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On the internal motion of electrons. II.

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1. A simple intuitive model for the pole-dipole particle (PDP) treated in Part I will be proposed (viz., the two-mass model).

2. It will be shown that when the two-mass model is treated relativistically, it will be identical to the PDP that is computed from the approximate solution to the gravitational equations.

3. The case in which the tensor $n_{\alpha\beta}$ that was introduced in I does not vanish will be examined.

In what follows, several considerations that are supplementary to the general theory of the pole-dipole particle (i.e., electron) that was developed in Part I ([1]) of this article will added to that theory. In Part III, the connection between that theory and **Schrödinger**'s "zitterbewegung" of the **Dirac** electrons will be presented.

§ 1. The two-mass model for the pole-dipole particle.

It is simple, as well as instructive, to give an intuitive model for the pole-dipole particle. The new fundamental assumption consists of the fact that we shall introduce positive, as well as *negative*, masses.

First, let two free positive masses m_1 and m_2 be given. One dynamically-possible state of motion for the system is the one in which the two masses rotate about the rest center-of-mass that lies between them with an angular velocity that is given by the equilibrium of their Newtonian force of attraction and their centripetal force (e.g., a double-star system). Nothing in this will change, in reality, if we assume that one of the masses – say, m_1 – is negative. The difference will then consist of the fact that the motion will then result about a center-of-mass that lies *outside* of the masses, and that stationary orbits for free masses will be possible only when $m_1 > | m_2 |$. One will then easily see that for $m_1 > | m_2 |$, the acceleration of both masses will be centripetally-directed towards the center-of-mass, and for $m_1 < | m_2 |$, it will be centripetally directed away from it. (Observe that the Newtonian force of gravity between masses with dissimilar signs will be a mutual "repulsion," but acceleration and force will have opposite directions for negative masses.)

In that form, the structure is not specialized enough for our purposes. We then introduce a "rigid" connection between the two m_1 and m_2 , which will have the effect

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([1]) Zeit. Phys. **112** (1939), 512; referred to as I in what follows.

that the rest case will now become possible, as well, since the Newtonian reaction between the two masses will be cancelled by the rigid connection between the two masses. The general state of motion of the system (disregarding translation) will then be such that the system rotates about the center-of-mass of the masses with an *arbitrary* angular velocity.

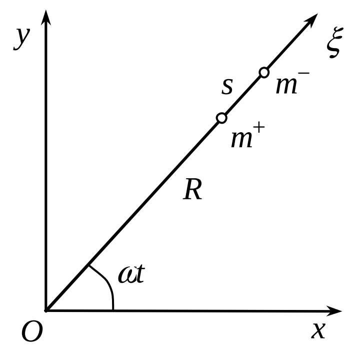

Figure 1. The coordinate representation of the two-mass model.

From now on, we would like to assume that we are concerned with two masses with nearly equal magnitudes m^+ and m^- and different signs, in such a manner that the size of the structure will be small in comparison to the distance to the center-of-mass (which corresponds to passing to an inextended structure that rotates about a center at a finite distance from it). We would now like to consider the motion of such a structure more closely.

We assume that $m^+ > | m^- |$ ([1]), and let s be the distance between the two masses, such that the center-of-mass O will lie on the side of m^+ at a distance R from it (Fig. 1), and the center-of-mass theorem will give:

(1) $$m^+ R = | m^- | (R + s).$$

It will then follow from the requirement that $s / R \ll 1$ that:

(1a) $$m^+ = | m^- | \gg m^+ - | m^- | .$$

Obviously, the structure corresponds to the assumption that one has a pole of strength:

(2a) $$m_0 = m^+ - | m^- |$$

and a dipole of moment:

(2b) $$p = m^+ s = | m^- | s.$$

We first consider *small* angular velocities ω for the system, so that we can use the formulas of non-relativistic mechanics. The kinetic energy will then become:

$$E_{\text{kin}} = \tfrac{1}{2} [m^+ R^2 - | m^- | (R + s)^2] \, \omega^2,$$

([1]) As we will establish later on, this assumption corresponds to a *positive* total energy for the particle.

and then, from (1) and (2):

(3) $$E_{\text{kin}} = -\tfrac{1}{2} m^+ R^2 s\, \omega^2 = -\tfrac{1}{2} p\, s\, \omega^2.$$

It is worth noting that $E_{\text{kin}} < 0$, so the total energy E will decrease during the motion; cf., (2*a*) and (3):

(3′) $$E = (m^+ - |\, m^- |)\, c^2 + E_{\text{kin}} = m_0\, c^2 - \tfrac{1}{2} p\, R\omega^2.$$

On the other hand, from the center-of-mass theorem (1), the momentum P will vanish:

(4) $$P = [m^+ R - |\, m^- |\, (R + s)]\ \omega = 0.$$

One finds from (1) and (2*b*) that the angular momentum J is:

(5) $$J = [m^+ R - |\, m^- |\, (R + s)]\ \omega = -\, m^+ R\, s\omega = -\, p\, R\ \omega.$$

