On the curvature of non-holonomic manifolds

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When one studies non-holonomic manifolds, one confirms that the usual method for arriving at curvature is no longer applicable to a $X_n^m$, due to the impossibility of constructing an infinitely-small, closed parallelogram (1). In the present Note, I propose to generalize the usual method for a $X_n^m$ by replacing the closed parallelogram with an infinitely-small closed cycle that is composed of an arbitrary parallelogram that is situated in $X_n^m$ and the vector that joins its final point to its initial point, in such a way that one returns to the starting point.

Consider a $X_n^m$ that is embedded in a $X_n$ with a local $(n – m)$-direction and let $x^j (\lambda, \mu, \nu = 1, 2, ..., n)$ be the holonomic parameters of $X_n$. If one introduces the non-holonomic parameters $x^k (i, j, k = 1, 2, ..., n)$ (2):

$$dx^k = A^k_\lambda dx^\lambda, \quad dx^\lambda = A^\lambda_k dx^k,$$

and if one denotes:

$$\Delta_{12} = \frac{d}{dx^i} \frac{d}{dx^j}$$

then one will have:

$$\Delta x^k = \Pi^k_y dx^i dx^j + A^k_\lambda \Delta x^\lambda,$$

in which:

$$\Pi^k_y = 2 A^k_\lambda \partial_i (A^\lambda_j) + \left( \partial_i = A^\lambda_i \frac{\partial}{\partial x^\lambda} \right),$$

in which the choice of the parameters $x^k$ and the operation $\Delta$ is completely arbitrary. Now, one can choose the $dx^k$ in such a manner (3) that the displacements in the $X_n^m$ are coupled by the $n – m$ conditions:


(2) J.-A. SCHOUTEN, loc. cit.

\[ dx^r = 0 \quad (r = m + 1, \ldots, n), \]

and that the other \( m \) differentials \( dx^\alpha (a, b, c, d, e, f = 1, 2, \ldots, m) \) are situated in \( X^m_n \).

Having said that, we shall suppose that the displacements \( d_1, d_2 \) that intervene in the operation are placed in \( X^m_n \) in such a way that:

\[(2) \quad d_1 x^r = 0, \quad d_2 x^r = 0, \quad \Delta x^r = 0.\]

Hence, the two displacements \( d_1, d_2 \) generate a parallelogram in \( X^m_n \) whose final point can agree with its initial point by displacing along the vector:

\[- \Delta x^j = \Delta x^j.\]

It is precisely the vector that completes the parallelogram considered to a closed cycle to which we shall appeal for our definition of curvature. Finally, if we write:

\[- \Lambda^j_{12} \Delta x^j = - \Lambda^k_{12} = \Lambda^k_{21}\]

then, by virtue of (2), the equations (1) will become:

\[(3) \quad \Delta x^j = \Pi^j_1 d x^b d x^d - \Delta^r, \quad \Delta' = \Pi_{bd}^j d x^b d x^d.\]

Now suppose that one introduces an arbitrary linear connection in \( X_n \) that is defined by the coefficients \( \Gamma_\mu^\nu, \Gamma'^\nu_\mu \). If one denotes the coefficients of that connection by \( \Lambda^j_\nu, \Lambda'^j_\nu \), when they expressed by means of the non-holonomic parameters \( x^k \) (1) – i.e., if one sets:

\[(4) \quad \nabla_i v_k = \partial_i v_k + \Lambda^k_j v^j d x^i, \quad \nabla_i w_k = \partial_i w_k + \Lambda'^k_j w^j d x^i,\]

then the non-holonomic connection that is induced (2) in the \( X^m_n \) by the given connection will be defined the coefficients \( \Lambda^c_{ab}, \Lambda'^c_{ab} \) (3).

Having said that, we propose to calculate the change \( D_{12} v^r \) that a vector \( v^r (v^r = 0) \) that is situated in \( X^m_n \) experiences during a circulation around our closed cycle. If the symbol

\[(1) \quad \text{HORÁK, loc. cit.}\]

\[(2) \quad \text{The covariant derivative of an affinor in } X^m_n \text{ is equal to the } X^m_n \text{-component of the covariant derivative in } X_n \text{.; see Schouten (loc. cit.).}\]

\[(3) \quad \text{See my Czech paper: “On a generalization of the notion of manifold,” Publication of the Science Faculty at the University of Masaryk, Brno, no. 86, 1927, pp. 2.}\]
$D v^c$ denotes the change that the same vector experiences during displacement along the parallelogram that is generated by $d x^b_1, d x^b_2$ then one obtains the total change $D v^c$ by adding to $D v^c$ the change that the vector $v^c$ experiences upon returning from the final point of the parallelogram to its initial point along the vector $\Delta^k$. One will obviously have:

$$D v^c = D v^c + \nabla_k v^c \Delta^k$$

then, from which one can easily infer by calculation and taking (3) and (4) into account:

$$D v^c = R^c_{bda} v^a_1 \frac{d x^b_1}{d x^d_1},$$

$$R^c_{bda} = 2 \partial_{[d} \Lambda^c_{\mu\rho]} + 2 \Lambda^c_{\mu[j} \Lambda^j_{\rho\rho]} + \Lambda^c_{\mu k} \Pi^k_{bd}.$$

We have then arrived at a definition of the affinor (5) as the quantity of curvature in $\mathcal{X}_n^m$. By the same reasoning, when applied to a covariant vector $w_a$, we will arrive at the equation:

$$D w_a = -R^c_{bda} w_c \frac{d x^b_1}{d x^d_1},$$

in which the quantity $R^c_{bda}$ results from (5) by replacing the $\Lambda^c_{ab}$ with $\Lambda^c_{ab}$. If $\mathcal{X}_n$ becomes an $\mathcal{A}_n$ then (5) will reduce to the affinor that Schouten (1) defined to be the curvature of the $\mathcal{A}_n^m$, and for a holonomic manifold, the $R^c_{bda}$, $R^c_{bda}'$ will take the form that was given by the author (2).

(1) Loc. cit.
(2) Loc. cit.