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On the internal motion of electrons. I.

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Upon attempting to bring the fact of electron spin into harmony with a point-like structure for the electron from the standpoint of classical theory, one will necessarily be led to introduce negative masses, along with positive ones. The consideration of a mass pole m and a mass dipole $\mathbf{p}$ presents itself as a simple model of the electron.

1. There is a close connection with the work of Lubanski on the foundations of approximate solutions to the field equations of gravitation (viz., retarded potentials) as a way of obtaining the equations of motion for a pole-dipole particle.

2. It will be shown that the solutions to the equations of motion exhibit something that corresponds to a circular motion of the particle with a constant (micro-) velocity and vanishing momentum (for particles that are macroscopically at rest). The equations of that circular motion admit a remarkable passage to the limit ($m \to \infty$, $\mathbf{p} \to 0$), under which, the micro-velocity of the particle will attain the speed of light, although the energy will remain finite. That suggests that we should bring that limiting case into agreement with the infinite (or practically infinite) self-energy of the electron charge. The case of translation will be examined.

3. The energy function of the pole-dipole particle exhibits a close analogy with the Hamiltonian operator of the Dirac electron (viz., both are bilinear forms in the components of the macro-momentum and micro-velocity of the particle). There are equations of motion in canonical form that also apply to the case of an external electromagnetic field.

4. An expression for the angular momentum that proves to be a constant of motion will be derived on the basis of an intuitive analysis of the energy distribution. The total angular momentum appears to split into two parts that one will refer to as the *internal angular momentum* (i.e., *spin*) and the *orbital angular momentum*. The center-of-mass theorem is satisfied. That will imply a far-reaching justification for the model of the electron as a "rotating mass point" that was recently described by the authors, in which a point-like mass with the speed of light describes an orbit with a radius $\hbar / 2\mu c$ (μ is the electron mass).

One of the authors recently proposed a model of the electron that sought to overcome the well-known problems with its dimensionality ([1]). According to that model, the electron is imagined to be a "mass point" that rotates on a circle of radius $\hbar / 2\mu c$ with a velocity of light ($\hbar$ is the Planck-Dirac constant, μ is its mass, and c is the speed of light) ([2]). Electron spin (of magnitude $\hbar / 2$) will become immediately understandable in that way, and in fact in a way that corresponds precisely to the "zitterbewegung" of electrons

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([1]) **H. Hönl**, Ann. Phys. (Leipzig) **33** (1938), 565; quoted as *loc. cit.* in what follows. Cf., also Naturwiss. **26** (1938), 408.

([2]) In *loc. cit.*, that electron model was introduced as Model II, as opposed to another Model I, in which the electron was imagined to be a rotating charge distribution that continuously fills up space and also has dimension $\hbar / 2\mu c$. From now on, Model I will be rejected as a basis for what follows in this article.

that is associated with the Dirac equation ([1]). Furthermore, a kinematic analysis of the motion of electrons, by analogy with the laws of motion for a symmetric top (*loc. cit.*), indicates that the electron model will reproduce the correspondingly correct probability for the recombination of electron-positron pairs (for a sufficiently general special case). That argument encouraged the authors to pursue the Ansatz thus-obtained further towards a corresponding description of the properties of the electron and to analyze it more deeply.

In regard to Lorentz's extended electron, the proposed electron model means a return to the point electron. In contradiction to the electron with a charge that continuously fills up space, one can raise only the generally-logical objection that one might expect that an "elementary particle" should have a point-like structure, since otherwise the introduction of extended electrons would directly contradict the quantum theory of wave fields. In fact, the quantum theory of electromagnetic fields implies that the quantity of charge in an arbitrarily-bounded volume must be an integer multiple (including zero) of an elementary charge *e*, which is compatible with only an unextended elementary particle. However, the demand that the electron should be point-like will lead to a fundamental problem in regard to the existence of angular momentum for the electron, which has been established by experiment. Namely, if one lets the dimensions of a particle go to zero then the angular momentum will also vanish, since the circumferential speed of the particle cannot exceed the speed of light. On the other hand, the introduction of a rotating mass point seems nearly impossible to justify dynamically.

Meanwhile, the internal contradictions for a particle of vanishing extension will come about only when the proper rotation must take place about an axis that goes through the center-of-mass, and that will be assumed to lie inside of the particle, since one tacitly supposes that the particle has a positive mass distribution throughout it. Thus, in order to remove the difficulties, it will often be necessary to admit not only positive masses, but *negative* ones, as well. The fact that this theoretical possibility will imply an actual, practicable path will be shown by the following considerations collectively. For the time being, we would like to restrict ourselves to a simple examination that indeed delivers a crude, but all the more characteristic, picture of the expected consequences of the introduction of negative masses.

We consider a structure that is composed of a positive point-mass, along with an equally-large negative one – viz., a *mass* (*gravitational*, resp.)-*dipole.* We define the dipole moment by the vector:

(1) $$\mathfrak{p} = m\,(\mathfrak{r}_+ - \mathfrak{r}_-),$$

in which $\mathfrak{r}_+$ and $\mathfrak{r}_-$ are the position vectors of the point-masses + m and – m, respectively. If such a particle performs a translatory motion then the resulting momentum will often be zero. On the other hand, a pure translation will correspond to a *non-vanishing angular momentum* $\mathfrak{L}$, in general. In fact (by considering the relativistic variation of mass), it will be:

$$\mathfrak{L} = \frac{m}{\sqrt{1-\beta^2}}\{[\mathfrak{r}_+, \dot{\mathfrak{r}}_+] - [\mathfrak{r}_-, \dot{\mathfrak{r}}_-]\},$$

([1]) **E. Schrödinger**, Berl. Ber. (1930), 146.

or when one introduces the translational velocity $\mathfrak{v} = \dot{\mathfrak{r}}_+ = \dot{\mathfrak{r}}_-$:

(2) $$\mathfrak{L} = \frac{1}{\sqrt{1-\beta^2}}[\mathfrak{p}, \mathfrak{v}] \qquad \left(\beta = \frac{|\mathfrak{v}|}{c}\right),$$

which is an expression that will vanish in the special case of $\mathfrak{p} \parallel \mathfrak{v}$. Observe that the angular momentum is completely independent of the reference point. If the dipole executes a rotation about the midpoint of the line that connects both point-masses then the angular momentum relative to the midpoint will vanish. In that case, a *resultant momentum:*

$$\mathfrak{B} = \frac{m}{\sqrt{1-\beta^2}}(\dot{\mathfrak{r}}_+ - \dot{\mathfrak{r}}_-)$$

will remain. Upon introducing the angular velocity $\mathfrak{o}$ of the dipole about its corresponding midpoint:

$$\dot{\mathfrak{r}}_+ = [\mathfrak{o}, \mathfrak{r}_+], \qquad \dot{\mathfrak{r}}_- = [\mathfrak{o}, \mathfrak{r}_-],$$

one will get:

(3) $$\mathfrak{B} = \frac{1}{\sqrt{1-\beta^2}}[\mathfrak{o}, \mathfrak{r}] .$$

The mass-dipole will then define a sort of counterpart to the simple mass-pole (i.e., point-mass): During force-free motion (i.e., translation), the dipole will possess only angular momentum, while the simple pole will have only momentum without angular momentum. By contrast, for a dipole, a rotation of the system will be required to maintain the constancy of the momentum; i.e., in general, an external force will be necessary, just as it will be for the pole to conserve its angular momentum.

In what follows, we shall systematically examine the motion of a point-like particle that arises from considering a mass-pole and dipole, and it will be shown that such a particle will make it possible to give a far-reaching interpretation of the properties of an electron. In fact, it will imply that this structure possesses not only different mechanical properties from the usual mass-points (as was indicated just now in the example of the simple dipole), but it will also enable the reaction of the gravitational field that it produces to be "steered." In other words: It will allow equations of motion for the pole-dipole particles to be derived from the requirement that the gravitational field must satisfy the field laws. Those equations of motion can be compared to the characteristic equations for the Dirac electron, and a close analogy will emerge from that which can be pursued in fine detail (if one disregards a peculiarity that relates to the sign of the magnetic moment). On that basis, it will seem legitimate for us *to regard the gravitational pole-dipole particle as the classical model for the electron*, and that sense, to speak of the "internal motions of electrons" (viz., micro-motions). We emphasize that this description of the properties of electrons by the Dirac equations is not only complete, but simple, in principle. For that reason, there might also be a special attraction to pursuing the extent to which the properties of Dirac electrons can also be interpreted as "classical," in the sense of the correspondence principle.

§ 1. The derivation of the equations of motion for a pole-dipole particle from the method of retarded potentials.