We shall now compare those results with the consequences of the general theory of pole-dipole particles in I, in which we heuristically introduced the apparently-obvious assumption that $m = m_0$. Furthermore, we have assumed that $R\omega / c = \beta = 1$, so the mass m' that was introduced in I will have an order of magnitude of β^2 ; cf., [I, (58) and (48)]:

(6) $$m' = \frac{pR\omega^2 u_0^2}{c^2} = \frac{p}{R}\beta^2, \qquad u_0 = \frac{1}{\sqrt{1-\beta^2}}.$$

Consequently, from (2*a*) and (2*b*), the center-of-mass theorem (1) can also be written in the form:

(1′) $$p = m_0\, R,$$

which is identical with the relation (39) that was given in I:

$$p = (m - 2m')\, R,$$

up to magnitudes of order β^2. The expression (5) for angular momentum agrees with [I, (88)], and with the same accuracy ([1]). Furthermore, from [I, (50)], when we develop the energy up to order β^2, we will have [cf., (6)]:

$$E = (m - 2m')\, u_0\, c^2 = \frac{mc^2}{\sqrt{1-\beta^2}} - 2m'\, c^2 \approx mc^2 - \frac{3}{2} pR\omega^2$$

$$= \left(m - \frac{3}{2}\frac{p}{R}\beta^2 \right) c^2.$$

([1]) From [I, (88)], one will have $J = -\, cpu$, where $cu = R\omega u_0$, $u_0 = 1$.

As a comparison with (3′) will show, the agreement up to terms of order β^2 is also lost here. However, the discrepancy is only a result of the assumption that $m = m_0$, and will be removed by the rigorous relations between m and m_0 that will be given later on.

We would now like to examine the consequences of our model for *large* velocities (i.e., β arbitrarily close to 1). We will then have that the energy is:

$$\frac{1}{c^2}E = m^+u_0^+ - |\,m^-\,|\,u_0^- , \tag{7}$$

in which one can assume that, perhaps, $u_0^+ = u_0$, and is obtained from the series expansion:

$$u_0^- = u_0\left(1 + \frac{Rs\,\omega^2 u_0^2}{c^2} + \cdots\right)$$

[cf., I, eq. (81*b*)]. From that, we will then get:

$$\frac{1}{c^2}E = (m^+ - |\,m^-\,|)\,u_0 - m'\,u_0 = (m_0 - m')\,u_0 , \tag{7′}$$

as $s / R \to 0$, just as in (2*a*), (2*b*), and (6). The center-of-mass theorem (1) will now assume the form:

$$m^+u_0^+R = |\,m^-\,|\,u_0^-(R + s), \tag{8}$$

and similarly, that will express the vanishing of momentum when one multiplies it by ω. As for the angular momentum, it will follow from (8) that:

$$J = [\,m^+u_0^+R - |\,m^-\,|\,u_0^-(R + s)]\;\omega = -\,m^+u_0^+R\,s\;\omega = -\,c\,p\,\beta\,u_0 , \tag{9}$$

which is identical to [I, eq. (88)].

When one compares (7′) with [I, (50)]:

$$\frac{1}{c^2}E = (m - m')\,u_0 , \tag{10}$$

one will gather that the assumption that $m = m_0$ for the two-mass model will suppress an energy term $-\,m\,u_0\,c^2$; i.e., precisely the "field energy E_{II}" that was introduced in [I, eq. (83)]. By comparison, eq. (7′) and (10) will agree when one assumes that the following relation exists between m_0 and m:

$$m_0 = m - m'. \tag{11}$$

Furthermore, on the basis of (11), the center-of-mass theorem (8) will become:

(8′) $$|m^-|u_0^- s = (m^+u_0^+ - |m^-|u_0^-)R,$$

and from (2*b*), (7) and (10) will also assume the form (59) that was required in I:

(12) $$p = (m - 2m')\,R.$$

One can also transform the last equation into:

$$\frac{p}{R} + m' = m_0,$$

which, from (6), will turn into:

(13) $$R = \frac{p u_0^2}{m_0}.$$

As one confirms by a brief calculation, that relation is identical with formula [I, (60′)], and says that for a given particle (i.e., for a given m_0 and p), the R will not remain constant, but will increase with increasing angular velocity. The smallest value of R will be determined by the "static" relation (1) [(1′), resp.].

In the next section, we will show that, in general, eq. (11) actually represents the correct relation between m_0 and m (when one neglects the gravitational interaction of both masses). As far as that is concerned, we can say for now, in summation, that *our simple two-mass model reproduces all of the properties of the pole-dipole particle that were described in Part I.*

2. At this point, we would like to make a remark that has a very hypothetical nature, although it will lead to some remarkable consequences. The Newtonian interaction of two masses of the particles:

(14) $$E_{\text{pot}} = +\frac{k m^+ m^-}{s}$$

(k = gravitational constant) was not introduced expressly into our equations up to now. If we would now like to take that energy, which is always positive (since masses with different signs will repel) under consideration then some subsequent assumptions about the internal structure of the particle will suggest themselves: The particle shall originally be conceived to be a *pure dipole* – viz., two masses + M and – M, with a mutual separation of s – and its pole-term m_0 shall be attributed to only the interaction energy of that two masses:

(15) $$m_0 = \frac{kM^2}{sc^2}, \qquad p = Ms.$$

It will be easy to state a relation for M then. First, from (7) and (8), we will have:

(16) $$E = \frac{p u_0 c^2}{R} = \frac{M s\, u_0 c^2}{R} .$$

Furthermore, from (13) and (15), we will have:

$$\frac{p u_0}{R} = \frac{m_0}{u_0} = \frac{k M^2}{u_0 s c^2} .$$

The energy (16) can also be written:

(17) $$E = \frac{k M^2}{u_0 s} .$$

Upon multiplying (16) and (17), *s* will drop out, and one will get the energy in the form:

(18) $$E = \mu c^2 = \sqrt{\frac{k\, M^3 c^2}{R}} ;$$

when one solves this for *M*, one will get:

(18′) $$M = \sqrt[3]{\frac{R \mu^2 c^2}{k}} .$$

If one introduces the quantized angular momentum $\frac{1}{2}\hbar$ into the last formula, by way of its corresponding radius [I, (95)]:

(19) $$R = \frac{\hbar}{2\mu c} ,$$

then one will finally have:

(20) $$M = \sqrt[3]{\frac{\hbar \mu c}{2k}} .$$

For an electron ($m = 0.9 \times 10^{-27}$ g), that will give:

$$M \approx 6 \times 10^{-13} \text{ g!}$$

However, an approximate determination of the particle is not possible. Indeed, there will be an upper bound on *s*, because if one demands that $\beta \approx 1$ (i.e., $u_0 \gg 1$) for an electron then from [I, (52)], one must have $m_0 \ll M$. However, that is the only restriction on s, which will then actually remain undetermined within further limits ([1]).

We see from those considerations that, in principle, it is possible to imagine that our particle starts out as a pure dipole particle. However, one must emphasize that we have thus completely neglected the electrical field energy, and that it is, moreover, doubtful whether the Ansatz (14) will be justified by a more rigorous theory of gravitation. In any event, it is questionable whether the mass M in (20), with its very odd order of magnitude, has any physical meaning.

§ 2. A general theorem about the approximate solution of the gravitational equations. The use of pole-dipole particles.

1. In Part I, we obtained the equations of motion for the pole-dipole particle (PDP) from the method of the rigorous equations for the gravitational field φ_i^k. In that approximation, the gravitational potentials φ_i^k are computed as solutions to the equations [I, (9)]:

(21) $$\Box \varphi_i^k = -2\kappa T_i^k$$

(T_i^k is the matter tensor, and κ is the relativistic gravitational constant) that must, at the same time, satisfy the auxiliary condition [I, (10)]:

(22) $$\frac{\partial \varphi_i^k}{\partial x^k} = 0.$$

Equations (21) show that the gravitational potentials depend upon the corresponding components of the matter tensor linearly. The various parts of the material system that the gravitational field in question produces will contribute to the existing constituents of the gravitational potentials purely additively. It follows directly from this that in this approximate solution, each gravitational interaction that is produced between the various parts of the material system should be neglected. That remark will be made more precise in the following analysis.

Obviously, due to the auxiliary conditions (22), the components T_i^k cannot take on arbitrarily-given values. We will find the conditions for T_i^k when we differentiate eq. (21) with respect to x^k and sum over k:

$$\Box \frac{\partial \varphi_i^k}{\partial x^k} = -2\kappa \frac{\partial T_i^k}{\partial x^k},$$

which will then show that, because of (22), we will have:

([1]) For that matter, it remains for us to pass to the limit $s \to 0$, in the sense of the considerations that we derived in Part I in general (viz., inextended particles). However, such a passage to the limit can be realized in principle only within the context of a rigorous theory of gravitation.

(23) $$\frac{\partial T_i^k}{\partial x^k} = 0.$$

Conversely, when eq. (23) is satisfied, the conditions (22) for the φ_i^k themselves will be satisfied, as one will easily convince oneself. The conditions (23) for the T_i^k are then entirely equivalent to (22).

However, in the four-dimensional realm, with the metric $g_{\mu\nu} = \delta_{\mu\nu}$, the conditions (23) will be identical to the equations of motion in special relativity ([1]). One will then obtain the following theorem: The approximate solution for the gravitational equations yield the potentials for the gravitational field of a material system whose state of motion can be sufficiently described by the equations of special relativity. One can use that potential, e.g., to determine the motion of an external test particle in the gravitational field of the system considered. *However, as for the state of motion of the system in question itself, the results of the approximate solution to the gravitational equations will agree precisely with the equations of motion in the special theory of relativity.*

2. If we now apply that theorem to the PDP that was examined in Part I on the basis of the approximate solution to the gravitational equations then we will find that this particle can also be described from the standpoint of special relativity. In what follows, we will have to show that this new representation of the PDP is identical with the relativistically-treated two-mass model. As we shall see later on, that also suggests the possibility of finding a precise meaning for the characteristic quantities for the PDP that were introduced in Part I.

