is determined merely by the displacements of the system positions. It does not depend upon how the new succession of positions in time that arises from the displacements is run through. For that reason, it does not matter whether we let the variation of the motion be general, except for the unallowable conditions, or we restrict ourselves to the first or second of the special kinds of variations.

That and the contents of the previous paragraphs will then imply that **Hamilton**'s principle, as well as the principle of least action in the form above, will be equivalent to **d'Alembert**'s principle ([2]).

([1]) The analogy leads to the suggestion that **Voß** made (*loc. cit.*, pp. 286) that one should take the condition in the motion in the form:

$$\sum_{(\nu)} (\varphi_{\iota\nu}\, dx_\nu + \psi_{\iota\nu}\, dy_\nu + \chi_{\iota\nu}\, dz_\nu) = 0 \qquad (i = 1, 2, \ldots),$$

in which the functions φ, ψ, χ, ω include the coordinates and time. That will yield the equations of the virtual displacements when one replaces dt by 0 and dx_ν, dy_ν, dz_ν with δx_ν, δy_ν, δz_ν. A suitable example of that would be that of a ball that rolls without slipping on a plane, while the plane advances in a prescribed manner in time.

([2]) The derivation of the differential equations of motion is also given by **Hamilton**'s principle and the principle of least action then, while new coordinates can likewise be introduced. It was given in symbolic form by **Voß** (*loc. cit.*, pp. 263), while our concept of the variation of a motion will always permit an actual application of the principle. The derivation of the equations from the least-action principle has led to various discussion (cf., **Rodrigues**, Correspondance sur l'école impériale polytechnique publ. par Hachette, vol. III, pp. 159, and **A. Mayer**, Ber. d. K. Sächs. Ges. d. W. math.-phys. Cl. 1886, pp. 343). It is simplest

§ 4. – Modifications of the principles.

If a force function U exists then equation (5) will assume the form:

$$\delta' U = \sum_{(\iota)} \left(\frac{\partial U}{\partial x_\nu} \delta x_\nu + \frac{\partial U}{\partial y_\nu} \delta y_\nu + \frac{\partial U}{\partial z_\nu} \delta z_\nu \right).$$

Now, when U includes the time t, along with the coordinate, in the event that time is not varied, one will then have:

(12) $$\delta' U = \delta U,$$

and one can express **Hamilton**'s principle by means of the equation:

$$\delta \int (T + U)\, dt = 0.$$

If one is dealing with the variations that the least-action principle demands then one must have a force function U that is free of time if equation (12) is to be true. The variational condition (8) can then be expressed in such a way that the quantity $T - U$ should have the same value for two corresponding states of the actual and varied motion. Now, if time does not enter into the condition equations, in addition, which might then be differential equations of the form (1) or ordinary equations, then $T - U$ would be constant for the actual motion ([1]). One then calls $- U$ the potential energy and $T - U$, the total energy, and it is clear that the total energy will not change at all during either the motion or the variation. In that way, one will get the following restricted form of the least-action principle:

That form of the principle assumes that the actual motion obeys the law of the constancy of energy, and that motion will determine more precisely the fact that when one compares it with a motion that deviates slightly from it and has the same constant energy motion, it will fulfill the condition:

$$\delta \int T\, dt = 0.$$

In that way, the variations of positions will be virtual displacements, and the initial and final positions will remain unvaried. That more restrictive form is applicable when a

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to follow our path above backwards. One develops $\delta \int T\, dt = \int (\delta T \cdot dt + T \cdot \delta dt)$ for arbitrary coordinates and with the help of equation (8), one then replaces the quantity δdt that enters here explicitly and implicitly with an expression that is multiplied by dt and includes variations of position and their derivatives. One removes those derivatives by partial integration. An integral will arise that is analogous to the right-hand side of (6). When that integral is set to zero, while observing that the virtual displacements are independent of the individual system positions, that will yield the differential equations of motion.

([1]) Cf., § **5**. Cf., also **Voß**, *loc. cit.*, pp. 266.

force-function that is independent of time exists, and time does not appear in the condition equations, either ([1]).

§ 5. – The broader form of the principle of least action and the law of conservation of energy.

The restricted form of the last-action principle assumes the law of the constancy of energy, but not the broader form. We can also derive the law of constancy of energy from the broader form of the principle when we assume that there is a force function that is free of time, along with the condition equations. We assume that the law is unknown and we imagine that, e.g., the quantity $T - U$ continually increases, in the algebraic sense, during a time interval (t', t'') of the actual motion. Now, the positions that are taken when $t \leq t'$ and $t \geq t''$ might not be displaced. Every position C that is taken when $t' < t < t''$ will be varied into one C' that will itself be taken in the actual motion, but at a later time-point

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([1]) Insofar as one can already speak of a definite principle with **Maupertuis**, and which was shared by the ideas of others – in particular, **Euler** – I would not like to examine it here (cf., **A. Mayer**, *Geschichte des Princips der kleinsten Action*, 1877, and **Helmholtz**, *loc. cit.*). One finds a derivation of the least-action principle by **Lagrange** in his *Mécanique analytique* (2nd ed., 1811, t. I, pp. 296, *et seq.*). He assumed that the equation that expresses the law of conservation of energy will continue to be true under the variation, with no change in the constants that enter into it. In that way, he obtained a relation that corresponded to equation (8) in the present article. One must conclude from the stated assumption that **Lagrange** had the restricted form of the principle in mind at the cited place. However, if one introduces equation (8) directly as a variational condition then **Lagrange**'s method of proof will remain completely unchanged, and if one asks what the narrowest assumptions would be under which it is still true then one be led to a broader form of the principle. One will also find that form suggested in his earlier work in the *Miscellanea Taurinensia*, t. II, 1760-1761 (*Oeuvres*, 1867, t. 1, pp. 365, *et seq.*). Namely, it will be stated in no. XIII that the relation (U) that is presented in no. VIII, which is nothing but our equation (8), can be employed in the case of completely-arbitrary forces. Most of **Lagrange**'s followers have taken only the more restricted form of the principle, such as, e.g., **Hamilton** in the Philosophical Transactions of 1834, pp. 253. **Jacobi** has granted that form of the principle with another expression, in which he expressed the quantity $R\,dt$ under the integral in terms of space-elements and a constant that is nothing but the value of $T - U$, which is constant and unvaried here. (*Vorlesungen über Dynamik*, 1866, 6th Lecture) **Helmholtz**, in the cited work, was the first to distill the broader form of the principle from **Lagrange**'s works. As far as the relationship between that form of the least-action principle and **Hamilton**'s principle (Philosophical Transactions, 1835, pp. 99) is concerned, in contrast to **Helmholtz**, I find that both of them can be obtained from each other rigorously. Since both of them are equivalent to **d'Alembert**'s principle, they will also be consequences of each other. Nevertheless, neither of the two principles is subordinate to the other one, since they relate to different kinds of variations. However, both principles are implied when one specializes the integral principle that was contained in equation (7) in this article, in which, the variations of the motion were more general. Integral (7) has a close relationship to **Helmholtz**'s integral formula 2_b :

$$\delta \int \left[\lambda F + \left(1 + \frac{dt}{d\vartheta}\right) L \right] d\vartheta .$$

