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Vector fields in *n*-dimensional manifolds

by

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§§

Poincaré has proved that in general it is not possible to attach a tangent vector to each point of a continuously differentiable, closed, boundaryless surface of genus p in such a way that the resulting vector field is everywhere continuous. He has shown that the sum of the “indices” of the singularities that thus appear has the value $2 - 2p$, from which it follows that for $p \neq 1$ discontinuities must always be present [1]). Brouwer has extended this theorem to *n*-spheres. Here, as well, the sum of the indices of the singularities is independent of the special choice of vector field; it is 2 for even-dimensional spheres and 0 for the spheres of odd dimension [2]). These facts may follow from a roughly simultaneous unproved theorem of Hadamard that is an obvious generalization the work of Brouwer on the subject, that for *any* *n*-dimensional, closed, boundaryless manifold that lies in $(n + k)$-dimensional $(k \geq 1)$ Euclidian space the sum of the indices of a tangential vector field is a *topological invariant* of the manifold, such that, e.g., for the determination of the numbers that Brouwer gave for spheres, the consideration of *special* vector fields suffices [3]). (As Herr Brouwer has informed me, the work of Brouwer and Hadamard came about piecemeal from an exchange of ideas between the two authors.)

During an examination of the *curvatura integra* for closed hypersurfaces, I arrived at a proof of the theorem proposed by Hadamard for the case of $k = 1$ [4]); since, as he likewise discussed, not every *n*-dimensional closed manifold can be regularly embedded in the $(n + 1)$-dimensional Euclidian space, he therefore treated only a special case of the previous assertion.

In the present work, it will now be proved completely. The theorem will thus be sharpened in two directions: the one – inessential – sharpening consists in the fact that one can always make an embedding of the manifold in a space of higher dimension, which comes about easily for a suitable definition of vector field, in particular, the interpretation of a vector field as a “small transformation.” The second, however, will be

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[1]) Sur les courbes définies par les équations différentielles, 3. parties, chap. 13, Journ. de Math. (4) **1** (1885).

[2]) Über Abbilding von Mannigfaltigkeiten, Math. Ann. **71** (dated July 1910).

[3]) Note sur quelques applications de l’indice de Kronecker in Tannery, Introduction à la théorie des fonctions d’une variable II, 2nd ed. (1910), no. 42. – In this, the work of Poincaré, Dyck, and Brouwer was cited; in the questions that arose in the treatises of these three authors, Poincaré and Brouwer treated the special cases mentioned above, while Dyck indeed proved different versions of the theorems, but not the theorem formulated by Hadamard.

[4]) Über die Curvatura integra geschlossener Hyperflächen, Math. Ann. (1925).

the one that actually makes the sum of the indices that appear to be a topological invariant: It is equal to the *Euler characteristic* of the manifold, which was already to be expected from its present determination in the special cases. Thus, singularity-free vector fields are possible only when the characteristic is 0. The question arises whether conversely a singularity-free vector field can always be constructed in the case of vanishing characteristic, thus in the case of a closed, boundaryless manifold of odd dimension [5]). This question is answered in the affirmative by showing that the desired construction comes down to the solution of a certain "boundary-value problem for vector distributions" that I have treated in connection with other things [6]). One of the consequences of these facts is the theorem: "A manifold admits arbitrarily small fixed-point-free transformations into itself when and only when its characteristic has the value 0." In particular, any boundaryless, closed manifold of odd dimension admits transformations of that sort, while that is never the case for manifolds of even dimension, in general.

A comparatively broad class of spaces (§§ 1, 2) must be employed for the discussion of the – mostly known – concepts and facts that concern complexes, manifolds, and their representations. The connection between the index sum of the singularities of a vector field and Euler characteristic will be essentially treated in § 3; thus, the (n – 1)-dimensional structure that leads one back to the proofs for the n-dimensional *manifolds* is no longer a manifold, but a *"complex:"* the boundary complex of the manifold. This situation makes it necessary that in a complex one can no longer speak of the continuity of a vector distribution, so one must introduce a new concept: that of the *"complex-continuous vector field."* In § 4, a proof of the auxiliary construction that was made in § 3 will be added, and in § 5 the theorem will be given its ultimate formulation; It will be regarded, in the aforementioned way, as a fixed-point theorem for small transformations and conversely, on the basis of the solubility of the "boundary-value problem" in the likewise aforementioned way; furthermore, it will be shown that the numbers that appear as the "total curvatures" of closed hypersurfaces [4]) can be interpreted as Euler characteristics in many cases.

§ 1.

Complexes and their representations.

1. In ordinary n-dimensional space, let β^n simplexes $T^n_{v^n}$ [$v^n = 1, \dots, \beta^n$] be given; let their k-dimensional boundary simplexes be denoted by $T^k_{v^k}$ [$v^k = 1, \dots, \beta^k$]. The $T^n_{v^n}$ define a "complex representation" $\mathfrak{D}^n$ if, between the points of certain $T^n_{v^n}$, which will be said to be "linked to each other," associations of the following sort exist:

Let T^n_1, T^n_2 be linked to each other; there are then two simplexes T^k_1, T^k_2 [$0 \le k \le n$] that belong to T^n_1, T^n_2 whose points are related in a one-to-one and continuous way, such

[5]) See, e.g., B. H. Tietze, Über die topologischen Invarianten mehrdimensionale Mannigfaltigkeiten, Wiener Monatsch. für Math. u. Phys. **19** (1908), § 8.

[6]) Abbildungsklassen *n*-dimensionaler Mannigfaltigkeiten, Math. Ann. **96**.

that each $T_1^p\,[0 \leq p \leq k]$ of T_1^k corresponds to a T_2^p of T_2^k, while two points A_1, A_2 of T_1^n, T_2^n that do not belong to T_1^k, T_2^k are not associated with each other. This association is transitive – i.e.: if, on the one hand, A_1, A_2, and on the other hand, A_2, A_3 are associated points of T_1^n, T_2^n (T_2^n, T_3^n, resp.) then A_1 and A_3 are associated with each other.

As a result of transitivity, we can, for each p $[0 \leq p \leq n]$, divide the β^p simplexes $T_{\nu^p}^p$ into α^p groups $g_{\lambda^p}^p$ $[\lambda^p = 1, \ldots, \alpha^p;\ 1 \leq \alpha^p \leq \beta^p]$, such that the T^p that belong to g^p are associated with each other, and analogously the points A can be collected into the groups a. We call the groups a the "points" and the groups $g_{\lambda^p}^p$, the "simplexes" of the "complex C^n that is represented by $\mathfrak{D}^n$," and say that two points (simplexes, resp.) of $\mathfrak{D}^n$ that belong to the same group are "identical in C^n."

2. If one has $\beta_1^k = \beta_2^k$ for each k for two complex representations $\mathfrak{D}_1^n, \mathfrak{D}_2^n$ and one does not distinguish them with regard to the groupings $g_{\lambda^k}^k$ of their simplexes, but only with regard to the point associations within the simplexes $T_{\nu^k}^k$, then we call them "isomorphic;" two complexes C_1^n, C_2^n that can be represented isomorphically by $\mathfrak{D}_1^n, \mathfrak{D}_2^n$, resp. , may be mapped to each other in a one-to-one and continuous way such that k-dimensional complexes correspond to each other, as is prescribed by the isomorphism [7]), and we consider them to be indistinguishable from each other.

To each representation $\mathfrak{D}^n$ there is an "affine" representation that is isomorphic to it – i.e., one such that the maps between two associated simplexes to each other are affine; in order to obtain such a representation, one must, for every two simplexes $T_{\nu^k}^k$, perform only such affine maps that are uniquely determined by the corners under the association prescribed by means of $\mathfrak{D}^n$.

A representation $\mathfrak{D}^n$ is called "reduced" when $\alpha^n = \beta^n$ in it – i.e., when associations are given only for boundary points, but not for interior points of $T_{\nu^n}^n$. One may "reduce" any representation by omitting certain $T_{\nu^n}^n$, and we regard the complex represented by the reduced complex as not being distinct from the original one. In general, we shall focus on reduced, affine, complex representations in the sequel.

3. The (n – 1)-dimensional boundary simplexes $T_{\nu^{n-1}}^{n-1}$ of $\mathfrak{D}^n$ define, by maintaining the association prescribed for $\mathfrak{D}^n$, an (n – 1)-dimensional complex representation $\mathfrak{D}^{n-1}$. If $\mathfrak{D}^n$ is affine then $\mathfrak{D}^{n-1}$ is also affine, although, in general, $\mathfrak{D}^{n-1}$ is also not reduced for a reduced $\mathfrak{D}^n$. We call the complex C^{n-1} that is represented by $\mathfrak{D}^{n-1}$ the "boundary complex" of C^n.

[7]) H. Kneser, Die Topologie der Mannigfaltigkeiten (Anhang), Jahresbericht der Deutsch. Math. Ver. **34**, 1 – 4, Heft (1925). – There, only manifolds were considered, so the validity of the argument remains unchanged for complexes.

