"Ueber die Darstellung der Curven durch Krümmung und Torsion," J. f. d. reine u. angew. Math. **60** (1862), 182-187.

On the representation of curves by curvature and torsion

(By **R. Hoppe**)

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If the curvature and torsion of a curve are given as functions of the arc length then its form will be determined completely, and independently of its position. In order to find its equations in rectangular coordinates, one must perform six integrations, corresponding to the six constants that determine its position in space. That is easy in two cases:

1. When the torsion is equal to zero.

2. When the torsion has a constant ratio to the curvature.

In addition, there are still many simple relations between both variables that the integration admits. Although the general solution of the problem is probably impossible, it seems that there are some reductions to which it is susceptible that are interesting in their own right, namely, to the extent that it will become obvious from those reductions what their integrability depends upon in some special cases. I shall allow myself to make use of the terminology that I explained in my last article (v. **58**, pp. 374).

I.

If τ and ϑ denote the curvature and torsion angles of a curve (i.e., the integrals of the contingency angles between the tangents and the osculating planes), while *u*, *v*, *w* denote the cosines of the angles that the tangent, pole line, and principal normal define with any axis, then:

(1) $$u^2 + v^2 + w^2 = 1$$

due to the perpendicular relationships between those lines, and from known formulas, one will have:

$$\partial u = w\,\partial\tau, \qquad \partial v = -\,w\,\partial\vartheta, \quad \partial w = v\,\partial\vartheta - u\,\partial\tau,$$

from which, it will emerge that:

(2) $$v = \frac{\partial\tau}{\partial\vartheta}\left(u + \frac{\partial^2 u}{\partial\tau^2}\right), \qquad\qquad u = \frac{\partial\vartheta}{\partial\tau}\left(v + \frac{\partial^2 v}{\partial\vartheta^2}\right).$$

Depending upon whether one now eliminates v or u, along with w, from equation (1), one will get:

$$\left(\frac{\partial\tau}{\partial\vartheta}\right)^2\left(u+\frac{\partial^2 u}{\partial\tau^2}\right)^2+u^2+\left(\frac{\partial u}{\partial\tau}\right)^2=1, \tag{3}$$

or

$$\left(\frac{\partial\vartheta}{\partial\tau}\right)^2\left(v+\frac{\partial^2 v}{\partial\vartheta^2}\right)^2+v^2+\left(\frac{\partial v}{\partial\vartheta}\right)^2=1, \tag{4}$$

resp. Therefore, any curve will have a reciprocal relationship to a second one, in general, in such a way that the curvature angle of one of them will be the torsion angle of the other one, and the tangent to one of them will be a parallel to the pole line of the other one. The integration of one of the foregoing equations then implies the two cases of given functions:

$$\varphi(\tau,\ \vartheta) = 0, \quad \varphi(\vartheta,\ \tau) = 0$$

at the same time. For that reason, the following discussion might be restricted to a consideration of equation (3). Accordingly, τ shall be considered to be an independent variable, and the prime shall denote the differential quotient with respect to it.

After differentiating once and dropping the factor $u + u''$, equation (3) will give:

$$u''' + (1 + \vartheta'^2)\, u' - (u + u'') = 0. \tag{5}$$

The complete integral of this linear equation has the form:

$$u = a\, u_1 + b\, u_2 + c\, u_3\,,$$

but the constants are subject to a relation that one obtains from introducing them into equation (3). If one then chooses three systems of the a, b, c such that sum of the squares of the three corresponding values will be $u = 1$ then they will represent the cosines of the direction angles between the tangent and three rectangular axes, and will give the coordinates as functions of τ after multiplying by the differential of the arc length, expressed in terms of τ, and a subsequent integration.

Since the solution to the problem is then complete without the arc length entering into the calculation, upon integrating equation (3) for any special relation between τ and ϑ, one will get a class of curves whose common feature is just that relation, while the function that expresses the arc length in terms of τ arbitrarily will require that their forms must be different. The deviation can exist here only in the extension of the individual arc length elements, such that each class will easily appear to be related.

II.

For the further transformation of the differential equation, one sets:

$$u = q\int \frac{\partial\tau}{q^2}\sqrt{q^2+q'^2}\ ,$$

and upon solving this for q, it will emerge that:

$$q = \sqrt{1-u^2}\, e^{i\int \frac{\partial\tau}{1-u^2}\sqrt{1-u^2-u'^2}}\ .$$

After dropping the factor:

$$\left(\frac{q'}{q\sqrt{q^2+q'^2}} + \int \frac{\partial\tau}{q^2}\sqrt{q^2+q'^2}\right)^2,$$

equation (3) will go to:

(6) $$\left(\frac{q+q''}{\vartheta'}\right)^2 + q^2 + q'^2 = 0.$$

One can also obtain that equation as a special integral of equation (5) by setting the integration constant equal to 0 [which must have the value of 1 according to equation (3)] and setting the corresponding value of u equal to q. The foregoing derivation shows just what the relationship is between the real roots of equation (3) and the useless ones in equation (5).