Namely, if one carried out the variation under the integral here, in which one leaves λ unvaried, varies dt, and (see the last rem. in § **2**) replaces δF with a form of work (viz., the $-\delta' U$ in this article) then an integral will come about that will coincide with one-half of our integral (7) for $L = T$, $\lambda = -1/2$, $\vartheta = t$. In order to be able to set $\vartheta = t$ *after the variation*, one must only regard ϑ as the time that is required to reach a certain position under the actual motion.

that lies between t' and t''. For that reason, since time does not appear in the condition equations, those displacements will be virtual displacements ([1]). From the variational conditions, in the form that they can now be expressed, where a force function that is free of time exists, $T - U$ must have the value for the position C of the varied motion that is had for the position C' of the actual motion. Now, we have imagined that $T - U$ increases from C to C' under the actual motion. $T - U$, and therefore the *vis viva* T, as well, must be smaller for the varied motion than it was for the actual one. We can perhaps assume that the ratio of these two *vis vivas* is like $\varepsilon^2 : 1$, where $\varepsilon < 1$. Now, under the transition to the system position C', all velocities of the varied motion will have the ratio $\varepsilon : 1$ with the ones that occur for the actual motion, since the system paths of both motions coincide. One now compares two small intervals of the varied and actual motion, and indeed intervals for which position are traversed that differ only slightly from C', and which are not regarded as associated under the variation then. The elapsed times would then relate to each other like $1 : \varepsilon$, while the partial integrals will relate like $\varepsilon : 1$. Hence, the partial integral $\int T\,dt$ that extends from t' to t'' and represents the "action" will be reduced by the chosen variation. A more precise consideration will show that the other part of the integral will not vary. That will imply a reduction in the total integral, and it can be shown that this reduction will generally have the same order as the variations of the coordinates and the quantity $1 - \varepsilon$. If the position of the system were displaced in the opposite sense then that would yield an enlargement of $\int T\,dt$ by the variation. One would then think of the quantity $T - U$ as non-increasing for the actual motion, if one would not like to contradict the principle of least action; naturally, the same thing would be true if it were decreasing. $T - U$ is constant.

§ 6. – Inequivalence of the true and varied motions.

Everywhere, we have observed the condition that the variations of the positions must be virtual displacements. Something else would happen if we were to demand that the varied motion should satisfy the same condition equations as the actual one. For example, if the condition equations are given in the form (9) – i.e., as ordinary equations:

$$\omega_\iota (x_1, y_1, z_1, \ldots, x_r, y_r, z_r, t) = 0 \qquad (i = 1, 2, \ldots),$$

then the last demand would imply that:

$$\omega_\iota (x_1 + \delta x_1, , \ldots, z_r + \delta z_r, t + \delta t) = 0,$$

and thus that one would also have $\delta\omega_\iota = 0$. However, an application of mechanical principles would call for equations (11):

([1]) Cf., **G. Kirchhoff**'s *Mechanik*, pps. 25 and 34. The relationship between virtual and actual displacements that **Hertz** expressed in no. 111 is based upon the fact that he did not include time in his condition equations.

$$\delta\omega_\iota - \frac{\partial\omega_\iota}{\partial t}\,\delta t = 0 \qquad (i = 1, 2, \ldots).$$

Indeed, those equations will agree with $\delta\omega_\iota = 0$ when:

$$\frac{\partial\omega_\iota}{\partial t} = 0,$$

i.e., when time does not enter into (9), and likewise when $\delta t = 0$, i.e., when **Hamilton**'s principle should be applied. By contrast, for the principle of least action, one should observe the aforementioned difference when time enters into the condition equations (9). The actual and varied motions are not equivalent in this case.

That inequivalence will also appear with **Hamilton**'s principle ([1]) when the condition equations are given as differential equations in the form (1) into which time does not enter. That will be illuminated by the example in the next paragraph. Here, it will only be stressed that the inequivalence of the motions will vanish again when one treats **Hertz**'s holonomic material systems. The conditions in that case can be assumed to also have the form (2):

$$d\Phi_\iota = 0 \qquad (i = 1, 2, \ldots).$$

It says that $\Phi_1, \Phi_2, \ldots$ should remain constant under the motion without those values having to be prescribed initially. Now, should the varied motion satisfy the same conditions, one could choose other constant values $\Phi_1, \Phi_2, \ldots$ for that motion, *per se.* However, that is excluded by the fact that one does not vary the initial and final positions. Now, one sees that the result would be the same if the variation were performed according to the equations:

$$\delta\Phi_\iota = 0 \qquad (i = 1, 2, \ldots).$$

However, the latter equations arise from the condition of motion when one replaces the coordinate differentials with coordinate variations. Those equations will then correspond here to the true demand that the variations of position must be virtual displacements. Now, that explains why the conception of the principles of **Maupertuis** and **Hamilton** that **Hertz** chose brought with it the restriction to holonomic systems. Namely, **Hertz** assumed that the varied was possible – i.e., as one that satisfied the same conditions as the actual path ([2]).

([1]) **C. Neumann** had already emphasized that fact for rolling motion, which I first noted during its publication. Compare Ber. d. Sächs. Ges. d. W. math.-phys. Cl., 1888, pp. 34, and especially the words: "By contrast, the fictitious motion will correspond to the character of the system in general."

([2]) Cf., nos. 347, 358, 110, 112, 113.

§ 7. – Special conditions of motion for a point.