In what follows, we shall occupy ourselves with a given particle of vanishing extension by the introduction of a mass-pole and a mass-dipole, and indeed without considering any external forces, to begin with. In that case, we shall base our considerations on the well-known approximate solutions to the strong-field equations for gravitation. It will be shown that those approximate solutions will suffice completely, since they imply the equations of motion for particles from considerations that pertain to large distances from those particles. Thus, we shall first closely follow the method of retarded potentials that was employed by Lubanski ([1]), and for the convenience of the reader, we shall describe it briefly here once more:

1. Let:

$$x_1 = ix, \qquad x_2 = iy, \qquad x_3 = iz, \qquad x_4 = ct \tag{4}$$

be the coordinates of the space-time continuum, in which x, y, z mean the usual spatial coordinates, and t means time, and let $g_{\mu\nu}$ be the fundamental metric tensor of the line element:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{5}$$

$ds^2 > 0$ for the element of a time-like line; s will then have the meaning of proper time along that world-line. In a space-time region that is free from gravitational forces, it can be assumed that:

$$g_{\mu\nu} = g^{\mu\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{for} \quad \mu = \nu, \\ 0 & \text{for} \quad \mu \neq \nu. \end{cases} \tag{6}$$

For sufficiently-weak gravitational fields (i.e., sufficiently-large distances from the matter that produces the field), one can set:

$$g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}, \tag{7}$$

in which $\gamma_{\mu\nu}$ should be envisioned as a tensor field on the background of the pseudo-Euclidian metric that has the normal values (6) for $g_{\mu\nu}$. If one introduces the "gravitational potential":

$$\varphi_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\, \delta_{\mu\nu}\, \gamma^\rho_\rho \tag{8}$$

then the field equations for gravitation will demand that when one neglects all higher potentials, the $\varphi_{\mu\nu}$ must satisfy equations of Poisson type:

([1]) **Lubanski**, Acta Phys. Pol. **6** (1937), 356; cited as L in what follows. In addition, cf. also, **M. Mathisson**, also in *ibid.* **6** (1937), 167. The goal of the Lubanski paper was to re-establish the equations of motion for a material system that Mathisson had posed.

(9)
$$\begin{cases} \Box\varphi_{\mu\nu} = -2\kappa T_{\mu\nu}, \\ \left(\Box \equiv \sum_{\mu} \frac{\partial^2}{(\partial x^\mu)^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right), \end{cases}$$

($T_{\mu\nu}$ is the matter tensor, and κ is the relativistic gravitational constant), when one imposes the *auxiliary conditions* ([1]):

(10)
$$\frac{\partial \varphi_{\mu\nu}}{\partial x^\mu} = 0.$$

Hence, the $\varphi_{\mu\nu}$ will have to satisfy the system of simultaneous equations:

(9′)
$$\Box\varphi_{\mu\nu} = 0, \qquad \frac{\partial \varphi_{\mu\nu}}{\partial x^\mu} = 0$$

outside the particles, and when one excludes external forces.

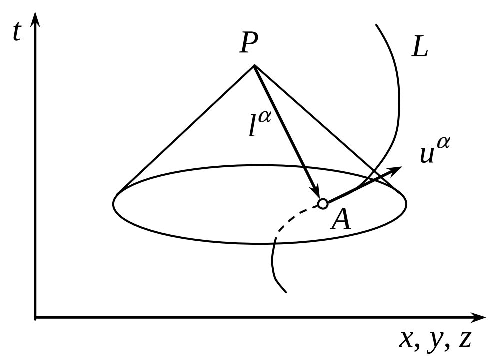

Figure 1. The past-directed light-cone at P: L is the world-line of the particle. A is the intersection of L with the light-cone (i.e., the reduction point).

2. Let us consider the time-like world-line L that the motion of our particle describes. Let X_α be the coordinates of an arbitrary point A along L. L will be determined uniquely when the X_α are expressed as functions of proper time s: $X_\alpha = X_\alpha(s)$. From (4) and (5), the components of the velocity:

(11)
$$u_\alpha = \frac{dX_\alpha}{ds}$$

will then satisfy the requirement that:

(12)
$$u_\alpha u^\alpha = 1.$$

([1]) Enzykl. d. math. Wiss., Bd. V, pp. 736, *et seq.*

Furthermore, let P (x_α) be an arbitrary. The "light-cone" that emanates from P and is directed into the past will meet the world-line L at the point A; the components of the vector PA will then be (cf., Fig. 1):

$$l_\alpha = X_\alpha - x_\alpha \,. \tag{13}$$

A function $n = n$ (P) that is single-valued in all space will now be defined with the help of l_α and u_α by the relation:

$$n = l^\alpha u_\alpha \,. \tag{14}$$

We would next like to explain the meaning of n:

From (13) and (4), we will have explicitly:

$$l^1 = i\,(X - x), \quad l^2 = i\,(Y - y), \quad l^3 = i\,(Z - z). \tag{13$'$}$$

We will then have the velocity components:

$$u_1 = i\,\beta_x\,u_4\,, \quad u_2 = i\,\beta_y\,u_4\,, \quad u_3 = i\,\beta_z\,u_4\,, \quad u_4 = \frac{1}{\sqrt{1-\beta^2}}\,, \tag{15}$$

with:

$$\beta_x = \frac{v_x}{c}\,, \qquad \beta_y = \frac{v_y}{c}\,, \qquad \beta_z = \frac{v_z}{c}\,, \qquad \beta = \frac{v}{c}\,,$$

if $v_x = dx / dt$, v_y , v_z are the components of the velocity $\mathbf{v}$ that is defined in the usual way. It will follow from this and (14) that:

$$n = u_4\,[l^4 - \{(X - x)\,\beta_x + (Y - y)\,\beta_y + (Z - z)\,\beta_z\}]. \tag{14$'$}$$

Since l_α lies on the light-cone, we will then have:

$$l^\alpha\,l_\alpha = 0, \qquad (l^4)^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2 = r^2,$$

in which r is the *retarded* distance between the point P and A, in the usual sense. Since the vector l_α is directed into the past, moreover, such that:

$$l^4 = -\,r,$$

one can, as a consequence of (14), and by means of:

$$(X - x)\,\beta_x + (Y - y)\,\beta_y + (Z - z)\,\beta_z = r\,\beta_r\,,$$

also write:

$$n = -\,u_4\,r\,(1 + \beta_r). \tag{16}$$

Hence, β_r will be the component (divided by c) of the velocity in the direction of the space-like vector $\mathbf{r} = PA$. Disregarding the factor $-\,u_4$, n will then prove to be identical with the denominator in the retarded potential for an electrical potential charge. The use of n in the formulas that follow for the potential (in place of r in the corresponding expression for the static potential) will then lead to one the electrodynamical effects that are analogous to the retarded effect for gravitational phenomena.

3. In order to pursue our considerations further, we will require a single differential rule that we would like to derive briefly here. Let P' ($x_\alpha + \delta x_\alpha$) be a point that is infinitely close to P (x_α). P' will then correspond to a certain point A' on L that is shifted from A along L by:

$$\delta s = \frac{\partial s}{\partial x_\nu}\,\delta x_\nu = \frac{\partial s}{\partial x^\nu}\,\delta x^\nu .$$

(Because of the single-valued association of A to P, s will become a single-valued function of P.) Now, from (11), we will have:

$$\delta X^\alpha = u^\alpha\,\delta s,$$

such that the change in l^α will be:

(17) $$\delta l^\alpha = \delta X^\alpha - \delta x^\alpha = u^\alpha\,\frac{\partial s}{\partial x^\nu}\,\delta x^\nu - \delta x^\alpha .$$

The vector that is constructed at P' with the components $l^\alpha + \delta l^\alpha$ will lie on the light-cone that emanates from the point P'. From that, we will have:

$$(l^\alpha + \delta l^\alpha)\,(l_\alpha + \delta l_\alpha) = 0,$$

which will further imply that:

(18) $$l^\alpha\,\delta l_\alpha = l_\alpha\,\delta l^\alpha = 0.$$

Now , upon considering (14), the introduction of (17) into (18) will imply that:

$$l_\alpha\left(u^\alpha\,\frac{\partial s}{\partial x^\nu}\,\delta x^\nu - \delta x^\alpha\right) = \left(n\,\frac{\partial s}{\partial x_\nu} - l_\nu\right)\delta x^\nu = 0,$$

such that, due to the arbitrariness in δx^ν, one will get:

(19) $$\frac{\partial s}{\partial x_\nu} = \frac{l_\nu}{n} .$$

From that and (17), one can further prove that:

(20) $$\frac{\partial l^\alpha}{\partial x^\beta} = -\delta^\alpha_\beta + \frac{u^\alpha l_\beta}{n} .$$

Finally, if a function $f(s)$ on L is given then when we insist upon the one-to-one correspondence between the point P and the point A, we will find from (19) that:

$$\frac{\partial f}{\partial x^\nu} = \dot{f}\,\frac{l_\nu}{n}, \qquad \dot{f} = \frac{df}{ds}. \tag{21}$$

We will get:

$$\frac{\partial n}{\partial x^\nu} = l^\alpha \frac{\partial u_\alpha}{\partial x^\nu} + u_\alpha \frac{\partial l^\alpha}{\partial x^\nu} = -u_\nu + \frac{l_\nu}{n}(1 + l^\alpha \dot{u}_\alpha) \tag{22}$$

from (20) and (21), just as we did for (12) and (14).

4. On the basis of the exactly-derived differential rules (21) and (22), the following fundamental theorem, can now be stated (by a long computation that we shall omit):

If f is an arbitrary function of s then the differential equation:

$$\Box\left(\frac{f}{n}\right) = 0 \tag{23}$$

will be satisfied everywhere except for the line L.