4. If one subdivides each $T^n_{\nu^n}$ of $\mathfrak{D}^n$ into finitely many sub-simplexes in such a way that the various $T^k_{\nu^k}$ $[1 \leq k \leq n]$ of the resulting decomposition, as long as they are associated with each other, are "identical in C^n" with each other then what results "by subdivision" of $\mathfrak{D}^n$ (C^n, resp.) is a representation $\mathfrak{D}^n_1$ of a complex C^n_1. As is well-known, C^n and C^n_1 have the same "Euler characteristic;" in the above notation for C^n this is defined to be $\sum_{k=0}^{n}(-1)^k\alpha^k$. Under the decomposition carried out on the $T^n_{\nu^n}$ what results at the same time by subdivision of $\mathfrak{D}^{n-1}$ (C^{n-1}, resp.) is a representation $\mathfrak{D}^{n-1}_1$ of the boundary complex C^{n-1}_1 of C^n_1.

Let $\mathfrak{A}^n$ be an affine representation. Thus, any representation that results from subdivision $\mathfrak{A}^n_1$ is also affine. One can perform an arbitrarily dense subdivision of a given affine representation $\mathfrak{A}^n$ as follows: Let *m* be an arbitrarily large whole number. One divides each edge $T^1_{\nu^1}$ into *m* equal parts, and through each such point *A*, one intersects planar spaces that are parallel to those T^{n-1} that belong to the same T^n, but do not include *A*. In this way, each T^k will be divided into finitely many arbitrarily small convex polyhedra P^k, and these decompositions of T^k are "identical in C^n" with each other. One now further divides the P^k into simplexes, and still further, while observing the association that is present, such that a subdivision of C^n results [8]). – Thus, the following remark is important for a later application: We refer to two polyhedra as indistinguishable from each other in "shape and position" when they can be converted into each other by means of a dilatation and a translation – Thus, in an $(x_1, \ldots, x_\nu)$-coordinate system, by a transformation $x'_\nu = c\, x_\nu + a_\nu$ $[\nu = 1, \ldots, n]$ – then in shape and position only *finitely* many polyhedra come into consideration for the P^n, independently of *m*. In fact, if we introduce an affine coordinate system into – e.g. – T^n_1, whose sides are $T^{n-1}_1, \ldots, T^{n-1}_{n+1}$, such that the corner that is opposite the side T^{n-1}_{n+1} is the null point, the edges are the axes through it, and the remaining *n* corners are the unit points on the axes, then a P^n that belongs to T^n_1 is a part of a "parallelepiped" Π whose edges are parallel and proportional to the unit line segments of the coordinate system – namely, of length 1/*m* – thus, we have a structure that is independent of shape and position of *m*; indeed, P^n is one of the pieces of Π that one obtains when one intersects the planar space through each corner of Π that is parallel to T^{n-1}_{n+1}, which are likewise determined in shape and position from now on. Now, let this decomposition of this P^n into simplexes in shape and position likewise be prescribed from now on [9]). – This consequence is true for each individual $T^n_{\nu^n}$; thus, one shows that one can give a representation $\mathfrak{A}^n_1$ of $\mathfrak{A}^n$ by an arbitrarily dense subdivision (i.e., a subdivision with arbitrarily large *m*), whose

[8]) Hadamard, loc. cit., no. 10, footnote 2).

[9]) One links the center of mass of each P^k $[2 \leq k \leq n]$ with each corner of P^k and with the center of mass of each P^l $[2 \leq l < k]$ that belongs to the boundary of P^k.

simplexes are henceforth restricted in regard to shape and position to finitely many given possible cases, which are determined from $\mathfrak{A}^n$ alone.

5. Let T_1^n be a simplex of $\mathfrak{D}^n$ and let T_1^{n-k} $[k \geq 1]$ be a boundary simplex of T_1^n. T_1^{n-k} belongs to k simplexes T_κ^{n-k} $[\kappa = 1, \ldots, k]$; the planar $(n-k)$-dimensional space E^{n-k} that includes the T_1^{n-k} is the intersection of the k $(n-1)$-dimensional planar spaces E_κ^{n-1} that include the T_κ^{n-k}. Each E_κ^{n-1} decomposes the n-dimensional space into two parts: We call the one that contains T_1^n the "positive side" of E_κ^{n-1}. We call the intersection of the positive sides of E_κ^{n-1} $[\kappa = 1, \ldots, k]$ the "interior" of the "k-fold angle W_k^{n-1}" defined by E_κ^{n-1} whose vertex is E^{n-k}; the interior, when one includes the boundary, is the "closed angle space" W_k^n. (Thus, one understands W_1^n to mean the positive half of the space that is determined by a E^{n-1}.) Each W_k^n will be bounded by k closed angle spaces W_{k-1}^{n-1}, which belong to the representation $\mathfrak{D}^{n-1}$ of the boundary complex C^{n-1} defined by $\mathfrak{D}^n$.

6. Let $\mathfrak{A}^n$ be a reduced affine representation of C^n, $\mathfrak{A}^{n-1}$, the associated affine (non-reduced) representation of the boundary complex C^{n-1}, and $\mathfrak{A}_1^{n-1}$, a reduced affine representation of C^{n-1} in a planar space F^{n-1}. Let E_1^{n-1} be the planar space that includes the boundary simplex T_1^{n-1} of $\mathfrak{A}^n$, P_1, a point of T_1^{n-1}, and $\mathfrak{w}_1$, a ray of E_1^{n-1} that emanates from P_1. If $T_2^{n-1}, \ldots, T_r^{n-1}$ are boundary simplexes of $\mathfrak{A}^n$ that are identical with T_1^{n-1} in C^n and $P_2, \ldots, P_r$ are the points of them that are identical with P_1 then the rays $\mathfrak{w}_2, \ldots, \mathfrak{w}_r$ that begin at P_ρ and lie in the E_ρ^{n-1} are defined by means of the affine association between the T_ρ^{n-1} $[\rho = 1, \ldots, r]$ that are included in E_ρ^{n-1}. The r rays defined in $\mathfrak{A}^n$ correspond in $\mathfrak{A}_1^{n-1}$, by means of the affine and transitive association, to precisely one $\mathfrak{w}^*$ of F^{n-1}, which emanates from the point p of the simplex t^{n-1} of $\mathfrak{A}_1^{n-1}$ that corresponds to P_1, which is the image of the T_ρ^{n-1}. If P_1 and $\mathfrak{w}_1$ simultaneously belong to many $(n-1)$-dimensional boundary simplexes T^{n-1} of $\mathfrak{A}^n$ then the ray $\mathfrak{w}_1$ and the rays $\mathfrak{w}_2, \ldots, \mathfrak{w}_m$ $[m \geq r]$ of $\mathfrak{A}^n$ that are identical with them in C^n correspond to many rays F^{n-1}, which therefore all lie in *boundary* spaces of $\mathfrak{A}_1^{n-1}$ and are mapped to each other by means of the affine and transitive relation between the boundary spaces $\mathfrak{A}_1^{n-1}$.

7. Let $k \geq 1$, let T^{n-k} be a boundary simplex of $\mathfrak{A}^n$, P, a point of T^{n-k}, E^{n-k}, the planar space that includes T^{n-k}, W_k^n, the k-fold angle that belongs to E^{n-k} as a vertex, $\mathfrak{u}$, a ray that is based at P and directed into the interior of W_k^n, $\bar{\mathfrak{u}}$, the ray diametrically opposite to $\mathfrak{u}$, and e^2, a two-dimensional half-plane spanned by $\mathfrak{u}$ and $\bar{\mathfrak{u}}$. e^2 intersects each of the k boundary spaces E_κ^{n-1} $[\kappa = 1, \ldots, k]$ that include E^{n-k} in a ray $\mathfrak{w}_\kappa$. If T_κ^{n-1} are the boundary simplexes that belong to the E_κ^{n-1} then each T_κ^{n-1} corresponds to a simplex t_κ^{n-1} of $\mathfrak{A}_1^{n-1}$, and on each of them there is a t_κ^{n-k} that is the image of T^{n-k}, and to each t_κ^{n-k} there is an angle $(w_{k-1}^{n-1})_\kappa$ of $\mathfrak{A}_1^{n-1}$; at each t_κ^{n-k} there is an image point p_κ of *P*, and each $\mathfrak{w}_\kappa$ corresponds to a ray $\mathfrak{w}_\kappa^*$ of P^{n-1} based at p_κ. We consider the directions of these $\mathfrak{w}_\kappa^*$ more closely; there are two cases to distinguish:

I. (Main case): e^2 has only the point *P* in common with any two of the E_κ^{n-1}; each of the rays $\mathfrak{w}_\kappa$ then belong to only one T_κ^{n-1}; thus, no $\mathfrak{w}_\kappa^*$ lies in a $(n-2)$-dimensional boundary space of $\mathfrak{A}_1^{n-1}$. If one rotates $\mathfrak{u}$ in e^2 into the position $\overline{\mathfrak{u}}$ then let $\mathfrak{w}_1$ be the *first* intersection with an E_κ^{n-1}; $\mathfrak{w}_1$ is then the *only* $\mathfrak{w}_\kappa$ that belongs to the boundary of W_k^n, since all other $\mathfrak{w}_\kappa$ point to the *exterior* of W_k^n. Thus, $\mathfrak{w}_1^*$ points to the *interior* of $(w_{k-1}^{n-1})_1$, while every other $\mathfrak{w}_\kappa^*$ is directed to the *exterior* of its $(w_{k-1}^{n-1})_\kappa$.