It would be appropriate to explain the foregoing by way of an example. Since only the difficulties in the variation should be discussed here, it would seem permissible for me to choose a simple (although probably not realizable) motion. Furthermore, it belongs to the ones that **Hertz** allowed ([1]). The motion of a material point upon which no force acts shall be constrained by the condition equation:

$$\varphi(x, y, z)\, dx + \psi(x, y, z)\, dy + \chi(x, y, z)\, dz = 0. \tag{13}$$

The point will then be forced to move along a given surface element at each location. The direction cosines of the surface element that belongs to x, y, z will have the ratios:

$$\varphi(x, y, z) : \psi(x, y, z) : \chi(x, y, z).$$

Equation (13) can be integrated only in special cases in the form:

$$\omega(x, y, z) = \text{const.}$$

In those cases, we call equation (13) integrable. A function Ω (x, y, z) will then exist such that when it multiplies the left-hand side of the equation, it will go to a total differential. In order for that to happen, Ω must satisfy the conditions:

$$\frac{\partial(\Omega\cdot\varphi)}{\partial y} = \frac{\partial(\Omega\cdot\psi)}{\partial x}, \quad \frac{\partial(\Omega\cdot\psi)}{\partial z} = \frac{\partial(\Omega\cdot\chi)}{\partial y}, \quad \frac{\partial(\Omega\cdot\chi)}{\partial x} = \frac{\partial(\Omega\cdot\varphi)}{\partial z},$$

which can be put into the forms:

$$\begin{aligned} \Omega\,(\varphi_2 - \psi_1) &= \Omega_1\,\psi - \Omega_2\,\varphi, \\ \Omega\,(\psi_3 - \chi_2) &= \Omega_2\,\chi - \Omega_3\,\psi, \\ \Omega\,(\chi_1 - \varphi_3) &= \Omega_3\,\varphi - \Omega_1\,\chi, \end{aligned}$$

in which the partial derivatives with respect to x, y, z are denoted by 1, 2, 3, resp. If one multiplies these equations by χ, φ, ψ and adds them then that will give:

$$\chi(\varphi_2 - \psi_1) + \varphi(\psi_3 - \chi_2) + \psi(\chi_1 - \varphi_3) = 0. \tag{14}$$

That is the integrability condition, which is not always fulfilled ([2]). It only when it is fulfilled that the material point subjected to the above condition will represent a holonomic system.

([1]) It was treated already by **Voß** (*loc. cit.*, pp. 280).

([2]) The integrability condition that we found is also sufficient (cf., **A. Mayer**, Math. Ann., Bd. 5, pp. 450 to 452 and *Theorie der Transformationsgruppen* by **Lie**, with the collaboration of **F. Engel**, 1888, first Sections, pp. 90 to 93).

§ 8. – Varying the path.

The variation of a motion basically involves only the paths. Let us consider a path that obeys equation (13). The application of mechanical principles requires variations of position that are virtual displacements; i.e., they correspond to the equation:

(15) $$\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z = 0.$$

Since that is true for the displacements of all places along the original path, one will also have:

(16) $$d\,(\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z) = 0.$$

By contrast, if one would like to vary in such a way that the varied path satisfies the same condition as the original one then equation (13) must be true for two small corresponding pieces of both paths. Subtracting the two equations thus-obtained will yield:

(16) $$\delta(\varphi\,dx + \psi\,dy + \chi\,dz) = 0.$$

The behavior of the two requirements that were posed here for the variation will become clearer when we look for the variations that fulfill both requirements. If equations (16) and (17) were developed and then subtracted from each other then the result would be the equation:

(18) $$(\varphi_2 - \psi_1)(\delta x\,dy - \delta y\,dx) + (\psi_2 - \chi_1)(\delta y\,dz - \delta z\,dy) + (\chi_1 - \varphi_3)(\delta z\,dx - \delta x\,dz) = 0.$$

Equation (13), together with the relation (13) that exists for the original path, gives the proportion:

$$(\delta x\,dy - \delta y\,dx) : (\delta y\,dz - \delta z\,dy) : (\delta z\,dx - \delta x\,dz) = \chi : \varphi : \psi.$$

However, that proportion is compatible with (18) only when either the integrability condition (14) is fulfilled or:

(19) $$\delta x : \delta y : \delta z = dx : dy : dz.$$

The latter case represents an entirely special variation, namely, a variation of the path; such a variation corresponds to the one that applied in § **5**. However, equation (15) admits a more general type of solution. Likewise, equation (17) – i.e., ([1]):

([1]) If one initially assumes that the variations are finite and that the varied path should satisfy the same condition (13) as the original one then that would actually imply the validity of the equation:

$$\varphi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(x + \delta x) + \psi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(y + \delta y)$$
$$+ \chi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(z + \delta z) = 0,$$

in which σ is any variable that one can make depend upon a variable point along the original path. The equation in the text will emerge from this equation after one subtracts:

$$\left(\frac{\partial\varphi}{\partial x}\delta x+\frac{\partial\varphi}{\partial y}\delta y+\frac{\partial\varphi}{\partial z}\delta z\right)dx+\left(\frac{\partial\psi}{\partial x}\delta x+\frac{\partial\psi}{\partial y}\delta y+\frac{\partial\psi}{\partial z}\delta z\right)dy$$

$$+\left(\frac{\partial\chi}{\partial x}\delta x+\frac{\partial\chi}{\partial y}\delta y+\frac{\partial\chi}{\partial z}\delta z\right)dx + \varphi\, d\,\delta x + \psi\, d\,\delta y + \chi\, d\,\delta z = 0$$

can be satisfied by variations that vanish at the end points of the path and do not satisfy the proportion (19) along the path.

If the integrability condition is fulfilled then the two requirements will lead to different type of variations. One can illustrate those variations roughly as follows: From (13), a surface element will belong to each point of the original path. Those planes will envelop a developable surface α. The varied path will always run parallel to the original one, so the two will collectively define a narrow ribbon. Now, under the variations that correspond to mechanical principles, the segment that consists of the components δx, δy, δz will lie in the surface element that belongs to the point x, y, z, and thus the surface α, as well; that is not the case for the other variations. For that reason, one can regard the ribbon approximately as something that is cut from α in the former case, while in the latter case, it will generally make a finite angle with the developable surface α along the original path.

§ 9. – Equations of motion. True and geodetic paths.

We now define the differential equations for the motion of the material point. We would like to apply the principle of least work in its restricted form. If s means the arc length of the path then, from the conservation of energy, the velocity:

$$\frac{ds}{dt} = c \tag{20}$$

will be constant for the actual motion. One must think of the varied motion as having the same constant velocity. Now, that principle will imply that:

$$\frac{2}{mc}\,\delta\int T\,dt = \delta\int c\,dt = \delta\int ds = 0.$$

Upon developing that, one will find:

$$\varphi(x, y, z)\frac{d}{d\sigma} + \psi(x, y, z)\frac{d}{d\sigma} + \chi(x, y, z)\frac{d}{d\sigma} = 0$$

and neglecting certain terms. One must then regard the variations and their derivatives as small quantities of first order and omit terms of higher order.

$$\delta\int ds = \int \delta\, ds = \int \frac{dx\,\delta\, dx + dy\,\delta\, dy + dz\,\delta\, dz}{ds}$$
$$= \int\left(\frac{dx}{ds} d\,\delta x + \frac{dy}{ds} d\,\delta y + \frac{dz}{ds} d\,\delta z\right) = 0.$$

One splits the three parts in the last integral and partially integrates. Afterward, due to the vanishing of the initial and final variations, one will get:

(21) $$\int\left(-\frac{d^2x}{ds^2}\delta x + \frac{d^2y}{ds^2}\delta y + \frac{d^2z}{ds^2}\delta z\right) ds = 0.$$

