"Über die innere Bewegung des Elektrons. III," Zeit. Phys. **116** (1940), 153-183.

On the internal motion of electrons. III.

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The correspondence between the Dirac electron and the pole-dipole particle will be examined. The parallel correspondence can be pursued in an especially clear way through the distinction between the even and odd parts of the Dirac electron and through the calculation of typical expectation values for them (Section 1). That method will be applied to the details of the macro-motion and the micro-motion (i.e., zitterbewegung) of the electron (Section 2), proper angular momentum (Section 3), and the energy function (Section 4). In all cases, there is a very precise correspondence that suggests that one can regard the pole-dipole particle as the classical model for the Dirac electron.

This third part of the examination of the internal motion of electrons pursues the objective of confirming in detail the correspondence that was described in the first two parts between the physical systems that are characterized by pole-dipole particles ([1]) and the **Dirac** electron. In any event, the simple relativistic mass-point *cannot* be considered to be the corpuscular counterpart to the Dirac electron, since it has no internal degrees of freedom that would correspond to the electron spin. The pole-dipole particle behaves quite differently, since its internal rotational motion possesses a proper angular momentum that makes it possible to draw a parallel with electron spin. We shall now investigate the extent to which that correspondence reaches in detail; i.e., the extent to which the pole-dipole particle can serve as a classical model for the Dirac electron. *We glimpse that the problem for the model is to present a corpuscular counterpart to the wave picture of* ***Dirac****'s wave mechanics for the spinning electron* that behaves in relation to it in the manner that the simple (relativistic) mass-point relates to the **de Broglie-Schrödinger** matter waves.

We would like to confine ourselves to *force-free* motion in the comparison that is to be carried out. In that case, the *Dirac electron* will be described in a well-known way by the wave equation:

$$H\psi + \frac{\hbar}{i}\frac{\partial \psi}{\partial t} = 0, \tag{1}$$

with the Hamiltonian operator:

$$H = c\,(\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3) + \alpha_4\, \mu_0\, c^2, \tag{2}$$

([1]) **Hönl** and **Papapetrou**, Part I: Zeit. Phys. **112** (1939), 512; Part II: Zeit. Phys. **114** (1939), 478 (referred to as I and II in what follows).

in which one has the commutation relations:

$$P_k\, x_1 - x_1\, P_k = \frac{\hbar}{i}\,\delta_{kl} \qquad (k, l = 1, 2, 3), \tag{3a}$$

$$\alpha_\kappa\, \alpha_\lambda + \alpha_\lambda\, \alpha_\kappa = 2\,\delta_{\kappa\lambda} \qquad (\kappa; \lambda = 1, 2, 3), \tag{3b}$$

($E_0 = \mu_0\, c^2$ is the rest energy of the electron). As **Breit** ([1]) first remarked and **Schrödinger** ([2]) followed through with in detail, the well-known "trivial" connection between impulse and velocity in the mechanics of point-masses is abolished for the Dirac electron. Namely, if one takes the time derivative x_k then one will get:

$$\frac{dx_k}{dt} = \frac{i}{\hbar}(H\, x_k - x_k\, H) = c\,\alpha_k \tag{4}$$

from the general prescription for taking the time derivative of an operator and from (2), (3a), and (3b); i.e., a quantity whose eigenvalues are $\pm c$. The expectation value of dx_k / dt will then generally possess an order of magnitude c in any case. That will be true independently of the expectation value for the impulse, and for a wave group with a sufficiently-sharply-defined macrovelocity ([3]), that impulse will be connected with that velocity in the usual manner for a relativistic point-mass. **Schrödinger** has interpreted that discrepancy between microvelocity and macrovelocity by saying that the Dirac electron does not move rectilinearly, but performs a kind of *zitterbewegung* (jittering motion), in such a way that the center-of-mass of the charge cloud moves back and forth with a very small computable amplitude and an oscillatory speed of light, and therefore advances with a macrovelocity that is very small in some situations. All of the details of that zitterbewegung are obtained by integrating equation (4) in the sense of the operator calculus (Section 1).

The behavior of the Dirac electron now exhibits a striking similarity with the behavior of the *pole-dipole particle*, which one can describe classically. From I, that particle shall be characterized by the following system of equations ([4]):

$$\frac{d}{ds}\left(\frac{P_\alpha}{c}\right) \equiv \frac{d}{ds}(m\, u_\alpha - 2\dot{\pi}^\beta\, u_\beta\, u_\alpha + \dot{\pi}^\alpha) = 0 \qquad (\alpha, \beta = 1, 2, 3, 4), \tag{5}$$

$$\pi_\alpha \dot{u}_\beta - \pi_\beta \dot{u}_\alpha = 0. \tag{6}$$

([1]) **G. Breit**, Proc. Amer. Acad. **14** (1928), 553.

([2]) **E. Schrödinger**, Berl. Ber. (1930), 418; *ibid.* (1931), 63 (cited as A and B in what follows).

([3]) The "macrovelocity" is then sharply-defined when one is dealing with a (monochromatic) wave group with sharply-defined impulse components. We shall avoid using the word "group velocity" for it, since the center-of-mass of the wave group takes part in precisely the "zitterbewegung."

([4]) In that article, we especially considered the system of equations that arises from the general eq. [I, (43)] by setting the tensor $n_{\alpha\beta}$ equal to zero; cf., also Section II, Section 3. The four-vector p_α that was introduced there is referred to as π_α here.

P_α is then the energy-impulse vector, so (5) will express the conservation of energy and impulse (force-free motion). The u_α are the velocity components dx_α / ds of the particle, in which the derivatives of the coordinates x_α are taken with respect to proper time s, such that one will have $u^\alpha u_\alpha = 1$ at every point of the path (in particular, $u_4 = 1 / \sqrt{1-\beta^2}$, when $\beta = v / c$ is the speed of the particle divided by c). Correspondingly, the derivatives that are denoted by a dot will be understood to mean derivatives with respect to proper time. The internal structure of the particle will characterized by its mass pole m (scalar) and its dipole moment π_α (four-vector).

Now, as was shown in Part I, the system of equations (5), (6) possesses solutions that correspond to a *circular motion* of the pole-dipole with constant velocity, while at the same time, the resultant impulse **P** will vanish (i.e., a particle that is macroscopically at rest). The macro-impulse **P** and the internal (micro-) velocity of the particle are then to be regarded as also being coupled by that to a great extent. If one overlays the circular motion of the "rest" particle with a translation then the *constant* impulse **P** will assume the magnitude that belongs to its macroscopic velocity, in the sense of the relativistic mechanics of point-masses (P_α = four-vector). It is, moreover, remarkable that the formulas for the circular motion admit the *passage to the limit: microvelocity $v \to$ speed of light c* without the rest mass and impulse moment becoming infinite in the process ([1]). In what follows, that limiting case shall always be invoked when one compares things with the Dirac electron.

The following conception of **Schrödinger**'s zitterbewegung will then be based upon our model: In that model, the zitterbewegung of the coordinates of the Dirac electron that remains when one solves for the macro-motion will be represented by a periodic orbiting of the particle around a circular path that is initially oriented arbitrarily; the electron spin comes about with the speed of light by the orbiting of the particle. *We will show that this model-based conception of the proper properties of the Dirac electron will largely justify the motion that Schrödinger discussed for force-free motion, up to the more complicated details.* For the time being, we would like to satisfy ourselves by establishing that in the limiting case of $v \to c$, the model will yield a frequency $\left(= \frac{\hbar}{2\mu_0 c}\right)$ and radius $\left(= \frac{2\mu_0 c^2}{\hbar}\right)$ for the circular orbit that coincide with the frequency and amplitude, resp., of the coordinate oscillation for the Dirac electron when one assumes that the internal impulse moment (spin) in the model is quantized in units of $\frac{1}{2}\hbar$ (cf., Section 2).

Naturally, the concrete intuitive character of the model goes far beyond the **Dirac** electron ([2]). That can already be expressed by, e.g., the fact that the measured valued of

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([1]) Moreover, that passage to the limit sheds a peculiar light on the problem of the *self-energy* of the particle that is due to its electric charge. Namely, it shows that the infinite self-energy of the point-charge of the particle is not only consistent with a finite rest mass, but that the going to infinity of the mass pole m is even the condition for the internal (micro-) velocity of the particle to attain the speed of light (asymptotically). (I, Section 2.2)

([2]) The fundamental impossibility of verifying the statements of the model by experiments shall not be discussed here. In that regard, it is, however, of great interest that, from **Schrödinger**, the geometric configuration of a system with rest mass μ_0 can be observed with a linear precision of $l_0 = \hbar / 2\mu_0 c^2$, at best. It emerges from that imprecision that it will be impossible to make any concrete statement about the details of the internal motion. On that, cf., **Schrödinger**, Berl. Ber. (1931), 238; in particular, equations (5) any velocity *component* (eigenvalue of $c\alpha_k$) is capable of taking on only the values $\pm c$ for the Dirac electron, while in our model, all intermediate values between $+ c$ and $- c$ will be possible, and the extreme values still depend upon the orientation of the orbital plane ([1]). One should then perhaps expect that the model must break down for the details of the spin motion. However, the fact that this is in no way the case, and the fact that, in contrast, it underscores a very far-reaching correspondence will emerge in the following sections.

The physical systems that are defined, on the one hand, by (1), (2), and (3a, b), and (5) and (6), on the other, seem to have no similarity with each other, on first glance. However, if one eliminates the $\dot{\pi}_\alpha$ from equations (5) then one will get the energy $c / i\, P_4$ in the form:

$$H^* = v_1\, P_1 + v_2\, P_2 + v_3\, P_3 + \frac{u_0}{u_4}\mu_0\, c^2, \tag{7}$$

in which u_0 represents the value of u_4 for a particle that is macroscopically at rest [I, eq. (70)]. Now, when one observes (4) and regards u_0 / u_4 as a parallel to α_4, the expression (7) will, in fact, exhibit a striking analogy with the Hamilton operator H of the Dirac electron. In the final Section 4, we will go further than that and see that this formal analogy is based upon a deeper-lying correspondence. One would also have to expect corresponding relationships for other operators – e.g., position coordinates, spin – (Section 2 and 3).