Furthermore, let:

$$q = r\, e^{-\int \frac{r\,d\tau}{4r'}}\ ,$$

from which, one has inversely:

$$r = \sqrt{q}\, e^{-\frac{1}{2}\int \frac{dq}{q}\sqrt{q^2+q'^2}}\ .$$

After dropping the factor:

$$q^2\left(1+\frac{r^2}{4r'^2}\right)^2,$$

equation (6) will go to:

$$\frac{1}{\vartheta'^2}\left(\frac{r''}{r}+\frac{1}{4}\right)^2 + \frac{r'^2}{r^2} = 0,$$

from which:

(7) $$r'' \pm \vartheta' r' + \tfrac{1}{4} r = 0.$$

The possible transformations of this equation are known. One gets the simplest form by the substitution:

$$r = \frac{p}{\sigma}, \qquad \tau = 2\int \frac{\partial\sigma}{\sigma^2} e^{\pm i\vartheta}\ ,$$

namely, the following one:

$$\frac{\partial^2 p}{\partial\sigma^2} + \frac{p}{\sigma^4} e^{\pm 2i\vartheta} = 0.$$

III.

If a special solution of equation (3) is known then the complete solution to the problem will be guaranteed by reducing it to a second-order linear equation. One can then arrive at it along a much shorter path than by means of equation (7). Namely, let:

$$u = \cos\varphi\cos\psi, \qquad v = \sin\varphi\cos\psi, \qquad w = \sin\psi$$

be the cosines of the direction angle of the tangent. If w is further known as a function of τ, and one sets:

$$\psi' = \sin\omega$$

then ψ and ω will also be known. One finds the following values for the cosines of the direction angle of the principal normal by a differentiation:

$$\begin{aligned} u' &= -\varphi'\sin\varphi\cos\psi - \cos\varphi\sin\psi\sin\omega, \\ v' &= \varphi'\cos\varphi\cos\psi - \sin\varphi\sin\psi\sin\omega, \\ w' &= \cos\psi\sin\omega, \end{aligned}$$

whose square-sum is:

$$1 = \varphi'^2\cos^2\psi + \sin^2\omega.$$

The value:

$$\varphi = \int \frac{\cos\omega}{\cos\psi}\partial\tau$$

that one gets from this permits one to know u, v, and consequently:

$$x = \int u\,\partial s, \qquad y = \int v\,\partial s, \qquad z = \int w\,\partial s,$$

as functions of τ. The missing integration constants, two of which will be contained in complete values of ψ and one of which will be included in φ, are easy to derive by an orthogonal substitution.

The expressions for u, v that one gets by way of equation (7) are much more complicated than the ones that were just derived. Moreover, I would not like to verify the identity of those two sets of expressions in some other way, for the sake of brevity.

IV.

I shall present the solutions for four of the simplest cases in their simplest special form. Let the cosines of the direction angle of the pole line be denoted by l, m, n.

1. If $\vartheta' = 0$ then, as is known:

$$u = \cos\tau, \qquad v = \sin\tau, \qquad w = 0,$$

$$u' = -\sin\tau, \qquad v' = \sin\tau, \qquad w' = 0,$$

$$l = 0, \qquad m = 0, \qquad n = 1.$$

If one determines the constant factor of the integral according to equation (3) then that will yield:

$$u = \cos\lambda\cos\frac{\tau}{\cos\lambda}, \qquad v = \cos\lambda\sin\tau, \qquad w = \sin\lambda,$$

$$u' = -\sin\frac{\tau}{\cos\lambda}, \qquad v' = \cos\frac{\tau}{\cos\lambda}, \qquad w' = 0,$$

$$l = -\sin\lambda\cos\frac{\tau}{\cos\lambda}, \quad m = -\sin\lambda\sin\frac{\tau}{\cos\lambda}, \quad n = \cos\lambda.$$