The variations are determined in such a way that δx, δy, δz represent virtual displacements; i.e., they satisfy the equation:

$$\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z = 0.$$

The left-hand side of that equation is multiplied by $\lambda \cdot ds$ and then added under the last integral. In that way, one first gets:

$$\int\left\{\left(\lambda\varphi - \frac{d^2x}{ds^2}\right)\delta x + \left(\lambda\psi - \frac{d^2y}{ds^2}\right)\delta y + \left(\lambda\chi - \frac{d^2z}{ds^2}\right)\delta z\right\} ds = 0,$$

and from that:

(22) $$\begin{cases} \dfrac{d^2x}{ds^2} = \lambda\varphi, \\ \dfrac{d^2y}{ds^2} = \lambda\psi, \\ \dfrac{d^2z}{ds^2} = \lambda\chi. \end{cases}$$

Since λ means an unknown variable here, the content of equations (22) consists of just the proportions:

(23) $$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} = \varphi : \psi : \chi.$$

However, from a known theorem, the second differential quotients $d^2 x / ds^2$, $d^2 y / ds^2$, $d^2 z / ds^2$ behave like the direction cosines of the normals to the path that lie in the osculating plane. That normal is then identical with the normal to the surface element that is assigned to the point x, y, z by way of (13). *Hence, the osculating plane at each* *point of the curve will be perpendicular to the surface element that is assigned to that point* ([1]).

Equations (20) and (22) correspond to the two statements in **Hertz**'s basic laws ([2]) that the differential equations (22), together with (13), determine what **Hertz** called the *straightest path* ([3]).

We have just now determined the actual path with the help of the equation:

$$\delta \int ds = 0. \tag{24}$$

We now present the same equation, but with a different way of picturing the variation. We shall now no longer demand that variations of the positions should be virtual displacements, but we will demand that the varied path should satisfy the same differential equation (13) that we prescribed for the varied path. That will generally pose an entirely different problem in the calculus of variations from which the actual paths of material points will not generally emerge. In the problem, one must subject the variations to the condition (17); i.e., to the equation:

$$\delta\varphi \cdot dx + \delta\psi \cdot dy + \delta\chi \cdot dz + \varphi \cdot d\delta x + \psi \cdot d\delta y + \chi \cdot d\delta z = 0. \tag{25}$$

If one now develops (24) as before then that will yield (21) again. One must add λ times (25) under the integral in the latter equation ([4]). After one has partially integrated part of the terms under the integral, one will then get, in the known way:

$$\frac{d^2x}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\chi}{\partial x}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\varphi) = 0,$$

$$\frac{d^2y}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial y}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\chi}{\partial y}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\psi) = 0,$$

$$\frac{d^2z}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial z}\frac{dx}{ds} + \frac{\partial\psi}{\partial z}\frac{dy}{ds} + \frac{\partial\chi}{\partial z}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\psi) = 0.$$

One can also give those equations the form:

([1]) Cf., **Voß**, pp. 280. If one treats the varied path as if it lived on the developable surface α of the previous paragraph then the condition $\delta\int ds = 0$ would yield the actual path in the form of the *geodetic line* on the surface α in the ordinary sense of the term. However, in that way, one would arrive directly at the geometric property that is expressed in the text.

([2]) No. **309**.

([3]) Cf., **Hertz**, no. **155**d. Since we have derived those paths from the least-action principle, it would not mean anything here to say that they are the "straightest" paths.

([4]) In regard to that rule from the calculus of variations, cf., **Scheefer**, Math. Ann. **55** (1885), pp. 555 *et seq*. and **A Mayer**, Ber. d. Sächs. Ges. d. Wiss. (1885), (1895), and Math. Ann, **26** (1886).

$$
(26)\qquad \left\{
\begin{aligned}
&\frac{d^2x}{ds^2}+\varphi\frac{d\lambda}{ds}-\left(\frac{\partial\psi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\lambda\frac{dy}{ds}-\left(\frac{\partial\chi}{\partial x}-\frac{\partial\varphi}{\partial z}\right)\lambda\frac{dz}{ds}=0,\\
&\frac{d^2y}{ds^2}+\psi\frac{d\lambda}{ds}-\left(\frac{\partial\chi}{\partial y}-\frac{\partial\psi}{\partial z}\right)\lambda\frac{dz}{ds}-\left(\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}\right)\lambda\frac{dx}{ds}=0,\\
&\frac{d^2z}{ds^2}+\chi\frac{d\lambda}{ds}-\left(\frac{\partial\varphi}{\partial z}-\frac{\partial\chi}{\partial x}\right)\lambda\frac{dx}{ds}-\left(\frac{\partial\psi}{\partial z}-\frac{\partial\chi}{\partial y}\right)\lambda\frac{dy}{ds}=0.
\end{aligned}
\right.
$$

Together with (13), they determine the so-called *geodetic paths* ([1]).

Hertz showed that for his holonomic systems the geodetic paths coincided with the straightest ones – i.e., the actual ones ([2]). Namely, when the integrability condition is fulfilled, one can also think of equation (13) as being given in such a way that $\varphi\,dx + \psi\,dy + \chi\,dz$ will be a total differential. The relations then exist:

$$
\frac{\partial\varphi}{\partial y}=\frac{\partial\psi}{\partial x},\qquad \frac{\partial\varphi}{\partial z}=\frac{\partial\chi}{\partial x},\qquad \frac{\partial\psi}{\partial z}=\frac{\partial\chi}{\partial y},
$$

and if one recalls equations (26) then that means nothing but the validity of the proportion (23).

§ 10. – Manifold of the true and geodetic paths.

The actual path of the material point is determined completely when the initial position and direction are given. That follows on mechanical grounds, but also allows one to verify the actual paths from its geometric properties. One can then still choose the initial direction in the surface element that is associated with the point A arbitrarily. A simple infinitude of actual paths will then emanate from a well-defined location.

The geodetic paths will behave differently when the integrability condition is not fulfilled. They will be determined by equations (26), to which one adds (13). When (13) is differentiated, that will give:

$$
(27)\qquad
\begin{aligned}
&\varphi\frac{d^2x}{ds^2}+\psi\frac{d^2y}{ds^2}+\chi\frac{d^2z}{ds^2}+\left(\frac{\partial\varphi}{\partial x}\frac{dx}{ds}+\frac{\partial\varphi}{\partial y}\frac{dy}{ds}+\frac{\partial\varphi}{\partial z}\frac{dz}{ds}\right)\\
&+\left(\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial z}\frac{dz}{ds}\right)\frac{dy}{ds}+\left(\frac{\partial\chi}{\partial x}\frac{dx}{ds}+\frac{\partial\chi}{\partial y}\frac{dy}{ds}+\frac{\partial\chi}{\partial z}\frac{dz}{ds}\right)\frac{dz}{ds}=0.
\end{aligned}
$$

([1]) **Hertz**, no. 181a. **Voß**, pp. 282.
([2]) No. 190.