Obviously, the evidence for our assertion can be deduced from the following method: The relativistically-treated two-mass model represents a solution to eq. (23). We will now compute the gravitational potential for that system from (21) [(22) is itself satisfied, since (23) is satisfied], and we will compare that gravitational potential with the corresponding one in Part I for the PDP. With that potential, we will naturally envision the particle moving in the rest orbit, in particular.

We next sum the potentials of both masses m^+ and m^-, to which, we must add the potential of the tension that acts along the line that connects them. From [I, (24)] and [I, (25*a*)], the potentials of both masses are:

(24) $$(\varphi_{\alpha\beta})_{\text{mat}} = m^+ \frac{u_\alpha^+ u_\beta^+}{n^+} - |\,m^-\,| \frac{u_\alpha^- u_\beta^-}{n^-}\,.$$

We now decompose the magnitude of the larger mass m^+ according to (2*a*):

(25) $$m^+ = |\,m^-\,| + m_0\,,$$

([1]) Cf., **W. Pauli**, Enzyklopädie d. mathem. Wiss., Bd. V, pp. 682.

Through which, the two-mass system will reduce to a pole m and a dipole $(+| m^+ |, -|m^-|)$. From formula (24), we will now have:

(26) $$(\varphi_{\alpha\beta})_{\text{mat}} = (\varphi_{\alpha\beta})_{\text{pole}} + (\varphi_{\alpha\beta})_{\text{dipole}} ,$$

(26a, b) $$(\varphi_{\alpha\beta})_{\text{pole}} = m_0 \frac{u_\alpha u_\beta}{n} , \quad (\varphi_{\alpha\beta})_{\text{dipole}} = | m^- | \left(\frac{u_\alpha^+ u_\beta^+}{n^+} - \frac{u_\alpha^- u_\beta^-}{n^-} \right).$$

The mean values for the particle u_α, n were inserted into (26a), since the distinction between u_α^+, u_α^-, etc., as pole-terms is irrelevant [cf., assumption (1a)].

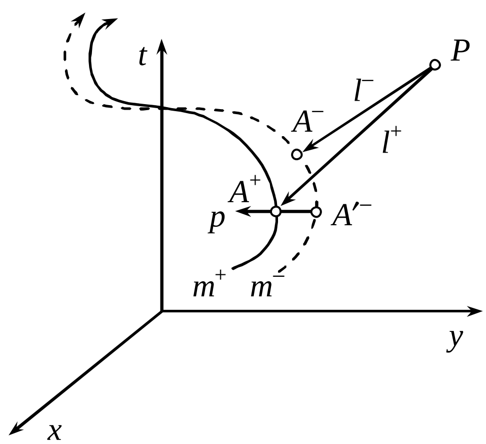

Figure 2. The world-line of the two–mass system:
A^+ and A^- are the retarded positions of the masses m^+ and m^-, resp., relative to P.
The dipole moment is in the direction of the "simultaneous" points A^+ and A'^-.

We now pass on to the computation of the dipole terms (26b). In Fig. 2, let $P(x_\alpha)$ be an arbitrary world-point that intersects the past-pointing light-cone of the world-line of both mass points at the world-point $A^+(X_\alpha^+)$ and $A^-(X_\alpha^-)$. From [I, (13)] and [I, (14)], and when we observe that u_α^+ and u_α^- are the velocity components that correspond to the point A^+ and A^-, resp., we will then have:

(27) $$l_\alpha^+ = X_\alpha^+ - x_\alpha , \quad n^+ = l_\alpha^+ u^{\alpha +} , \quad l_\alpha^- = X_\alpha^- - x_\alpha , \quad n^- = l_\alpha^- u^{\alpha -} .$$

We set:

(28) $$l_\alpha^+ - l_\alpha^- = \delta l_\alpha , \qquad u_\alpha^+ - u_\alpha^- = \delta u_\alpha , \qquad n^+ - n^- = \delta n,$$

and consider another point $A'^-(X'^-_\alpha)$ along the world-line from $| m^- |$ that has the same time coordinates as A^+:

(29) $$l_4^+ - l'^-_4 = X_4^+ - X'^-_4 = 0.$$

That point will serve to introduce the dipole moment four-vector:

(30) $$| m^- | (l_\alpha^+ - l_\alpha^-) = p_\alpha .$$

(Cf., Part I, in which one had $p_4 = 0$ for rest motion.) We now write (26*b*) in the form:

(31) $$(\varphi_{\alpha\beta})_{\text{dipole}} = \frac{|m^-|}{n^2}(n\, u_\alpha\, \delta u_\beta + n\, u_\beta\, \delta u_\alpha - u_\alpha\, u_\beta\, \delta n).$$

Our task is now to express the variations δu_α and δn as functions of the vectors u_α, p_α, l_α, and their time derivatives.