II. (Boundary case): e^2 has, in addition to *P*, another point in common with some of the E_κ^{n-1}, hence, a ray; thus, not all of the $\mathfrak{w}_\kappa$ are distinct from each other. The *k* rays $\mathfrak{w}_\kappa$ can be collected into *i* groups ($i < k$) in such a way that the rays of one group overlap in a ray $\mathfrak{w}'_j$ [$j = 1, \ldots, i$]. For the $\mathfrak{w}'_j$, the facts that were established for the $\mathfrak{w}_i$ in case I remain correct. If $\mathfrak{w}'_1$ is the *first* intersection of the rotated ray *u* with a E_κ^{n-1} and $\mathfrak{w}'_1$ is identical with only one $\mathfrak{w}_\kappa$ then the result of the argument in case I remain unchanged, that of the $\mathfrak{w}_\kappa^*$ [$\kappa = 1, \ldots, k$] precisely one of them – namely, $\mathfrak{w}_1^*$ – points into the *interior* of its $(w_{k-1}^{n-1})_1$, and every other $\mathfrak{w}_\kappa^*$ is directed to the *exterior* of its (w_{k-1}^{n-1}). By comparison, if $\mathfrak{w}'_1$ is identical with some $\mathfrak{w}_\kappa$ then this fact must be modified in such a way that certain $\mathfrak{w}_\kappa^*$ – say, $\mathfrak{w}_1^*, \ldots, \mathfrak{w}_m^*$ (namely, the ones that correspond to $\mathfrak{w}'_1$) – have *boundaries* that belong to their $(w_{k-1}^{n-1})_\kappa$, and indeed in such a way that they are mapped to each other by means of the affine, transitive association defined in $\mathfrak{A}_1^{n-1}$, while every other $\mathfrak{w}_\kappa^*$ [$\kappa = m + 1, \ldots, k$] points to the *exterior* of its $(w_{k-1}^{n-1})_\kappa$.

Before we utilize the facts thus established we must first consider a special complex.

§ 2.

Manifolds and their representations

1. A corner $T_{\nu^0}^0$ of a reduced representation of C^n is called a regular corner when the $T_{\nu^k}^k$ [$k = 1, \ldots, n$] that contain it, along with the points in C^n that are identical to it, are associated with each other like the adjacent simplexes and boundary simplexes of a certain simplex star of the *n*-dimensional Cartesian space. Thus, we understand a simplex star to mean an element S^n composed of finitely many simplexes in such a way that all simplexes of a corner have *A* in common, while all other corners lie on a sphere around A [10]); $T_{\nu^0}^0$ is called and "interior corner" or a "boundary corner" according to whether A lies in the interior or on the boundary of S^n.

A complex that possesses only regular corners – whether interior or boundary corners – and is, in addition, "connected," – i.e., one in which one can get from any T_1^n to any other T_2^n along a chain that links each T^n to the ones that follow – is called a (closed) "manifold" M^n. If M^n has only interior corners then one calls it "boundaryless" [11]); if M^n also has boundary corners then all "boundary points" define a finite number of closed boundaryless (n – 1)-dimensional manifolds [12]); thus, a point is called a boundary point of M^n when it belongs to a boundary simplex such that under each association of simplexes, the simplexes of a simplex star S^n corresponds to a simplex constructed from the boundary points of S^n .

A complex whose representation $\mathfrak{D}_1^n$ comes about by subdividing a representation $\mathfrak{D}^n$ of a manifold M^n is, as would follow from the definition, itself a manifold. It gives us nothing different from M^n.

2. We consider the *simultaneous* map of some simplexes of a representation in M^n that are bound together to a subset of an element in Cartesian space: First, let $T_{\nu^n}^n$ be the simplexes of an affine representation $\mathfrak{A}^n$ of M^n, T_0^0, a corner, and S_0^n, the associated simplex star. The association that exists between the k-simplexes Z_ρ^k $(0 \leq k \leq n)$ of S_0^n, on the one hand, and the simplexes $T_{\nu^k}^k$, on the other, as long as it is defined, may then be refined into a *map* in which one carries out that uniquely defined affine map from each simplex Z_ρ^k of S_0^n and the $T_{\nu^k}^k$ that is associated with it under the association of the corners of Z_ρ^k to those of $T_{\nu^k}^k$; in this way, S_0^n will be mapped in a one-to-one and continuous manner to that subset Σ_0^n of M^n that is represent in $\mathfrak{A}^n$ by all of the corners T_0^0 or a simplex T_i^n that contains the corner T_i^0 that is identical to it in M^n.

3. The subset Σ_0^n of M^n that is thus constructed in a piece of Cartesian space includes all simplexes that define the neighborhood of *point*, namely, the ones represented by T_0^0; we now seek an analogous map of the entire neighborhood of a *simplex* of a representation of M^n; we define:

An affine representation $\mathfrak{A}_1^n$ of M^n is called a "neighborhood representation" when each of its simplexes T_0^n gives rise to an element E_0^n of ordinary space with the following properties: If Ω_0^n is the "simplicial neighborhood of T_0^n" – i.e., the subset of M^n that is represented by the simplexes T_i^n $[i = 1, \ldots, m]$ in $\mathfrak{A}_1^n$ that are linked to T_0^n – then E_0^n can

[10]) This definition of simplex star deviates inessentially from the one that was given by Brouwer in the reference cited in [2]).

[11]) Obviously, M^n then has nothing but "interior" points in the ordinary sense; on this, cf., the report of H. Kneser cited in [7]).

[12]) Hadamard, loc. cit., no. 16.

be decomposed into $m + 1$ simplexes $z_{0,i}^n$ [$i = 0, 1, \ldots, m$] and mapped to Ω_0^n in a one-to-one and continuous manner, such that $z_{0,i}^n$ is affinely related to T_i^n [$i = 0, 1, \ldots, m$] [13]).

We show that one can present a neighborhood representation for each M^n: Let $\mathfrak{A}^n$ be the aforementioned affine representation, relative to which one has been given the $S_{\nu^0}^n$ and $\Sigma_{\nu^0}^n$ for the map described for a single ν^0 [$\nu^0 = 1, \ldots, \beta^0$]. We present a representation $\mathfrak{A}_1^n$ of M^n by subdivision by dividing each one-dimensional edge $T_{\nu^1}^1$ into $n + 1$ equal parts: Lay a planar $(n - 1)$-dimensional space that is parallel to the faces $T_{\nu^{n-1}}^{n-1}$ through the dividing points, and decompose the resulting convex polyhedron into simplexes. If t_0^n is a simplex of the representation $\mathfrak{A}_1^n$ and, say, T_0^n is the simplex of $\mathfrak{A}^n$ that belongs to t_0^n then there is an $(n - 1)$-dimensional face of T_0^n that has no point in common with t_0^n. In fact, if we introduce (as in § 1.4) an affine coordinate system $\xi_1, \ldots, \xi_n$ into T_0^n, whose null point is the corner of T_0^n that also intersects the faces $T_1^{n-1}, \ldots, T_n^{n-1}$, whose axes are the edges that emanate from the null point, and whose unit points on the axes are the remaining corners of T_0^n then the coordinates of each point of the simplexes of the representation $\mathfrak{A}_1^n$ that has a point in common with each of the faces $T_1^{n-1}, \ldots, T_n^{n-1}$ satisfy the inequalities:

$$\xi_i \leq \frac{1}{n+1} \qquad [i = 1, \ldots, n]; \qquad \sum_{i=1}^{n} \xi_i < 1;$$

this simplex thus possesses no point in common with the latter side T_{n+1}^{n-1} of T_0^n, which is defined by the equation $\sum_{i=1}^{n} \xi_i = 1$. Thus, to t_0^n there is a face of T_0^n – e.g., T_0^{n-1} – that t_0^n has no point in common with. If T_0^0 is the corner point of T_0^n that is opposite to T_0^{n-1} and S_0^n is the simplex star that belongs to T_0^0 then one can clarify the aforementioned one-to-one and continuous piecewise affine relation between S_0^n and the simplexes of $\mathfrak{A}^n$ that contain the corner that is identical with T_0^0 in M^n in t_0^n, as well as in *any* simplex of $\mathfrak{A}_1^n$ that is linked to t_0^n and is therefore in the "simplicial neighborhood" Ω_0^n of t_0^n – i.e., $\mathfrak{A}_1^n$ is a neighborhood representation.

We can now directly employ the aforementioned simplexes z_{μ^n,μ^n}^n in place of the $t_{\mu^n}^n$ for the representation of M^n, and thus, when we henceforth set $z_{\mu^n,\mu^n}^n = T_{\mu^n}^n$, in order to revert to our previous notation, obtain a neighborhood representation that is as follows:

[13]) In general, z_{ν^1,ν^2}^n then denotes the sub-simplex that is the image of $T_{\nu_2}^n$ when the element $E_{\nu_1}^n$ represents the simplex neighborhood of $T_{\nu_1}^n$,

To each simplex $T^n_{\mu^n}$ [14]), one attaches simplexes $z^n_{\mu^n,i}$ [i = 1, …, m_{μ^n}] to those boundary simplexes that do not represent any boundary point of M^n, which, together with $T^n_{\mu^n}$, define an element $E^n_{\mu^n}$, the one-to-one image of the simplex neighborhood $\Omega^n_{\mu^n}$ of $T^n_{\mu^n}$ in M^n; thus, any two simplexes $z^n_{\mu_1^n,\mu^n}$, $z^n_{\mu_2^n,\mu^n}$, that belong to two different elements $E^n_{\mu_1^n}, E^n_{\mu_2^n}$, and which likewise correspond to the piece of M^n that is represented by $T^n_{\mu^n} = z^n_{\mu^n,\mu^n}$ by the mediation of M^n, are affinely mapped to each other.