One can express $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, $\frac{d\lambda}{ds}$ in terms of x, y, z, λ, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$. The quantities x, y, z, λ can be determined as functions of s, with the help of the aforementioned equations when the initial values of x, y, z, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, λ are given for an initial value of s. On the other hand, if one performs the integration of (26) and (27) with any initial values whatsoever then due to equation (27) that will yield functions φ, ψ, χ that satisfy the condition:

$$\varphi\frac{dx}{ds}+\psi\frac{dy}{ds}+\chi\frac{dz}{ds} = C_1 ,$$

in which C_1 means a constant. Moreover, since equations (26), when multiplied by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, resp., and added will have the consequence:

$$\frac{d^2x}{ds^2}\frac{dx}{ds}+\frac{d^2y}{ds^2}\frac{dy}{ds}+\frac{d^2z}{ds^2}\frac{dz}{ds}+\left(\varphi\frac{dx}{ds}+\psi\frac{dy}{ds}+\chi\frac{dz}{ds}\right) = 0,$$

then the functions that one obtains must also fulfill the condition:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2 + 2C_1\,\lambda = C_2 .$$

C_2 is constant here. The initial values must now be chosen such that $C_1 = 0$ and $C_2 = 1$. In that way, it will be clear that as long as the initial position x, y, z is given, one can choose only the initial values of s, λ, and perhaps $\frac{dx}{ds}:\frac{dy}{ds}$ arbitrarily. Nothing can be said about the initial value of s, and one will thus introduce two constants into the geodetic path to be determined. However, that shows that the initial value of λ will influence the path only when the integrability condition (14) is not fulfilled ([1]). Therefore, if the quantity:

([1]) When one takes into account (13) and the fact that s means the arc length, equations (26) will imply the relation:

$$\left(\psi\frac{d^2x}{ds^2}-\varphi\frac{d^2y}{ds^2}\right)\frac{dz}{ds}+\left(\chi\frac{d^2y}{ds^2}-\psi\frac{d^2z}{ds^2}\right)\frac{dx}{ds}+\left(\varphi\frac{d^2z}{ds^2}-\chi\frac{d^2x}{ds^2}\right)\frac{dy}{ds}$$
$$= [(\varphi_2-\psi_1)\,\chi+(\psi_3-\chi_2)\,\varphi+(\chi_1-\varphi_3)\,\psi]\,\lambda.$$

If the expression (28) does not perhaps vanish for the entire path then λ will be a given function of position along a geodetic path λ. Two geodetic paths that emanate from the same location and belong to different initial values of λ will certainly be different then when the expression (28) is zero for the initial location, and therefore also for its neighborhood. However, if (28) vanishes for all values of x, y, z then the remark that was made at the conclusion of the last paragraph should be considered.

$$\varphi(\psi_3 - \chi_2) + \psi(\chi_1 - \varphi_3) + \chi(\varphi_2 - \psi_1) \tag{28}$$

at a location then a double infinitude of geodetic paths will emanate from the that place and only a simple infinitude of actual paths.

That result is analogous to the one that is found for the ball. For the ball, the motions that emanate from a given position and satisfy the minimum problem in the introduction will define a higher-dimensional manifold of those motions that the ball can perform starting from a given position without the action of forces.

§ 11. – Rolling motion of the ball. Condition equations.

We would now like to exhibit the differential equations for the rolling motion of the ball. Let ξ, η, ζ be the coordinates relative to a rectangular coordinate system that is fixed in space. The ball rolls without slipping on the fixed $\xi\eta$-plane Let x, y, and z be coordinates that refer to a rectangular coordinate system that is invariably coupled with the ball. That system shall have its origin at the center of the ball. Now, if ξ, η, ζ and x, y, z are the coordinates of the same spatial point then the equations:

$$\begin{aligned}
\xi &= \alpha + \alpha_1 x + \alpha_2 y + \alpha_3 z, \\
\eta &= \beta + \beta_1 x + \beta_2 y + \beta_3 z, \\
\zeta &= \gamma + \gamma_1 x + \gamma_2 y + \gamma_3 z,
\end{aligned}$$

and

$$\begin{aligned}
x &= \alpha_1(\xi - \alpha) + \beta_1(\eta - \beta) + \gamma_1(\zeta - \gamma), \\
y &= \alpha_2(\xi - \alpha) + \beta_2(\eta - \beta) + \gamma_2(\zeta - \gamma), \\
z &= \alpha_3(\xi - \alpha) + \beta_3(\eta - \beta) + \gamma_3(\zeta - \gamma).
\end{aligned} \tag{29}$$

In the first coordinate system, $\xi = \alpha$, $\eta = \beta$, $\zeta = \gamma$ are the coordinates of the center of the ball, and α, β, 0 are the coordinates of the point at which the ball contacts the $\xi\eta$-plane. γ is constant and equal to the radius a of the ball. The particle of the ball that is found at precisely the contact point must have the velocity 0 at that moment, since otherwise slipping would take place. Hence, the relations:

$$\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0$$

will be true for that particle at the moment of contact; i.e.:

$$\left\{\begin{aligned}
&\frac{d\alpha}{dt} + \frac{d\alpha_1}{dt}x + \frac{d\alpha_2}{dt}y + \frac{d\alpha_3}{dt}z = 0, \\
&\frac{d\beta}{dt} + \frac{d\beta_1}{dt}x + \frac{d\beta_2}{dt}y + \frac{d\beta_3}{dt}z = 0, \\
&\frac{d\gamma}{dt} + \frac{d\gamma_1}{dt}x + \frac{d\gamma_2}{dt}y + \frac{d\gamma_3}{dt}z = 0.
\end{aligned}\right. \tag{30}$$

Here, x, y, z mean those values that (29) will give when $\xi = \alpha$, $\eta = \beta$, $\zeta = 0$. One will then have to substitute:

$$x = -\gamma\gamma_1 = -a\gamma_1,$$
$$y = -\gamma\gamma_2 = -a\gamma_2,$$
$$z = -\gamma\gamma_3 = -a\gamma_3$$

in (30). In that way, one will get:

$$(31) \quad \begin{cases} \dfrac{d\alpha}{dt} = a\left(\gamma_1 \dfrac{d\alpha_1}{dt} + \gamma_2 \dfrac{d\alpha_2}{dt} + \gamma_3 \dfrac{d\alpha_3}{dt}\right), \\ \dfrac{d\beta}{dt} = a\left(\gamma_1 \dfrac{d\beta_1}{dt} + \gamma_2 \dfrac{d\beta_2}{dt} + \gamma_3 \dfrac{d\beta_3}{dt}\right), \\ \dfrac{d\gamma}{dt} = a\left(\gamma_1 \dfrac{d\gamma_1}{dt} + \gamma_2 \dfrac{d\gamma_2}{dt} + \gamma_3 \dfrac{d\gamma_3}{dt}\right). \end{cases}$$

Of these equations, the last one is fulfilled by itself, since γ is constant, and the right-hand side will vanish by means of the relations of the orthogonal coordinate transformation. The first two of equations (31), together with $\gamma = a$, are the conditions for pure rolling them ([1]).