One can then immediately state solutions to the first system of equations (9′) – viz., $\Box\varphi^{\alpha\beta} = 0$ – that behave in a "pole-like" fashion along the world-line L.

$$\varphi^{\alpha\beta} = \frac{m^{\alpha\beta}}{n}, \tag{24}$$

in which $m^{\alpha\beta}$, and similarly $\varphi^{\alpha\beta}$, are symmetric in α and β. The pole-like character of the solutions comes from the fact that, from (16), the $\varphi^{\alpha\beta}$ will possess singularities like $1 / r$ along L. However, the tensor $m^{\alpha\beta}$ will be severely restricted by the conditions (10) for the $\varphi^{\alpha\beta}$. In the case of a pole-like solution, we would like to content ourselves with the following result: It can be proved ([1]) that the $m^{\alpha\beta}$ must possess the form:

$$m^{\alpha\beta} = m\, u^\alpha u^\beta, \tag{25a}$$

and furthermore:

$$\dot{m} = 0, \qquad m\dot{u}^\alpha = 0. \tag{25b}$$

That result will finally justify envisioning the $\varphi^{\alpha\beta}$ in (24), when compared with (9), as the gravitational potential of a simple point mass (up to a factor of $\kappa / 2\pi$). The $m^{\alpha\beta}$ in (25a) will then be the matter tensor for the point-mass, while (25b) will express the conservation of mass and momentum (i.e., Galilean inertial motion).

([1]) L, *loc. cit.*, equation (7).

One will now arrive directly at an extensive class of solutions to the potential $\varphi^{\alpha\beta} = 0$ that no longer have a pole-like character by remarking that each solution will imply another solution when one differentiates it one or more times with respect to the coordinates x^λ and adds such an expression with arbitrary coefficients. We would like to proceed with that development up to only first order, and from (24), set:

$$\varphi^{\alpha\beta} = \frac{m^{\alpha\beta}}{n} + \frac{\partial}{\partial x^\lambda}\left(\frac{m^{\lambda,\alpha\beta}}{n}\right), \tag{26}$$

in general, in which $m^{\lambda,\alpha\beta}$ must be symmetric in α, β. We will have to refer to the second term in the right-hand side of (26) as the "dipole term," by analogy with the representation (by Maxwell) of multipoles in electrostatics; its precise physical meaning will then demand a special analysis.

First of all, with the Ansatz (26), the conditions (10) must be satisfied. By some lengthy, but elementary, computations, that are based upon the rules (19) through (21), one will find an expression for $\partial\varphi^{\alpha\beta} / \partial x^\beta$ that takes the form ([1]):

$$\frac{\partial\varphi^{\alpha\beta}}{\partial x^\beta} = \frac{A^\alpha}{n^3} + \frac{B^\alpha}{n^2} + \frac{C^\alpha}{n}, \tag{27}$$

in which:

$$A^\alpha = m^{\lambda,\alpha\beta}\left(g_{\lambda\beta} + 2u_\lambda u_\beta - 3\frac{u_\lambda l_\beta + u_\beta l_\lambda}{n}\right), \tag{27a}$$

$$\begin{aligned} B^\alpha = {} & \dot{m}^{\lambda,\alpha\beta}\left(-g_{\lambda\beta} + 2\frac{u_\lambda l_\beta + u_\beta l_\lambda}{n} - 3\frac{l_\lambda l_\beta}{n^2}\right) \\ & + m^{\lambda,\alpha\beta}\left(g_{\lambda\beta}\frac{l_\nu \dot{u}^\nu}{n} - 3\frac{u_\lambda l_\beta + u_\beta l_\lambda}{n}\frac{l_\nu \dot{u}^\nu}{n} + 6\frac{l_\lambda l_\beta}{n^2}\frac{l_\nu \dot{u}^\nu}{n} + \frac{\dot{u}_\lambda l_\beta + \dot{u}_\beta l_\lambda}{n}\right) + m^{\alpha\beta}\left(u_\beta - \frac{l_\beta}{n}\right), \end{aligned} \tag{27b}$$

$$\begin{aligned} C^\alpha = {} & \ddot{m}^{\lambda,\alpha\beta}\frac{l_\lambda l_\beta}{n^2} - 3\dot{m}^{\lambda,\alpha\beta}\frac{l_\lambda l_\beta}{n^2}\frac{l_\nu \dot{u}^\nu}{n} - m^{\lambda,\alpha\beta}\frac{l_\lambda l_\beta}{n^2}\left[\frac{l_\nu \ddot{u}^\nu}{n} - 3\left(\frac{l_\nu \dot{u}^\nu}{n}\right)^2\right] \\ & + \dot{m}^{\alpha\beta}\frac{l_\beta}{n} - m^{\alpha\beta}\frac{l_\beta}{n}\frac{l_\nu \dot{u}^\nu}{n}. \end{aligned} \tag{27c}$$

Since (10) is satisfied identically, we must have:

$$A^\alpha = B^\alpha = C^\alpha = 0. \tag{28}$$

In order to draw conclusions from equation (28), Lubanski decomposed the tensors $m^{\alpha\beta}$ and $m^{\lambda,\alpha\beta}$ according to the components of the velocity u^α. We shall skip over the

([1]) L, *loc. cit.*, equations (23) to (23c).

work that Lubanski did, due to the uniqueness of that decomposition, and content ourselves with a summary of the end results ([1]), in which, one can set:

$$m^{\alpha\beta} = {}^*m^{\alpha\beta} + q^\alpha u^\beta + q^\beta u^\alpha + m\, u^\alpha u^\beta, \tag{29a}$$

$$m^{\lambda,\,\alpha\beta} = n^{\lambda\alpha} u^\beta + n^{\lambda\beta} u^\alpha - p^\lambda u^\alpha u^\beta, \tag{29b}$$

with no loss of generality. The tensors ${}^*m^{\alpha\beta}$, $n^{\alpha\beta}$, p^α, q^α that were recently introduced into this are all orthogonal to u^α:

$$ {}^*m^{\alpha\beta} u_\beta = 0, \quad q^\alpha u_\alpha = 0, \qquad n^{\alpha\beta} u_\beta = 0, \qquad p^\alpha u_\alpha = 0; \tag{30}$$

furthermore, $n^{\alpha\beta}$ is antisymmetric in α, β, while ${}^*m^{\alpha\beta}$ is symmetric.

5. Of the quantities that appear in (29a) and (29b), m and p^λ are the ones that are closest to having an immediate physical meaning. In order to see that, it will suffice to consider the static case:

$$u^1 = u^2 = u^3 = 0, \qquad u^4 = 1,$$

and to do that, one must compare the value of φ^{44} that follows from the general solution to (9), namely:

$$\varphi^{\alpha\beta} = -\frac{\kappa}{2\pi}\int \frac{T^{\alpha\beta}_{\text{ret}}}{\rho}\, dV \tag{31}$$

with the one that is computed from (26). The component T^{44} of the matter tensor is equal to the mass density μ_0 in this case; (31) will then imply that:

$$\varphi^{44} = -\frac{\kappa}{2\pi}\int \frac{\mu_0}{\rho}\, dV\,, \tag{32}$$

in which ρ means the distance from the volume element dV to the reference point P, and the integration extends over the volume of the particle. The position of the volume element dV has the coordinates $X + \xi$, $Y + \eta$, $Z + \zeta$. Since the coordinates ξ, η, ζ relative to the reduction point A (X, Y, Z) are small in comparison to r, the following development will be possible:

$$\frac{1}{\rho} = \frac{1}{r} + \frac{\partial(1/r)}{\partial X^\lambda}\xi^\lambda + \ldots = \frac{1}{r} - \frac{\partial(1/r)}{\partial \xi^\lambda}\xi^\lambda + \ldots,$$

in which, we have set $\xi^1 = i\xi$, $\xi^2 = i\eta$, $\xi^3 = i\zeta$, in analogy with (4). With that, (32) will become:

$$\varphi^{44} = -\frac{\kappa}{2\pi}\left[\frac{1}{r}\int \mu_0\, dV - \frac{\partial(1/r)}{\partial x^\lambda}\int \mu_0\, \xi^\lambda dV + \cdots\right]. \tag{32$'$}$$

([1]) L, *loc. cit.*, the equation before (35) and equation (36).

On the other hand, the orthogonality conditions (30) imply that:

$$*m^{44} = 0, \qquad q^4 = 0, \qquad n^{4\lambda} = -\, n^{\lambda 4} = 0, \qquad p^4 = 0.$$

Thus, in the static case, from (29a) and (29b), we will have $m^{44} = m$ and $m^{\lambda,44} = -\, p^{\lambda}$, while, from (16), $n = -\, r$. From that and (26), we will get:

(26′)
$$\varphi^{44} = -\frac{m}{r} + p^{\lambda}\frac{\partial(1/r)}{\partial x^{\lambda}}.$$

A comparison of (26′) with (32′) will now imply that:

(33)
$$m = \frac{\kappa}{2\pi}\int \mu_0\, dV, \qquad p^{\lambda} = \frac{\kappa}{2\pi}\int \mu_0\, \xi^{\lambda} dV.$$

The vector p^{λ} then represents the static moment of the mass distribution (relative to the chosen reduction point), up to a factor of $\kappa / 2\pi$, or, when one passes to a point-like particle, the *dipole moment*, resp. That implies that m has the same meaning that it did in the case of a simple mass point.