The *negative-energy states*, which are anomalies of the Dirac electron, play an essential role in the following investigation. They first make it possible to carry out the comparison of the Dirac electron with the model of the orbiting mass-point in detail, and indeed without appealing to **Dirac**'s theory of positrons. Whether or not the choice of certain special representative wave functions that we shall base the comparison upon might not seem mandatory *a priori*, the results will allow one to recognize an even more striking analogy between the behavior of the model and the Dirac electron. The deeper physical basis for that correspondence can perhaps be sought in the fact that the relativity postulate upon which the unification with the principles of quantum theory in **Dirac**'s theory is based already represents such a narrow conceptual framework that any electron model that is free from internal contradictions and allows one to calculate electron spin in a relativistically-invariant way will agree with the Dirac electron ([2]).

The comparison that is carried out in what follows will be based upon the two cited papers by **Schrödinger** on the quantum dynamics of electrons. The results that are obtained will rest completely upon the "method of time-dependent operators" that will be introduced there, and which consists of the fact that the time-dependency of the wave

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and (10); cf., also **H. T. Flint** and **O. W. Richardson**, Proc. Roy. Soc. London (A) **117** (1928), 625 and 637.

([1]) From the model, one would expect the correspondence:

$$c^2 = v_1^2 + v_2^2 + v_3^2 \to c^2\,(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\,;$$

however, in reality, one has $\alpha_k^2 = 1$ for *every individual k*. Correspondingly, the measured value of the square of every velocity component, and therefore, the expectation value, as well, is always c^2 in the Dirac model.

([2]) The authors thank Herrn Prof. **W. Heisenberg** for a discussion of that viewpoint.

function ψ in equation (1) will be "inherited" from the operators. That method has the great advantage for us that, of all the quantum-mechanical methods, it comes closest to the viewpoint of corpuscular theory.

§ 1. General considerations regarding Schrödinger's zitterbewegung. Viewpoint for the comparison with the model.

1. The "equation of motion" for the arbitrary operator A:

$$\frac{\hbar}{i}\frac{dA}{dt} = HA - AH, \tag{8}$$

can be generally integrated in the sense of the operator calculus ([1]):

$$A(t) = e^{iHt/\hbar} A(0)\, e^{-iHt/\hbar}. \tag{9}$$

H is to be regarded as an operator in (9), as well as in (8); t is a c-number. $A(t)$ accordingly emerges from $A(0)$ by a special unitary transformation, which is entirely analogous to the way that motion can arise from a sequence of infinitesimal contact transformations in classical mechanics. The expectation value $\overline{A_t}$ of A at the time t is then:

$$\overline{A_t} = \int \psi^*(0) A(t)\, \psi(0)\, dx, \tag{10}$$

in which $\psi(0)$ means the wave function at time $t = 0$, which is assumed to be normalized:

$$\int \psi^*(0)\, \psi(0)\, dx = 1. \tag{10′}$$

[Integration over the coordinate space x_1, x_2, x_3 ; summation over the spin indices has been suppressed in the notation of (10) and (10′).]

Now, the distinction between *even* and *odd* operators plays a fundamental role in this method of time-dependent operators, and it is linked with the existence of states of positive and negative energy in the Dirac electron ([2]). We shall understand an even operator G ([†]) to mean one that takes a wave function ψ^+ that consists of an aggregate of eigenfunctions with *only* positive energies to other such wave functions, and likewise converts a wave function ψ^- that consists of eigenfunctions with *only* negative energies to another such wave function. By contrast, an odd operator U will take ψ^+ to ψ^- and ψ^- to ψ^+; in formulas:

$$G\,\psi^+ = \psi^{+\prime}, \quad G\,\psi^- = \psi^{-\prime}, \qquad\qquad U\,\psi^+ = \psi^{-\prime}, \quad U\,\psi^- = \psi^{+\prime}. \tag{11}$$

([1]) **E. Schrödinger**, A, § 2.
([2]) **E. Schrödinger**, B, § 2.
[†] Translators note: G = *gerade* = even, U = *ungerade* = odd.

It is easy to see that every arbitrary operator A can be split *uniquely* into an even and an odd part:

$$A = G + U. \tag{12}$$

Namely, if one represents the operator A by a matrix field $A(E, E')$ whose "elements" are ordered by the eigenvalues E, E' of the energy function H then that field will decompose into four separate regions I, II, III, IV, since $|E|$ ($|E'|$, resp.) is always $\geq \mu_0 c^2$ (Fig. 1): The matrix fields I and III by themselves will yield the representation of G, and II and IV by themselves will yield the representation of U.

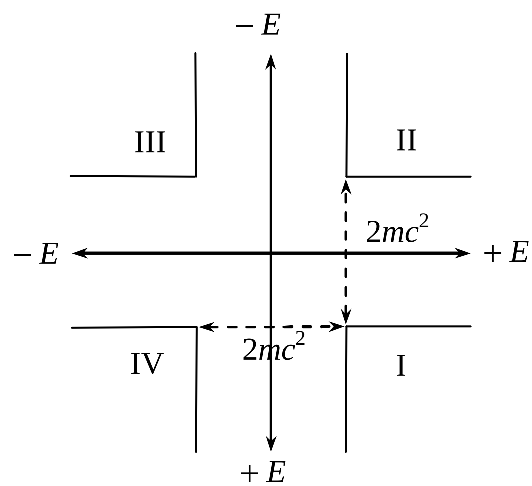

Figure 1. Matrix element (E, E') for an arbitrary operator $A = G + U$. The diagonals $E = E'$ ($E = -E'$, resp.) correspond to the representation of an operator that commutes (anticommutes, resp.) with H. (The locations in the interior of the central cross with beam width $2\mu_0 c^2$ are meaningless.)

A sufficient, but not necessary, condition for an operator A to belong to the even or odd type is that is must commute (anticommute, resp.) with H ([1]):

$$HG - GH = 0, \qquad HU + UH = 0. \tag{13}$$

Since the representation of H in the eigen-system contains only diagonal elements, which we can express symbolically by:

$$H(E, E') = E \cdot \delta(E - E')$$

(with the **Dirac** δ-function), we will get from (13) that:

([1]) According to **Schrödinger** (B), the necessary condition for an operator A to be even or odd is that a *finite* commutator (anticommutator, resp.) of A must vanish. Hence, the first commutator of A is $K_1 = HA - AH$, the second one, $K_2 = HK_1 - K_1H$, etc.; the first, second, etc. anticommutator of A is defined correspondingly. A general analytic prescription for splitting an operator into its even and odd parts was given by **W. Pauli**, *Handb. d. Phys.*, XXIV/1, pp. 229; however, such a decomposition is unnecessary for the calculation of expectation values (cf., *infra*).

$$\left.\begin{aligned} G(E,E')(E-E') &= 0, \\ U(E,E')(E+E') &= 0. \end{aligned}\right\} \tag{13′}$$

$G(E, E')$ can then possess non-vanishing elements only for $E' = E$, and $U(E, E')$, only for $E' = -E$, which agrees with our assertion. It will then follow from (9) that for all even (off, resp.) operators that commute (anticommute, resp.) with H, one will have:

$$G(t) = G(0) = \text{const.} \tag{14a}$$

or

$$U(t) = e^{+i\frac{2Ht}{\hbar}}\, U(0) = U(0)\, e^{-i\frac{2Ht}{\hbar}}, \tag{14b}$$

resp. The result (14a) is obvious; however, from (14b), U takes on a universal, apparent, "periodic" time-dependency of the operator with a "frequency" of $2H/\hbar$.

The "zitterbewegung" of the position coordinates x_k in the Dirac electron that was mentioned in the introduction is now based upon the fact that one part of x_k (which will be denoted by ξ_k, in what follows) can be split off that has odd type and anticommutes with H; its time-dependency has the type of (14b), accordingly. However, the arguments in this section show that *any* dynamical variable will perform a corresponding zitterbewegung that possesses a component that anticommutes with H; i.e., in general, any variable that is not a constant of the motion. However, in order to recognize the nature of that zitterbewegung more precisely, it is instructive to go from operators to expectation values. Furthermore, that shall happen in a general manner.

2. For what follows, we assume that U is Hermitian and anticommutes with H (which are two conditions that will be fulfilled by all of the operators that will be considered in what follows). Now, it should be remarked from the outset that the expectation value of U will always vanish when U is applied to a "pure positive" wave function (ψ^+) or a "pure negative" one (ψ^-), since, from (11), $U\psi$ will then possess the opposite character to ψ and the wave functions ψ^+ and ψ^- are orthogonal to each other. U can then possess a non-zero, time-dependent expectation value only for a "mixed" wave function $\psi = \psi^+ + \psi^-$.

Since the eigenfunctions of the force-free moving particle can be represented by:

$$a_\rho^\pm(p_1, p_2, p_3)\, e^{i(\mathbf{px})/\hbar},$$

in which $\mathbf{p}$ (p_1, p_2, p_3) means an eigenvector of the impulse and the amplitudes a_ρ^+ shall belong to a positive-energy state, while the amplitudes a_ρ^- shall belong to a state of negative energy ($\rho = 1, 2, 3, 4$), we set:

$$\psi_\rho(x) = \int \left\{ a_\rho^+(p) + a_\rho^-(p) \right\} e^{i(\mathbf{px})/\hbar} dp, \tag{15}$$

in the case of a general superposition, in which x and p means a triple of values x_1, x_2, x_3 (p_1, p_2, p_3, resp.) , and the integration extends over all of impulse space ($dp = dp_1\, dp_2\, dp_3$). With the abbreviation:

$$\psi(p) = e^{i(\mathbf{px})/\hbar}, \tag{16}$$

from (10), the expectation value $\overline{U_t}$ will become:

$$\overline{U_t} = \sum_\rho \iiint \left\{a_\rho^{+*}(p) + a_\rho^{-*}(p)\right\}\psi^*(p)\,dp\cdot U\left\{a_\rho^{+}(p') + a_\rho^{-}(p')\right\}\psi(p')\,dp'\,dx\,. \tag{17}$$