§ 12. – Character of the condition equations.

In order to understand the character of the conditions, we express the coefficients of the coordinate transformation by the **Euler** formulas ([2]):

$$(32) \quad \begin{cases} \alpha_1 = -\cos\varphi\cos f\cos\vartheta - \sin\varphi\sin f, \\ \beta_1 = -\sin\varphi\cos f\cos\vartheta + \cos\varphi\sin f, \\ \gamma_1 = \quad \cos f\sin\vartheta, \\ \alpha_2 = -\cos\varphi\sin f\cos\vartheta + \sin\varphi\cos f, \\ \beta_2 = -\sin\varphi\sin f\cos\vartheta - \cos\varphi\cos f, \\ \gamma_2 = \quad \sin f\sin\vartheta, \\ \alpha_3 = \quad \cos\varphi\sin\vartheta, \\ \beta_3 = \quad \sin\varphi\sin\vartheta, \\ \gamma_3 = \quad \cos\vartheta. \end{cases}$$

If those values were introduced into equations (31) then that would yield:

([1]) Cf., **Neumann**, Sächs. Ber. 1888, pp. 358.

([2]) Novi Commentarii Acad. Petrop. **15** (1770), pp. 75. For the geometric meaning of the angles φ, f, ϑ, cf., e.g., **Kirchhoff**'s *Mechanik*, 1877, pp. 43 and 44.

$$(33) \qquad \begin{cases} d\alpha = -\,a\sin\varphi\sin\vartheta\, df + a\cos\varphi\, d\vartheta, \\ d\beta = \quad a\cos\varphi\sin\vartheta\, df + a\sin\varphi\, d\vartheta. \end{cases}$$

Those equations are not completely integrable, since they are not integrable to begin with ([1]). The ball that can roll on a plane but not slide on it will then represent a non-holonomic material system.

§ 13. – New form of the conditions.

The momentary state of motion of the ball will now be regarded as a combined rotation about an axis that goes through the center and a displacement. Let p, q, r be the components of the angular velocity, and let u, v, w be the displacement velocity, and both of them are replaced the x, y, z axes. Those components are given by the equations ([2]):

([1]) That is, there is no function $\omega(\alpha, \beta, \varphi, f, \vartheta)$ whose differential vanishes because of equations (33) either. Namely, such a function must satisfy the partial differential equations [cf., **A. Mayer**, Math. Ann. **5** (1872), pp. 449 and **Lie** *Theorie der Transformationsgruppen*, Section I, pp. 91 and 92]:

$$\begin{aligned} &\frac{\partial\omega}{\partial\varphi} = 0, \\ &\frac{\partial\omega}{\partial f} - a\sin\varphi\sin\vartheta\frac{\partial\omega}{\partial\alpha} + a\cos\varphi\sin\vartheta\frac{\partial\omega}{\partial\beta} = 0, \\ &\frac{\partial\omega}{\partial\vartheta} + a\cos\varphi\frac{\partial\omega}{\partial\alpha} + a\sin\varphi\frac{\partial\omega}{\partial\beta} = 0. \end{aligned}$$

Those equations, which do not define a complete system (cf., **Clebsch**, Journal für reine und angewandte Math. **65**, pp. 258), can be extended to such a system and then show directly what one can also verify that they will be satisfied by only a constant.

The nonexistence of a function ω with the aforementioned property can be inferred from the fact that such a function would have to maintain a constant, as a result of the differential equations (33). However, one can go from each system of values $\alpha_1, \beta_1, \varphi_1, f_1, \vartheta_1$ to another $\alpha_2, \beta_2, \varphi_2, f_2, \vartheta_2$ in which the ball can roll without slipping from any position to any other without harming equations (33) as a result of the transition. Since that fact was also employed by **Hertz**, if should be explained to some degree. If w means a constant angle that is expressed in terms of arc length, and if ϑ is also assumed to be constant then equations (33) will be satisfied for:

$$\alpha = w\frac{a\sin\vartheta}{2\pi}\cos\varphi, \qquad \beta = w\frac{a\sin\vartheta}{2\pi}\sin\varphi, \qquad f = \frac{w}{2\pi}\varphi.$$

Here, φ can mean any function of time, and one will be dealing with a known motion. If φ runs through the interval from 0 to 2π then α, β and [cf., (32)] α_3, β_3, γ_3 will finally have the same values that they had to be begin with. One will obtain the same final position by a pure rolling that one would reach if one only rotated around the z-axis. Since f will then increase by w, w will be the magnitude of that rotation. The z-axis has no special direction. One can then replace every rotation around the center with just a rolling motion and produce the same final result. Now, in summary, it is clear that one can find a pure rolling transition from any initial position to any final position. Since that motion is composed of pieces, it would introduce discontinuities into the velocity that can, however, be eliminated by a small alteration.

([2]) CF., e.g., **Kirchhoff**'s *Mechanik*, pp. 50.

$$p = \alpha_3 \frac{d\alpha_2}{dt} + \beta_3 \frac{d\beta_2}{dt} + \gamma_3 \frac{d\gamma_2}{dt},$$

$$q = \alpha_1 \frac{d\alpha}{dt} + \beta_1 \frac{d\beta_3}{dt} + \gamma_1 \frac{d\gamma_3}{dt},$$

$$r = \alpha_2 \frac{d\alpha_1}{dt} + \beta_2 \frac{d\beta_1}{dt} + \gamma_2 \frac{d\gamma_1}{dt}$$

and

$$(34) \qquad \left\{ \begin{aligned} u &= \alpha_1 \frac{d\alpha}{dt} + \beta_1 \frac{d\beta}{dt} + \gamma_1 \frac{d\gamma}{dt}, \\ v &= \alpha_2 \frac{d\alpha}{dt} + \beta_2 \frac{d\beta}{dt} + \gamma_2 \frac{d\gamma}{dt}, \\ w &= \alpha_3 \frac{d\alpha}{dt} + \beta_2 \frac{d\beta}{dt} + \gamma_3 \frac{d\gamma}{dt}. \end{aligned} \right.$$