In what follows, we will show that the quantities $m^{\alpha\beta}$ and p^{α} depend upon $m^{\mu\nu}$ and p^{λ} and their derivatives (along with u^{ν} and $\dot{u}^{\nu}$); hence, they do not represent any independent variables of the system. Therefore, all that remains is to give meaning to $n^{\alpha\beta}$, which cannot be dealt with in such a simple manner as m and p^{λ}. It is only in the case that was treated by Lubanski (*loc. cit.*), in which $p^{\lambda} = 0$, that there is an argument (in regard to the $\varphi^{4\nu}$) that is analogous to the one above, and $n^{\alpha\beta}$ will represent the asymmetric energy-momentum tensor of the system, up to a factor.

However, one must emphasize that in the interesting case of a point-like pole-dipole particle, the *dipole moment* p^{λ} *must be non-zero in any case.* One must always choose the reduction point inside of the particle, since one demands that the elements of the world-line L must represent the instantaneous translational motion of the particles. However, the center-of-mass of a point-like pole-dipole system will always lie *outside* of the particle, so the vanishing of p^{λ} will be impossible ([1]).

In other words, due to the non-vanishing of p^{λ}, a simple meaning for $n^{\alpha\beta}$ will no longer be possible. *For that reason, we will simplify our problem by the assumption that the internal situation of the particle is already characterized sufficiently by m and* p^{λ}*, and therefore we will set* $n^{\alpha\beta} = 0$. Naturally, that assumption (just like the truncation of the development for $\varphi^{\alpha\beta}$ after the dipole terms) corresponds to just the principle of maximum simplicity, and it will be justified by the following result:

([1]) In contrast with the present situation, Lubanski's considerations relate to an extended, everywhere-positive, mass distribution, so its center-of-mass will lie in the interior of the particle. One can then let the reduction point A coincide with the center-of-mass, and therefore take $p^{\lambda} = 0$. However, in return for that, $n^{\alpha\beta}$ will be an actual, well-defined part of the physical arrangement of the system, since it will characterize its moment of momentum. We shall not go into that in detail, since a passage to a "small" particle, in the sense of Lubanski and Mathisson (*loc. cit.*) takes place here; cf., our introductory explanation, as well.

In the derivation of the equations of motion, we will first set $n^{\alpha\beta}$, as well as p^{λ}, to something non-zero, and subsequently set $n^{\alpha\beta} = 0$.

6. The computations that result from substituting the decomposition (29) into (27) are elementary, but lengthy. For that reason, we would like to restrict ourselves to the statement of the results of the computation. First of all, the first of conditions (28) – viz., $A^{\alpha} = 0$ – is satisfied identically. The second condition – viz., $B^{\alpha} = 0$ – implies that [1]:

(34) $$\left(\dot{n}^{\alpha\beta} + {}^*m^{\alpha\beta} + q^{\beta}u^{\alpha} - \dot{p}^{\beta}u^{\alpha} - p^{\beta}\dot{u}^{\alpha}\right)\left(u_{\beta} - \frac{l_{\beta}}{n}\right) = 0.$$

That equation shall be satisfied l_{β} identically. We denote the first factor by $Q^{\alpha\beta}$:

(35) $$Q^{\alpha\beta} = \dot{n}^{\alpha\beta} + {}^*m^{\alpha\beta} + q^{\beta}u^{\alpha} - \dot{p}^{\beta}u^{\alpha} - p^{\beta}\dot{u}^{\alpha},$$

such that (34) will go to:

(34′) $$Q^{\alpha} = Q^{\alpha\beta}u_{\beta} = Q^{\alpha\beta}\,\frac{l_{\beta}}{n}.$$

We now consider the vector $(l_1, 0, 0, 0)$, in particular. From (14) and (34′), we will then have:

$$Q^{\alpha} = \frac{Q^{\alpha 1}\,l_1}{l_1\,u^1} = \frac{Q^{\alpha 1}}{u^1}.$$

One similarly proves, upon considering the vectors $(0, l_2, 0, 0)$, …, that:

$$Q^{\alpha} = \frac{Q^{\alpha 1}}{u^1} = \frac{Q^{\alpha 2}}{u^2} = \frac{Q^{\alpha 3}}{u^3} = \frac{Q^{\alpha 4}}{u^4},$$

such that:

$$Q^{\alpha\beta} = Q^{\alpha}u^{\beta},$$

and from (35):

(36) $$Q^{\alpha}u^{\beta} = \dot{n}^{\alpha\beta} + {}^*m^{\alpha\beta} + q^{\beta}u^{\alpha} - \dot{p}^{\beta}u^{\alpha} - p^{\beta}\dot{u}^{\alpha}.$$

From the latter relation, upon multiplying it by u_{β} and recalling (12) and (30), one will next get:

(37) $$Q^{\alpha} = \dot{n}^{\alpha\beta}u_{\beta} - \dot{p}^{\beta}u_{\beta}\,u^{\alpha}.$$

Multiplying by u_{α} and considering (37) and (30) will give us:

(38) $$q^{\alpha} = \dot{p}^{\alpha} - \dot{p}^{\beta}u_{\beta}\,u^{\alpha} - \dot{n}^{\beta\alpha}u_{\beta}.$$

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(1) Some terms are rearranged in Lubanski [*loc. cit.*, equation (40)].

Upon substituting (37) and (38) in (36), it will follow that:

$$\dot{n}^{\alpha\beta} + \dot{n}^{\beta\nu} u_\nu \, u^\alpha - \dot{n}^{\alpha\nu} u_\nu \, u^\beta + {}^*m^{\alpha\beta} - p^\beta \dot{u}^\alpha = 0. \tag{39a}$$

${}^*m^{\alpha\beta}$ will now imply the symmetry property of (39a) directly. Upon switching β and α in (39a), it will follow that:

$$-\, \dot{n}^{\alpha\beta} + \dot{n}^{\alpha\nu} u_\nu \, u^\beta - \dot{n}^{\beta\nu} u_\nu \, u^\alpha + {}^*m^{\alpha\beta} - p^\alpha \dot{u}^\beta = 0. \tag{39b}$$

Upon adding (39a) and (39b), one will get:

$${}^*m^{\alpha\beta} = \tfrac{1}{2}(p^\beta \dot{u}^\alpha + p^\alpha \dot{u}^\beta), \tag{40}$$

whereas the complete relation in regard to Lubanski will emerge upon subtracting:

$$\dot{n}^{\alpha\beta} + \dot{n}^{\beta\nu} u_\nu \, u^\alpha - \dot{n}^{\alpha\nu} u_\nu \, u^\beta + \tfrac{1}{2}(p^\alpha \dot{u}^\beta - p^\beta \dot{u}^\alpha) = 0. \tag{41}$$

All that remains is to satisfy the third of the conditions (28), namely, $C^\alpha = 0$. Upon considering (38) and (40), we will get, after some computation:

$$2\dot{q}^\alpha + \dot{m} u^\alpha + m \dot{u}^\alpha - \ddot{p}^\alpha = 0, \tag{42}$$

which is the expansion of a corresponding relation in Lubanski. That equation can be integrated immediately:

$$2\, q^\alpha + m\, u^\alpha - \dot{p}^\alpha = \text{constant},$$

or, if we recall the meaning of q^α that corresponds to (38):

$$\frac{1}{c} P^\alpha \equiv m\, u^\alpha + 2\dot{n}^{\alpha\nu} u_\nu - 2\,\dot{p}^\nu u_\nu \, u^\alpha + \dot{p}^\alpha = \text{const.} \tag{43}$$

Equation (43) expresses the *law of conservation of energy momentum*, as one will recognize upon passing to the simple mass point ($p^\alpha = 0$, $n^{\alpha\beta} = 0$, $P^\alpha = mc\, u^\alpha$). P^α is the generalized energy-momentum vector:

$$P_1 = i\, P_x\,, \qquad P_2 = i\, P_y\,, \qquad P_3 = i\, P_z\,, \qquad P_4 = \frac{E}{c}, \tag{44}$$

when P_x, P_y, P_z are the usual components of the momentum, and E is energy. As for m and p^α in (43), one understands that they should have the values in (33), without the factor $\kappa / 2\pi$.

7. According to our assumption in Section 5, we finally set $n^{\alpha\beta} = 0$ and get the following *system of dynamical equations of motion* from (41) and (43):

(45) $$p^\alpha \dot{u}^\beta - p^\beta \dot{u}^\alpha = 0,$$

(46) $$\frac{d}{ds}\left(\frac{P^\alpha}{c}\right) \equiv \frac{d}{ds}(m\,u^\alpha - 2\,\dot{p}^\nu u_\nu\, u^\alpha + \dot{p}^\alpha) = 0,$$

upon which we will base our further considerations.

§ 2. Solving the equations of motion for the case of circular orbits.