First, we write δu_α in the form:

(32) $$\delta u_\alpha = u_\alpha^+ - u_\alpha^- = (u_\alpha^+ - u_\alpha'^-) + (u_\alpha'^+ - u_\alpha^-) .$$

($u_\alpha'^-$ are the components of the velocity at the position $A_\alpha'^-$.) In order to determine the first terms in (32), we differentiate (30):

(33) $$dl_\alpha^+ - dl_\alpha'^- = \frac{dp_\alpha}{| m^- |}$$

and divide by ds^+ :

(34) $$u_\alpha^+ - \frac{dl_\alpha'^-}{ds^+} = \frac{\dot{p}_\alpha}{| m^- |} .$$

However, we now have:

$$(ds^+)^2 = dl_\alpha^+\, dl^{\alpha+} , \qquad (ds^-)^2 = dl_\alpha'^-\, dl'^{\alpha-} ,$$

with the auxiliary condition (33):

$$dl_\alpha^+ = dl_\alpha'^- + \frac{dp_\alpha}{| m^- |},$$

in which the second term is very small in comparison to the first one. It follows from this by simple calculation that:

(35) $$ds^+ = ds' - \left(1 + \frac{u_\nu\, \dot{p}^\nu}{| m^- |}\right),$$

up to terms of order one in $\dot{p}_\alpha / | m^- |$, and from (34), that:

(36) $$u_\alpha^+ - u_\alpha'^- = \frac{1}{| m^- |}\left[\dot{p}_\alpha - u_\alpha \cdot u_\nu\, \dot{p}^\nu\right].$$

The second term in (32) includes the magnitude of the velocity along the world-line. Hence, if $\delta s'$ is the four-dimensional line element (A'^-, A^-) then we will have:

(37) $$u'^{-}_{\alpha} - u^{-}_{\alpha} = - \dot{u}_{\alpha}\, \delta s' .$$

For the determination of $\delta s'$, one must consider that A^{+}, as well as A^{-}, lies on the light-cone at P:

$$l^{+}_{\alpha}\, l^{\alpha +} = 0, \qquad l^{-}_{\alpha}\, l^{\alpha -} = 0.$$

Hence, from the first of eqs. (28), it will follow that:

(38) $$l_{\alpha}\, \delta l^{\alpha} = 0.$$

Now, due to (28) and (30), we will have:

(39) $$\delta l^{\alpha} = (l^{+}_{\alpha} - l'^{-}_{\alpha}) + (l'^{-}_{\alpha} - l^{-}_{\alpha}) = \frac{p_{\alpha}}{|\, m^{-} \,|} - u_{\alpha}\, \delta s'.$$

Thus, (38) will become:

$$\frac{l_{\nu}\, p^{\nu}}{|\, m^{-} \,|} - l_{\nu}\, l^{\nu}\, \delta s' = 0,$$

from which, it will follow that:

$$\delta s' = \frac{l_{\nu}\, p^{\nu}}{n\, |\, m^{-} \,|}, \qquad u'^{-}_{\alpha} - u^{-}_{\alpha} = - \dot{u}_{\alpha}\, \frac{l_{\nu}\, p^{\nu}}{n\, |\, m^{-} \,|},$$

and by substituting (36) and (40) into (32):

(41) $$|\, m^{-} \,|\, \delta u_{\alpha} = \dot{p}_{\alpha} - u_{\alpha} \cdot u_{\nu}\, \dot{p}^{\nu} - \dot{u}_{\alpha}\, \frac{l_{\nu}\, p^{\nu}}{n} .$$

Now, we also compute δn from (27) and (28):

(42) $$\delta n = l^{+}_{\alpha}\, u^{\alpha +} - l^{-}_{\alpha}\, u^{\alpha -} = l_{\alpha}\, \delta u^{\alpha} + u_{\alpha}\, \delta l^{\alpha}.$$

We will get the quantities δu^{α} from (41), while the δl_{α} are obtained from (39) and the first of eq. (40):

$$\delta l_{\alpha} = \frac{1}{|\, m^{-} \,|} \left(p_{\alpha} - u_{\alpha}\, \frac{l_{\nu}\, p^{\nu}}{n} \right).$$

Finally, when we consider [I, (22)], we will get:

(42*a*) $$|\, m^{-} \,|\, \delta n = l_{\nu}\, \dot{p}^{\nu} - n\, u_{\nu}\, \dot{p}^{\nu} - \frac{\delta n}{\delta x^{\nu}}\, p^{\nu} .$$

If we introduce the values (41) and (42*a*) into (31) then we will finally get, from simple calculations:

(43) $$(\varphi_{\alpha\beta})_{\text{dipole}} = -\frac{\partial}{\partial x^\lambda}\left(\frac{p^\lambda u_\alpha u_\beta}{n}\right) + \frac{\dot{p}_\alpha u_\beta + u_\alpha \dot{p}_\beta}{n} - u_\nu \dot{p}^\nu \frac{u_\alpha u_\beta}{n},$$

in which use was also made of the differential rule [I, (21)].