4. This "distinguished neighborhood representation" of M^n, which we would again like to denote by $\mathfrak{A}^n$, is suitable for the investigation of certain *transformations* of M^n:

A single-valued continuous map of M^n onto itself or a subset of itself is called a "neighborhood transformation" of M^n, relative to $\mathfrak{A}^n$, when each point of M^n that is represented by a point of $T^n_{\mu^n}$ goes to a point of the simplex neighborhood $\Omega^n_{\mu^n}$ of $T^n_{\mu^n}$.

For example, suppose that the transformations f_i are a series of transformations f_1, f_2, … that converge uniformly to the identity in all of M^n as neighborhood transformations, relative to any arbitrary distinguished neighborhood $\mathfrak{A}^n$ with a certain index that is independent of $\mathfrak{A}^n$; we occasionally express this by saying that an "arbitrarily small transformation" of M^n is a neighborhood transformation, relative to any distinguished normal neighborhood.

If f is a neighborhood transformation relative to $\mathfrak{A}^n$ then one defines, in a unique manner, a single-valued and continuous map f_{μ^n} of each simplex $T^n_{\mu^n}$ onto a point set that belongs to the element $E^n_{\mu^n}$. We assume that f has at most finitely many fixed points and that they correspond to only *interior* points of $T^n_{\mu^n}$. If we now attach to each point P of $T^n_{\mu^n}$ the vector $\mathfrak{v}(P)$ that points to the point $f_{\mu^n}(P)$ then this vector field $\mathfrak{V}$ is, in a certain sense, single-valued and continuous in all of M^n, except for the fixed points [15]). It has, by the use of the notations of § 1, the following properties, among others:

A. $\mathfrak{V}$ is single-valued and continuous on each individual $T^n_{\mu^n}$ [μ^n = 1, …, β^n], except for at most finitely many points that lie in its interior.

B. Let P_0 be a boundary point of T^n_0 that belongs to a boundary simplex T^{n-k}_0 [$1 \leq k \leq n$]. Let T^{n-k}_ρ [ρ = 1, …, r] be the boundary simplexes of the other $T^n_{\mu^n}$ that are identical to T^{n-k}_0, P_ρ, the points of T^{n-k}_ρ that are identical with P_0, and let $(W^n_k)_\rho$ [ρ = 1, …, r], the k-fold angle whose vertex is T^{n-k}_ρ. Then, one of the following two cases will appear:

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[14]) Thus, μ^n now denotes an index that runs from 1 to α^n, just as ν^n did in § 1.

[15]) From on, we shall, unless expressly stated to the contrary, consider only the continuity of direction, but not the length of the vectors; zero loci of the vector field then amount to singularities.

I. (Main case): Of the $r + 1$ vectors $\mathfrak{v}(P_\rho)$, *precisely one of them points to the interior* of its $(W_k^n)_\rho$, while *all others are directed to the exteriors* of their $(W_k^n)_\rho$.

II. (Boundary case): *Some* of the $\mathfrak{v}(P_\rho)$ are attached to the *boundaries* of their $(W_k^n)_\rho$, and are constructed by means of the affine and transitive associations that exist between the boundary spaces, while *the remaining* $\mathfrak{v}(P_\rho)$ point *to the exteriors* of their $(W_k^n)_\rho$.

As one knows, case II appears when and only when P_0 and $f_0(P_0)$ belong to the same boundary simplex.

§ 3.

Complex-continuous vector fields

In the formulation of properties A and B of the vector field $\mathfrak{V}$ that was given in the conclusion of the previous paragraph, no use was made of the fact that we have a *neighborhood representation* of a *manifold* before us. If we have a reduced *affine* representation $\mathfrak{A}^n$ of an arbitrary *complex* C^n then none of the aforementioned properties will become meaningless when we replace M^n with C^n. We may therefore define:

An association $\mathfrak{V}$ of vectors $\mathfrak{v}(P)$ to the points P of the reduced, affine representation $\mathfrak{A}^n$ of the complex C^n is called a "*complex-continuous* vector field on C^n (relative to $\mathfrak{A}^n$)" when it satisfies the requirements A and B. [See the "Appendix" at the conclusion of this paper.]

1. Of the properties of complex-continuous vector fields that we will be occupied with in the sequel, let us first establish: If $\mathfrak{A}_1^n$ is a complex representation that comes about by subdivision of $\mathfrak{A}^n$ then $\mathfrak{V}$ is also complex-continuous relative to $\mathfrak{A}_1^n$, assuming that no singular point of $\mathfrak{V}$ lies on a boundary simplex of the representation $\mathfrak{A}_1^n$. One convinces oneself of the validity of this assertion by establishing that $\mathfrak{V}$ has property B, not only, as assumed, on the boundaries of the representation $\mathfrak{A}_1^n$, but also on the new boundaries that come about by subdivision, on which $\mathfrak{V}$ is continuous in the ordinary sense.

2. A second important property of the complex-continuous vector fields concerns the "projection of the complex-continuous vector field $\mathfrak{V}$ onto the boundary complex." One understands this to mean: C^n, $\mathfrak{A}^n$, and $\mathfrak{V}$ have the meanings as all along, but let $\mathfrak{A}^{n-1}$ be the non-reduced representation of the boundary complex C^{n-1} that is defined for $\mathfrak{A}^n$, let $\mathfrak{A}_1^{n-1}$ be a reduced affine representation of C^{n-1}, let $T_{\nu^k}^k$ [$k = 0, \ldots, n$; $\nu^k = 1, \ldots, \beta^k$; $\beta^n = \alpha^n$] be the simplexes $\mathfrak{A}^n$, and let $t_{\lambda^k}^k$ [$k = 0, \ldots, n - 1$; $\lambda^k = 1, \ldots, \gamma^k$; $\gamma^{n-1} = \alpha^{n-1}$] be the

simplexes of $\mathfrak{A}_1^{n-1}$. On the boundary of $T_{\nu^n}^n$, let there be given a field $\mathfrak{U}_{\nu^n}$ of vectors $\mathfrak{u}(P)$ with the following properties:

a) $\mathfrak{u}(P)$ is directed into the interior of $T_{\nu^n}^n$.

b) If P lies on a T^{n-2} then the directions $\mathfrak{u}(P)$ and $\mathfrak{v}(P)$ *do not* agree.

c) There are at most finitely many points P at which the directions of $\mathfrak{u}(P)$ and $\mathfrak{v}(P)$ do agree.

We shall postpone for a moment the discussion of whether such vector fields $\mathfrak{U}_{\nu^n}$ always exist.

On each T^{n-1} of $T_{\nu^n}^n$, we now focus on the points P at which $\mathfrak{v}(P)$ is either directed to the positive side of the planar space E^{n-1} that contains T^{n-1} or lies in E^{n-1} – in which $\mathfrak{v}(P)$ thus belongs to the "closed angle space W_1^n" in question – and project these $\mathfrak{v}(P)$ from $\mathfrak{u}(P)$ onto E^{n-1}; i.e., we present those vectors $\mathfrak{w}(P)$ at which E^{n-1} will be intersected by the half-plane e^2 that is spanned by the $\mathfrak{u}(P)$, $\mathfrak{v}(P)$, and the vector $\overline{\mathfrak{u}}(P)$ thus, the stated sequence of vectors in e^2 is always the following one: $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, $\overline{\mathfrak{u}}$. This construction will be possible only at the points P considered at which $\mathfrak{u}(P)$ and $\mathfrak{v}(P)$ agree, which are at most finite in number. The vector $\mathfrak{w}(P)$ now corresponds to either (cf., § 1.6) precisely one vector $\mathfrak{w}^*$ in $\mathfrak{A}_1^{n-1}$ or it corresponds to several $\mathfrak{w}^*$ that lie in the boundary spaces of $\mathfrak{A}_1^{n-1}$ and are affinely mapped to each other. We call the totality $\mathfrak{W}^*$ of the vectors $\mathfrak{w}^*$ thus produced in $\mathfrak{A}_1^{n-1}$ a "projection of the field $\mathfrak{B}$" and assert that it represents a *complex-continuous* vector field on C^{n-1}. In fact: That $\mathfrak{W}^*$ possesses the properties A and B that are characteristic of complex-continuous vector fields follows from described construction of $\mathfrak{W}^*$, as well as the fact that the demand B on $\mathfrak{B}$ is fulfilled for $k = 1$, in particular. That $\mathfrak{W}^*$ possesses property B for every $k^* \leq n - 1$ is obtained from the fact that $\mathfrak{B}$ possesses this property for every $k = k^* + 1$, as well as the behavior of projected vectors that was discussed in § 1.7, by which, in particular, $\mathfrak{w}^*$ belongs to its closed angle space w_{k-1}^{n-1} when and only when $\mathfrak{w}$ is the *first* intersection of the vector $\mathfrak{u}$ rotated into e^2 with a boundary space E^{n-1} of W_k^n, and therefore, when $\mathfrak{v}$ belongs to the closed angle space .