1. We shall next examine solutions to the equations of motion for which the momentum of the system vanishes:

(47) $$P_1 = P_2 = P_3 = 0.$$

We will refer to such a system as a particle that is "macroscopically at rest," and we would like to show that the equations of motion in that case will admit an "internal" motion of the particle that corresponds to a *circular orbit* of a pole-dipole singularity that advances with constant velocity. The particle will then be characterized by its "proper" mass m and the magnitude of its dipole moment $\mathbf{p}$. We would like to omit a consideration of the general trajectory types that are consistent with the equations of motion here.

We then set v = const. for the circular orbit; it will follow that u_4, which we would like to denote u_0, will also be constant:

(48) $$u_4 = u_0 = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{\beta}{c}.$$

The path lies in the xy-plane with the coordinate origin as its midpoint; x and y, as well as u_1 and u_2, will then be periodic functions of time, while z and u_3 will vanish.

From (45), the four-vector p^α and the four-acceleration $\dot{u}^\alpha$ will be parallel to each other. Hence, the space-like vector $\mathbf{p}$ will have the direction of the radius vector (viz., it will be the same or the opposite), and its magnitude will be constant, on the grounds of symmetry, while p^4 will vanish. We will then have:

$$\dot{p}^\nu u_\nu = -\ (\dot{\mathbf{p}}\,\mathbf{u}) = \text{constant}, \qquad p^4 = 0.$$

We would like to denote the constant that arises in that way by m'. It is often Lorentz-invariant, and it will be shown that it is always positive; we then define:

(49) $$m' = \dot{p}^\nu u_\nu\,.$$

As for the energy E_0 of the particle, from (44) and (46), we will now have:

$$\frac{E_0}{c^2} = \frac{P_4}{c} = (m - 2m')\, u_0 \,. \tag{50}$$

On the other hand, upon multiplying (46) by u_α and recalling (49), one will prove that:

$$\frac{1}{c} P^\alpha u_\alpha = m - 2m' + \dot{p}^\alpha u_\alpha = m - m', \tag{51}$$

in general, so in the present case, due to (47) and (48):

$$\frac{E_0}{c^2} u_0 = m - m'. \tag{52}$$

We will then get the following double equation for the energy from (50) and (52):

$$\frac{E_0}{c^2} = (m - 2m')\, u_0 = \frac{m - m'}{u_0}. \tag{53}$$

Since we must have $u_0 > 1$, we then infer from this that:

$$\tfrac{1}{2} m \geq m' > 0. \tag{54}$$

The equality sign relates to the limiting case of $u_0 = \infty$, $\beta = 1$, which we shall discuss in detail later on. One further shows from (52) that since $m' > 0$, from (54), the rest mass of the particle $\mu = E_0 / c^2$ that appears will always be smaller than the "proper" mass m that belongs to the pole term:

$$\mu = \frac{E_0}{c^2} < m. \tag{55}$$

The introduction of the dipole terms along with the pole term will then lead to a reduction of the energy, in general.

Meanwhile, the radius R and velocity v of the circular orbit still remain undetermined. We would like to show that for a given particle (i.e., for a given m and $\mathbf{p}$), either of those quantities can be derived from the other one. We have:

$$x = R \cos \omega t, \qquad y = R \sin \omega t, \tag{56}$$

$$v_x = \frac{dx}{dt} = - R\omega \sin \omega t, \qquad v_y = \frac{dy}{dt} = R\omega \cos \omega t, \tag{56$'$}$$

and thus, from (15):

$$u_1 = - i\, \frac{R\omega u_0}{c} \sin \omega t, \qquad u_2 = i\, \frac{R\omega u_0}{c} \cos \omega t. \tag{56$''$}$$

Due to the fact that $m' > 0$ and from (54′), **p** will be directed towards the interior of the orbit:

(57) $$p_x = -p\cos\omega t, \qquad p_y = -p\sin\omega t.$$

In fact, with that choice of sign, one can show that:

(57′) $$\dot{p}_1 = i\,\dot{p}_x = i\,\frac{dp_x}{dt}\cdot\frac{dt}{ds} = i\,\frac{p\omega u_0}{c}\sin\omega t, \qquad \dot{p}_2 = -i\,\frac{p\omega u_0}{c}\cos\omega t,$$

and from that and (49) and (56″), it will follow that:

(58) $$m' = \frac{Rp\omega^2 u_0^2}{c^2};$$

thus, we have $m' > 0$. If one inserts (56″) and (57′) into $P_1 = 0$ or $P_2 = 0$, according to (46) then that will immediately yield:

(59) $$p = (m - 2m')\,R.$$

Now, from (53), we will have:

$$m - 2m' = \frac{m}{2u_0^2 - 1},$$

and it will follow from this and (59) that:

(60) $$2u_0^2 - 1 = \frac{mR}{p},$$

and if one recalls (48) then:

(60′) $$\frac{1+\beta^2}{1-\beta^2} = \frac{mR}{p}.$$

R and β will not be determined uniquely then until yet another dynamical quantity emerges (e.g., the angular momentum); cf., Section 4, as well.

By the way, equation (59) expresses the fact that a circular motion about the center-of-mass of the system will result. One occasionally clarifies that fact by noting that, from (50), the moving particle will have a "dynamical" mass of $m - 2m'$ (instead of the "proper" pole mass). (59) will then describe the vanishing of the total moment of the mass distribution relative to the center of the circular orbit, where one will observe that **p** contradicts that. However, we will obtain a more precise statement of the center-of-mass theorem later on. (Section 4.3).

2. The equation of circular motion admits a remarkable *passage to the limit in which the speed of the particle approaches the speed of light.* It can be inferred from (54) that u_0 will become infinite (i.e., $\beta = 1$) when $m' \to m / 2$. If m remains finite then from the second equation in (53), the energy E_0 will go to zero. For a finite, non-zero energy, one must then have $m \to \infty$! Similarly, from (59), the dipole moment **p** will go to zero when one adheres to a finite radius R. (A finite radius R will be required – e.g. – for a finite angular momentum; cf., Section 4. For a finite **p**, from (60), $R \to \infty$ as $\beta \to 1$.)

The relations shed a special light upon the problem of the *self-energy* of the electrically-charged elementary particle. From the arguments that were raised in the Introduction in favor of an inextended structure for elementary particles, it would seem plausible to attribute an elementary *point* charge to the charged elementary particle (and possibly higher electric multipoles, as well). However, from the previous discussion, the self-energy of an electric point charge will always be regarded as being infinite. Indeed, the authors showed in a paper that appeared recently ([1]) that the self-energy of an electric point charge will prove to be finite when one consider the gravitational force; however, the upper limit to that energy lies so high that for atomic dimensions, it can be regarded as practically infinite (namely, as much as a factor of 10^{21} higher than the rest energy of the electron!). It now seems appropriate to reconcile the fact that m will become infinite for a particle that moves with the speed of light (and finite rest energy, just the same) with the infinite or practically-infinite self-energy of the charge of the particle. We now see that by taking into account a small dipole term that vanishes in the limiting case, the principle of the self-energy of the charge will no longer represent an insurmountable obstacle, since a finite rest energy E_0 and a finite orbital radius for the particle are both compatible with an infinite self-energy.

We now return to the previously-sketched picture of the structure of the electron (assuming an infinite self-energy for the charge), in which the "mass-point" (mass $\mu = E_0 / c^2$) rotates around a "virtual center" (viz., center-of-mass) at a distance that is well-defined (by the angular momentum) at the speed of light ([2]). The objections that were raised in the name of mechanics and electrodynamics that we initially deferred can now be seen to actually weaken by the introduction of the dipole moment.

By the way, it is interesting to us to discuss the minor deviations from the strict limiting case $\beta = 1$ that are obtained when one replaces m with the upper limit $ec^2/\sqrt{k}$ (k is the Newtonian gravitational constant) for the self-energy of an electric point-charge e, as computed in the theory of gravitation ([3]). For the electron charge, $e = 4.77 \times 10^{-10}$ esu, so the mass that corresponds to that limit will be $m = 1.85 \times 10^{-6}\ g$ (but the electron mass is $\mu = 0.9 \times 10^{-27}\ g$!). With $m' = m / 2$, $\beta \approx 1$, we will next get from (52) that:

$$\mu = \frac{m}{2\mu_0}, \qquad u_0 = \frac{1}{\sqrt{1-\beta^2}} \approx \frac{m}{2\mu},$$

and from that:

$$\beta \approx 1 - \frac{2\mu^2}{m^2} = 1 - 5 \times 10^{-43}.$$

([1]) **H. Hönl** and **A. Papapetrou**, Zeit. Phys. **112** (1939), 65.
([2]) *Loc. cit.*, pp. 569; cf., the Introduction that paper, as well.
([3]) **H. Hönl** and **A. Papapetrou**, *loc. cit.*

The deviation from the velocity of light will be immeasurably small then. The magnitude of the associated dipole moment **p** will then follow from (50) and (59):

$$p = \frac{\mu R}{u_0} \approx \mu R \cdot \frac{2\mu}{m} \approx 10^{-21} \cdot \mu R.$$

The dipole moment **p** then proves to be smaller than the moment of the mass μ relative to the center of the orbit by a factor whose order of magnitude is 10^{-21}.