Now, since U is odd, one will have:

$$\sum_\rho \int a_\rho^{+*}(p)\psi^*(p)U\,a_\rho^{+}(p')\psi(p')\,dx = \sum_\rho \int a_\rho^{-*}(p)\psi^*(p)U\,a_\rho^{-}(p')\psi(p')\,dx = 0,$$

and due to the Hermiticity of U [cf., (16)]:

$$\sum_\rho \int a_\rho^{-*}(p)\psi^*(p)U\,a_\rho^{-}(p')\psi(p')\,dx = \left[\sum_\rho \int a_\rho^{+*}(p)\psi^*(p)U\,a_\rho^{-}(p')\psi(p')\,dx\right]^*.$$

Hence, (17) will become:

$$\overline{U_t} = \sum_\rho \iiint a_\rho^{+*}(p)\psi^*(p)\,dp\,U\,a_\rho^{-}(p')\psi(p')\,dp'dx + \text{conj.} \tag{17′}$$

Moreover, if one considers (14b) (viz., the anticommutation of U and H), as well as the fact that $a_\rho^{-}(p')\,\psi(p')$ is an eigenfunction of H with the eigenvalue $-\varepsilon(p')$, and if the eigenvalues of H are denoted by:

$$E = \pm c\sqrt{\mu_0^2\,c^2 + p_1^2 + p_2^2 + p_3^2} = \pm\,\varepsilon(p),$$

then (17′) will become:

$$\overline{U_t} = \sum_\rho \iiint a_\rho^{+*}(p)\psi^*(p)\,dp\,U(0)\,a_\rho^{-}(p')\psi(p')\,e^{i2\varepsilon(p')t/\hbar}\,dp'dx + \text{conj.} \tag{17″}$$

Since the operator $U(0)$ acts upon the spin variable ρ, it can be represented by a matrix $U^0_{\rho\rho'}$, such that:

$$\overline{U_t} = \sum_\rho \sum_{\rho'} \iiint a_\rho^{+*}(p)\psi^*(p)\,dp\,U^0_{\rho\rho'}\,a_\rho^{-}(p')\psi(p')\,e^{i2\varepsilon(p')t/\hbar}\,dp'dx + \text{conj.} \tag{17‴}$$

At this point, we would like to make the assumption that the operator U depends upon only the quantities α_κ and P_k (but not upon the x_k) ([1]), which is an assumption that will be fulfilled for all of the special operators that are considered in what follows. Due to the orthogonality relation:

$$\int \psi^*(p)\psi^*(p')dx = (2\pi\hbar)^3 \cdot \delta(p_1 - p_1')\,\delta(p_2 - p_2')\,\delta(p_3 - p_3') \qquad (16')$$

the integrations over x and p' in (17) can be performed immediately. One will then get ([2]):

$$\overline{U_t} = (2\pi\hbar)^3 \sum_\rho \sum_{\rho'} \int a_\rho^{+*}(p)\,U_{\rho\rho'}^0(p)\,a_\rho^-(p)\,e^{i2\varepsilon(p)t/\hbar}dp + \text{conj.}, \qquad (18)$$

which will imply the dependency of the matrix element $U_{\rho\rho'}^0(p)$ on the p_k when one replaces every P_k with its eigenvalue p_k .

An analogous argument that pertains to the even part G will lead to the result that:

$$\bar{G} = (2\pi\hbar)^3 \sum_\rho \sum_{\rho'} \int \left\{ a_\rho^{+*}(p)\,G_{\rho\rho'}^0(p)\,a_{\rho'}^+(p) + a_\rho^{-*}(p)\,G_{\rho\rho'}^0(p)\,a_{\rho'}^-(p) \right\} dp\ , \qquad (19)$$

in which $G_{\rho\rho'}^0(p)$ is defined in a manner that corresponds to $U_{\rho\rho'}^0(p)$. (Here, as well, it will be assumed that G depends upon only α_κ and P_k .)

The expressions (18) and (19) admit an extension that we will make frequent use of in what follows: Let an operator $A = G + U$ be given, whose parts G and U depend upon only α_κ and P_k , so the expectation values $\bar{G}$ and $\overline{U_t}$ can be calculated in such a way that one applies the schema in formulas (18) and (19) simply to A, instead of G (U, resp.) ([3]):

$$\bar{G} = \overline{A}^g = (2\pi\hbar)^3 \sum_\rho \sum_{\rho'} \int \left\{ a_\rho^{+*}(p)\,A_{\rho\rho'}^0(p)\,a_{\rho'}^+(p) + a_\rho^{-*}(p)\,A_{\rho\rho'}^0(p)\,a_{\rho'}^-(p) \right\} dp\,, \qquad (20a)$$

([1]) More precisely: in the form of a sum of products of powers:

$$\prod_\kappa \alpha_\kappa^{n_\kappa} \cdot \prod_k P_k^{m_k}\ ,$$

whose matrix elements are represented by:

$$\left(\prod_\kappa \alpha_\kappa^{n_\kappa} \right)_{\rho\rho'} \cdot \prod p_k'^{m_k}\,\delta(p_k' - p_k'')$$

(p-representation). Upon introducing that representation in place of $U_{\rho\rho'}^0$ in (17‴), a further integration over p'' will be needed. (One must then write p'', instead of p', in the part of the integration in (17) that follows from $U_{\rho\rho'}^0$.)

([2]) More precisely: over x, p', and p''; cf., the previous remark.

([3]) In fact, one has, in general, for $A = G + U$, with $\psi = \psi^+ + \psi^-$:

$$\int \psi^*\, G\, \psi\, dx = \int (\psi^{+*}\, G\, \psi^+ + \psi^{-*}\, G\, \psi^-)\, dx = \int (\psi^{+*}\, A\, \psi^+ + \psi^{-*}\, A\, \psi^-)\, dx,$$
$$\int \psi^*\, U\, \psi\, dx = \int (\psi^{+*}\, U\, \psi^+ + \psi^{-*}\, U\, \psi^-)\, dx = \int (\psi^{+*}\, A\, \psi^- + \psi^{-*}\, A\, \psi^+)\, dx,$$

from which the assertion will emerge immediately.

$$\overline{U_t} = (2\pi\hbar)^3 \sum_{\rho} \sum_{\rho'} \int a_{\rho}^{+*}(p)\, A_{\rho\rho'}^{0}(p)\, a_{\rho}^{-}(p)\, e^{i2\varepsilon(p)t/\hbar} dp + \text{conj.}, \tag{20b}$$

[with the corresponding meaning for the matrices $A_{\rho\rho'}^{0}$ as $G_{\rho\rho'}^{0}$ ($U_{\rho\rho'}^{0}$, resp.)]. We would like to refer briefly to the expectation values of A that are constructed in that way as *even and odd expectation values*. In the event that A depends upon only the α_k and P_k (but not, by contrast, on the x_k), those expectation values can be calculated without the algebraic decomposition of A into $G + U$ (1).

The general outline of zitterbewegung can now be read off from the expressions (18) and (20b) with no further assumptions. One first sees that $\overline{U_t}$ ($\overline{A_t^0}$, resp.) will always vanish when the positive and negative spectral domains $a^+(p) \neq 0$ and $a^-(p) \neq 0$ overlap *at least partially* (2). Moreover, the time-dependency of U_t is clearly *almost-periodic*, in general, since the time factor in the integrand will be integrated over p. It is only in the *limiting case* when one is dealing with the superposition of two *monochromatic wave systems of equal and opposite energies* that $\overline{U_t}$ will be *strictly periodic*, and indeed with a frequency that corresponds to *twice* the energy of the wave system. However, in the individual cases, the form of the zitterbewegung can prove to be very different depending upon the composition of the wave group.

3. We would now like to turn to the application of these general considerations to the comparison between the Dirac electron and our model that we would like to pursue. One would initially expect that, in general, an analogy exists between the analytical expressions that characterize the dynamical state quantities of the model and the corresponding operators of quantum-mechanical systems in a manner that is similar to the way that the dynamical variables of many systems can be carried over to quantum mechanics directly (e.g., the **Bohr** model of the atom). However, if one goes from the operators to the expectation values then one will encounter the fundamental difficulty that the concept of a well-defined physical situation that would be represented by the wave function ψ in quantum mechanics is foreign to classical dynamics. One would then expect only that a model might correctly give the characteristic outline of certain *types* of situations, in some approximation. That will imply a general viewpoint for the following investigation:

Any dynamical variable $A^*(t)$ of our model (3) – e.g., coordinate, proper impulse moment, terms in the eigenfunction – proves to be a function of time for an arbitrary, translating particle, from which a certain periodically-oscillating part can be split off that is due to the internal rotational motion. One can suspect *that this splitting of $A^*(t)$ has its quantum-mechanical analogue in the aforementioned splitting of the corresponding operator A into even and odd parts*, and then seek to perform that splitting with the most precise analogy that is possible. The separation of the periodically-oscillating part of

(1) It is easy to see that the conditions (13) will always be fulfilled for operators A that depend upon only α_k and P_k (in the form of products of powers), and one will therefore have (14a) and (14b).
(2) Cf., on that, **L. de Broglie**, *L'électron magnetique*, Paris, 1924, pp. 299.
(3) The quantities that pertain to the model will be denoted with an * in what follows.

$A^*(t)$ will then become *unique* in the cases examined, since the correspondence with the even and odd parts of A is, in fact, very precise. As a uniqueness *criterion*, we shall now demand that the periodic part of A^* that remains after splitting – viz., the "even" part G^* – must correspond *exactly* with the corresponding expectation value $\overline{G}$ for the Dirac electron for a wave group with sufficiently sharply-defined impulse and sharply-defined (positive) energy. However, such a splitting of A^* will, on the other hand, already approach the form of the corresponding operator A, and can then be performed in each case in a completely casual way ([1]).

On the other hand, since, as we have seen, the time function $\overline{U_t}$ of the expectation value of $U(t)$ depends upon the basic wave function ψ in an entirely essential way, there would generally be no unambiguous sense to speaking of an "amplitude." It might be helpful for one to then go from $U(t)$ to its square U^2, which is even and time-independent; the calculation of U^2 for a wave group with a sharp impulse would then clearly give a measure for the "amplitude" of $U(t)$. However, U^2 would still not encompass all of the detailed behavior of $\overline{U_t}$ (characteristic of this is the fact that the time-dependency of $U(t)$ will drop out when one forms U^2, for which no analogue exists in the classical model!).