3. We now go on to the computation of the parts of the gravitational potential in which the tension originates. Since the mass m^+ lies in the interior of the orbit (cf., Fig. 1), a force of pressure K whose magnitude is equal to the centripetal force that acts upon m^+ (the centripetal force that acts upon m^-, resp.) will exist along the line that connects the two masses:

(44) $$K = \left|\frac{d\mathbf{P}}{dt}\right| = m^+ u_0^+ R\omega^2.$$

We will assume that the tension in question acts only at a small distance from the connecting line; e.g., over a small cross-section f. The force of pressure $p_{\xi\xi}$ will then be in the direction ξ of the line that connects the masses; i.e., the instantaneous position of the orbital radius (Fig. 1):

$$p_{\xi\xi} = \frac{K}{f}.$$

Thus, the following relation will be true:

$$\int p_{\xi\xi}\, dv = \int p_{\xi\xi}\, f\, dv = K s,$$

or, with the value (44) for K, and upon considering (2*b*):

(45) $$\int p_{\xi\xi}\, dv = p u_0^+ R\omega^2.$$

All of the remaining components of the tension will vanish, as long as one remains in a Cartesian coordinate system that has ξ as a coordinate axis.

The components of the tension tensor are given immediately for the rest coordinate system, x, y, z on the basis of the rule that they must transform like the products of the corresponding coordinates:

(46*a*) $$p_{11} = -p_{xx}, \quad p_{12} = -p_{xy}, \quad p_{22} = -p_{yy}.$$

In order to avoid a detailed discussion of the constant factors (κ, etc.) that were left out of the integration of (21), we shall now remark that the potential (26*a*) of the mass pole is given in terms of the matter tensor by the following integration:

$$(\varphi_{\alpha\beta})_{\text{pole}} = \int \rho_0\, dv_0 \cdot \frac{u_\alpha u_\beta}{n} = \frac{1}{c^2 n}\int T_{\alpha\beta}\, dv_0 = \frac{u_0}{c^2 n}\int T_{\alpha\beta}\, dv.$$

dv_0 is the volume element in the coordinate system, relative to which, the element of matter is instantaneously at rest, and the factor $u_0 = dv_0 / dv$ corresponds to the Lorentz contraction itself. Correspondingly, we have that the gravitational potential of the tension is:

(48) $$(\varphi_{\alpha\beta})_{\text{tens.}} = \frac{u_0}{c^2 n}\int p_{\alpha\beta}\, dv .$$

If we now replace u_0^+ with u_0 in (45) then it will follow from (45), (46), and (46*a*) that the various components are:

$$\varphi_{11} = -\frac{1}{c^2 n} p u_0^2 R\, \omega^2 \cos^2 \omega t, \qquad \varphi_{22} = -\frac{1}{c^2 n} p u_0^2 R\, \omega^2 \sin^2 \omega t,$$
$$\varphi_{12} = -\frac{1}{c^2 n} p u_0^2 R\, \omega^2 \sin \omega t \cos \omega t,$$

which one can also summarize as:

(48*a*) $$(\varphi_{\alpha\beta})_{\text{tens.}} = \frac{p_\alpha \dot{u}_\beta + \dot{u}_\alpha p_\beta}{2n},$$

on the basis of the kinematic relations [I, (56″)] and [I, (57)].

Upon adding (26*a*), (43), and (48*a*), we will get the total gravitational potential of the two-mass system as:

(49) $$\varphi_{\alpha\beta} = -\frac{\partial}{\partial x^\lambda}\left(\frac{p^\lambda u_\alpha u_\beta}{n}\right) + \frac{\dot{p}_\alpha u_\beta + u_\alpha \dot{p}_\beta}{n} + \frac{p_\alpha \dot{u}_\beta + \dot{u}_\alpha p_\beta}{n} + (m_0 - \dot{p}^\nu u_\nu)\frac{u_\alpha u_\beta}{n} .$$

On the other hand, from [I, (26)] and [I, (29)], with $n_{\alpha\beta} = 0$, the potential of the PDP will be:

(50) $$\varphi_{\alpha\beta} = -\frac{\partial}{\partial x^\lambda}\left(\frac{p^\lambda u_\alpha u_\beta}{n}\right) + \frac{{}^*m_{\alpha\beta}}{n} + \frac{q_\alpha u_\beta + u_\alpha q_\beta}{n} + \frac{m u_\alpha u_\beta}{n},$$

in which, from [I, (38)] and [I, (40)], q_α and ${}^*m_{\alpha\beta}$ are taken to be:

(50′) $$q_\alpha u_\beta + u_\alpha q_\beta = \dot{p}_\alpha u_\beta + u_\alpha \dot{p}_\beta - 2\dot{p}^\nu u_\nu \cdot u_\alpha u_\beta ,$$

(50″) $${}^*m_{\alpha\beta} = \tfrac{1}{2}(p_\alpha \dot{u}_\beta + \dot{u}_\alpha p_\beta).$$

A comparison of (49) and (50) will now show that those two expressions will agree precisely when we set:

(51) $$m_0 = m - \dot{p}^\nu u_\nu = m - m'.$$

However, (51) is identical with the relation (11) that was given in Part I. *We thus come to the conclusion that the relativistically-treated two-mass model is completely equivalent to the PDP that is computed from the approximate solutions to the gravitational equation.*

Therefore, the difficulty that was connected with the "field energy" E_{II} that was introduced in Part I will also vanish ([1]). For that reason, that energy term must be introduced only because the quantity m was erroneously identified with the primary pole term m_0 there ([2]). Naturally, the validity of all of the formulas in Part I remain unchanged, except that one must give the quantity m the corresponding charged meaning (51). Here, we would like to transform only formula [I, (52)] according to (51); we will get:

$$\frac{E_0}{c^2} = \frac{m_0}{u_0} = m_0\sqrt{1-\beta^2}\,, \tag{52}$$

which now represents a peculiar counterpart to the relation:

$$\frac{E_0}{c^2} = m\,u_0 = \frac{m}{\sqrt{1-\beta^2}}$$

for the energy of a simple mass-pole. [One must, however, observe that (52) is valid only for the orbital motion of the particle "at rest."]