3. *The case of vanishing momentum.* – The case of $\mathbf{P} \neq 0$ will reduce to the case of $\mathbf{P} = 0$ by a Lorentz transformation. Let the latter be thought of as defining a system of particles that are "macroscopically at rest," with which we now associate a "moving" coordinate system x', y', z' that moves in an arbitrary direction relative to the internal orbit with a speed of $w = \beta^* c$. If we choose the direction of translation to be the x (x', resp.) direction then the Lorentz transformation will read:

(61) $$x = \frac{x' + wt'}{\sqrt{1-\beta^{*2}}}, \; y = y', \;\; z = z', \;\; t = \frac{t' + wx'/c^2}{\sqrt{1-\beta^{*2}}}.$$

Since the P_α constitute a four-vector, the transformation (61) must obviously take the energy-momentum components in the "rest system," namely:

(62) $$P'_x = P'_y = P'_z = 0, \qquad \frac{P'_4}{c} = \frac{E_0}{c^2} = \mu,$$

to:

(63) $$P_x = \frac{\mu w}{\sqrt{1-\beta^{*2}}}, \qquad P_x = P_x = 0, \quad \frac{E_0}{c^2} = \frac{\mu}{\sqrt{1-\beta^{*2}}}.$$

The "macroscopic" energy-momentum components thus behave precisely like a simple mass point of rest mass μ, regardless of the internal motion of the particle and its complicated structure.

We would now like to seek a detailed understanding of formula (63) by considering a special case, namely:

a) *Translation in the orbital plane*. – According to (61), when we write u_0, instead of u'_4, we will have:

$$u_x = \frac{u'_x + w u_0/c}{\sqrt{1-\beta^{*2}}}, \qquad u_y = u'_y, \qquad u_z = u'_z = 0, \qquad u_4 = \frac{u_0 + w u'_0/c}{\sqrt{1-\beta^{*2}}},$$

$$p_x = \frac{p'_x}{\sqrt{1-\beta^{*2}}}, \qquad p_y = p'_y, \qquad p_z = p'_z = 0, \qquad p_4 = \frac{w\,p'_x / c}{\sqrt{1-\beta^{*2}}}.$$

The invariance of $\dot{p}^\nu u_\nu$ then follows from this (since $ds = ds'$):

$$\dot{p}^\nu u_\nu = \dot{p}'^\nu u'_\nu = m',$$

in which m' has the meaning (58) for the orbit (in the co-moving system). Furthermore, from (46), the momentum proves to be:

$$\frac{1}{c}P_x = (m - 2m')\,u_x + \dot{p}_x = (m - 2m')\,\frac{u'_x + w u_0 / c}{\sqrt{1-\beta^{*2}}} + \frac{\dot{p}'_x}{\sqrt{1-\beta^{*2}}}, \tag{64}$$

while, on the other hand:

$$\frac{1}{c}P'_x = (m - 2m')\,u'_x + \dot{p}'_x = 0. \tag{64′}$$

It will follow from the last two relations that:

$$P_x = \frac{(m-2m')\,u_0 w}{\sqrt{1-\beta^{*2}}}, \tag{64″}$$

which is identical with the first equation in (63), due to (50). Similarly, the energy will prove to be:

$$\frac{1}{c^2}E = (m - 2m')\,u_4 + \dot{p}_4 = (m - 2m')\,\frac{u_0 + w u'_x / c}{\sqrt{1-\beta^{*2}}} + \frac{w\,\dot{p}'_x / c}{\sqrt{1-\beta^{*2}}}, \tag{65}$$

which likewise agrees with (63), due to (64′) and (50).

In (64) and (65), the actually-constant magnitudes P_x and E appear to be decomposed into two variable magnitudes whose fluctuations will cancel out. We will also find similar relations for angular momentum.

b) *Translation perpendicular to the orbital plane.* – The computations take an especially simple form in this case. If we choose the direction of translation to be the *z*-axis then, from (61) (with $u'_z = 0$, $p'_z = p'_4 = 0$), we will have:

$$u_x = u'_x, \qquad u_y = u'_y, \qquad u_z = \frac{w u_0 / c}{\sqrt{1-\beta^{*2}}}, \qquad u_4 = \frac{u_0}{\sqrt{1-\beta^{*2}}},$$

$$p_x = p'_x, \qquad p_y = p'_y, \qquad p_z = 0, \qquad p_4 = 0.$$

u_4 will then be constant, just as u_0 is, while p^α will remain simultaneously unchanged. It will now follow directly from (46) that:

(66a) $$\frac{1}{c}P_z = (m - 2m')\,u_z \quad = \frac{(m-2m')u_0 w/c}{\sqrt{1-\beta^{*2}}},$$

(66b) $$\frac{1}{c}E = (m - 2m')\,u_4 \quad = \frac{(m-2m')u_0}{\sqrt{1-\beta^{*2}}},$$

which will agree with (63), due to (50) (if one switches z and x).

§ 3. The Hamiltonian function and external forces.

The energy function H of our system offers especial interest, since that function can be compared to the Hamiltonian operator of Dirac wave mechanics for the electron, which expresses the known of the properties of the electron completely.

1. To that end, we eliminate $\dot{p}^\alpha$ from the energy function that is inferred from (46):

(67) $$\frac{H}{c^2} = \frac{P_4}{c} = (m - 2\dot{p}^\nu u_\nu)u_4 + \dot{p}_4$$

and introduce the momentum components P^α, in place of them (along with the velocity components u_α). In order to do that, we can appeal to the relation (51), which is valid in general:

(51) $$\frac{1}{c}P^\alpha u_\alpha = m - m',$$

which will go to:

(51′) $$\frac{E_0 u_0}{c^2} = m - m'$$

when one specializes to the rest system ($\mathbf{P} = 0$).

It will follow from (51) and (51′) that:

(68) $$c\,P^\alpha u_\alpha = E_0\,u_0\,.$$

We decompose the left-hand side into spatial and temporal parts, while keeping in mind the reality conditions that correspond to (4):

(68′) $$c\,P_4\,u_4 - c\,(\mathbf{P}\,\mathbf{u}) = E_0\,u_0\,.$$

From (67) and (68′), we will now get:

(69) $$H\,u_4 = c\,(\mathbf{P}\,\mathbf{u}) + E_0\,u_0 \,.$$

Upon dividing by u_4, in which we consider:

$$u_x = \frac{dx}{ds} = \frac{dx}{dt}\cdot\frac{dt}{ds} = v_x\frac{u_4}{c}, \qquad \ldots,$$

and inserting $E_0 = \mu c^2$, what will arise can be written comprehensively and definitively as:

(70) $$H = v_x\,P_x + v_y\,P_y + v_z\,P_z + \frac{u_0}{u_4}\mu\,c^2.$$

This expression for the Hamiltonian function has a noteworthy analogy to the Hamiltonian operator for the force-free motion of a Dirac electron, namely:

(71) $$H_{\text{op.}} = c\,(\alpha_x\,P_x + \alpha_y\,P_y + \alpha_z\,P_z) + \alpha_0\,\mu\,c^2,$$

in which P_x , P_y , P_z must be replaced with the operators $\frac{\hbar}{i}\frac{\partial}{\partial x}$, $\frac{\hbar}{i}\frac{\partial}{\partial y}$, $\frac{\hbar}{i}\frac{\partial}{\partial z}$ and α_x , α_y , α_z , α_0 mean matrices that fulfill the well-known commutation relations. The correspondence will become complete when one observes that, from **Breit** ([1]) and **Schrödinger** ([2]):

(72) $$\frac{dx}{dt} = \frac{i}{\hbar}(H_{\text{op.}}\,x - x\,H_{\text{op.}}) = c\alpha_x, \ldots;$$

i.e., the $c\alpha_x$, $c\alpha_y$, $c\alpha_z$ are the components of the "micro-velocity" of the particle. In (70), as well as in (71), we then have a *bilinear form in the macro-momenta and micro-velocities* of the particles. One must accordingly make the u_0 / u_4 in the mass term correspond to the matrix α_0 , which can be conceived as a "time-dependent operator" from **Schrödinger**, as can α_x , α_y , α_z . The eigenvalues of the α matrices are $\pm$ 1, so the observable values of the components of the micro-velocity will then be $\pm c$. As we saw in Section 2.2, nothing stands in the way of the micro-velocity of the pole-dipole particle approaching arbitrarily close to the speed of light ([3]).

One must also immediately confirm that the equations of motion exist in *canonical form*, if one imagines that the x, P_x, … in (70) are pairs of canonically-conjugate variables. In fact, we will then have:

(73) $$\frac{dx}{dt} = \frac{\partial H}{\partial P_x} = v_x, \ldots, \qquad \frac{dP_x}{dt} = -\frac{\partial H}{\partial x} = 0, \ldots$$

([1]) **G. Breit**, Proc. Am. Acad. **14** (1928), 553.
([2]) **E. Schrödinger**, *loc. cit.*
([3]) One must certainly observe the difference that, in the pole-dipole model, the velocity components can take on all values between + c and – c, while in quantum theory, the observable values for each velocity components can only be $\pm c$.