For the comparison with the model-specific quantity $U^*(t)$, one will have to choose a wave function that leads to the most representative expectation value $\overline{U_t}$ that is possible. That being the case, we would like to regard the strongly-time-periodic limiting case that was considered above and consists of the superposition of two monochromatic wave trains with *equal impulse and equal, but opposite, energy.* The choice of wave function will still not generally be established uniquely with that definition, and we will have to make an even narrower choice later on (Section 2).

In fact, we shall now show that for a suitably-chosen special wave function in the most important special cases that will be examined, for a *small* particle velocity v' (more precisely, up to the terms that are *linear* in $\beta' = v' / c$) ([2]), *one will get precise agreement between the behavior of* $\overline{U_t}$ *and the model-related quantity* $U^*(t)$ [assuming that one goes to the limiting case for the model that was mentioned in the introduction for which the internal (micro-) velocity of the particle attains the speed of light]. Noticeable deviations first appear for large particle velocities and grow with increasing particle velocity until finally the correspondence breaks down more and more in the neighborhood of the speed of light ([3]).

([1]) By contrast, it does not seem justified from the outset to perform the separation of the periodic part U^* of A^* in such a way that its *temporal* mean will vanish, since no precise analogue for that exists in quantum dynamics. Since $\overline{U_t}$ is only *almost*-periodic, in general, the temporal mean of $\overline{U_t}$ does not need to vanish. (The sequence of integrations over p and t is not arbitrary, in general.) The *operator* notation (14b) gives only the *impression* that the "temporal mean of $U(t)$" must vanish.

([2]) The "particle velocity" v' is the velocity that belongs to the eigenvalue of the impulse p' (the expectation value of the macrovelocity would be equal to 0 for the special wave function that we have assumed, cf., Section 2).

([3]) One might perhaps be inclined suspect that the increasingly noticeable lack of correspondence with increasing particle velocity is a contradiction to the relativity postulate; cf., however, concluding remarks to Section 2.

§ 2. Macrovelocity and microvelocity. Coordinate oscillation.

1. According to **Schrödinger**, the **Breit** equation (4):

$$\frac{dx_k}{dt} = c\alpha_k \tag{4}$$

can be integrated on the basis of the remark that it is not, in fact, the α_k themselves, but probably, as one confirms immediately, the quantities:

$$\eta_k = \alpha_k - c\, H^{-1}\, P_k \tag{21}$$

that anticommute with H:

$$H\eta_k + \eta_k\, H = 0. \tag{21′}$$

Hence:

$$\eta_k = \eta_k^0\, e^{-i2Ht/\hbar} = e^{i2Ht/\hbar}\eta_k^0\,, \tag{22}$$

in which the η_k^0 mean the "initial values" of the operators η_k at the time t = 0 (i.e., integration constants). Upon substituting (22) in (21), one will get, after repeated integration:

$$x_k = a_k + c^2\, H^{-1}\, P_k\, t - \frac{c\hbar}{2i}\eta_k^0\, H^{-1}e^{-i2Ht/\hbar}\,, \tag{23}$$

with a_k as further integration constants. It is significant that *no* integration constants that would correspond to the impulse of the particle appear in (23). Moreover, the expectation value of the impulse at time t = 0 is determined from the initial wave function $\psi = \psi(0)$.

From (23), the coordinate x_k can then be decomposed into two components of an essentially different character:

$$x_k = \tilde{x}_k + \xi_k\,, \tag{24}$$

$$\tilde{x}_k = a_k + c^2\, H^{-1}\, P_k\, t, \tag{24a}$$

$$\xi_k = -\frac{c\hbar}{2i}\eta_k^0\, H^{-1}e^{-i2Ht/\hbar} = -\frac{c\hbar}{2i}\eta_k\, H^{-1} = \frac{c\hbar}{2i}H^{-1}\eta_k\,. \tag{24b}$$

That decomposition corresponds to the splitting of x_k into an even and an odd part. The even part ([1]) $\tilde{x}_k$ increases linearly in time. One sees that most simply from the original definition (10) of the expectation values; for a wave with a sharply-defined propagation vector (viz., impulse) and positive energy, one will then get:

([1]) One easily confirms that the *second* commutator of $\tilde{x}_k$ vanishes, so from the definition in rem. 1 on pp. 6, $\tilde{x}_k$ will be even. The *first* commutator of $\tilde{x}_k$ will then be the component $\tilde{v}_k$ of the macroscopic velocity of the particle, up to the factor $\hbar / i$.

$$\overline{c^2 H^{-1} P_k} = c^2 \frac{\sqrt{1-\beta'^2}}{\mu_0 c^2} \cdot \frac{\mu_0 v_k'}{\sqrt{1-\beta'^2}} \int \psi^* \psi \, dx = v_k' , \tag{25}$$

and correspondingly:

$$\overline{x_k}^g = a_k' + v_k' t . \tag{25′}$$

However, the calculation will require a special examination in the limiting case of strongly-periodic expectation values $\overline{x_k}^u = \overline{\xi}_k$.

Next, the "amplitude" of the oscillation ξ_k can be estimated when one goes from ξ_k to its square ξ_k^2, which is even, and then calculate its expectation value with the wave function of the particle at rest. Since η_k can be replaced with α_k for the rest motion, one will then get directly from (24b), (21′), and (3b):

$$\overline{\xi_k^2} = \left(\frac{c\hbar}{2i}\right)^2 \overline{\eta_k H^{-1} \eta_k H^{-1}}^g = \left(\frac{c\hbar}{2i}\right)^2 \overline{\alpha_k^2 H^{-2}}^g = \left(\frac{\hbar}{2\mu_0 c}\right)^2 . \tag{26}$$

From (22) [(24b), resp.], the associated frequency will be:

$$\omega = \frac{2E_0}{\hbar} = \frac{2\mu_0 c}{\hbar} , \tag{26′}$$

if $E_0 = \mu_0 c^2$ is the rest energy of the particle.

It is very remarkable that the expressions (26) and (26′) for the expectation value of the amplitude-squared of the coordinate oscillation and the frequency, resp., correspond precisely with the corresponding expressions for the model ([1]). In fact, from [I, (91)], the magnitude of the impulse moment will be $\mu_0 cR$, if R denotes the radius of the orbit. For a particle with a quantized impulse moment of magnitude $\frac{1}{2}\hbar$, one will then have:

$$R = \frac{\hbar}{2\mu_0 c} , \qquad \omega = \frac{c}{R} = \frac{2\mu_0 c^2}{\hbar} . \tag{27}$$

2. We now want to link the more precise discussion of the correspondence between the Dirac electron and the model to formulas (20a) and (20b) for the even and odd expectation values of an operator. The choice of a suitable customized solution to the wave equation will prove decisive in that.

([1]) One should therefore observe that the expectation value (26) of ξ_k^2 is independent of the index k. Naturally, that argument cannot be interpreted intuitively in the model. However, cf., the sharpening of the correspondence in what follows by comparing the *odd* expectation values $\overline{\xi_k}$ and $c\overline{\alpha_k}^u$ with the quantities in the model.

We next notate the solutions of the Dirac wave equation that will be required in what follows. If we make the Ansatz of a *plane wave:*

$$\psi_\rho(x, t) = a_\rho \cdot e^{i(p_1x_1+p_2x_2+p_3x_3-Et)/\hbar}, \tag{28}$$

and employ the usual representation for the α_k matrices:

$$\left.\begin{aligned}
&\alpha_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} & & & i \\ & & -i & \\ & i & & \\ -i & & & \end{pmatrix}, \\
&\alpha_3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}\right\} \tag{29}$$

then the Dirac equation (1), (2) will belong to a system of equations for the amplitudes a_ρ in a well-known way, which will possess non-zero solutions if and only if the energy parameter assumes one of the eigenvalues:

$$E = \pm\, c\,\sqrt{\mu_0^2\,c^2 + p_1^2 + p_2^2 + p_3^2} = \pm\,\varepsilon.$$

Any two of the coefficients a_ρ are freely available (which will be denoted by upper-case letters in what follows), while the other two are determined by them. That implies the following cases:

a) *Positive energy*: $E > 0$ ($\varepsilon = E$):

$$a_1 = A, \qquad a_2 = B, \qquad a_3 = \frac{p_3A+(p_1+ip_2)B}{\varepsilon/c+\mu_0c}, \qquad a_4 = \frac{(p_1-ip_2)A-p_3B}{\varepsilon/c+\mu_0c}. \tag{30a}$$

b) *Negative energy:* $E < 0$ ($\varepsilon = -E$):

$$a_1 = -\frac{p_3C+(p_1+ip_2)D}{\varepsilon/c+\mu_0c}, \qquad a_2 = -\frac{(p_1-ip_2)C-p_3D}{\varepsilon/c+\mu_0c}, \qquad a_3 = C, \qquad a_4 = D. \tag{30a}$$

Finally, we record the eigenvalue of energy, the x_3-component of the impulse moment J, and the magnetic moment M for a rest particle that has only one of the coefficients A, B, C, D non-zero:

Eigenfunction	E	J_3	M_3
ψ_1 $(A \neq 0)$……..	$+ \mu_0 c^2$	$-\frac{1}{2}\hbar$	$+\dfrac{c\hbar}{2\mu_0 c}$
ψ_2 $(B \neq 0)$……..	$+$	$+$	$-$
ψ_3 $(C \neq 0)$……..	$-$	$-$	$-$
ψ_4 $(D \neq 0)$……..	$-$	$+$	$+$

We now go on to the definition of the *odd* expectation values for the microvelocity $c\,\alpha$, which, from (21), will belong predominantly to the odd type for small impulse. In general, from (21) and (21′):

$$\overline{\alpha_k}^{\,u} = \overline{\eta_k}\,, \tag{31}$$

since the operator of macrovelocity $c^2 H^{-1} P_k$ is even

The choice of *special wave function* shall now result in the fact that one superimposes solutions of the Dirac equation with positive and negative energy (Section 1) that correspond to particles with the same sense of orbiting; i.e. (for the same sign on the charge), they possess *magnetic moments that point in the same direction.* Namely, it proves that with precisely that choice of wave function, one can recognize an especially striking analogy between the Dirac electron and the model of orbiting mass-points ([1]).