3. We now relate the indices of the singularities of $\mathfrak{B}$ to the indices of the singularities of $\mathfrak{W}^*$. Let s_{ν^n} be the sum of the indices of those singularities of $\mathfrak{B}$ that lie in $T_{\nu^n}^n$ and let $s^n = \sum_{\nu^n=1}^{\alpha^n} s_{\nu^n}$ then be the sum of the indices of all singularities of $\mathfrak{B}$; furthermore, let s^{n-1} be the sum of all indices of all singularities of $\mathfrak{W}^*$, let a_{ν^n} be the sum of the coincidence indices [4]) of the two maps of the boundary of $T_{\nu^n}^n$ onto the sphere of directions, which will be mediated by $\mathfrak{U}_{\nu^n}$ and the boundary field $\mathfrak{B}_{\nu^n}$ associated with $\mathfrak{B}$

(in this sequence!), and let $a = \sum_{\nu^n=1}^{\alpha^n} a_{\nu^n}$ be the sum of all of these coincidence indices. Now, the number a may be determined in two ways: A singularity of $\mathfrak{W}^*$ exists where and only where $\mathfrak{U}_{\nu^n}$ and $\mathfrak{B}_{\nu^n}$ have coincidence loci. The index of such a coincidence is equal to the index of the singularity of the field of projected vectors $\mathfrak{w}$, and thus, also equal to the index of the singularity of $\mathfrak{W}^*$, assuming that one orients the $(n - 1)$-dimensional boundary space E^{n-1} that the point considered belongs to in such a way that a positively oriented system of axes of E^{n-1}, together with a vector of $\mathfrak{U}_{\nu^n}$ as the *last* axis, defines a *negative* system of axes for *n*-dimensional space [16]); in our case, however, the indicatrix of E^{n-1} is determined to be the boundary indicatrix of $T^n_{\nu^n}$; i.e., an *n*-fold system of axes that is defined in the manner just described is *positively* oriented [2]). It follows that the coincidence index of $\mathfrak{U}_{\nu^n}$ and $\mathfrak{B}_{\nu^n}$ is equal and opposite to the index of the singularity of $\mathfrak{W}^*$ at the corresponding point, and is therefore:

$$(1) \qquad a = - s^{n-1}.$$

On the other hand, $a = \sum_{\nu^n=1}^{\alpha^n} a_{\nu^n}$ is to be determined in the following way: a_{ν^n} is the sum of the coincidence indices of maps of the boundary of $T^n_{\nu^n}$ onto the sphere of directions that is mediated by $\mathfrak{U}_{\nu^n}$ and $\mathfrak{B}_{\nu^n}$. The map mediated by $\mathfrak{U}_{\nu^n}$ has the degree $(-1)^n$, since all vectors $u(P)$ are directed into the interior of $T^n_{\nu^n}$, and it is thus continuous when one establishes that its starting points go to vectors that point to a fixed interior point. The map mediated by $\mathfrak{B}_{\nu^n}$ has the degree s_{ν^n}. Thus, one has the equation [4]):

$$(2) \qquad a_{\nu^n} = (-1)^{n-1} \cdot (-1)^n + s_{\nu^n} = -1 + s_{\nu^n},$$

and from this, what follows upon summing is a second value for *a:*

$$(3) \qquad a = \sum_{\nu^n=1}^{\alpha^n} a_{\nu^n} = -\alpha^n + s^n.$$

Comparing the two values of *a* gives:

$$(4) \qquad s^n = \alpha^n - s^{n-1}.$$

4. We now begin the proof of the following theorem:

[16]) For the proof, cf., the paper cited in [4]) of § 1.

Theorem I. *The index sum of the singularities of a complex-continuous vector field on C^n is equal to the Euler characteristic of C^n multiplied by* $(-1)^n$.

We give the proof by going from $n - 1$ to n.

First, let $n = 1$. $C^n = C^1$ is therefore a system of α^1 line segments whose corners are composed of α^0 groups; the corners that belong to the one group are identical in C^1 and represent a point of this complex. (We can think of this identification as being carried out in – say – three-dimensional space by fastening them together.) The complex-continuous vector field consists of vectors that lie in the lines that the line segments belong to and possess singularities in the interior of the line segments with the index sum s^1. At each of the α^0 points of the complex that are represented by the β^0 corners of the line segment it exhibits precisely one line segment that is directed into its interior. Thus, if $-a$ [17]) is the number of all corner vectors that are directed into the interior of the line segments then one has:

(1*) $$a = -\alpha^0.$$

We determine a in a second way, in which we consider each of the line segments $T^1_{\nu^1}$ individually: A singular location for the 1-dimensional vector field $\mathfrak{B}$ is – in a reasonable application of the definitions that pertained to n dimensions – to be understood as having the index + 1 in the event that all of the vectors in its neighborhood point outward, the index – 1 in the event that all of the vectors in its neighborhood point inward, and the index 0 in the event that all vectors in its neighborhood have the same direction (and the singularity is therefore removable). Singularities with other indices do not occur for n = 1. Let s_{ν^1} be the sum of the indices of all singularities of $\mathfrak{B}$ on $T^1_{\nu^1}$ and let $-a_{\nu^1}$ the number of corner vectors that point into the interior of $T^1_{\nu^1}$; one then has $s_{\nu^1} = -1$, 0, or + 1, according to whether $-a_{\nu^1}$ = 2, 1, or 0, resp. In any case, one thus has:

(2*) $$a_{\nu^1} = -1 + s_{\nu^1}.$$

Summing gives:

(3*) $$a = \alpha^1 + s^1,$$

and from this, it follows, by comparison with (1*), that:

(4*) $$s^1 = \alpha^1 - \alpha^0 = -(\alpha^0 - \alpha^1).$$

For $n = 1$, this is the relation that we asserted in our theorem. We now assume that it has been proved for $n - 1$. Then, if C^n is a complex and $\mathfrak{B}$ is a complex-continuous vector field on it, such that one can construct a vector field $\mathfrak{U}_{\nu^n}$ with the properties a), b), c) stated above in 2, then, since $\mathfrak{W}^*$ is complex-continuous and the theorem is true for the boundary complex C^{n-1}, since one must have:

[17]) Notations and signs are chosen by specifying the agreement with the n-dimensional case.

$$s^{n-1} = (-1)^{n-1} \cdot \sum_{k=0}^{n-1} (-1)^k \alpha^k ,$$

the stated relation follows from (4):

$$s^n = \alpha^n - (-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k \alpha^k = (-1)^n \sum_{k=0}^{n-1} (-1)^k \alpha^k . \tag{5}$$

Thus, we do not know whether one can always construct the field $\mathfrak{U}_{\nu^n}$. However, since the complex that arises by subdivision of C^n has the same Euler characteristic as C^n, Theorem I is proved completely, as long as the validity of the following Lemma is proved, which will happen in the next paragraph:

If $\mathfrak{A}^n$ is a reduced, affine representation of the complex C^n and $\mathfrak{V}$ is a complex-continuous vector field on it then one can, by subdividing $\mathfrak{A}^n$, present a representation $\mathfrak{B}^n$ of $\mathfrak{A}^n$ and a complex-continuous vector field $\mathfrak{P}$ in $\mathfrak{B}^n$ whose singularities are identical with those of $\mathfrak{V}$ relative to position and index, in such a way that a vector field $\mathfrak{U}_{\nu^n}$ can be constructed in each *n*-dimensional simplex $t^n_{\lambda^n}$ of $\mathfrak{B}^n$ that possesses the properties a), b), c) relative to $\mathfrak{P}$.

§ 4.

Completion of the proof of the theorem on the index sum of the singularities of a complex-continuous vector field

In order to preserve $\mathfrak{B}^n$ and $\mathfrak{P}$ in the desired manner, we first remove the vectors of $\mathfrak{V}$ that are based in the interiors of the simplexes $T^n_{\nu^n}$ of $\mathfrak{A}^n$, and replace them with a new vector field $\mathfrak{P}$ that has the same boundary field $\mathfrak{V}_{\nu^n}$ and the same singularities with the same indices as $\mathfrak{V}$, but is *analytic* in a certain neighborhood $Q(P_\rho)$ of the singular point P_ρ – naturally, it is itself removed; the fact that there is such a $\mathfrak{P}$ will be shown in another place [18]). $\mathfrak{P}$ is complex-continuous on $\mathfrak{A}^n$, since has the same boundary field as the complex-continuous field $\mathfrak{V}$; $\mathfrak{P}$ is therefore (from § 3.1) also complex-continuous in each complex representation $\mathfrak{B}^n$ that arises by subdivision of $\mathfrak{A}^n$, as long as none of the singular points lie on a boundary simplex of $\mathfrak{B}^n$. Now, if γ is an arbitrary positive number then we present a representation $\mathfrak{B}^n(\gamma)$ by subdivision of $\mathfrak{A}^n$ that fulfills the following conditions, except for the aforementioned consideration of the singular loci:

[18]) § 5, problem 4, supplement to the work cited in [6]).

$\mathfrak{B}^n(\gamma)$ is a sufficiently fine decomposition that 1. Each simplex t^n of $\mathfrak{B}^n(\gamma)$ that contains a singular point P_ρ lies completely in the analytic neighborhood $Q(P_\rho)$. 2. The fluctuation of the vector direction of $\mathfrak{B}$ at each t^n that does not lie completely in a $Q(P_\rho)$ is smaller than γ; when 1 is already satisfied, condition 2 is always fulfilled by further subdivision, as a result of the uniform continuity of $\mathfrak{P}$ outside of $Q(P_\rho)$. 3. $\mathfrak{B}^n(\gamma)$ shall have the property that each of the simplexes t^n coincide with one of finitely many simplexes in shape and position (§ 1.4), which is henceforth determined by $\mathfrak{A}^n$; that the fulfillment of 3 is compatible with an arbitrary refinement of the subdivision was shown in § 1.4.