Furthermore, the relations exist ([1]):

$$(35) \qquad \begin{aligned} &\frac{d\alpha_1}{dt} = \alpha_2\, r - \alpha_3\, q, \quad &&\frac{d\alpha_2}{dt} = \alpha_3\, p - \alpha_1\, r, \quad &&\frac{d\alpha_3}{dt} = \alpha_1\, q - \alpha_2\, p, \\ &\frac{d\beta_1}{dt} = \beta_2\, r - \beta_3\, q, &&\frac{d\beta_2}{dt} = \beta_3\, p - \beta_1\, r, &&\frac{d\beta_3}{dt} = \beta_1\, q - \beta_2\, p, \\ &\frac{d\gamma_1}{dt} = \gamma_2\, r - \gamma_3\, q, &&\frac{d\gamma_2}{dt} = \gamma_3\, p - \gamma_1\, r, &&\frac{d\gamma_3}{dt} = \gamma_1\, q - \gamma_2\, p. \end{aligned}$$

If one now replaces the quantities $\frac{d\gamma}{dt}$ in (34) with 0 and $\frac{d\alpha}{dt}, \frac{d\beta}{dt}$ with the right-hand sides of (31), but afterwards replaces the quantities:

$$\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}, \qquad \frac{d\beta_1}{dt}, \frac{d\beta_2}{dt}, \frac{d\beta_3}{dt}$$

with the right-hand sides of (35) then, after one has employed the relations that pertain to orthogonal coordinate transformations, one will get:

$$(36) \qquad \begin{aligned} u &= a\,(\gamma_3\, q - \gamma_2\, r), \\ v &= a\,(\gamma_1\, r - \gamma_3\, p), \\ w &= a\,(\gamma_2\, p - \gamma_1\, q). \end{aligned}$$

([1]) *Ibidem.*

When rolling without slipping takes place, those equations will represent the connection that exists between rotation and displacement ([1]). The components *p*, *q*, *r* of the angular velocity can be chosen arbitrarily.

§ 14. – Equations of motion.

With these preparations, the differential equations of motion ([2]) can be exhibited. I shall employ **Hamilton**'s principle. Since no forces are active, one must set:

$$\int \delta T \cdot dt = 0,$$

in which the type of variation must be observed. Since the system of coordinates *x*, *y*, *z* is thought of as fixed in the ball, the *vis viva T* will be a function of *p*, *q*, *r*, *u*, *v*, *w* that is given once and for all, and one will get:

$$\int\left(\frac{\partial T}{\partial p}\delta p+\frac{\partial T}{\partial q}\delta q+\frac{\partial T}{\partial r}\delta r+\frac{\partial T}{\partial u}\delta u+\frac{\partial T}{\partial v}\delta v+\frac{\partial T}{\partial w}\delta w\right)dt = 0. \tag{37}$$

The variations of the velocity components that enter here must be expressed. The variation of the motion will once more come about in such a way that initially each of the original positions that the ball can run through will take on a small displacement. The displacement will be decomposed into a rotation that takes place around the center and a parallel displacement. The rotation and displacement have the components p', q', r' and u', v', w' with respect to the *x*, *y*, *z* axes. Now, the variations of the velocity components are represented by the formulas ([3]):

$$(38)\qquad \begin{aligned} \delta p &= \frac{dp'}{dt} + qr' - q'r, \\ \delta q &= \frac{dq'}{dt} + rp' - r'p, \\ \delta r &= \frac{dr'}{dt} + pq' - p'q, \end{aligned}$$

and

$$\delta u = \frac{du'}{dt} + vr' - v'r + w'q - wq',$$

([1]) It is not difficult to derive those equations geometrically.

([2]) We could also employ **Neumann**'s general equations for rolling motion, Ber. d. Sächs. Ges. 1888, pps. 36 and 39.

([3]) **Kirchhoff**, pp. 58 and 59.

(39) $$\delta v = \frac{dv'}{dt} + wp' - w'p + u'r - ur',$$

$$\delta w = \frac{dw'}{dt} + uq' - u'q + v'p - vp'.$$

The derivation of these formulas is based upon a commutation of the symbols d / dt and δ ([1]). Such a commutation is allowed when time t is not varied; however, that is the type of variation that is required by **Hamilton**'s principle.

One now introduces the right-hand sides of (38) and (39) for δp, δq, δr, δu, δv, δw in (37). If one takes into account the fact that the variations should vanish for the beginning and end of the interval in question then one will get, after certain partial integrations:

(40) $$\int\left\{\left(-\frac{d}{dt}\frac{\partial T}{\partial p} + r\frac{\partial T}{\partial q} - q\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial v} - v\frac{\partial T}{\partial w}\right)p'\right.$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial q} + p\frac{\partial T}{\partial r} - r\frac{\partial T}{\partial p} + u\frac{\partial T}{\partial w} - w\frac{\partial T}{\partial u}\right)q'$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial r} + q\frac{\partial T}{\partial p} - p\frac{\partial T}{\partial q} + v\frac{\partial T}{\partial u} - u\frac{\partial T}{\partial v}\right)r'$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial u} + r\frac{\partial T}{\partial v} - q\frac{\partial T}{\partial w}\right)u'$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial v} + p\frac{\partial T}{\partial w} - r\frac{\partial T}{\partial u}\right)v'$$

$$\left.+\left(-\frac{d}{dt}\frac{\partial T}{\partial w} + q\frac{\partial T}{\partial u} - p\frac{\partial T}{\partial v}\right)w'\right\}dt = 0.$$

Up to now, the condition that constrains the motion has not been used in this paragraph. Since the ball must roll without slipping, the displacements that correspond to the variations, which are virtual displacements, must also consist of a pure rolling. Each of those small displacements decomposes into a rotation and a displacement, and the components of such a rotation and displacement must be coupled by relations. Those relations are analogous to (36), and they are the following ones:

$$u' = a\,(\gamma_3\, q' - \gamma_2\, r'),$$
$$v' = a\,(\gamma_1\, r' - \gamma_3\, p'),$$

([1]) **Kirchhoff**, pp. 58.