For a macroscopic particle *at rest* and a positive sense of orbiting, the wave function will then be determined by the superposition of the special states a_ρ^+ and a_ρ^- (cf., Table 1):

$$\left.\begin{aligned} a_\rho^+ &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{2}}(1,0,0,0)\cdot\{\delta(p_1)\,\delta(p_2)\,\delta(p_3)\}^{1/2}, \\ a_\rho^- &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{2}}(0,0,0,1)\cdot\{\delta(p_1)\,\delta(p_2)\,\delta(p_3)\}^{1/2}, \end{aligned}\right\} \tag{32}$$

in which the normalization is performed in such a way that ([2]):

$$(2\pi\hbar)^3 \sum_\rho \int\int_{-\infty}^{+\infty}\int \left\{\left| a_\rho^+(p)^2 \right| + \left| a_\rho^-(p)^2 \right|\right\} dp_1\, dp_2\, dp_3 = 1. \tag{33}$$

([1]) As a foundation for that *special* choice of wave function, one can cite the fact that one can think of the classical model (viz., the "pole-dipole particle") as being composed of a positive and a negative mass that rotate in the same sense (cf., Part II). However, we do not place any significant weight on that argument, since the physical sense of the superposition of positive and negative-energy states is hardly clear; that foundation can also seem less compelling in a different regard. By the superposition of other special eigen-solutions of equal and opposite energy, one will get, in part, quite bizarre results (e.g., linear oscillations, instead of circular ones).

([2]) The normalization condition (33) is equivalent to the normalization (10′) of the ψ-function, as one easily confirms upon observing (15), (16), and (16′).

With that, one immediately gets from (20b) and (32), and when one observes that only the (1, 4) element of the matrix α_k will give a contribution:

$$\left.\begin{aligned}
\overline{c\,\alpha_1}^{\,u} &= \tfrac{1}{2}c\left(e^{i2\varepsilon_0 t/\hbar}+e^{-i2\varepsilon_0 t/\hbar}\right)=c\cdot\cos\frac{2\varepsilon_0 t}{\hbar},\\
\overline{c\,\alpha_2}^{\,u} &= \tfrac{1}{2}c\left(e^{i2\varepsilon_0 t/\hbar}-e^{-i2\varepsilon_0 t/\hbar}\right)=-c\cdot\sin\frac{2\varepsilon_0 t}{\hbar},\\
\overline{c\,\alpha_3}^{\,u} &= 0.
\end{aligned}\right\} \tag{34a}$$

At the same time, the eigenvalues of even type will vanish:

$$\overline{c\,\alpha_1}^{\,g} = \overline{c\,\alpha_2}^{\,g} = \overline{c\,\alpha_3}^{\,g} = 0. \tag{34b}$$

For the special choice of wave function (28), (32), *the expectation values* (34a) *of the microvelocity* $c\boldsymbol{\alpha}$ *will then correspond to a circular motion of a particle in the* (x_1, x_2)-*plane that possesses the speed of light at all points of the orbit.*

We would also like to calculate the expectation values for the odd operator of the coordinate oscillation ξ_k explicitly in a corresponding way. Since we can replace η_k with α_k and H with $\mu_0\,c^2\alpha_4$ for the rest motion, from (24b), we will have:

$$\xi_k = \frac{c\hbar}{2i}H^{-1}\eta_k \to = \frac{c\hbar}{2i}\frac{\alpha_4^{-1}}{\mu_0 c^2}\;\alpha_k = \frac{\hbar}{2\mu_0 c}\frac{\alpha_4\,\alpha_k}{i}. \tag{35}$$

Now, with the matrix representation (29), one will have:

$$\alpha_4\,\alpha_1 = i\begin{pmatrix} & & & -i\\ & & -i & \\ & i & & \\ i & & & \end{pmatrix},\quad \alpha_4\,\alpha_2 = i\begin{pmatrix} & & & 1\\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},\quad \alpha_4\,\alpha_3 = i\begin{pmatrix} & & -i & \\ & & & i\\ i & & & \\ & -i & & \end{pmatrix}. \tag{36}$$

Thus, from (24b), (32), and (35), one will have:

$$\left.\begin{aligned}
\overline{\xi_1} &= \frac{\hbar}{2\mu_0 c}\cdot\frac{1}{2}\left(-i\,e^{i2\varepsilon_0 t/\hbar}+i\,e^{i2\varepsilon_0 t/\hbar}\right)=\frac{\hbar}{2\mu_0 c}\cdot\sin\frac{2\varepsilon_0 t}{\hbar},\\
\overline{\xi_2} &= \frac{\hbar}{2\mu_0 c}\cdot\frac{1}{2}\left(\,e^{i2\varepsilon_0 t/\hbar}+e^{i2\varepsilon_0 t/\hbar}\right)\quad=\frac{\hbar}{2\mu_0 c}\cdot\cos\frac{2\varepsilon_0 t}{\hbar},\\
\overline{\xi_2} &= 0.
\end{aligned}\right\} \tag{37}$$

These expectation values, in turn, correspond to the model precisely. Obviously, one has [cf., (34a), (34b), and (37)]:

$$\frac{d\overline{x_k}^u}{dt} = \frac{d\overline{\xi_k}}{dt} = \overline{c\,\alpha_k}^u, \qquad \frac{d\overline{x_k}^g}{dt} = 0. \tag{38}$$

The **Breit** relation (4) is thus in complete harmony with the model.

3. With that, an analogue of the coordinate oscillation for *translating* particles shall be examined. One finds the square of ξ_k from a simple computation ([1]):

$$\xi_k^2 = \left(\frac{\hbar c}{2}\right)^2 H^{-2}\,(1 - c^2 H^{-2}\, P_k^2). \tag{39}$$

The associated (even) expectation value for a wave with a sharply-defined impulse will be:

$$\overline{\xi_k^2} = \left(\frac{\hbar}{2\mu_0 c}\right)^2 (1 - \beta'^2)(1 - \beta_k'^2). \tag{39′}$$

if $\beta' = v' / c$ and $\beta_k' = v_k' / c$ are the macroscopic velocity (velocity components, resp.) of the particle. Obviously, $\overline{\xi_k^2}$ grows with increasing translatory velocity of the particle in *all* spatial directions, but generally most strongly in the direction of translation itself.

In order examine the phenomena that appear under translation more closely, we again consider the *odd* eigenvalues of $c\alpha_k$ (the expectation values of ξ_k, resp.). Once more, we correspondingly superimpose states of positive and negative energy with sharply-defined impulses and equal (positive) senses of orbiting in the (x_1, x_2)-plane. The translation initially results in the x_1-direction ($p_1 \neq 0$), so, from (30a) and (30b), we can set:

$$\left.\begin{aligned} a_\rho^+ &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{N}}\left(1, 0, 0, \frac{p_1}{\varepsilon / c + \mu_0 c}\right)\cdot\{\delta(p_1 - p_1')\,\delta(p_2)\,\delta(p_3)\}^{1/2}, \\ a_\rho^- &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{N}}\left(\frac{-p_1}{\varepsilon / c + \mu_0 c}, 0, 0, 1\right)\cdot\{\delta(p_1 - p_1')\,\delta(p_2)\,\delta(p_3)\}^{1/2}, \end{aligned}\right\} \tag{40}$$

in analogy with (32), in which, due to (33):

$$M = 2\left(1 + \frac{p_1'^2}{(\varepsilon' / c + \mu_0 c)^2}\right), \qquad \varepsilon' = c\,\sqrt{\mu_0^2\, c^2 + p_1'^2}\,. \tag{40′}$$

([1]) **E. Schrödinger**, A, eq. (18), *et seq.*

Hence, it will follow from the matrix representation (29) that:

$$\overline{c\,\alpha_1}^u = \frac{1}{N}\cdot c\left(1-\frac{p_1'^2}{(\varepsilon'/c+\mu_0 c)^2}\right)e^{i2\varepsilon t/\hbar} + \text{conj.},$$

$$\overline{c\,\alpha_2}^u = \frac{1}{N}\cdot ci\left(1+\frac{p_1'^2}{(\varepsilon'/c+\mu_0 c)^2}\right)e^{i2\varepsilon t/\hbar} + \text{conj.},$$

$$\overline{c\,\alpha_3}^u = 0.$$

After a simple intermediate calculation, one will have:

$$\frac{1-\dfrac{p_1'^2}{(\varepsilon'/c+\mu_0 c)^2}}{1+\dfrac{p_1'^2}{(\varepsilon'/c+\mu_0 c)^2}} = \sqrt{1-\beta'^2}\,, \tag{41}$$

if β' is the velocity that belongs to p_1' divided by c. One will then get:

$$\overline{c\,\alpha_1}^u = c\,\sqrt{1-\beta'^2}\cdot\cos\frac{2\varepsilon' t}{\hbar}, \qquad \overline{c\,\alpha_2}^u = c\,\cdot\sin\frac{2\varepsilon' t}{\hbar}, \qquad \overline{c\,\alpha_3}^u = 0. \tag{42}$$

Translation along the x_2-direction yields nothing new (except for the obvious difference that $\sqrt{1-\beta'^2}$ appears as a factor in $\overline{c\,\alpha_2}^u$, instead of $\overline{c\,\alpha_1}^u$), By contrast, the translation along the x_3-direction ($p_3 \neq 0$) that then results "perpendicular to the orbital plane is of interest. According to (30a) and (30b), we set:

$$\left.\begin{aligned} a_\rho^+ &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{N}}\left(1, 0, \frac{p_1}{\varepsilon/c+\mu_0 c}, 0\right)\cdot\{\delta(p_1)\,\delta(p_2)\,\delta(p_3-p_3')\}^{1/2}, \\ a_\rho^- &= \frac{(2\pi\hbar)^{-3/2}}{\sqrt{N}}\left(0, \frac{-p_1}{\varepsilon/c+\mu_0 c}, 0, 1\right)\cdot\{\delta(p_1)\,\delta(p_2)\,\delta(p_3-p_3')\}^{1/2}, \end{aligned}\right\} \tag{43}$$

with the corresponding meaning for N and ε' as in (40′). It then follows from the matrix representation (29), since the normalization factor N drops out in each case:

$$\overline{c\,\alpha_1}^u = c\cdot\cos\frac{2\varepsilon' t}{\hbar}, \qquad \overline{c\,\alpha_2}^u = -\,c\cdot\sin\frac{2\varepsilon' t}{\hbar}, \qquad \overline{c\,\alpha_3}^u = 0. \tag{44}$$

The amplitudes of the odd expectation values $\overline{c\alpha_k}^u$ (k = 1, 2, 3) will not be influenced by translation perpendicular to the orbital plane then.