Finally, we can give the ultimate conclusion of the development in this particular section: The "classical Dirac equation" [I, (70)] can be founded upon the principle of special relativity exclusively without making any actual use of the theory of gravitation.

§ 3. Several supplementary remarks about Part I. The tensor $n_{\alpha\beta}$.

1. The classical Dirac equation [I, (70)] came about under the simplifying assumption that $n_{\alpha\beta} = 0$. It is interesting to remark that this simplification is not necessary, and that one will also arrive at [I, (70)] precisely in the general case. In order to that, one must start with the general equation [I, (43)], multiply it by u_α, and sum over α:

$$\frac{1}{c}P^\alpha u_\alpha = m - m' + 2\dot{n}^{\alpha\nu}u_\nu\,u_\alpha\,,$$

in which use was made of the relations [I, (12)] and [I, (49)]. However, as a result of the antisymmetry of $n_{\alpha\beta}$, one will now have:

$$\dot{n}^{\alpha\nu}u_\nu\,u_\alpha = 0.$$

([1]) Cf., the remarks at the beginning of this section, in which all gravitational interactions between different particles of the system in question were omitted from the approximate solution so that energy that would be due to the gravitational field would not arise.

([2]) From [I, (33)], the relation $m_0 = m^+ - |\,m^-\,| = m$ is completely valid for the static case ($u_1 = u_2 = u_3 = 0$); however, in the general case, that relation must be replaced with (51).

One will then get:

$$\frac{1}{c} P^{\alpha} u_{\alpha} = m - m',$$

which is identical to [I, (51)]. [I, (70)] will then follow from this the same calculations that were given in Part I. One will then come to the result that *the classical Dirac equation represents the characteristic equation for the generalized pole-dipole particle.*

We would now like to examine how the expression for *angular momentum* can be generalized for non-vanishing $n_{\alpha\beta}$. That question can be answered most simply by saying that one makes use of the constancy of the total angular momentum of the particles [cf., I, (89), *et seq.*]. We must now differentiate the angular momentum with respect to proper time:

$$\frac{d}{ds} (J_{ik})_{\text{orb.}} = \frac{d}{ds} (x_i P_k - x_k P_i) = u_i P_k - u_k P_i ,$$

from [I, (43)]. When we considers [I, (41)], after some simple calculations, we will get:

$$\frac{d}{ds} (J_{ik})_{\text{orb.}} = - c \frac{d}{ds} (2 n_{ik} + p_i u_k - p_k u_i).$$

It will follow from this that the internal angular momentum (i.e., spin) of the particle is:

$$J_{ik} = c (2 n_{ik} + p_i u_k - p_k u_i). \tag{53}$$

[I, (88)] is then completed by the term $2 c n_{ik}$. The assignment of the spatial tensor components J_{ik} is a result of the choice of [I, (4)] of real and imaginary coordinates:

$$J_{12} = - J_z , \qquad J_{23} = - J_x , \qquad J_{31} = - J_y . \tag{53′}$$

2. We would now like to generalize the expression for a particle that orbits in the rest space that was treated in Part I, by assuming that the tensor $n_{\alpha\beta}$ does not vanish. We must make the following obvious assumption about the spatial components of $n_{\alpha\beta}$:

$$n_{12} = \text{const.} \neq 0, \qquad n_{23} = n_{31} = 0, \tag{54}$$

from which, the angular momentum of the particle will remain perpendicular to the orbital plane. Furthermore, we assume that the direction of p_{α} is exactly as it was in Part I, so eqs. [I, (57) and (58)] will still remain valid with no changes. Next, the remaining components of $n_{\alpha\beta}$ can also be determined from the orthogonality relations (30). One will then confirm immediately that [I, (41)] is satisfied identically. Finally, the energy-momentum equation [I, (43)] will lead to the following generalization of the relations [I, (59)] and [I, (50)]:

(55) $$(m - 2m')\,R = p + 2n_{12}\,\beta\,u_0\,, \qquad \mu = \frac{pu_0}{R} + \frac{2n_{12}\beta}{R}\,.$$

From (53) and (53′), when one infers p from the first of eq. (55), the angular momentum of that particle will now be:

(56) $$J = J_z = -\,\mu\,v\,R - \frac{2c\,n_{12}}{u_0^2}\,.$$

Formula (56) admits a remarkable application: Namely, one can remove the difficulty that was spoken of at the end of I in regard to the sign of the angular momentum by introducing a suitable value for n_{12} . To that end, one has only to set:

$$J = -\,\mu\,v\,R - \frac{2c\,n_{12}}{u_0^2} = +\,\mu\,v\,R,$$

from which, it will follow that:

(57) $$n_{12} = -\,\mu\,R\,\beta\,u_0^2\,.$$

One will further deduce from (55) that:

(57′) $$p = \mu\,R\,u_0\,(1 + \beta^2), \qquad m = m u_0^2\,(1 + \beta^4).$$

One can arrive once more at a particle that moves with the speed of light, and for that reason, it will possess a finite value for μ and J. Now, from (57) and (57), all of the three quantities n_{12}, p, and m will be infinite. However, we would not like to discuss that possibility any further, since it seems to offer no real advantages.