We now prove that for a sufficiently small γ one can attach a vector field $\mathfrak{U}_{\lambda^n}$ to the boundaries of the $t^n_{\lambda^n}$ in the desired manner. In order to determine such a γ, we first focus on a simplex τ^n_ρ; let E^{n-1}_ν [$\nu = 1, \dots, n + 1$] be the planar spaces that bound τ^n_ρ, whose positive sides are defined as in § 1.5. We understand the phrase "the negative star of directions σ_ρ of τ^n_ρ" to mean a system of $n + 1$ unit vectors $\mathfrak{a}_n$ attached to a fixed point O of space, which are directed such that $\mathfrak{a}_n$ [$\nu = 1, \dots, n + 1$] does not point to the positive side of E^{n-1}_ν; thus, it either points to the negative side of E^{n-1}_ν or it is parallel to E^{n-1}_ν. The σ_ρ define an $(n - 1) \cdot (n + 1)$-dimensional closed set S_ρ. Among the $\frac{1}{2} n \cdot (n + 1)$ angles between each two directions of a σ_ρ there is a largest one $m(\sigma_\rho)$; thus, angle quantities must be measured so that they always lie between 0 and π, inclusive. $m(\sigma_\rho)$ is always positive. If one had $m(\sigma_\rho) = 0$ then that would mean that all vectors $\mathfrak{a}_\nu$ of a σ_ρ overlap in a single vector α, and that this vector a would be directed to the positive side for no E^{n-1}_ν; however, that is impossible, since an oriented line that is parallel to $\mathfrak{a}$ and goes through an interior point of τ^n_ρ is directed to the positive side of that E^{n-1}_ν, through which it enters τ^n_ρ. One therefore always has $m(\sigma_\rho) > 0$. On the other hand, $m(\sigma_\rho)$ is a continuous function on the closed set S_ρ so it attains its lower limit γ_ρ at some point; hence, one also has $\gamma_\rho > 0$.

We now define γ to be the smallest of the r numbers $\gamma_1, \dots, \gamma_r$ and must prove that one can construct a vector field $\mathfrak{U}$ with the properties a), b), c) (cf., § 3), after establishing the subdivision $\mathfrak{B}^n(\gamma)$ either (Case α) on each simplex t^n, on whose boundary [19]) the fluctuation of $\mathfrak{P}$ is smaller than γ, or also (Case β) on each simplex t^n, on whose boundary $\mathfrak{P}$ is analytic.

We begin with Case α: t^n_0 thus has the property that the angle between any two vectors that are attached to the part $\mathfrak{P}_0$ of $\mathfrak{P}$ that is found on its boundary is smaller than γ; we then assert that there is, among its boundary spaces $F^{n-1}_1, \dots, F^{n-1}_{n+1}$, at least one of them such that *all* vectors of $\mathfrak{P}_0$ are directed to its positive side. Otherwise, this would

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[19]) It suffices to consider $\mathfrak{P}$ on the boundary of the t^n.

allow one to define a negative direction star σ for t_0^n from the vectors of $\mathfrak{P}_0$, and this would likewise be a negative direction star for that τ_ρ^n that agrees with t_0^n in shape and position; one would then have $m(\sigma) \geq \gamma_\rho \geq \gamma$, contrary to the fact that the fluctuation of $\mathfrak{P}_0$ is less than γ. Therefore, let all vectors of $\mathfrak{P}_0$ be directed to – say – the positive side of F_1^{n-1}. Let A be an interior point of the boundary simplexes t_1^{n-1} of t_0^n that belong to F_1^{n-1} and let g be a ray that emanates from A and is directed into the interior of t_0^n; let $t_2^{n-1}, \ldots, t_{n+1}^{n-1}$ be the remaining $(n - 1)$-dimensional boundary simplexes of t_0^n, let $\bar{\mathfrak{P}}_0$ be the part of $\mathfrak{P}_0$ that belongs to it, M, the (possibly empty) set of points of g in which g will be cut out from the rays determined by the vectors of $\bar{\mathfrak{P}}_0$. A does not belong to M, since otherwise the ray of $\bar{\mathfrak{P}}_0$ that contains A would not be directed to the positive side of F_1^{n-1}. M is, however, closed; thus, there are points on g in the interior of t_0^n that do not belong to M; let B be such a point. We now next define the field $\mathfrak{U}_0$ to be constructed on the boundary of t_0^n from the $t_2^{n-1}, \ldots, t_{n+1}^{n-1}$ by the demand that these vectors all go through B. It then certainly fulfills conditions a), b), c) there if it is everywhere directed to the interior of t_0^n and has no point of coincidence at all with $\bar{\mathfrak{P}}_0$. We must now construct $\mathfrak{U}_0$ at the interior points of the simplexes t_1^{n-1}, on whose boundary, it is already established. If we consider $\mathfrak{U}_0$ as pointing to the positive side of F_1^{n-1} on this boundary and $\mathfrak{P}_0$ as pointing to the positive side of F_1^{n-1} on all of t_0^n then we can determine $\mathfrak{U}_0$ at the interior points of t_1^{n-1} by the following prescription: If P is an interior point of t_1^{n-1} that is different from A then let $\bar{P}$ be the intersection point of the ray AP with the boundary of t_1^{n-1}. Let $\mathfrak{p}(P)$, $\mathfrak{p}(\bar{P})$, $\mathfrak{u}(\bar{P})$ be the vectors of $\mathfrak{P}_0$ ($\mathfrak{U}_0$, resp.) attached to P ($\bar{P}$, resp.), $\mathfrak{q}(\bar{P})$, the projection of the vector $\mathfrak{p}(\bar{P})$ from the vector $\mathfrak{u}(\bar{P})$ onto E_1^{n-1} (i.e., as before, the intersection of with the half-plane spanned by $\mathfrak{u}(\bar{P})$, $\mathfrak{p}(\bar{P})$, and the vector $\bar{\mathfrak{u}}(\bar{P})$ that is diametrically opposite to $\mathfrak{u}(\bar{P})$), and let $\mathfrak{q}(P)$ be the vector that is attached to P and parallel to $\mathfrak{q}(\bar{P})$. The vector $\mathfrak{u}(P)$ to be defined shall now be the vector of the two-dimensional angle between 0 and π that is spanned by $\mathfrak{p}(P)$ and $\mathfrak{q}(P)$, this angle being divided up such that the angle ratio $\sphericalangle\{\mathfrak{p}(P), \mathfrak{u}(P)\} : \sphericalangle\{\mathfrak{u}(P), \mathfrak{q}(P)\}$ is equal to the product of the angle ratio $\sphericalangle\{\mathfrak{p}(\bar{P}), \mathfrak{u}(\bar{P})\} : \sphericalangle\{\mathfrak{u}(\bar{P}), \mathfrak{q}(\bar{P})\}$ and the line segment ratio $AP : \overline{AP}$; at A itself, one shall $\mathfrak{u}(A) = \mathfrak{p}(A)$. Moreover, the field $\mathfrak{U}_0$ that is defined on the entire boundary of t_0^n satisfies all requirements: It is continuous, everywhere directed into the interior, and has a single coincidence point A with $\mathfrak{P}_0$.

Case α is therefore dealt with, and we then go on to case β, by assuming that $\mathfrak{P}_0$ is analytic on the boundary of t_0^n. Let K^n be a solid ball that lies completely in the interior of t_0^n. There then exists a positive angle δ such that every angle is greater than δ whose vertex and one side belong to the boundary of t_0^n, while the other side contains a point of K^n. We divide the boundary simplexes t_2^{n-1}, …, t_{n+1}^{n-1} into sub-simplexes s_ρ^{n-1} that are small enough that that fluctuation of $\mathfrak{P}_0$ at each individual is smaller than δ; then, if *one* vector of $\mathfrak{P}_0$ that belongs to a point of s_ρ^{n-1} points to a point of K^n then *all* $\mathfrak{P}_0$-vectors of s_ρ^{n-1} point to the *interior* of t_0^n. The rays that are established by the vectors of $\mathfrak{P}_0$ that are attached to the (n – 2)-dimensional boundary simplexes s_σ^{n-2} of s_ρ^{n-1} define a finite number of a *analytic,* (n – 2)-*dimensional* hypersurface pieces; thus, there are certain points in K^n that do not lie on any hypersurface; let C be one such point. If we then define $\mathfrak{U}_0$ on the s_σ^{n-2} by the demand that the vectors $\mathfrak{u}(P)$ point to C then no coincidence point with $\mathfrak{P}_0$ is present there. We encounter this definition for those s_ρ^{n-1} in which no ray that belongs to $\mathfrak{P}_0$ points to a point of K^n; in the remaining s_ρ^{n-1} all vectors $\mathfrak{u}(P)$ point to the interior of t_0^n, and the same is true for the vectors of $\mathfrak{U}_0$ that are already attached to its boundary. Thus, in each one of them, by the procedure with which we treated the simplex in case α we can construct vectors $\mathfrak{u}(P)$ that go inward from them and which are continuously linked to the vectors of $\mathfrak{U}_0$ that are already present on the boundary of s_ρ^{n-1}, and coincide with the field $\mathfrak{P}_0$ at precisely one point in the interior of s_ρ^{n-1}.