The explicit calculation of the expectation values can be omitted here, since, from (4) and (24), one has, in full generality [cf., (38)]:

$$\frac{d\xi_k}{dt} = \overline{c\alpha_k}^u . \tag{38′}$$

By employing the special wave functions (40) and (48), the translation parallel to x_1 will then be:

$$\overline{\xi_1} = \frac{\hbar}{2\mu_0 c}(1-\beta'^2)\sin\frac{2\varepsilon' t}{\hbar}, \qquad \overline{\xi_2} = \frac{\hbar}{2\mu_0 c}\sqrt{1-\beta'^2}\cos\frac{2\varepsilon' t}{\hbar}, \qquad \overline{\xi_3} = 0, \tag{45a}$$

and parallel to x_3 :

$$\overline{\xi_1} = \frac{\hbar}{2\mu_0 c}\sqrt{1-\beta'^2}\sin\frac{2\varepsilon' t}{\hbar}, \qquad \overline{\xi_2} = \frac{\hbar}{2\mu_0 c}\sqrt{1-\beta'^2}\cos\frac{2\varepsilon' t}{\hbar}, \qquad \overline{\xi_3} = 0. \tag{45b}$$

[Cf., the even expectation values $\overline{\xi_k}$, equation (39′).]

The even expectation value $\overline{\frac{dx_k}{dt}}^g$ also vanishes in the case of translation, since the states of positive and negative energy make equal and opposite contributions. (The impulse and velocity are oppositely direct for the states of negative energy.)

4. It is of especial interest to refer to the differences from the classical models. According to (45a) and (45b), the frequency of the coordinate oscillation of the Dirac electron will be $2\varepsilon'/\hbar$, so it has been enlarged by a factor of $\frac{\varepsilon'}{\varepsilon_0} = \frac{1}{\sqrt{1-\beta'^2}}$ in comparison to the rest particle ([1]). On the other hand, the intuitive model is comparable to a translating clock, and accordingly, its frequency has been slowed down (time dilatation) by a factor of $\sqrt{1-\beta'^2}$ in comparison to a classical particle at rest. Obviously, that discrepancy is directly connected with the fundamental wave-particle duality and cannot be omitted by any change in the model. Likewise, from (45a) and (45b), the Dirac electron can also have no "normal" Lorentz contraction, as in the intuitive circular path model. Both kinds of deviations have an order of magnitude of β'^2, and one will then establish that *the correspondence (in the narrow sense) between the Dirac electron and the intuitive model will be lost when one goes to at least quadratic terms in* β'*.*

([1]) One must recall that energy and frequency transform the same way under a Lorentz transformation for a de Broglie-Dirac wave ($\varepsilon = h\nu$ is Lorentz invariant!).

On the other hand, in the example of the coordinate oscillation and velocity, it will become especially clear that one cannot direct one's attention to the characteristic limiting cases of the motion that correspond to the limiting case of even and odd expectation values *at the same time, but only in succession.* Thus, forming the even expectation values of x_k will yield a motion that is uniform and rectilinear, and will exhibit no sort of oscillation. By contrast, the "zitterbewegung" will show itself when one forms the odd expectation values, and for a special limiting case, it will assume precisely the same character that corresponds to the intuitive picture of the orbiting particle. In that regard, the Dirac electron will necessarily differ fundamentally from any intuitive model in which translation and internal rotation can differ simultaneously. One should not regard the fact that for large translational velocities there are discrepancies between the expectation values for the Dirac electron and the corresponding quantities for the model as a fundamental lack of a *special* model then. Here, and in the following, we will have to content ourselves with the proof that the properties of the model then *separately* approach the emergent processes of macroscopic and microscopic motion of the Dirac electron in certain limiting cases (e.g., special eigenfunctions, slow macroscopic motion) asymptotically.

§ 3. Proper impulse moment (1).

1. Spin is the true characteristic of the Dirac electron. Its existence as an impulse moment that is supplementary to the orbital impulse moment $[x, P]$ is implied by the fact that a conservation law (2) emerges from (2), (3), (8), and (29):

$$J = [x, P] + \frac{\hbar}{2} s = \text{const.}, \tag{46}$$

in which one sets:

$$[\alpha, \alpha] = 2is, \tag{47}$$

to abbreviate. Let:

$$J_B = [x, P], \qquad S = \frac{\hbar}{4i}[\alpha, \alpha] = \frac{\hbar}{2} s, \tag{48}$$

so J_B is the orbital impulse moment of the particle and S is its proper impulse moment (i.e., spin moment). It follows further from the definition (47) that:

$$s_1^2 = s_2^2 = s_3^2 = 1, \qquad [s, s] = 2is, \tag{49}$$

as well as the fact that the s_k have the eigenvalues ± 1; that further implies that the measured values of the components of the spin moment are $\pm\frac{1}{2}\hbar$ in any spatial direction.

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(1) In the following two sections, the quantities x, P, s, S, … *with no* index will always mean spatial vectors the components x_k, P_k, s_k, S_k, … (k = 1, 2, 3).

(2) **E. Schrödinger**, A, pp. 424, *et seq.*

Now, it is known that for force-free translatory motion, the orbital impulse moment and spin moment are by no means individually constant, but only their sum $J_B + S$ is constant. Hence, the oscillations of those two components must cancel out.

The oscillations of J_B and S, which we denote by ΔJ_B and ΔS, can now be given immediately in explicit form. Namely, since ξ is the part of x that is "periodic" in time, one will have ([1]):

$$\Delta J_B = -\Delta S = [\xi, P]. \tag{50}$$

The spin oscillation is connected with the coordinate oscillation ξ immediately with that. The associated differential law reads:

$$\frac{dJ_B}{dt} = -\frac{dS}{dt} = \frac{d}{dt}[x, P] = \left[\frac{dx}{dt}, P\right], \tag{51}$$

or also, if one recalls (4):

$$\frac{dS}{dt} = -c\,[\alpha, P]. \tag{51$'$}$$

The right-hand side of (50) can also be given as a function of time in operator notation when one correspondingly substitutes (24b) for ξ and observes the definition (21) of η. One will then get:

$$\Delta S = \frac{c\hbar}{2i}\,[\alpha, P]\,H^{-1} = \frac{c\hbar}{2i}[\alpha, P]_0\,H^{-1}\,e^{-2Ht/\hbar},$$

in which $[\alpha, P]_0$ is the initial value of the operator $[\alpha, P]$. That gives the general integral of (51′) as:

$$S = \tilde{S} + \frac{c\hbar}{2i}[\alpha, P]_0\,H^{-1}\,e^{-2Ht/\hbar}, \tag{52}$$

in which $\tilde{S} = \frac{\hbar}{2}\tilde{s}$ means the constant part of S.

As long as the macroscopic velocity $c^2 H^{-1} P$ (precisely: the expectation value of that operator for a sharply-defined impulse and energy) is small compared with the speed of light, the amplitude of the variable part of S will also be small compared with the constant one, and indeed, it will have order of magnitude β'. It emerges from (51) that the oscillation must prove to be "perpendicular" to P, such that:

$$(S, P) = (\tilde{S}, P) = \text{const.} \tag{53}$$

([1]) In *loc. cit.*, equation (50) was derived by integration, in the sense of the operator calculus, while here we shall be content to use a more intuitive foundation that makes the comparison with the model even simpler.

2. It will now be shown that all of the analogous laws of oscillation can be derived for the proper impulse moment.

In Part I, it was shown that a conservation law for the total impulse moment of the system can be inferred from equations (5) and (6) in any event; namely, in analogy with (45) and (38), one will have ([1]):

$$J^* = J_B^* + S^* = [x^*, P^*] + c\,[\pi, u] = \text{const.} \tag{54}$$

Since the first component J_B^* once more represents the orbital impulse moment, it is assumed that the second component S^* will correspond to the spin of the Dirac electron. We then expect that the proper moment obey the following correspondence:

$$c\,[\pi, u] \rightarrow \frac{\hbar}{2} s. \tag{55}$$

A confirmation of that suggestion next lies in the fact that a law of oscillation is true for J_B^* and S^* that is analogous to the one for J_B and S. Namely, it follows immediately from (54) that:

$$\frac{dJ_B^*}{dt} = -\frac{dS^*}{dt} = \frac{d}{dt}[x^*, P^*] = \left[\frac{dx^*}{dt}, P^*\right], \tag{56}$$

which corresponds precisely to (51). One further reads off from (56) that the oscillation of S^* proves to be perpendicular to P^*, so:

$$(S^*, P^*) = \text{const.} \tag{57}$$

which is analogous to (53). One can also set:

$$\Delta J_B^* = -\Delta S^* = \Delta\,[x^*, P^*] = [\xi^*, P^*] \tag{58}$$

here, corresponding to (59), as long as one introduces:

$$\xi^* = x^* - \tilde{x}^* \tag{59}$$

as the difference between the *simultaneous* position vectors x^* and $\tilde{x}^*$ of the particle and its center-of-mass, respectively. (54) will then follow immediately from uniform, rectilinear motion of the particle.

In summary, one can say that the formal laws of oscillation for the electron spin and the internal angular impulse in the model correspond to each other completely ([2]).

([1]) [I, eqs. (89) and (90)]. – Here, we characterize the variables that relate to the model by an *, to distinguish them from the corresponding quantities that also appear for the Dirac electron.

([2]) Moreover, that is true independently of whether the orbital velocity of the particle in the model attains the speed of light or possesses a smaller, arbitrary value.