3. By comparison, it is interesting to examine whether it is possible to get a classical analogue of the Dirac particle that has vanishing dipole moment and non-vanishing tensor $n_{\alpha\beta}$. From the general formulas (55) and (56), it seems that this possibility will actually exist. Namely, with $p = 0$, one will get:

(58) $$\mu = \frac{2n_{12}\beta}{R}\,, \qquad J = -\,2c\,n_{12}\,;$$

furthermore, from the first of eq. (55):

(58′) $$m = \frac{2n_{12}\beta\,u_0}{R} = \mu\,u_0$$

(which is certainly not tantamount to the particle possessing a primary pole-term that is analogous to m_0 for the PDP). We eliminate n_{12} in (58) and get:

(59) $$J = -\frac{\mu v R}{\beta^2}.$$

It will follow from this that one can again pass to the limit $\beta \to 1$ for a finite μ and J, from which, one will once more find that the result that one obtains for a PDP is:

(60) $$J = -\mu\, c\, R,$$

precisely as one had in Part I. With that, one comes to the remarkable conclusion that one can obviously get a "classical Dirac particle" for $p \neq 0$, $n_{12} = 0$, just as one does for $p = 0$, $n_{12} \neq 0$ ([1]).

4. Finally, an interpretation for the tensor $n_{\alpha\beta}$ shall be sought. It is obtained from the following consideration: The existence of a rest particle ($u_1 = u_2 = u_3 = 0$, $u_0 = 1$) is compatible with equations [I, (41)] and [I, (43)], from which, m and p_α will vanish, and only the tensor $n_{\alpha\beta}$ will be non-zero. We can assume the following simplest form for it:

(61) $$n_{12} = -n_{21} = \text{const.} \neq 0, \qquad \text{all other } n_{\alpha\beta} = 0,$$

which also satisfies the orthogonality conditions [I, (30)]. We can now determine the gravitational potential for such a particle, which will then give us information about the associated matter tensor by way of (21); i.e., information about the internal structure of the particle. The gravitational potentials can then be calculated from [I, (26)] and [I, (29)]. One will next get from [I, (29*a*)], with [I, (38)] and [I, (40)], that:

$$m_{\alpha\beta} = 0.$$

One will then further deduce from [I, (29*b*)] that the only non-zero component of $m_{\lambda,\alpha\beta}$ is:

$$m_{1,42} = m_{1,42} = n_{12}, \quad m_{2,41} = m_{2,41} = -n_{12}.$$

The only non-vanishing gravitational potential will finally follow from this when one considers that for a rest particle, from [I, (16)], one will have $n = -r$, so:

(62) $$\varphi_{14} = \varphi_{41} = n_{12}\,\frac{\partial(1/r)}{\partial x_2}, \qquad \varphi_{24} = \varphi_{42} = -n_{12}\,\frac{\partial(1/r)}{\partial x_1}.$$

Conversely, from (62), one will have the means to get (21) from the fact that the only non-zero components of T are $T_{14} = T_{41}$ and $T_{24} = T_{42}$ for our particle; i.e., the only non-zero components of the momentum are in the xy-plane. However, those components are not pole-like at the position of the particle, but decompose into the form of a double

([1]) The possibility that a more thorough discussion might imply that one model (presumably $p \neq 0$, $n_{12} = 0$) is superior to the other one is certainly not excluded from this.

source. Our particle will then represent a *momentum vortex* in the *xy*-plane, to some extent. Obviously, that momentum vortex will correspond to an angular momentum of $2n_{12}$, which agrees with formula (53).

That momentum vortex can be described by the motion of a simple material system. One realization of it consists of the simultaneous rotation of two material rings that lie close to each other, with opposite mass densities and angular velocities, in such a way that the spatial components $T_{\alpha\beta}$, as well T_{44} , are equal to zero, and only the momentum components are non-zero. (For just *one* ring, all of the components $\rho_0\, c^2\, u_\alpha\, u_\beta$ of the matter tensor will be non-zero.) One will then come to a seemingly-complicated picture of the momentum vortex, which contradicts the simplest-possible picture of mass dipoles. That difference will make it apparent that one must assume that p_α and $n_{\alpha\beta}$ are simultaneously non-zero. With that, our assumption in I that we should set $n_{\alpha\beta} = 0$ due to the non-vanishing of the dipole moment seems to have been subsequently justified.

Athens and Stuttgart, in August, 1939.

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