$$w' = a\,(\gamma_2\,p' - \gamma_1\,q').$$

If one introduces those values of u', v', w' into (40) then, since the components p', q', r' of the rotation are arbitrary, one will get:

$$(41)\qquad \left\{\begin{aligned}
&\frac{d}{dt}\frac{\partial T}{\partial p} - a\gamma_3\frac{d}{dt}\frac{\partial T}{\partial v} + a\gamma_2\frac{d}{dt}\frac{\partial T}{\partial w} - r\frac{\partial T}{\partial q} + q\frac{\partial T}{\partial r}\\
&\quad - a\,(\gamma_3\,r + \gamma_2\,q)\frac{\partial T}{\partial r} + (-w + a\gamma_2\,p)\frac{\partial T}{\partial v} + (v + a\gamma_2\,q)\frac{\partial T}{\partial w} = 0,\\[2ex]
&\frac{d}{dt}\frac{\partial T}{\partial q} - a\gamma_1\frac{d}{dt}\frac{\partial T}{\partial w} + a\gamma_3\frac{d}{dt}\frac{\partial T}{\partial u} - p\frac{\partial T}{\partial r} + r\frac{\partial T}{\partial p}\\
&\quad - a\,(\gamma_1\,p + \gamma_3\,r)\frac{\partial T}{\partial v} + (-u + a\gamma_3\,q)\frac{\partial T}{\partial w} + (w + a\gamma_1\,q)\frac{\partial T}{\partial u} = 0,\\[2ex]
&\frac{d}{dt}\frac{\partial T}{\partial r} - a\gamma_2\frac{d}{dt}\frac{\partial T}{\partial u} + a\gamma_1\frac{d}{dt}\frac{\partial T}{\partial v} - q\frac{\partial T}{\partial p} + p\frac{\partial T}{\partial q}\\
&\quad - a\,(\gamma_2\,q + \gamma_1\,p)\frac{\partial T}{\partial w} + (-v + a\gamma_1\,r)\frac{\partial T}{\partial u} + (u + a\gamma_2\,r)\frac{\partial T}{\partial v} = 0.
\end{aligned}\right.$$

Those are the desired differential equations that will determine the motion, in combination with the condition (26) ([1]).

§ 15. – Special case.

We now assume that the ball has its center of gravity at its center, without actually being homogeneous. The coordinate system x, y, z shall be defined by the principal axes that are constructed at the center of gravity. The *vis viva* is then inferred from the equation:

$$2T = (u^2 + v^2 + w^2)\,M + Pp^2 + Qq^2 + Rr^2,$$

in which M means the mass, and P, Q, R mean the principal moments of inertia. Equations (41) then take on the form:

([1]) The derivatives $\partial T / \partial p$, etc. in these equations are defined by partially-differentiating a function that represents the *vis viva* for a rolling and slipping motion of the ball. Namely, those derivatives emerge from the calculation of δT and the *vis viva* $T + \delta T$ of the varied motion cannot be calculated from an expression that is valid for a pure rolling motion. That fact was overlooked in the development of the special **Neumann** formulas, which relate to rolling on a fixed plane. Those formulas (Ber. d. Sächs. Ges. math.-phys. Cl. 1888, pp. 42 and 1885, pp. 368) need to be corrected then.

$$
(42)\qquad \left\{\begin{aligned}
&P\frac{dp}{dt}-aM\left(\gamma_3\frac{dv}{dt}-\gamma_2\frac{dw}{dt}\right)+(R-Q)\,qr+aM\{\gamma_2(pv-qu)+\gamma_3(pw-ru)\}=0,\\
&Q\frac{dq}{dt}-aM\left(\gamma_1\frac{dw}{dt}-\gamma_3\frac{du}{dt}\right)+(P-R)\,rp+aM\{\gamma_3(qw-rv)+\gamma_1(qu-pv)\}=0,\\
&R\frac{dr}{dt}-aM\left(\gamma_2\frac{du}{dt}-\gamma_1\frac{dv}{dt}\right)+(Q-P)\,p\,q+aM\{\gamma_1(ru-pw)+\gamma_2(rv-qw)\}=0.
\end{aligned}\right.
$$

Differentiation of (36) will yield:

$$
(43)\qquad \begin{aligned}
\frac{du}{dt}&= a\left(\frac{d\gamma_3}{dt}q-\frac{d\gamma_2}{dt}r+\gamma_3\frac{dq}{dt}-\gamma_2\frac{dr}{dt}\right),\\
\frac{dv}{dt}&= a\left(\frac{d\gamma_1}{dt}r-\frac{d\gamma_3}{dt}p+\gamma_1\frac{dr}{dt}-\gamma_3\frac{dp}{dt}\right),\\
\frac{dw}{dt}&= a\left(\frac{d\gamma_2}{dt}p-\frac{d\gamma_1}{dt}q+\gamma_2\frac{dp}{dt}-\gamma_1\frac{dq}{dt}\right).
\end{aligned}
$$

Those equations serve to make the quantities $\frac{du}{dt},\frac{dv}{dt},\frac{dw}{dt}$ known by way of (42). Afterwards, one replaces $\frac{d\gamma_1}{dt},\frac{d\gamma_2}{dt},\frac{d\gamma_3}{dt}$ with the right-hand sides of (35), and furthermore, *u*, *v*, *w* with the right-hand side of (36) and ultimately obtains:

$$
(44)\qquad \left\{\begin{aligned}
&[P+a^2M(\gamma_2^2+\gamma_3^2)]\frac{dp}{dt}-a^2M\gamma_1\gamma_2\frac{dq}{dt}-a^2M\gamma_1\gamma_3\frac{dr}{dt}=(Q-R)\,qr,\\
&[Q+a^2M(\gamma_3^2+\gamma_1^2)]\frac{dq}{dt}-a^2M\gamma_2\gamma_3\frac{dr}{dt}-a^2M\gamma_2\gamma_1\frac{dp}{dt}=(R-P)\,rq,\\
&[R+a^2M(\gamma_1^2+\gamma_2^2)]\frac{dr}{dt}-a^2M\gamma_3\gamma_1\frac{dp}{dt}-a^2M\gamma_3\gamma_1\frac{dq}{dt}=(P-Q)\;pq.
\end{aligned}\right.
$$

We now have equations that are linear in $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$ and have a positive determinant. That will yield expressions for $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$ in terms of the quantities γ_1, γ_2, γ_3, *p*, *q*, *r*, and except for the relation:

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1,$$

they are arbitrary for the initial state.

The simplest case is the one in which:

$$P = Q = R,$$

such as, e.g., the homogeneous ball. That will then imply equations (44):

$$\frac{dp}{dt} = \frac{dq}{dt} = \frac{dr}{dt} = 0;$$

i.e., one has a rotational axis that is fixed in the ball and a uniform rotation about it. With the help of the latter equations and the relations (43), (35), and (36), that will now yield:

$$\frac{d}{dt}(\alpha_1 u + \alpha_2 v + \alpha_3 w) = 0,$$

$$\frac{d}{dt}(\beta_1 u + \beta_2 v + \beta_3 w) = 0.$$

That means that the center moves with a uniform, rectilinear motion.

From now on, we shall once more assume that the moments of inertia P, Q, R are different, but that the initial state is such that $p = q = 0$. The rotational axis is one of the principle axes at the outset. Now, equations (44) imply that $p = q = 0$ and r remains constant, and the motion in this case will proceed as it does for the homogeneous ball. That can also be assumed. Namely, if one imagines that the initial state obeys the condition, but the ball is completely free and subject to no forces, then the motion will proceed as described. Hence, if the initial state corresponds to a pure rolling on the plane then the same thing will be true for all of the states that follow it. If one adds the constraint that slipping is prohibited then that will not change anything in regard to the motion.

Tübingen, May 1896.