The first system of equations is analogous to (72), while the second one expresses the conservation of momentum for force-free motion. One must emphasize that here equations (73) do not represent an equivalent substitute for the equations of motion (46), since they do not lead to the defining equations for momentum as they would in the dynamics of point-masses.

2. We would like to employ the canonical equations (73) for the purpose of arriving at equations for the motion of a pole-dipole particle that carries a charge e in an electromagnetic field by formally extending them ([1]). In order to do that, we introduce the vector potential $\mathfrak{A}$ and the scalar potential Φ, from which the electric and magnetic field strengths $\mathfrak{E}$ and $\mathfrak{H}$, resp., of the external field will be given by ([2]):

$$\mathfrak{H} = \mathrm{rot}\,\mathfrak{A}, \qquad \mathfrak{E} = -\,\mathrm{grad}\,\Phi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t}. \tag{74}$$

We now introduce the momentum of the particle in an electromagnetic field by analogy with the corresponding expression for the charged mass point, if we recall (46) and (49):

$$\Pi = (m - 2m')\,\mathbf{u} + \dot{\mathbf{p}} + \frac{e}{c}\mathfrak{A} \tag{75}$$

and the energy function, by expanding (70):

$$H = \left(\Pi - \frac{e}{c}\mathfrak{A}\right)\mathbf{v} + \frac{u_0}{u_4}\mu c^2 + e\,\Phi. \tag{76}$$

From that, we will now regard x, Π_x, … as pairs of canonically-conjugate variables. As before, the first system of canonical equations will yield:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \Pi_x} = v_x, \;\ldots \tag{73′}$$

It will follow that the second system is:

$$\frac{d\Pi_x}{dt} = -\frac{\partial H}{\partial x} = \frac{e}{c}\left(v_x\frac{\partial \mathfrak{A}_x}{\partial x} + v_y\frac{\partial \mathfrak{A}_y}{\partial x} + v_z\frac{\partial \mathfrak{A}_z}{\partial x}\right) - e\frac{\partial \Phi}{\partial x}, \;\ldots \tag{73b′}$$

([1]) The introduction of the electromagnetic field goes strictly beyond our assumption in Section 1 that $T^{\alpha\beta} = 0$ (outside the particle). The equations of motion (78) derived below then demand a firmer foundation from the standpoint of pure field theory.

([2]) We shall denote vectors that relate to the external field – e.g., $\mathfrak{A}$, $\mathfrak{E}$, $\mathfrak{H}$ – by German letters and the ones that relate to the particle by boldface print. [Trans. note: In the original, it was an overhead arrow.]

If one preserves the definition (46) of the variables P_x, P_y, P_z, as before, then one can show that on the one hand, from (75):

$$\text{(77)} \qquad \frac{d\Pi_x}{dt} = \frac{dP_x}{dt} + \frac{e}{c}\frac{d\mathfrak{A}_x}{dt}$$

$$= \frac{dP_x}{dt} + \frac{e}{c}\left(v_x \frac{\partial\mathfrak{A}_x}{\partial x} + v_y \frac{\partial\mathfrak{A}_x}{\partial y} + v_z \frac{\partial\mathfrak{A}_x}{\partial z} + \frac{\partial\mathfrak{A}_x}{\partial t}\right), \ldots$$

It will then follow from ($73b'$) and (77) that:

$$\frac{dP_x}{dt} = e\left(-\frac{\partial\Phi}{\partial x} - \frac{1}{c}\frac{\partial\mathfrak{A}_x}{\partial t}\right) + \frac{e}{c}\left[v_y\left(\frac{\partial\mathfrak{A}_y}{\partial x} - \frac{\partial\mathfrak{A}_x}{\partial y}\right) - v_z\left(\frac{\partial\mathfrak{A}_x}{\partial z} - \frac{\partial\mathfrak{A}_z}{\partial x}\right)\right], \ldots$$

or, when one recalls (74):

$$\text{(78)} \qquad \frac{d\mathbf{P}}{dt} = e\,\mathfrak{E} + \frac{e}{c}[\mathbf{v}, \mathfrak{H}].$$

That is, in fact, the expected extension of the equations of motion (46) to the case in which an electromagnetic field is present, in which the Lorentz force appears as the resulting external field.

We will pursue the analogy between the pole-dipole particle and the Dirac electron further in a later part of this article, along with the applications of the equations of motion (78).

§ 4. Angular momentum and the center-of-mass theorem.

One can, in principle, take two paths for the introduction of angular momentum: One can proceed by stating an expression for angular momentum using formal analogies; e.g., with the corresponding expressions for the Dirac electron (viz., orbital angular momentum plus spin), such that the constancy of angular momentum will represent an important foundation as a consequence of the equations of motion. One can also proceed with an intuitive analysis of the mass (energy, resp.) distribution of the system, and from there, inductively seek to arrive at a general expression for angular momentum. We would like to take the latter informative path, and from the outset, we must stick to it in order to also find the (distributed) energy of the gravitational field from the energy of the original mass distribution ([1]).

([1]) The distinction between the energy of matter (which corresponds to the matter tensor $T_{\mu\nu}$) and the energy of the gravitational field is based upon the fact that a conservation law of the form:

$$\frac{\partial}{\partial x_\mu}(T_\nu^\mu + t_\nu^\mu) = 0$$

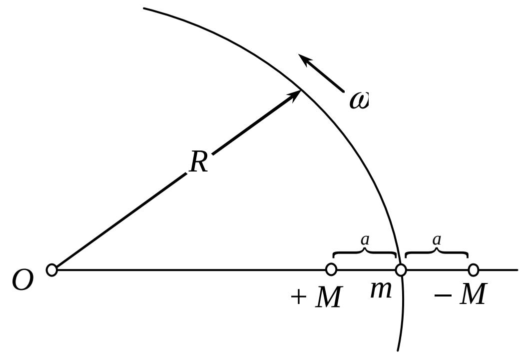

Figure 2. Schematic arrangement of the masses m, $\pm M$ along the radius of the particle orbit.

1. We consider a particle that is macroscopically at rest, and which we introduce as being represented by a pole m and a dipole **p**, and for the sake of illustration, we imagine that the dipole is realized by two masses $\pm M$ on both sides of m at a distance of a (cf., Fig. 2):

$$p = 2Ma, \qquad a \leq R. \tag{79}$$

Since **p** is directed towards the interior of the orbit $-M$ will lie outside of the circle of radius R, and $+M$ will lie inside of it.

We must complete that simple picture of the mass distribution by the addition of a suitable field energy, in such a way that the behavior of the structure will be in harmony with the equations of motion (45) and (46).

We first compute the energy E_I of the original mass distribution on the basis of our intuitive model:

$$\frac{1}{c^2} E_I = m\, u_0 + M\,(u_0^+ - u_0^-)\,, \tag{80}$$

in which are the $u_0^\pm$ are the corresponding velocity-dependent magnitudes for the masses $\pm M$. We will then have:

$$u_0 = \frac{1}{\sqrt{1 - R^2\omega^2 / c^2}}\,, \tag{81}$$

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will be true in the general theory of relativity, in which the t_ν^μ depend upon only the components of the gravitational field. In general, the gravitational energy density t_ν^μ cannot be localized uniquely, since the t_ν^μ do not transform like the components of a tensor of rank two. By comparison, integral theorems of the type:

$$\bar{P}^\mu = \int (T_\nu^\mu + t_\nu^\mu)\, dx_1\, dx_2\, dx_3 = \text{const.}$$

will be true, in which $\bar{P}^\mu$ do behave like the components of a four-vector under coordinate transformations. Cf., **W. Pauli**, Enzykl. d. math. Wiss., Bd. V, pp. 740, *et seq.*

One must further presume that one can also formulate angular momentum and center-of-mass theorems for the total system, in general (i.e., independently of the coordinate system), although evidence of that has yet to come to light up to now.

(ω is the orbital frequency), and correspondingly:

$$u_0^+ = \frac{1}{\sqrt{1-(R-a)^2\omega^2/c^2}} = u_0\left(1-\frac{Ra\omega^2}{c^2}u_0^2+\cdots\right), \tag{81a}$$

$$u_0^- = \frac{1}{\sqrt{1-(R+a)^2\omega^2/c^2}} = u_0\left(1+\frac{Ra\omega^2}{c^2}u_0^2+\cdots\right). \tag{81b}$$

From that, E_I will become:

$$\frac{1}{c^2}E_I = m\,u_0 - \frac{2aRM\,\omega^2}{c^2}u_0^2 + \ldots, \tag{80$'$}$$

and upon considering (79) and (58) in the limit as $a \to 0$:

$$\frac{1}{c^2}E_I = (m - m')\,u_0\,. \tag{82}$$

A comparison with (50) will show that the material part E_I of the energy by no means subsumes the total energy; we must further add the field energy E_{II} to it:

$$\frac{1}{c^2}E_{II} = \frac{1}{c^2}(E_0 - E_I) = -\,m'\,u_0\,. \tag{83}$$

Remarkably, E_{II} depends upon only m'; i.e., from (49) and (54′), the energy of the field will be produced by the angular motion of the dipole and will always be negative.