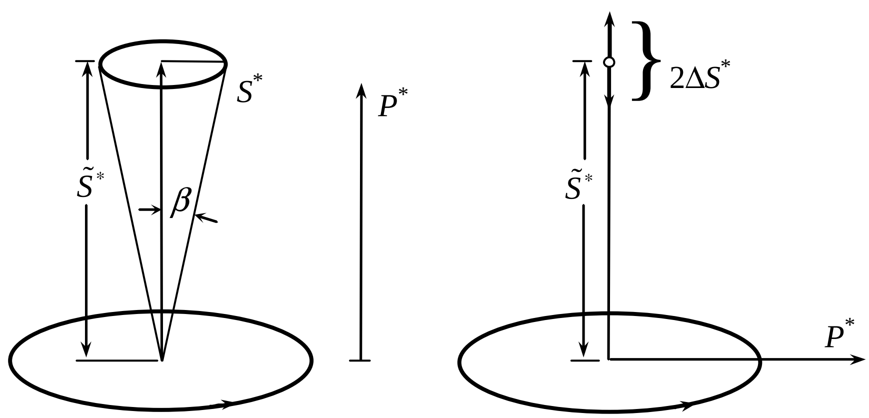
3. Equation (58) makes it possible to form an immediate picture of the type of oscillations of the proper impulse moment in the model. For a translation in the x_3-direction ($P_3^* = P^*$), one will have:

$$\Delta S_1^* = -\xi_2^* P^*, \qquad \Delta S_2^* = \xi_1^* P^*, \qquad \Delta S_3^* = 0. \tag{58a}$$

Now, since ξ_1^* and ξ_2^* are periodic functions with a phase difference of $\pi / 4$, S^* will uniformly describe a "cone of precession" around the x_3-axis whose vertex angle, which one obtains from the magnitude of ξ^*, will possess the magnitude β for small translational velocities (cf., Fig. 2a). By contrast, for a translation in the x_1-direction ($P_1^* = P^*$), one will have:

$$\Delta S_1^* = 0, \qquad \Delta S_2^* = 0, \qquad \Delta S_3^* = \xi_2^* P^*, \tag{58b}$$

and therefore only ΔS_3^* is non-zero. The oscillation of S^* here consists of a time-periodic oscillation with a magnitude of S^* and no change in direction. The amplitude of the spin variables is also equal to β here, up to the factor $\hbar/2$ (again, except for terms of higher order in β) (Fig. 2b).



(a). $P^* = P_3^*$ (b). $P^* = P_1^*$

Figure 2. Oscillations of the proper impulse moment S^* of the model under translation.

(a). Translational direction perpendicular to the orbital plane ($P^* = P_3^*$): S^* describes a cone of precession.

(b). Translational direction lies in the orbital plane ($P^* = P_1^*$): S^* oscillates in magnitude, but not direction.

The expectation value $\bar{S}^u$ of the Dirac electron exhibits an analogous behavior.

4. We would now like to extend the foregoing considerations by calculating the expectation values of the spin variable for the Dirac electron. Since $\tilde{S}$ in (52) is even (as a constant of the motion), and ΔS is odd (since it anticommutes with H, is a time-"periodic" function, resp.), the even and odd expectation values of the components of S will be:

$$\overline{S_k}^g = \overline{\tilde{S}_k}\,, \qquad \overline{S_k}^u = \overline{\Delta S_k}\,, = -\,\overline{[\xi, P]_k}\,. \tag{60}$$

Now, from (48), one has:

$$S_1 = \frac{\hbar}{2}\frac{\alpha_2\alpha_3}{i}, \quad S_2 = \frac{\hbar}{2}\frac{\alpha_3\alpha_1}{i}, \quad S_3 = \frac{\hbar}{2}\frac{\alpha_1\alpha_2}{i}, \tag{61}$$

or, in matrix form:

$$S_1 = \frac{\hbar}{2}\begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix}, \quad S_2 = \frac{\hbar}{2}\begin{pmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{pmatrix}, \quad S_3 = \frac{\hbar}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{61′}$$

The expectation values can also be calculated effortlessly for the case of translation from that.

We once more base the calculation of the expectation values upon the special wave functions (40) and (43). If we consider a translation in the x_3-direction then (43) will imply that:

$$\overline{S_1}^g = \overline{S_2}^g = 0, \qquad \overline{S_3}^g = \mp\frac{\hbar}{2}\frac{1}{N}\left(1 + \frac{p_3'^2}{(\varepsilon'/c + \mu_0 c)^2}\right) = \mp\frac{\hbar}{2}; \tag{62a}$$

by contrast, for a translation along x_1, (40) will imply:

$$\overline{S_1}^g = \overline{S_2}^g = 0, \qquad \overline{S_3}^g = \mp\frac{\hbar}{2}\frac{1}{N}\left(1 - \frac{p_1'^2}{(\varepsilon'/c + \mu_0 c)^2}\right) = \mp\frac{\hbar}{2}\sqrt{1-\beta'^2}\,. \tag{62a}$$

The expectation value relates to a wave group of positive negative energy, and in fact, the upper sign corresponds to a wave function with an amplitude of a_ρ^+, while the lower sign corresponds to a wave function with an amplitude of a_ρ^-; correspondingly the normalization factor in that is replaced with:

$$N = 1 + \frac{p_3'^2}{(\varepsilon'/c + \mu_0 c)^2} \qquad \text{or} \qquad = 1 + \frac{p_1'^2}{(\varepsilon'/c + \mu_0 c)^2}\text{, resp.} \tag{62b}$$

It is just as simple to get the odd expectation values for the wave groups with sharply-defined impulse and equal, but opposite, energy. For a translation along x_3, (61′) implies that:

$$\left.\begin{aligned}
\overline{S_1}^u &= \frac{\hbar}{2}\frac{1}{N}\cdot\frac{-2p_3'}{\varepsilon'/c+\mu_0 c}e^{i2\varepsilon' t/\hbar}+\text{conj.} = -\frac{\hbar}{2}\beta'\cdot\cos\frac{2\varepsilon' t}{\hbar},\\
\overline{S_2}^u &= \frac{\hbar}{2}\frac{1}{N}\cdot\frac{-2ip_1'}{\varepsilon'/c+\mu_0 c}e^{i2\varepsilon' t/\hbar}+\text{conj.} = +\frac{\hbar}{2}\beta'\cdot\sin\frac{2\varepsilon' t}{\hbar},\\
\overline{S_3}^u &= 0.
\end{aligned}\right\} \tag{63a}$$

Likewise, for the translation along x_1:

$$\left.\begin{aligned}
&\overline{S_1}^u = 0, \qquad \overline{S_2}^u = 0,\\
&\overline{S_3}^u = \frac{\hbar}{2}\frac{1}{N}\cdot\frac{2p_1'}{\varepsilon'/c+\mu_0 c}e^{i2\varepsilon' t/\hbar}+\text{conj.} = +\frac{\hbar}{2}\beta'\cdot\cos\frac{2\varepsilon' t}{\hbar},
\end{aligned}\right\} \tag{63b}$$

One sees that for small translational velocities, the odd expectation values of the components of S_k that are calculated in that way – i.e., their oscillations – will coincide precisely with the corresponding oscillations for the model-based quantities, up to terms of order β'. It is first for large translational velocities that one will get deviations, as we already know from the coordinate oscillations. Moreover, as a comparison of (50) and (58) will show immediately, the deviations are connected directly with the behavior of $\overline{\xi}_k$ for large translational velocities.

5. In what follows, we would like to consider the correspondence between the *even* expectation values (62a) and (62b) and the model-related expressions in more detail.

We initial direct our attention to the case of *rest motion* for the model. For that, ξ^* will be identical with radius vector r^* of the orbit (referred to the center of the circular orbit). In I, it was shown, on the basis of (5) and (6), that the spin moment of the system could be expressed in that case as:

$$S^* = -\mu_0\left[r^*, \frac{dr^*}{dt}\right], \tag{64}$$

in which dr^* / dt is the "microvelocity" of the particle (whose magnitude equals c), and thus $\mu_0\, dr^* / dt$ will be the "micro-impulse" of the particle ([1]).

It is natural to ask whether the constant (i.e., even) part of the spin moment can also be understood in terms of the micro-motion of the particle for the Dirac electron, in such a way that one can combine the "lever arm" ξ with a suitably-chosen micro-impulse in the form of a vector product. **Schrödinger** proved the following connection in that regard ([2]):

$$2\tilde{S} = \hbar\tilde{s} = \left[\xi, \frac{\eta H}{c}\right] = -\left[\xi, \frac{H\eta}{c}\right], \tag{65}$$

([1]) Cf., [I, eq. (91)].
([2]) **E. Schrödinger**, A, eq. (30).

as one can confirm easily upon the basis of (21) and (24b). We would like to bring (65) into a somewhat different form that will ease the comparison with the model. One sees immediately that $H\eta / c$ actually represents a kind of micro-impulse, if one rewrites dx / dt from (24), (4), and (24a):

$$\frac{d\xi}{dt} = \frac{d(x-\tilde{x})}{dt} = c\,(\alpha - c\,H^{-1}P) = c\,\eta. \tag{66}$$

(65) then goes to:

$$2\tilde{S} = -\left[\xi, \frac{H}{c^2}\frac{d\xi}{dt}\right] = \frac{H}{c^2}\left[\xi, \frac{d\xi}{dt}\right]. \tag{67}$$

Now, the quantity $\frac{H}{c^2}\frac{d\xi}{dt}$ obviously means the "micro-impulse" of the system, since H / c^2 is indeed the mass, and $d\xi / dt$ is the "microvelocity" of the Dirac electron. The analogy with (64) and (67) immediately comes to mind. However, two aspects of (67) [(65), resp.] are worthy of note: First of all, the factor 2 on the left (it is an immediate consequence of the commutation relations for the α_k); second, its fragility in regard to sign (whether a positive or a negative sign appears before the bracket of the vector product depends upon the position of the "mass factor" H / c^2). It is significant that the factor of 2 from the Dirac model does *not* appear in the intuitive model. By contrast, μ_0 appears with a *negative* sign in (64) ([1]), while on the other hand the sign remains undetermined in the corresponding expression (67). The correspondence will even exist in this case, as well, as one could only expect.