Thus, case β is also dealt with, the validity of the lemma formulated at the end of the previous paragraph is shown, and Theorem I is proved completely.

§ 5.

Fixed points of small transformations and singularities of continuous vector fields in closed manifolds

We now make some applications of Theorem I and restrict ourselves exclusively to the case in which $C^n = M^n$ is a closed manifold (with or without boundary).

Let each point P of M^n be associated with a neighborhood $U(P)$ such that it will be entirely represented in each $E_{\mu^n}^n$ that is the image of the simplex neighborhood of the simplexes that contain P when one establishes a definite "distinguished neighborhood representation" $\mathfrak{A}^n$ of M^n – in the terminology of § 2; this condition is certainly fulfilled for sufficiently small neighborhoods $U(P)$. Now, let f be a single-valued and continuous map of M^n onto a point set that belongs to M^n and is "small" enough that P, as well as the image $f(P)$, belongs to the neighborhood $U(P)$; moreover, f has no fixed points on the boundary, in the event that M^n has a boundary. Then f is a "neighborhood transformation" relative to $\mathfrak{A}^n$ and generates a complex-continuous vector field whose singularities, which, since they are interior points of M^n, we may assume appear only in the interior of $T_{\mu^n}^n$ [20]), are identical with the fixed points of *f* in position and index. From Theorem I, it then follows that:

Theorem II. *The sum of the indices of the fixed points of a sufficiently small transformation of the closed manifold M^n into itself is, assuming that at most finitely many fixed points appear, equal to the Euler characteristic of M^n multiplied by* $(-1)^n$.

From this, one obtains:

Theorem IIa. *Any sufficiently small transformation of a manifold with an Euler characteristic that is different from* 0 *into itself possesses at least one fixed point.*

We now pose the question of whether there are then arbitrarily small transformations with at most finitely many fixed points in any M^n. One recognizes that this question can be answered affirmatively as follows – always while employing the notation of § 2: Let $T_1^n, \ldots, T_{\alpha^n}^n$ be the simplexes of $\mathfrak{A}^n$ and $E_1^n, \ldots, E_{\alpha^n}^n$, the elements that represent the simplex neighborhoods of $T_{\alpha^n}^n$. On the boundary of T_1^n, one defines a continuous field of non-vanishing vectors whose endpoints belong to E_1^n, and which also determines the lengths; by means of the affine maps that exist between the subsets of the various $E_{\mu^n}^n$ they correspond to vectors on certain boundary simplexes of certain of $T_1^n, \ldots, T_{\alpha^n}^n$. We attach these vectors to the points that they belong to in such a way that now a subset of the of the boundary simplexes of $T_1^n, \ldots, T_{\alpha^n}^n$ possess vectors. We now attach a field of vectors to the *entire* boundary of T_2^n whose endpoints lie in E_2^n and which possibly includes ones that are attached to certain boundary simplexes of T_2^n; that this attachment of vectors is always possible was shown in the paper "Abbildingsklassen *n*-dimensionaler Mannigfaltigkeiten" 6) (§ 5.2, 5.3). We then proceed for $n = 3, 4, \ldots, \alpha^n$ until the boundaries of all $T_{\mu^n}^n$ are completely possessed with vectors. We then choose a point P_{μ^n} in the interior of each $T_{\mu^n}^n$ and associate each point *P* of $T_{\mu^n}^n$ that is different from it with that vector PP' that is parallel to the vector of that boundary point $\bar{P}$ of $T_{\mu^n}^n$ onto which the P will be projected from P_{μ^n}, and whose length behaves in relation to the stated boundary vector as the line segment $P_{\mu^n}P$ does in relation to the line segment $P_{\mu^n}\bar{P}$; we associate the point *P* itself with a vanishing vector. In this way, a vector field with the singularities P_{μ^n} is defined. By the prescription that each point shall go to that point of the vector PP' that divides the line segment PP' in the ratio $t : 1 - t$, a neighborhood

[20]) To each representation $\mathfrak{A}^n$ of M^n there is a representation that is homeomorphic to it, in the sense of combinatorial topology – i.e., one that comes about by the decomposition and combination of simplexes – in which finitely many prescribed interior points of M^n will be represented by *interior* points of the *n*-dimensional simplexes.

transformation f_t is defined for each t between 0 and 1. The family of f_t converges uniformly to the identity when t approaches 0; each of these maps has the points P_{μ^n}, and only these points, as fixed points.

Thus, there are arbitrarily small transformations of M^n with finitely many fixed points. We infer a consequence from this: If M_1^n is a manifold that is homeomorphic M^n – i.e., one that can be mapped onto M^n in a one-to-one and continuous way – then a transformation with finitely many fixed points can be constructed in M^n such that each point moves so slightly from its starting point that this map is a neighborhood transformation, not only relative to a representation $\mathfrak{A}^n$ of M^n, but also relative to a representation $\mathfrak{A}_1^n$ of M_1^n. Now, since the index of a fixed point is a topological invariant of the transformation in question [4]), this yields, on the grounds of Theorem II, the following well-known:

Theorem III. *Homeomorphic manifolds have the same Euler characteristics.*

This theorem is one of the classical and simplest theorems of *combinatorial* topology 5), in which one regards two manifolds as homeomorphic when their representations possess subdivisions that are isomorphic to each other (cf., § 1). The proof carried out above is valid for topology in the broader sense in which one already designates two manifolds as being homeomorphic when they can be mapped onto each other in a one-to-one and continuous way. Theorem III has also been proved by Alexander [21]) from this general viewpoint.

We now pursue further the question raised above of the existence of arbitrarily small transformations with finitely many fixed points: Is it possible to give an arbitrarily small transformation that possesses fixed points with prescribed indices $q_1, \ldots, q_m$ at the prescribed interior locations $Q_1, \ldots, Q_m$ $(m \geq 0)$, with only the condition that its sum be equal to the characteristic of M^n multiplied by $(-1)^n$? This is, in fact, always possible [22]). The points $P_1, \ldots, P_{\alpha^n}, Q_1, \ldots, Q_m$ may then be included in an element F that belongs to M^n [23]), and in it, a further element F_1 can be given that includes the stated points in the interior. We now choose – with the notation above – t sufficiently small that that the image of F_1 under f_t lies completely in F. Let F' be a topological image of F that belongs to ordinary space, F_1', the image of F_1 in it, and let $P_1', \ldots, P_{\alpha^n}', Q_1', \ldots, Q_m'$ be the images of $P_1, \ldots, P_{\alpha^n}, Q_1, \ldots, Q_m$, resp. The map f_t corresponds to a map f_t' of F_1' onto a subset of F'; its fixed points are $P_1', \ldots, P_{\alpha^n}'$, the associated indices are, due to their topological invariance, the same as the corresponding indices under the map f_t. The vectors that point from the boundary points of F_1' to the image points under the map f_t'

[21]) J. W. Alexander II, A proof of the invariance of certain constants of Analysis Situs, Trans. of the Am. Math. Soc. **16** (1915). – There, the invariance of the Betti numbers was proved for the topology in the broader sense. Since the Euler characteristic is expressible through the Betti numbers (cf., e.g., Tietze, loc. cit.) Theorem III is thus proved; cf., also H. Kneser, loc. cit., footnote 2 on pp. 12.

[22]) We assume that $n \geq 2$.

[23]) See the paper cited in footnote [6]) of § 2.

thus define a map of the boundary of F_1' onto the direction sphere whose degree is $(-1)^n \cdot c$. On the grounds of the solubility [22]) of a "boundary-value problem for vector distributions" (see the paper on mapping classes cited above in [6]), § 5.4), we can, since one also has $\sum_{\mu=1}^{m} q_\mu = (-1)^n c$, extend these boundary vectors to a continuous vector field that is defined in all of F_1' in such a way that its vectors vanish at the Q_μ' ($\mu = 1, \ldots, m$), and only there, and that the singularities of the direction field at these points possess the indices q_μ. Above, all, we can choose the vectors of this field to be so small that their endpoints all lie in the interior of F'. By the prescription that each point of F_1' shall go to the endpoint of the vector that is attached to it, F_1' will be mapped to a subset of F' in such a way that this map g' agrees with f_t' on the boundary and has fixed points at the Q_μ' with the indices q_μ, but is fixed-point free at the remaining points. The map g' corresponds to an analogous map g in F_1. If we now replace f_t with g in the interior of F_1, while we leave f_t unchanged in the exterior and on the boundary of F_1, then we have constructed a map with the desired properties. We have thus proved:

Theorem IV. *If* $Q_1, \ldots, Q_m$ ($m \geq 0$) *are arbitrary interior points if the manifold* M^n *and* $q_1, \ldots, q_m$ *are arbitrary whole numbers whose sum is equal to the characteristic of* M^n *multiplied by* $(-1)^n$ *then there are arbitrarily small transformations of* M^n *into itself that possess fixed points at the* Q_μ ($\mu = 1, \ldots, m$) *with the indices* q_μ, *but are fixed-point-free at the remaining ones* [22]).