3. If one sets $w = \tau/\alpha$ then that will imply the relation:

$$\tau^2 + \vartheta^2 = \alpha^2 - 1,$$

which will be fulfilled by the values:

$$\tau = \sqrt{\alpha^2 - 1}\sin\lambda, \qquad \vartheta = -\sqrt{\alpha^2 - 1}\cos\lambda,$$

and when one uses the method that was described in the previous section, one will find the following expressions:

$$u = \frac{1}{\alpha}\sin\alpha\lambda\sin\lambda + \cos\alpha\lambda\cos\lambda,$$

$$v = -\frac{1}{\alpha}\cos\alpha\lambda\sin\lambda + \sin\alpha\lambda\cos\lambda,$$

$$w = \frac{\sqrt{\alpha^2 - 1}}{\alpha}\sin\lambda,$$

$$u' = -\frac{\sqrt{\alpha^2 - 1}}{\alpha}\sin\alpha\lambda, \qquad v' = -\frac{\sqrt{\alpha^2 - 1}}{\alpha}\cos\alpha\lambda, \qquad w' = \frac{1}{\alpha},$$

$$l = \frac{1}{\alpha}\sin\alpha\lambda\cos\lambda - \cos\alpha\lambda\sin\lambda,$$

$$m = -\frac{1}{\alpha}\cos\alpha\lambda\cos\lambda - \sin\alpha\lambda\sin\lambda,$$

$$n = \frac{\sqrt{\alpha^2 - 1}}{\alpha}\cos\lambda\,.$$

4. If one sets $\omega = \psi$ then one will have:

$$\varphi = \tau, \qquad \vartheta = 2\psi, \qquad \tan\frac{\vartheta}{4} = e^{\tau},$$

so:

$$u = \cos\tau\cos\frac{\vartheta}{2}, \qquad v = \sin\tau\cos\frac{\vartheta}{2}, \qquad w = \sin\frac{\vartheta}{2},$$

$$u' = -\sin\tau\cos\frac{\vartheta}{2} - \cos\tau\sin^2\frac{\vartheta}{2},$$

$$v' = \cos\tau\cos\frac{\vartheta}{2} - \sin\tau\sin^2\frac{\vartheta}{2},$$

$$w' = \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2},$$

$$l = \left(\sin\tau - \cos\tau\cos\frac{\vartheta}{2}\right)\sin\frac{\vartheta}{2},$$

$$m = -\left(\cos\tau + \sin\tau\cos\frac{\vartheta}{2}\right)\sin\frac{\vartheta}{2},$$

$$n = \cos\frac{\vartheta}{2}.$$

V.

Geometrically, the problem can be reduced in the following way: If one describes a sphere around the coordinate origin with the linear unit as its radius and draws, along with the tangent to the curve, one the one hand, its pole line and on the other, two parallel radii then they will describe two curves on the spherical surface that are parallel to each other, because the running point of one of them will lie on the corresponding normal plane of the other one. By switching the curvature and torsion of the original curve, each of those spherical curves will go to the other one.

If one distinguishes the element of that spherical curve by the index 1 then its equations will be:

$$x_1 = u, \qquad y_1 = v, \qquad z_1 = w,$$

and differentiating this will give:

$$u_1\,\partial s_1 = u'\partial\tau, \qquad v_1\,\partial s_1 = v'\partial\tau, \qquad w_1\,\partial s_1 = w'\partial\tau.$$

Forming the sum of the squares will show that:

$$(8) \qquad \begin{cases} s_1 = \tau, \\ u_1 = u', \quad v_1 = v', \quad w_1 = w'. \end{cases}$$

A second differentiation yields:

$$u_1'\,\partial\tau_1 = (l \tan\lambda - u)\,\partial\tau,$$

along with two analogous equations, and the sum of their squares will reveal that:

$$(9) \qquad \begin{cases} \cos\lambda\,\partial\tau_1 = \partial\tau, \\ u_1' = l\sin\lambda - u\cos\lambda, \quad \text{etc.}, \end{cases}$$

and a third differentiation, while referring to equations (9), will give:

$$l_1\,\vartheta_1' - u_1 = \lambda'\,(l\cos\lambda + u\sin\lambda)\cos\lambda - u',$$

or, from equations (8):

$$l_1\,\vartheta_1' = \lambda'\,(l\cos\lambda + u\sin\lambda)\cos\lambda,$$

along with two analogous equations, and summing their squares will yield:

$$\vartheta_1' = \lambda'\cos\lambda,$$

or, from equation (9):

$$\partial\vartheta_1 = \partial\lambda.$$

Moreover, equation (9) can also be written:

$$s_1' = \cos\lambda,$$

which is a quantity that expresses the curvature radius of the spherical curve.

That implies that the arc length of the spherical curve represents the curvature angle, it curvature radius represents the cosine of the curvature width, and its torsion angle represents the curvature width of the original curve itself. Now, if ϑ also given in terms of τ by:

$$\lambda = \arctan\,\vartheta'$$

then the curvature radius of the spherical curve will also be known as a function of its arc length. The problem that was posed initially will then come down to this one: Calculate a spherical curve from the relation between curvature and arc length.

Berlin, 5 January 1861.