Equation (64) cannot be generalized in any simple way for the case of the macroscopically *moving* particle, even though the corresponding relation (67) is true for the Dirac electron, in general. The basis for that discrepancy is to be found in the fact that the **Schrödinger** operator ξ corresponds to the vector $r^* = x^*$ in the model *precisely* only for $P = 0$, and even for slow velocities, it will correspond only approximately. However, $\tilde{S}$ is even, and one must then expect that a rigorous correspondence with the model will exist for it, as well (as has been true for all even operators, up to now); i.e., that the relations (62a, b) that were established for the Dirac electron (for positive energy) will also be true for the model with no changes. The fact that this suggestion is actually true can be proved most simply on the basis of a Lorentz transformation with the use of a theorem on the dependency of the position of the center-of-mass of a closed, material system on the reference system in relativistic mechanics ([2]). In that way, the model-based conception of ξ as the difference between the position vector of the particle and its center-of-mass, which depends upon the reference system, will be corroborated [cf., (24) and (59)]. However, we shall avoid presenting the proof of that here again.

([1]) The classical model behaves as if a *negative* mass $-\mu_0$ orbits at a distance of $|r^*|$ with a velocity of dr^* / dt; cf., [I, Section 4.2].

([2]) **A. Papapetrou**, Praktika Akad. Athen **14** (1939), 540.

§ 4. Energy functions.

1. We have already referred to the analogy between the Hamiltonian function (2) of the Dirac electron and the energy function (7) of our model in the introduction. That analogy will emerge even more clearly when we now decompose the individual terms in the Hamiltonian (2) into their even and odd components, and calculate the associated distinguished expectation values.

To that end, we decompose the Hamiltonian (2):

$$H = H_1 + H_2\,, \qquad H_1 = c\,(\alpha, P), \qquad H_2 = \alpha_4\,\mu_0\,c^2. \tag{68}$$

The splitting of H_1 into an even and an odd component will come about when one substitutes the η_k that corresponds to (21) for α_k. One will then get:

$$H_1 = c^2\,P^2\,H^{-1} + c\,(\eta_0, P)\;e^{-i2Ht/\hbar}. \tag{69}$$

With the introduction of:

$$\eta_4 = \alpha_4 - \mu_0\,c^2\,H^{-1}, \tag{70}$$

which anticommutes with H and is therefore "periodic," H_2 can be split correspondingly:

$$H_2 = (\mu_0\,c^2)^2\,H^{-1} + \mu_0\,c^2\,\eta_4^0\,e^{-i2Ht/\hbar} \tag{71}$$

(η_0 and η_4^0 are the initial values of η and η_k, resp.). Now, it follows from (69) and (71) that:

$$\begin{aligned} H &= \{c^2\,P^2 + (\mu_0\,c^2)^2\}\,H^{-1} + c\,(\eta, P) + \eta_4\,\mu_0\,c^2 \\ &= H + c\,(\eta, P) + \eta_4\,\mu_0\,c^2, \end{aligned}$$

or

$$\Delta H_1 + \Delta H_2 = c\,(\eta, P) + \eta_4\,\mu_0\,c^2 = 0. \tag{72}$$

The sum of the odd parts ΔH_1 and ΔH_2 in H is zero then, as it must be, since H is constant. From (72) and (70), we can also put ΔH_1 and ΔH_2 into the form:

$$\Delta H_1 = \left(\frac{d\xi}{dt}, P\right), \quad \Delta H_2 = (\alpha_4 - \mu_0\,c^2\,H^{-1}), \tag{73}$$

which are expressions that can be carried over to the model immediately.

2. In the context of the model, one can decompose H^* in eq. (7) analogous to (68):

$$H^* = H_1^* + H_2^*\,, \qquad H_1^* = \left(\frac{dx^*}{dt}, P^*\right), \qquad H_2^* = \frac{u_0}{u_4}\,\mu_0\,c^2. \tag{74}$$

The correspondence between H_1^* and H_1 is now implied directly by the **Breit** relation (4). For H_2^*, one will correspondingly shift:

$$\frac{u_0}{u_4} \to \alpha_4 . \tag{75}$$

The oscillations ΔH_1^* and ΔH_2^* of H_1^* and H_2^*, resp. will be, analogous to (73):

$$\Delta H_1^* = \left(\frac{d\xi^*}{dt}, P^*\right), \qquad \Delta H_2^* = \left(\frac{u_0}{u_4} - \frac{\mu_0 c^2}{E}\right)\mu_0\, c^2. \tag{76}$$

(E = energy constant). From (74) and the meaning of $d\xi^* / dt$ as the microvelocity, one easily confirms that the oscillations ΔH_1^* and ΔH_2^* will cancel each other, in analogy with (72):

$$\Delta H_1^* + \Delta H_2^* = 0. \tag{77}$$

3. It only remains to be proved that the even expectation values of H_1 and H_2 coincide *exactly* with the constant parts $H_1^* - \Delta H_1^*$ and $H_2^* - \Delta H_2^*$, resp., in the model for arbitrary translational velocities, while they will coincide with the oscillating parts ΔH_1^* and ΔH_2^*, resp., only for *small* translational velocities. In that way, the parallel correspondence (75) will also be fully ensured. Now, one has, in fact:

$$\overline{H_1}^g = \overline{c^2 P^2 H^{-1}} = \frac{\mu_0 v'^2}{\sqrt{1-\beta'^2}} = \sum_{k=1}^{3} v_k'\, p_k' \qquad \to H_1^* - \Delta H_1^*, \tag{78a}$$

$$\overline{H_2}^g = \overline{(\mu_0 c^2)^2 H^{-1}} = \mu_0 c^2 \sqrt{1-\beta'^2} \qquad \to H_1^* - \Delta H_1^*, \tag{78b}$$

in which $v' = \beta' c$ is the group velocity for a wave group with a sharply-defined impulse p' and positive energy ([1]).

([1]) In the previous presentation of the correspondence between the Hamiltonian operator for the Dirac electron (2) with the classical **Lorentz** electron energy:

$$E = \frac{\mu_0 c^2}{\sqrt{1-\beta^2}} = \sum_{k=1}^{3} v_k \frac{\mu_0 v_k}{\sqrt{1-\beta^2}} + \mu_0 c^2 \sqrt{1-\beta^2}\ ,$$

it was causally suggested that since one can deal with the velocity components v_k in parallel with the matrices $c\ \alpha_k$, one must also let $\sqrt{1-\beta^2}$ correspond to α_4. [**G. Breit**, *loc. cit.*; cf., also **V. Fock**, Zeit. Phys. **55** (1929), 127.] We can now make that concept more precise by means of our distinction between expectation values of even and odd type. If one restricts oneself to the *even* expectation values of the terms in the Dirac operator H then, from (78a) and (78b), one will led to precisely the **Lorentz** electron energy (in

As far as the odd expectation values are concerned, for a translation along x_3, as in (43), and with the matrix representation (29), one will get immediately that:

$$\overline{\Delta H_2} = \mu_0\, c^2 \cdot \overline{\alpha_4}^u = 0. \tag{79}$$

That will correspond to the fact that one also has $\Delta H_2^* = 0$ in the model, since [cf., (76)] u_0 / u_4 , as well as $\mu_0\, c^2 / E$, will assume the value $\sqrt{1-\beta^2}$. $\overline{\Delta H_1}^u$ and ΔH_1^* will correspondingly vanish. For a translation along x_1, (40) and (40′) will yield, after a simple calculation:

$$\frac{\overline{\Delta H_2}}{\mu_0 c^2} = \frac{1}{N}\cdot\frac{-2p_1'}{\varepsilon'/c+\mu_0 c^2}e^{i2\varepsilon' t/\hbar} + \text{conj.} = -\,\beta' \cos\frac{2\varepsilon' t}{\hbar}. \tag{80}$$

The relative amplitudes of oscillation $\overline{\Delta H_1}$ and $\overline{\Delta H_2}$ will then possess the magnitude β'. On the other hand, one will get $\overline{\Delta H_2^*}$ when one reverts to (76) with the help of a Lorentz transformation to the circular motion of the model in its rest system (with the coordinate system that is coupled with the center-of-mass). Let $u_1^0 = \beta_1^0 u_0$ be the x_1-component of the four-velocity of the particle in its rest system the (76) will imply, with a Lorentz transformation, that:

$$\left.\begin{aligned}\frac{\Delta H_2^*}{\mu_0 c^2} &= \frac{u_0}{u_4} - \sqrt{1-\beta'^2} = u_0\,\frac{\sqrt{1-\beta'^2}}{u_0+\beta' u_1^0} - \sqrt{1-\beta'^2},\\ &= -\sqrt{1-\beta'^2}\,\frac{\beta'\beta_1^0}{1+\beta'\beta_1^0}.\end{aligned}\right\} \tag{81}$$

Now, one will have $\beta_1^0 = \cos\dfrac{2E_0\, t_0}{\hbar}$ for a particle that rotates with the speed of light, in which t_0 is the time coordinate. $E_0 = \mu_0\, c^2$ is the energy in the rest system. On the other hand: $t_0 \approx t$, $E_0 \approx E'$, up to quantities in β'^2. For slow translational velocity, one will then have:

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which one must naturally replace v_k' with the sharply-defined v_k , and β' with β). It will then be more reasonable to associate the summands in the **Lorentz** energy expressions with the *even* parts of the operators H_1 and H_2 . The decomposition of H_1 and H_2 into even and odd parts:

$$H_1 = c\,(\alpha - \eta, P) + c\,(\eta, P), \qquad H_2 = c\,(\alpha_4 - \eta_4)\,\mu_0\, c^2 - \eta_4\, \mu_0\, c^2$$

[corresponding to (69) and (71)] will imply the following association:

$$v_k \to (\alpha_\kappa - \eta_\kappa) = c^2\, P_1\, H^{-1},$$
$$\sqrt{1-\beta^2} \to \alpha_4 - \eta_4 = \mu_0\, c^2\, H^{-1}.$$

$$\frac{\Delta H_2^*}{\mu_0 c^2} \approx -\beta' \cos \frac{2E' t}{\hbar}. \tag{82}$$

One sees from this that asymptotic agreement with the expectation value (80) for **Dirac**'s theory will be achieved once more.

Erlangen and *Athens*, May 1940.