A special case of this theorem is:

Theorem IVa. *Any manifold whose characteristic is* 0 *admits arbitrarily small fixed-point-free transformations into itself.*

Since the characteristic is 0 for any *boundaryless* closed manifold of odd dimension, one has, in particular:

Theorem IVb. *Any closed, boundaryless manifold of odd dimension admits arbitrarily small fixed-point-free transformations into itself.*

We now consider vector field that are continuous in the ordinary sense: In a neighborhood $U(P)$ of each point P of M^n, let a Cartesian coordinate system on a set be distinguished in such a way that the coordinates of any two coordinates systems (belonging to the same or different points) go over to each other on a common piece by continuously differentiable transformations; boundary manifolds of M^n shall be continuously differentiable in these coordinate systems. In order for the examination of the indices of such vector fields to lead directly back to the consideration of our complex-continuous vector fields, we must possess a representation of M^n in which the boundaries of each individual simplex $T^n_{\mu^n}$ also belong to a planar space relative to one of the distinguished coordinate systems of M^n. The existence of such a representation is, in itself, self-explanatory. We restrict ourselves, in order to avoid the difficulty thus hinted at, to the special case in which M^n is a Riemannian manifold; i.e., at each point, a symmetric matrix (g_{ik}) $(i, k = 1, \dots, n)$ is given that depends continuously on the point relative to any distinguished coordinate system, and whose associated quadratic form $\sum_{i,k=1}^{n} g_{ik}\, dx_i\, dx_k = ds^2$ is positive definite, and its values do not change when one goes from one distinguished coordinate system to another one. In any such Riemannian manifold, each sufficiently small vector now corresponds to a displacement of the point to which it is attached, and each sufficiently small displacement to a vector at the point in question. With that, it follows from Theorems II and IV:

Theorem V. *The sum of the indices of a vector field in a Riemannian manifold is equal to the characteristic multiplied by* $(-1)^n$; *one can always* [22]) *construct a vector field with prescribed singularities and indices, as long as their sum is equal to the stated number. A singularity-free vector exists when and only when the characteristic is* 0; *in particular, such a vector field can be attached to any boundaryless, closed manifold of odd dimension.*

Among the Riemannian manifolds that are thus treated are included, e.g., the ones that are embedded in the $(n + k)$-dimensional Euclidian space $(k \geq 0)$ in a differentiable way. Thus, the case $k = 0$ includes the submanifolds of space that are bounded by finitely many continuously differentiable $(n-1)$-dimensional closed, boundaryless hypersurfaces. Moreover, it includes the Clifford-Klein manifolds, as well as many others in which a Riemannian metric may be defined. As an example, let, perhaps, the complex projective space Z_k be mentioned; i.e., the totality of all ratios $z_0 : \dots : z_k$ of complex, not all vanishing numbers. In it, a metric may be defined [24]) with the line element:

$$ds^2 = \frac{1}{\left(\sum_{i=0}^{k} z_i \bar{z}_i\right)^2} \begin{vmatrix} \sum_{i=0}^{k} z_i \cdot \bar{z}_i & \sum_{i=0}^{k} z_i \cdot d\bar{z}_i \\ \sum_{i=0}^{k} dz_i \cdot \bar{z}_i & \sum_{i=0}^{k} dz_i \cdot d\bar{z}_i \end{vmatrix}.$$

We linger for a moment on the case of submanifolds of n-dimensional space that bounded by closed hypersurfaces; let M^n be bounded by closed, boundaryless hypersurface M^{n-1}. The vectors of a field of the type considered belonging to all such M^n are then, everywhere on M^{n-1}, directed either into the interior of M^n or tangentially to M^{n-1}. The map of M^{n-1} that these vectors provides has degree $(-1)^n \cdot c$ on the direction sphere, when c is again the characteristic of M^n. The map that is diametrically opposite to this map, which will be mediated by a field of vectors that are nowhere directed into the

[24]) In the paper cited in [4]) of § 5, I have given, in a simple manner, an arbitrarily small transformation (a vector field, resp.) in Z_k with the index sum $k + 1$, and, in addition, in a somewhat circumstantial way, showed that the characteristic has the value $k + 1$; this determination of the characteristic is, moreover, superfluous, in the basis of Theorem V.

interior of M^n thus has the degree $(-1)^n \cdot (-1)^n \cdot c = c$. This degree is the "curvatura integra" of M^n [4]). With that, we have proved:

Theorem VI. *The curvatura integra of a continuously differentiable Jordan hypersurface that lies in n-dimensional space and is bounded by an n-dimensional manifold is equal to the characteristic of the bounding manifold.*

I have previously only proved this theorem for the special case that the bounding manifold is an *element.* Furthermore, I obtained in the stated place: The $2k$-dimensional closed, not necessarily Jordan, continuously differentiable hypersurface m of the $(2k + 1)$-dimensional Euclidian space is a "model" for the two-sided, closed, boundaryless manifolds M^{2k}; its curvatura integra $C(m)$ is then a topological invariant of M^{2k}, and the index sum of the singularities of each vector field tangential to m is, assuming that only finitely many singularities are present, equal to $2C(m)$. From this, it follows, moreover:

Theorem VII. *The curvatura integra of a closed, not necessarily Jordan, continuously differentiable hypersurface in* $(2k + 1)$-*dimensional space that is a model of a two-sided, closed, boundaryless manifold* M^{2k} *is equal to one half the characteristic of* M^{2k}.

From this, one further deduces (cf., the many papers cited earlier), since the curvatura integra is always a whole number:

Theorem VIII. *A closed, boundaryless, two-sided manifold* M^n *with odd characteristic possesses no continuously differentiable hypersurface in* $(n + 1)$-*dimensional Euclidian space as a model, not even when one allows self-intersections.*

The simplest example of such an M^n is the four-dimensional manifold that is defined to be the 'complex projective plane" Z_2 (cf., footnote 24).

An analogue of Theorem VIII is the fact that a $2k$-dimensional, closed manifold M^{2k} that defines the complete boundary of a closed M^{2k+1} always has an *odd* characteristic, namely, twice the characteristic of M^{2k+1} [25]). An M^{2k} with odd characteristic – thus, e.g., Z_2 – can therefore never be embedded in a simply-connected, not necessarily homeomorphic to ordinary space, closed $(2k + 1)$-dimensional space R^{2k+1} – at least, not in the sense of combinatorial topology, i.e., such that it will be represented by a subset of the boundary complex of a representation of R^{2k+1} – since they will then define the boundary of each of the two subsets into which they must divide R^{2k+1} [26]).

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[25]) This follows from the fact that the boundaryless $(2k+1)$-dimensional manifold that comes about under the identification of corresponding boundary points of two exemplars of M^{2k+1} possesses the characteristic 0; cf., Dyck, Beiträge zur Analysis Situs II, Math. Ann. **37** (1890).

[26]) H. Kneser, Ein topologischer Zerlegungssatz, Koninkl. Akad. v. Wetenschapen te Amsterdam Proc. **27**, Sept. 1924.

Appendix

I will draw your attention to the fact that the concept of "complex-continuous vector field," upon the use of which the results of the paper above rest essentially, is not defined sufficiently clearly and has given rise to misunderstanding. I thus formulate this definition again, but more thoroughly than before:

Let $\mathfrak{A}^n$ be a reduced, affine representation of the complex C^n. An association $\mathfrak{B}$ of vectors $\mathfrak{v}(P)$ with the points P of $\mathfrak{A}$ is called a C^n (relative to $\mathfrak{A}^n$) "complex-continuous vector field" when the following conditions are fulfilled:

A. $\mathfrak{B}$ is single-valued and continuous in the interior *and on the boundary* of each individual $T^n_{\mu^n}$ $[\mu^n = 1, \ldots, \beta^n]$, except for at most finitely many points that lie in the interior.

B. Let P_0 be a boundary point on $T^n_{\mu^n_0}$, and let T^n_0 $[1 \leq k \leq n]$ be *any* boundary simplex – *not necessarily one of lowest dimension* – that P_0 belongs to. Let T^{n-k}_ρ $[\rho = 1, \ldots, r]$ be the boundary simplexes of the other $T^n_{\mu^n_0}$ that are to be considered to be identical with T^{n-k}_0 in C^n, P_ρ, the points of T^{n-k}_ρ that corresponding to P_0, and $(W^n_k)_\rho$ $[\rho = 1, \ldots, r]$, the k-fold angle whose vertex is the planar space E^{n-k}_ρ to which T^{n-k}_ρ belongs.

Then one of the following two cases enters in:

I. (Main case) Of the $r + 1$ vectors $\mathfrak{v}(P_\rho)$, precisely one of them points into the interior of its $(W^n_k)_\rho$, while all others are directed to the exterior of its $(W^n_k)_\rho$.

II. (Boundary case) One of the vectors $\mathfrak{v}(P_\rho)$ – say, $\mathfrak{v}(P_0)$ – belongs to the boundary of its $(W^n_k)_0$. Thus, among the vectors $\mathfrak{v}^*$ that correspond to the vector $\mathfrak{v}(P_0)$ under the affine and transitive association of the vectors that exists between the boundary spaces, there can be one or more that likewise belongs to $\mathfrak{B}$. *However, one does not need for all of these* $\mathfrak{v}^*$ *to belong to* $\mathfrak{B}$ – in contrast to the special case of the vector field on a manifold that was continuous in the ordinary sense that was given at the end of § 2. All remaining vectors $\mathfrak{v}(P_\rho)$ that are not vectors of $\mathfrak{v}^*$ point to the exteriors of their $(W^n_k)_\rho$.

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