We employ a corresponding consideration for the momentum. Analogous to (80), we have the material system:

$$\frac{1}{c}\mathbf{P}_I = m\,\mathbf{u} + M\,(\mathbf{u}^+ - \mathbf{u}^-). \tag{84}$$

Now, we have that the magnitude of $\mathbf{u}$ is:

$$u = \frac{v}{c}u_0 = \frac{R\omega}{c}u_0\,, \tag{85}$$

and correspondingly, if we recall (81a) and (81b) then the magnitudes of $\mathbf{u}^+$ and $\mathbf{u}^-$ will be:

$$u^+ = \frac{(R-a)\omega}{c}u_0^+ = u\left(1-\frac{a}{R}-\frac{Ra\omega^2}{c^2}u_0^2+\cdots\right), \tag{85a}$$

$$u^- = \frac{(R+a)\omega}{c}u_0^- = u\left(1+\frac{a}{R}+\frac{Ra\omega^2}{c^2}u_0^2+\cdots\right). \tag{85b}$$

From that, we will see that in the limit as $a \to 0$, (84) will go to:

$$\frac{1}{c}\mathbf{P}_I = (m - m')\,\mathbf{u} - \frac{p}{R}\,\mathbf{u}\,. \tag{84′}$$

The expression (84′) obviously *does not* come from (82) when one multiplies (82) by the velocity $\mathbf{u} / u_0$. The reason for that lies in the fact that the mass distribution I is *not pole-like*. In fact, from (3), the extra term in (84′) whose magnitude is $pu / R = p\,\omega\,u_0 / c$ will correspond to the momentum that originates precisely in the rotation of the dipole. From (59), we can also write:

$$\frac{1}{c}\mathbf{P}_I = m'\,\mathbf{u}. \tag{86}$$

Since the total momentum $\mathbf{P}$ will vanish for the rest motion, the momentum $\mathbf{P}_{II}$ for the field energy that is associated with $\mathbf{P}_I$ must be:

$$\frac{1}{c}\mathbf{P}_{II} = -\,m'\,\mathbf{u}. \tag{87}$$

A comparison of (87) and (83) will now show that the field energy must be regarded as being *pole-like at the position of the particle.*

We are therefore in a position to give an immediate expression for the angular momentum. The pole-like constituents m and E_{II} provide no contribution to the angular momentum $\mathbf{J}$ relative to the (instantaneous) position of the particle. All that will remain is the contribution from the dipole, which, from the considerations that were introduced in the context of (2), will be represented by:

$$\mathbf{J}_0 = c\,[\mathbf{p}\;\mathbf{u}]. \tag{88}$$

For another reference point, relative to which the instantaneous position of the particle will be denoted by the vector $\mathbf{r}$, we will have an expression:

$$\mathbf{J} = c\,[\mathbf{p}\;\mathbf{u}] + [\mathbf{r}\;\mathbf{P}], \tag{89}$$

instead of (88), and it will often remain valid for the general case of a macroscopically-moving particle, as well. Therefore, (88) will then be independent of the reference point in the case of the rest motion.

From (89), the total angular momentum is seen to decompose into two pieces that one can rightfully identify as the *internal angular momentum (i.e., spin)* and the *orbital angular momentum.*

We would now like to state that the quantity that is defined by (89) rigorously represents a constant of the motion. When one observes that $\mathbf{P}$ = const., and from (45), that $\mathbf{p}$ is parallel to $\dot{\mathbf{u}}$, differentiating (89) with respect to proper time will show that:

$$\dot{\mathbf{J}} = c\,[\mathbf{p}\;\mathbf{u}] + [\dot{\mathbf{r}}\;\;\mathbf{P}].$$

Now, we have from (46) that:

$$[\dot{\mathbf{r}}\ \mathbf{P}] = [\mathbf{u}\ \mathbf{P}] = c\,[\mathbf{u}\ \dot{\mathbf{p}}],$$

from which:

(90) $$\dot{\mathbf{J}} = 0, \qquad \mathbf{J} = \text{const.}$$

The constancy of the angular momentum for the force-free motion defines another justification for the Ansatz (89).

By comparison, neither the intrinsic not the orbital angular momentum will be constant in the case of non-vanishing momentum. These relations are very similar to the ones that **Schrödinger** discussed for the force-free motion of the Dirac electron ([1]).

2. There exists a peculiarity with regard to the *sign* of the angular momentum. Since **p** is directed towards the interior of the orbit (§ 2), in the case in which we imagine that the circular orbit of the rest motion takes place in the *xy*-plane in the positive sense of the particle [in the sense of equations (56)], from (88), the *z*-component of the angular momentum will become negative:

$$J = J_z = -\,cpu.$$

We substitute p in this from (59) and replace $m - 2m'$ with μ / u_0 , from (50), and then get:

(91) $$J = -\,\mu\, v\, R,$$

in which v is the speed of the particle. The angular momentum of the rest motion will therefore have precisely the same magnitude as when the particle mass m rotates at a distance R with a velocity of v that has the *opposite* sign! From the complexity of the internal structure of the particle, such behavior should not need to astonish anyone. By the way, we can also assert that directly from the form of the moment of momentum relative to the center of the circular orbit. We first find the part of the angular momentum of the material system that is associated with m, **p** by recalling (84) and (79):

$$\frac{1}{c} J_I = m\, uR + M\,[u^+ (R - a) - u^- (R + a)] = \frac{1}{c} P_I R - p\, u;$$

in addition, we will have a contribution from the field energy, for which we will have:

$$\frac{1}{c} J_{II} = \frac{1}{c} P_{II} R = -\,\frac{1}{c} P_I R,$$

due to its pole-like structure, and the assertion will be proved once more.

3. It is now easy to confirm that the *center-of-mass theorem* is satisfied for the motion of the particle. We consider the case of rest motion and first determine the

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([1]) Cf., the Introduction to this paper.

distance x from the center of mass of the mass distribution E_I to the midpoint from the form of the momentum itself. From (80) and (79), we will have:

$$x\,\frac{E_I}{c^2} = m\,u_0\,R + M\,[u_0^+(R-a) - u_0^-(R+a)] = R\frac{E_I}{c^2} - p\,u_0\,.$$

Due to (59) and (82), we can write this as:

$$x\,\frac{E_I}{c^2} = m'\,u_0\,R, \tag{92a}$$

such that, from (82) and (54′):

$$x = \frac{m'}{m-m'}R \;\leq R. \tag{93}$$

The center-of-mass of E_I will approach the position of the particle with increasing angular velocity (for constant R) in the interior of the orbit and will coincide with it when the particle attains the speed of light. On the other hand, due to the pole-like character of E_{II}, the moment of E_{II} relative to the center of the orbit is, when one recalls (83):

$$P\,\frac{E_{II}}{c^2} = -\,m'\,u_0\,R. \tag{92b}$$

It will follow from (92*a*) and (92*b*) that:

$$x\,E_I + R\,E_{II} = 0; \tag{92c}$$

i.e., the total moment will vanish relative to the center of the orbit. That center will then be, at the same time, the center-of-mass of the particle. With that, we have resolved the paradox that a force-free particle should rotate about a center that lies outside of itself.

4. The foregoing considerations subsequently provide a far-reaching justification for the formerly-stated electron model of the "rotating mass point" (see the Introduction). The model, which was naïve at the time, was much simpler than the one that we treated here, since the complications of the pole-dipole structure of the particle were therefore ignored. However, only when we had introduced that structure was it possible to give the dynamical foundation for the former purely-hypothetical circular motion. The dimensions of the orbit (for the electron at rest) prove to be in agreement with the naïve model, from the present computations, when we also introduce the basic assumption that the micro-velocity of the particle is the velocity of light, as was suggested by the Dirac equation (Section 3.1), and the momentum of momentum is assumed to be quantized with the value $\frac{1}{2}\hbar$. Equation (91) will then imply that the magnitude of the angular momentum is:

$$J = \mu\,c\,R = \tfrac{1}{2}\hbar\,, \tag{94}$$

and therefore, the radius of the orbit will be:

$$R = \frac{\hbar}{2\mu c}, \tag{95}$$

as before. The rotational frequency will then be:

$$\omega = \frac{c}{R} = \frac{2\mu c^2}{\hbar}, \tag{96}$$

and from (94) and (96), the energy will be:

$$E_0 = \mu c^2 = \omega J, \tag{97}$$

which is an expression that was based upon the considerations of *loc. cit.*

It is especially remarkable that the sign of the angular momentum for the present refined model is the opposite of the one that emerged in the naïve model; hence, from that, a negative mass will rotate. However, the sign of the angular momentum (the sense of rotation, resp.) cannot be inferred from the mere mechanical behavior of the particle. Others related it to the particle simultaneously carrying a charge. One will then get the ratio of the magnetic moment to the mechanical moment of momentum from the new expression for the opposite sign to the one in the naïve model; i.e., if we regard our particle as a model for the electron, it will contradict the experiments (e.g., magneto-mechanical effects, the Landé *g*-formula). We shall not discuss the matter of how to possibly remove that contradiction any further here. If one attributes an electric dipole (of magnitude one magneton and located at the position of the particle) to the particle, along with its electric charge, then one can imagine that is it, however, likely that the contradiction is connected with the general difficulty that one will encounter when one attempts a step-by-step quantum formulation of electrodynamics.

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